

Online Appendix to “Attention Oligopoly” by Andrea Prat and Tommaso Valletti

Summary

The model used for the main results is highly stylized. This appendix probes its robustness in three ways. First, it lists all explicit and implicit assumptions. Second, it relaxes one of the key assumptions – ads are sold through a second-price auction – and replaces it with two more general approaches. In the first, the platforms can use any take-it-or-leave-it offers. In the second approach, the offer stage is modeled as an abstract transfer utility cooperative game. Third, the appendix develops a general model that removes the following assumptions: (i) There is one incumbent only; (ii) There is one entrant only; (iii) Each platform sells one ad only; (iv) Being exposed to one ad provides the consumer with perfect information.

A1. Discussion of Modeling Choices

We discuss below the major explicit and implicit assumptions that we have made:

- *Ad Targetability.* The model assumes attention brokers have knowledge about their users and are able to understand what product k a particular consumer is interested in. This assumption is meant to capture an idealized situation where digital platforms collect large amounts of browsing data and process it through machine learning, perhaps the limit point of the long-term trajectory of technological progress, as described by Larry Page, one of the founders of Google, in 2000: “Artificial intelligence would be the ultimate version of Google. The ultimate search engine that would understand everything on the web. It would understand exactly what you wanted, and it would give you the right thing. We’re nowhere near doing that now. However, we can get incrementally closer to that, and that is basically what we work on.”¹

This assumption brings two major advantages. First, it makes the model tractable: without it, we would have asymmetric information between platforms and/or sellers which would make the analysis laborious and opaque. Second, we see some value in characterizing Larry Page’s asymptotic scenario, especially in a fast changing world. Future research could revisit our set-up and extend the analysis to a world where platforms have asymmetric information.

- *Individual Pricing.* Firms’ expected profits (π_E, π_1, π_2) and consumers’ expected utilities (u_1 and u_2) are in reduced form. This is consistent with situations where firms can target prices to

¹<http://www.artificialbrains.com/google>

individual consumers or situations where prices are uniform. In the latter case, both products are available at a fixed price and the only potential effect of the ad is to inform the consumer about the entrant. If that happens, the expected profit of the incumbent goes down because she is less likely to make a sale, the expected profit of the entrant goes up because he is more likely to make a sale, and the expected utility of the consumer goes up because his consideration set is larger. In the former case, the retail product firms may be able to tailor prices to individual consumers (e.g., product firms can engage in first-degree price discrimination). This corresponds, for instance, to the price discrimination case analyzed by Armstrong and Vickers (2019). The assumptions made above would hold a fortiori, because the incumbent would raise her price when she has less competition.

- *Knowledge about the Set of Platforms.* The results in Sections 3 and 4 do not require the platforms to know whether a specific consumer is using other platforms. This is because the mechanism is given: a second-price auction. Where that knowledge matters is for firms that participate in the auctions and in Section 5 when we discuss the effect of a merger. We do not know how prevalent that is true in practice, though we observe that Facebook acquired in 2013 the data analytics firm Onavo and then used its analytics platform to monitor competitors (including individual usage of several apps). The stream of data about app traffic obtained via Onavo, influenced Facebook to make various business decisions and acquisitions, including its acquisition of the photo-sharing app Instagram and messaging platform WhatsApp that were exploding in popularity. In 2021, Facebook announced the purchase of Giphy, another photo app with a database of GIFs. Because Giphy is integrated into other platforms (e.g., Twitter, TikTok, Signal, Slack) and other prominent messaging, productivity and social services, Giphy’s data could provide Facebook with similar market intelligence that Onavo did for years.
- *One Incumbent, One Entrant.* The model can also be extended beyond the assumption that there is one incumbent and one entrant; see Section A3 below. The pre-emption condition is unchanged, but it is more challenging to provide a closed-form characterization of the competitive equilibrium because an equilibrium in pure strategies may fail to exist and there is no general characterization of mixed-strategy equilibria outside specific cases (Szentes and Rosenthal, 2003).
- *One Ad per Platform.* We assume that each platform has only one ad to sell. Section A3 analyzes this extension.
- *Full Informativeness of Ads.* Seeing one ad – whether from a generalist channel or a digital platform – is sufficient to become aware of the entrant’s product. One possible extension of this paper is to assume some degree of inattention on the part of consumers, so that exposure to an ad does not guarantee knowledge. This would create scope for the platforms to put more than

one ad for sale. Section A3 relaxes this assumption.

