ONLINE APPENDIX
The Commitment Benefit of Consols in Government Debt Management

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A. Proofs of Section I

Proof of Lemma 1

The necessity of these conditions is proved in the text. To prove sufficiency, let the government choose the associated level of debt $\left\{\{b_{t,k}\}_{k=1}^{\infty}\right\}_{t=0}^{\infty}$ which satisfies (9) and a tax sequence $\{\tau_t\}_{t=0}^{\infty}$ which satisfies (8). Let bond prices satisfy (8). (9) given (1) implies that (3) and (4) are satisfied. Therefore household optimality holds and all dynamic budget constraints are satisfied along with the market clearing, so the equilibrium is competitive.■

B. Proofs of Section III

Proof of Lemma 2

Note that if $b_k = b\ \forall k$, then from Assumption 1, the solution under commitment admits $\{c_t, n_t\} = \{c^*(b), n^*(b)\}\ \forall t$, and this solution can be implemented with $b'_k = b$ given (18) – (19). Since the MPCE satisfies the same constraints of the problem under commitment plus additional constraints regarding sequential optimality, it follows that

$$W(\mathbf{B}) = \frac{u(c^*(b), n^*(b))}{1-\beta} \geq V(\mathbf{B}) \tag{B.1}$$

if $b_k = b\ \forall k$. Now consider optimal policy under the MPCE in (12) – (14) given $b_k = b\ \forall k$. A government has the option of choosing $c = c^*(b)$ and $n = n^*(b)$ together with $b'_k = b\ \forall k$. This satisfies the resource constraint (13) and the implementability constraint (14). Therefore, it follows that

$$V(\mathbf{B}) \geq u(c^*(b), n^*(b)) + \beta V(\mathbf{B}). \tag{B.2}$$

Equations $(B.1)$ and $(B.2)$ imply that

$$V(\mathbf{B}) = W(\mathbf{B}). \tag{B.3}$$

By Assumption 1, $W(\mathbf{B})$ is uniquely characterized by $\{c_k, n_k\} = \{c^*(b), n^*(b)\}$ $\forall k$. Therefore, it follows that any solution to (12) – (14) given $b_k = b\ \forall k$ admits $c = c^*(b)$ and $n = n^*(b)$.■

Conditional on $\mathbf{B}$, if a solution admits $b'_k = b_k$, then this means that $\mathbf{B}$ is an absorbing state with $\mathbf{B} = \mathbf{B}'$ and consumption and labor are constant and equal to some $\{c, n\}$ from that period onward. Therefore, $h^k(\mathbf{B}') = \beta^k u_c(c, n)\ \forall k \geq 1$ for $h^k(\mathbf{B}')$ defined in (11). As such, (14) can be rewritten as

$$u_c(c,n)\,c + u_n(c,n)\,n - u_c(c,n)\,b_1 + u_c(c,n)\sum_{k=1}^{\infty}\beta^k\left(b'_k - b_{k+1}\right) = 0 \tag{B.4}$$

which combined with (21) and the fact that $b'_k = b_k$ implies that

$$u_c(c,n)\,c + u_n(c,n)\,n = u_c(c,n)\,\widehat{b}. \tag{B.5}$$

Now consider the solution to the following problem given $\widehat{b}$:

$$\max_{c,n}\frac{u(c,n)}{1-\beta} \text{ s.t. } c + g = n \text{ and } (B.5). \tag{B.6}$$

It is necessary that $V(\mathbf{B})$ be weakly below the value of $(B.6)$. This is because the solution to $V(\mathbf{B})$ also admits a constant consumption and labor (as in the program in $(B.6)$) and since the constraint set in $(B.6)$ is slacker, since the program ignores the role of government debt in changing future policies. Note furthermore that the value of $(B.6)$ equals $W(\{b_k\}_{k=1}^{\infty})\,|_{b_k=\widehat{b}\ \forall k}$, where this follows from Assumption 1. Therefore,

$$V(\mathbf{B}) \leq W(\{b_k\}_{k=1}^{\infty})\,|_{b_k=\widehat{b}\ \forall k}. \tag{B.7}$$

Now consider the welfare of the government in the MPCE if, instead of choosing $b'_k = b_k\ \forall k$, it instead chooses $b'_k = \widehat{b}\ \forall k$ with $c = c^*(\widehat{b})$ and $n = n^*(\widehat{b})$. It follows from Lemma 2 that under this perturbation, $h^k(\mathbf{B}') = \beta^k u_c(c^*(\widehat{b}), n^*(\widehat{b}))\ \forall k \geq 1$, which implies that the resource constraint (13) and implementability constraint (14) are satisfied under this deviation. Because the continuation value associated with this deviation is $W(\{b_k\}_{k=1}^{\infty})\,|_{b_k=\widehat{b}\ \forall k}$, it follows that for this deviation to be weakly dominated:

$$W(\{b_k\}_{k=1}^{\infty})\,|_{b_k=\widehat{b}\ \forall k} \leq V(\mathbf{B}). \tag{B.8}$$

Given $(B.7)$ and $(B.8)$, it follows that $W(\{b_k\}_{k=1}^{\infty})\,|_{b_k=\widehat{b}\ \forall k} = V(\mathbf{B})$. Therefore, given $\mathbf{B}$, there exists another solution to $(12)-(14)$ with $b'_k = \widehat{b}\ \forall k$ which achieves the same welfare.■

Proof of Lemma 4

Before proving this lemma, define c^{laffer} as:

$$c^{laffer} = \arg\max_{c} \left\{ c + \frac{u_n(c, c+g)}{u_c(c, c+g)}(c+g) \right\}, \tag{B.9}$$

and let b^{laffer} correspond to the value of the maximized objective in $(B.9)$. It follows that a solution to $(15) - (17)$ exists if $b_k = b\ \forall k \geq 1$ if $b \leq b^{laffer}$.

Given this definition, we can proceed to prove this lemma by contradiction. By Lemma 3,

$$V(\mathbf{B}) = W\left(\{b_k\}_{k=1}^{\infty}\right)|_{b_k = \widehat{b}\ \forall k} = \frac{u(c^*(\widehat{b}), n^*(\widehat{b}))}{1-\beta} \tag{B.10}$$

for $\widehat{b}$ defined in (21). Now suppose that $b_1 \neq \widehat{b}$. Given the definition of $\widehat{b}$, this means that $\widehat{b} \in (\underline{b}, \overline{b})$ and that $\widehat{b} \leq b^{laffer}$. We consider two cases separately.

Case 1. Suppose that $\widehat{b} < b^{laffer}$, and suppose that the government locally deviates to $b_k' = \widetilde{b} \neq \widehat{b}\ \forall k$ so that from tomorrow onward, consumption is $c^*(\widetilde{b})$ and labor is $n^*(\widetilde{b})$, where this follows from Lemma 2. This means