- *Second-Price Auctions.* We assume that each platform sells its ad through a second-price auction. In practice, platforms employ n -price mechanisms, like Google’s AdWords. In Section A2, we show that our main result is robust to alternative selling mechanisms used. We consider a situation where, instead of using second-price auctions, platforms can make any take-it-or-leave-it offer.
- *Random Order of Auctions.* We assume that platforms hold their ad auctions sequentially. If auctions were simultaneous, we would face the kind of non-existence of pure-strategy equilibria discussed in Szentes and Rosenthal (2003). Given that auctions are held sequentially, our equilibrium characterization holds for any platform order. We choose a symmetric random order as the most agnostic option.
- *Set of Digital Platforms.* As mentioned in the introduction, the set of digital platforms that serve a particular individual includes those platforms that: (i) have information about the preferences of that individual; and (ii) are able to target ads to that individual. This set arguably includes social media like Facebook and search engines like Google if the individual uses them. It does not probably include other digital media, like online newspapers or stream services, because either (i) or (ii) or both fail. At the current state of technology, those outlets are best represented as more traditional “mass” media, like television and newspapers who sell ads to a bulk of generic “eyeballs” who are all shown the same ad. Of course, if technology were to change in the future, the set of digital platforms may change too and our model would apply to whatever the relevant sets are.

We discuss (i) and (ii) immediately below.

- *Difference between Targeted and Non-Targeted Ads.* The model posits a stark difference between media platforms that have perfectly targetable ads and media platforms that have non-targeted ads. The truth is that all platforms have some information about their users and some leeway to target the ads: even newspapers have some sense of who is more likely to read a particular section and target ads accordingly (e.g., hotel ads in the travel section). However, digital platforms have a threefold advantage: they have access to user behavior data, which provides them with accurate information about user preferences, they can customize ads individually, and they can sell them individually. Currently, only some social media platforms and search engines are able to achieve this triple advantage. In the future, the set of platforms with this capability may increase. Our paper applies to whichever set of platforms has this targeting capability.
- *No Information Synergies between Platforms.* Although this restriction is an immediate consequence of the knowledge and targetability assumption above, it should be highlighted separately.

Our idealized platforms already know everything about their users. Hence, a merger between two or more platforms cannot increase their knowledge base or enhance their ability to target individual customers. Future empirical research should try to assess how less than omniscient platforms might make information gains or reduce information processing costs if they merge.

- *Bundling across consumers.* We assume that platforms do not sell ads to bundles of consumers. There are circumstances where a platform can increase its profit without merging by selling ads to bundles of heterogeneous consumers rather than selling them separately. One example is when there are two consumers and three platforms. Consumer A uses only platform 1, while consumer B uses all three platforms. Suppose that, when sold separately, one of the ads to consumer B goes to the entrant. By bundling the two consumers together, Platform 1 may be able to monopolize the markets of both consumers. Obviously, just like in other industrial organization models of bundling (see Chapter 11 of Belleflamme and Peitz, 2015), there are also circumstances where bundling is not optimal. It may make no difference (like when the two consumers are identical) or possibly hurt the bundling platform (when the two consumers face different entrants). Future work should explore the interaction between ad bundling across consumers and ad bundling across platforms.
- *Non-strategic Consumers.* In our model social platform usage patterns are unaffected by advertising. Our consumers choose to use a certain set of platforms for pure consumption value, without taking into account that they may receive useful information about products or they may be charged different prices depending on the set of platforms they utilize. While a fully rational consumer should weigh these factors before checking his or her Facebook page, introspection suggests that myopia is not an entirely unrealistic assumption. Future research could add a platform usage selection stage to the present model.
- *Similar Industries.* We assume that u_1 , u_2 , π_1 , and π_2 are the same across different industries. The model could be easily extended to industry-specific values.
- *Reduced-Form Payoffs.* The payoffs of firms, platforms and consumers are expressed in reduced form as u_1 , u_2 , π_1 , π_2 , and π_E . Recall that these payoffs are expressed per consumer. They can be microfounded in a number of ways – under the knowledge and targetability assumption above. For instance, assume the consumer derives utility V_I from the incumbent’s product and utility $V_E \in (V_I, 2V_I)$ from the entrant’s product. If the consumer is only aware of the incumbent, the incumbent will charge a price V_I and the consumer will buy from the incumbent. If the consumer is also aware of the entrant, he will buy from the entrant at price $V_E - V_I$. This yields $u_1 = 0$, $u_2 = V_I$, $\pi_1 = V_I$, $\pi_2 = 0$, and $\pi_E = V_E - V_I$, which satisfies all the assumptions above. In particular, entry increases consumer welfare and total welfare but decreases industry profits.

While this is the simplest microfoundation, one could also allow the consumer to buy multiple items of both products or endow him with stochastic preferences.²

- *Incumbent/Entrant Product Familiarity.* The model can be extended beyond our extreme assumption that all consumers are aware of the incumbent’s product and unaware of the entrant’s product. There may be segments of consumers that are unaware of the incumbent’s product, aware of the entrant’s product, or both.
- *Focus on Consumer Welfare rather than Total Welfare.* In line with standard competition policy practice, our key metric will be consumer welfare. One could instead focus on total welfare, which also includes platform and producer profits. A set of standard assumptions would guarantee that consumer welfare and total welfare go in the same direction.

A2. Alternative Ad Selling Mechanism

The model was based on the assumption that platforms use a simple selling mechanism. The selling order is randomized and each platform uses a second-price auction. This section asks whether our key equilibrium characterization (Proposition ??) is robust to alternative specifications of the mechanism space.

We approach the question in two ways: a more concrete one – we suggest a different mechanism and show that it yields the same result – and a more abstract one – we consider a corresponding cooperative game and show that the same condition determines the presence of a stable coalitional partition.

Starting with the more concrete approach, the auction mechanism assumed in the baseline model may be criticized because it assigns a purely passive role to platforms: they always run the same kind of auction, thus forgoing potential gains they may make by exploiting their market power. Here we examine what can be seen as the polar opposite case: suppose every platform can make a take-it-or-leave-it offer to one of the two producers, with the understanding that, if that producer rejects the offer, the ad will go to the other producer. For every platform, the mechanism is therefore $(z_i, t_i) \in \{I, E\} \times \mathfrak{R}^+$, where z_i is the identity of the producer the offer is made to and t_i is the amount requested. Platforms have commitment power and the commitment is observed by everyone: they each announce (z_i, t_i) .³

²This microfoundation is similar to the model in the discrimination case considered by Armstrong and Vickers (2019).

³The analysis of the no-commitment case is very simple.

If platforms cannot commit to a price and $n \geq 2$, the monopoly equilibrium never exists.

To see this, note that in a monopoly equilibrium, the incumbent must start the last auction holding $n - 1$ ads. In that case, the last platform would charge her the whole surplus: $\pi_1 - \pi_2$. Predicting this, the incumbent would be willing to pay at most zero in any of the previous auctions, implying that previous platforms would prefer to sell to the entrant, who is willing to pay any price up to π_E to avoid the monopoly outcome.

As before, we focus on a segment with n platforms. The game is composed of three stages:

- The order of platforms is randomized and observed by all players: 1, 2, ... n .
- In the announcement stage, platforms announce their mechanisms simultaneously.
- In the acceptance stage, the mechanisms selected by the platforms are played publicly in order 1, 2, 3, ..., n .⁴

We focus on the set of pure-strategy subgame perfect equilibria. As in the baseline case, we have a monopoly equilibrium when the incumbent buys all ads, while an entry equilibrium is the set of complementary cases where at least one ad is purchased by the entrant.

Proposition 1 *A monopoly equilibrium exists if and only if $n \leq \bar{n}$.*

Let us illustrate the Proposition by re-visiting the examples in Figure 1. In Example 1, there is a symmetric equilibrium where each platform demands $t_i = 7/3$ from the incumbent. If any of the platforms tried to demand more, the incumbent would rather not buy any ad. If a platform sold an ad to the entrant instead of the incumbent, the most it would get is 2.⁵

In Example 2, there is no equilibrium where all the ads go to the incumbent. If such an equilibrium existed, at least one of the platforms would have to charge a price below 2, but then that platform would rather switch to selling its ad to the entrant. Instead, there is an equilibrium where all ads are sold to the entrant for zero.

The more abstract approach shows that the $n < \bar{n}$ condition captures a strategic feature of the environment we are considering. We move from non-cooperative to cooperative game theory and we show that the $n < \bar{n}$ condition determines whether the game has a stable outcome where only the incumbent advertises or not.

Assume that instead of playing the auction described in Section 2, the incumbent, the entrant, and the n platforms are engaged in a transferable utility game (TUG). The set of players is $\{I, E, 1, \dots, n\}$. The characteristic function v has the following properties:

- The coalition that includes the incumbent and all platforms gets $v_{I,1,\dots,N} = \pi_1$ for itself, while the coalition that includes only the entrant gets $v_E = 0$.

Obviously, if $n = 1$, the monopoly equilibrium will arise.

Thus, the consumer welfare analysis of the no-commitment case is exactly the same as in the baseline with $\bar{n} \in (0, 1)$. The effect on consumer welfare of a merger is also immediately derived: a merger matters if and only if it leads to $n = 1$.

⁴The assumption that the acceptance stage is sequential guarantees the existence of a pure-strategy equilibrium. If the acceptance stage was simultaneous, we conjecture that the proposition could be re-written as: A pure-strategy equilibrium where all ads are sold to the incumbent exists if and only if $n \leq \bar{n}$.

⁵Obviously, this is not the only equilibrium: there is a continuum of asymmetric equilibria with $t_1 + t_2 + t_3 = 7$ and $t_i \geq 2$ for all platforms.

- A coalition that includes the incumbent and a set of platforms P that does not include all platforms gets $v_{I,P} = \pi_2$.
- A coalition that includes the entrant and a non-empty set of platform P gets $v_{E,P} = \pi_E$.
- Coalitions including both E and I are not feasible: the two producers cannot directly cooperate because they cannot collude (if they could, the solution would be simple: the incumbent would pay the entrant an amount between π_E and $\pi_1 - \pi_2$ to stay out).
- Any other coalition gets zero.

This is not a superadditive transferable utility game because the grand coalition is not feasible, hence the concept of core is not applicable. However, one can ask whether the game has a stable partition of the players into coalitions. A partition is stable if there exists an imputation x – a vector of payoffs to players that is compatible with the characteristic function – such that no coalitional deviation can guarantee a strictly higher payoff for each of its members.

Proposition 2 *If $n < \bar{n}$, the only stable partition of the TUG is $\{\{I, 1, \dots, n\}, \{E\}\}$. If $n > \bar{n}$, the set of stable partitions is empty.*

The intuition for this result generalizes the intuition for the two non-cooperative games we considered: the sequential auctions and the take-it-or-leave-it offers. Namely, if $n < \bar{n}$ the incumbent’s monopoly profit π_1 is large enough for the incumbent to pay at least π_E to every platform and leave at least π_2 for herself. If instead the condition fails, the incumbent is not willing to pay at least π_E to every platform and that leaves at least one of platform open to switching to the entrant.

A3. General Model

The goal of this section is to relax four assumptions that were made in the baseline case: (i) There is one incumbent only; (ii) There is one entrant only; (iii) Each platform sells one ad only; (iv) Being exposed to one ad provides the consumer with perfect information. In a stylized extension, we will allow for any number of incumbents, a large number of potential entrants, any number of ads per platform, and imperfectly informative ads.

Consider a consumer segment with n platforms. Each platform shows k ads. We focus on one consumer who is interested in a product made by a retail industry with q incumbents and a large number of potential entrants.

Every platform runs a $k + 1$ th price auction (similar to the one used by Google). Each firm submits a bid and can buy at most one ad on that platform. The highest k bidders receive an ad and

they all pay the bid of the $k + 1$ th bidder. All auctions are run simultaneously. Thus, each firm submits a n -vector of non-negative bids to all platforms.

The probability that a consumer sees any specific ad is given by $p \in [0, 1]$. We assume that this probability is independent across ads and platforms.

There are a large number of entrants. The payoff of each entrant from a specific consumer is π_E if the consumer learns about the entrant's product and zero otherwise. The consumer learns about the product if she sees an ad by the entrant.

There are q incumbents. To make the problem interesting, assume that $q \geq k$. The incumbent's utility depends on the number of entrants the consumer is aware of. With no entrants, it is π_H , with at least one entrant, it is π_L , with $\pi_H > \pi_L$.⁶⁷

The probability of entry – namely the probability that the consumer becomes aware of at least one entrant – depends on the number of ads bought by entrants. By the independence assumption above, it does not depend on which platforms the ads appear on, or which set of entrants get ads (or whether multiple ads are bought by the same entrant).⁸ If m ads end up with entrants, the probability that the consumer becomes aware of at least one entrant is:

$$P_m = 1 - (1 - p)^m.$$

We say we are in a *pre-emption equilibrium* if all the ads from all the platforms are bought by incumbents. We restrict attention to pure-strategy equilibria where every ad receives at least one full-value bid from some entrant, where a full-value bid is one that equals the additional payoff the entrant would receive if he won the ad.⁹

The following result is a partial extension of Proposition ?? to this more general environment:

Proposition 3 *A pre-emption equilibrium exists if and only if the following condition is satisfied*

$$p\pi_E \leq \frac{P_K (\pi_H - \pi_L)}{K}.$$

where $K = \lceil nk/q \rceil$ (the smallest integer that is at least as large as nk/q).

To understand the proposition, we consider two special cases. In what follows we assume that $\frac{nk}{q}$ is an integer.

⁶The analysis can be extended to a payoff that depends on the number of successful entrants as long as the marginal effect of entry is decreasing in the number of entrants.

⁷One could also assume that there are other firms outside the industry that are interested in buying these spaces, presumably to engage in non-targeted advertising. That would lead to a model that combines the present set-up with that of Section 6.

Alternatively, one could consider a cost of running ads.

⁸Alternative formulations can be accommodated at a higher notation cost.

⁹This restriction eliminates equilibria where the auction winner pays less than the losers' valuation. In auctions with one object, these equilibria are usually ruled out by invoking weak dominance. However, with multiple simultaneous auctions, weak dominance is not applicable.

1. One incumbent, perfectly informative ads, one ad per platform. When $q = 1$, $k = 1$ and $p = 1$, the no-deviation condition in Proposition 3 boils down to the monopolization condition in our baseline Proposition

$$n \leq \frac{\pi_H - \pi_L}{\pi_E}.$$

2. Perfectly informative ads. When $p \rightarrow 1$, Proposition 3 boils down to the monopolization condition in our baseline Proposition

$$\frac{nk}{q} \leq \frac{\pi_H - \pi_L}{\pi_E}.$$

This condition is similar to the baseline case, but the number of independent platform is multiplied by the ratio between k and q . A higher number of ads per platform makes monopolization harder. A higher number of incumbents makes it easier (but it is important to keep in mind that a higher number of oligopolists is also likely to modify $\pi_H - \pi_L$ and consumer welfare).

As mentioned above, Proposition 3 is only a partial extension of Proposition ???. Both propositions provide a necessary and sufficient conditions for pre-emption. However, only the latter characterizes what happens when the pre-emption equilibrium does not exist. The problem is that, if there are multiple ads, when the pre-emption does not exist, all (reasonable) equilibria involve mixed strategies: entry occurs with positive probability but we do not know how to characterize that probability. The general result is therefore qualitative: entry increases consumer welfare but we cannot say by how much in a general way. One could of course compute the mixed-strategy equilibrium numerically for specific instances.

This difficulty is not specific to our framework. It is a manifestation of a general issue when there multiple auctions. One can show that an equilibrium exists (e.g., Simon and Zame, 1990). However, a general analytical characterization does not exist. Szentes and Rosenthal (2003) provide a solution with three objects and two bidders for first- and second-price auctions: the complexity of solving that case explains a general characterization has yet to be found.

References

- [1] Armstrong M. and J. Vickers (2019), “Discriminating Against Captive Customers”, *AER: Insights*, 1, 3, 257-272.
- [2] Belleflamme, P. and M. Peitz (2015), *Industrial Organization: Markets and Strategies*, Cambridge University Press.
- [3] Simon, L. and W. Zame (1990), “Discontinuous Games and Endogenous Sharing Rules”, *Econometrica*, 58, 861-872.

- [4] Szentes, B. and R. Rosenthal (2013), “Beyond chopsticks: Symmetric equilibria in majority auction games”, *Games and Economic Behavior*, 45, 278-295.

A4. Proofs of Propositions Stated in the Online Appendix

Proposition 1

For the “if” part, consider the following candidate equilibrium: Each platform demands $(I, \frac{\pi_1 - \pi_2}{n})$. Given that $\frac{\pi_1 - \pi_2}{n} > \pi_E$, no platform gains by deviating and selling to the entrant. No platform gains by demanding a higher t_i , as that would induce the incumbent to reject all offers.

For the “only if” part, suppose for contradiction that $n > \bar{n}$ and there exists an equilibrium where the incumbent wins all the ads. Clearly, the entrant is paying zero to all platforms. Let (t_1, \dots, t_n) be the transfers that the incumbent is paying in equilibrium to the n platforms. Note that platform i could switch to $(E, \pi_E - \varepsilon)$, where ε is an arbitrarily small positive number. To guarantee that such deviation is not profitable, it must be that $t_i \geq \pi_E$. This implies that the total amount of transfers paid by the incumbent is $n\pi_E$. But that would be greater than the incumbent’s benefit $\pi_1 - \pi_2$ if $n > \bar{n} = \frac{\pi_1 - \pi_2}{\pi_E}$, yielding a contradiction.¹⁰

Proposition 2

It is easy to see that if the condition for monopolization ($n < \bar{n}$) is satisfied the corresponding TUG has a unique stable partition: $\{\{I, 1, \dots, n\}, \{E\}\}$. For instance, this is sustained by the imputation $x_i = \pi_E$ for every platform $i = 1, \dots, n$, $x_I = \pi_1 - n\pi_E$ for the incumbent, and $x_E = 0$ for the entrant, which yields the same payoff vector as in the non-cooperative equilibrium. The incumbent forms a coalition with all platforms and offers each of them enough money to defend against a coalition of a subset of platforms with the entrant, which can achieve at most π_E .

One can also see that if the monopolization condition fails ($n > \bar{n}$), there exists no stable partition. The partition above, $\{\{I, 1, \dots, n\}, \{E\}\}$, is no longer stable because $v_{E,1,\dots,N}$ is not large enough to guarantee at least π_2 to the incumbent and π_E to each platform. No coalition of the form $\{E, X\}$ is stable because: (i) If the imputation to X is zero, there is a deviation to $\{I, 1, \dots, n\}$; (ii) If the imputation to X is positive, there is a deviation to $\{E, \bar{X}\}$ (because in turn any stable coalition $\{I, \bar{X}\}$ must have a zero imputation to \bar{X}).

¹⁰Note the set of equilibria is non-empty because there is a pure-strategy subgame perfect equilibrium where every platform offers $(E, 0)$. No platform has a strict incentive to deviate.

Proposition 3

“*Only If*”. In a monopoly equilibrium nk ads are bought by q incumbents. There must be an incumbent who in equilibrium buys at least $K = \lceil nk/q \rceil$ ads. An entrant who buys exactly one ad receives a payoff $p\pi_E$. Therefore, the bid on every ad must be at least $p\pi_E$. In equilibrium, an incumbent who buys K ads receives payoff $\pi_H - Kp\pi_E$. If the incumbent deviates by offering zero on all ads, her payoff would become $(1 - P_K)\pi_H + P_K\pi_L$. The deviation is profitable if

$$p\pi_E > \frac{P_K(\pi_H - \pi_L)}{K}.$$

“*If*”. We construct the following equilibrium. Some incumbents bid $p\pi_E$ on $K = \lceil nk/q \rceil$ ads, while the remaining incumbents buy $\lceil nk/q \rceil - 1$ ads (the number of incumbents who buy K ads is $nk - q(K - 1)$). Every ad receives a bid by exactly one incumbent and at least one entrant.¹¹

No entrant has a profitable deviation because $p\pi_E$ is the additional payoff they get from buying one ad and buying more than one ad generates a lower payoff per ad.

We check that no incumbent has a profitable deviation. Clearly no incumbent benefits by increasing her bid or bidding on more ads. If an incumbent bids on $K' < \tilde{K}$ ads instead of $\tilde{K} \in \{K, K - 1\}$ (or reduces her bid on the ads she is buying in equilibrium), entrants would win those auctions, and the incumbent’s payoff would become $(1 - P_{K'})\pi_H + P_{K'}\pi_L - K'p\pi_E$. Thus a deviation to K' is profitable if

$$(\tilde{K} - K')p\pi_E - P_{\tilde{K}-K'}(\pi_H - \pi_L) > 0,$$

namely

$$p\pi_E - \frac{P_{\tilde{K}-K'}(\pi_H - \pi_L)}{\tilde{K} - K'} > 0.$$

Note that because $P_{\tilde{K}-K'}$ exhibits decreasing differences it is then

$$\arg \min_{K'} \frac{P_{\tilde{K}-K'}}{\tilde{K} - K'} = 0.$$

Thus, if the inequality holds for some K' it also holds for $K' = 0$ and the necessary and sufficient condition for the deviation is

$$p\pi_E - \frac{P_{\tilde{K}}(\pi_H - \pi_L)}{\tilde{K}} > 0.$$

As $\tilde{K} \in \{K, K - 1\}$, by a similar argument, the necessary and sufficient condition for a deviation for some incumbent is

$$p\pi_E - \frac{P_K(\pi_H - \pi_L)}{K} > 0,$$

which yields the statement.

¹¹For instance, with $n = 3$ platforms, $k = 4$ ad slots, and $q = 7$ incumbents, it is $K = \lceil 12/7 \rceil = 2$. Hence $nk - q(K - 1) = 5$ incumbents buy 2 ads each, and the remaining 2 incumbents buy 1 ad each.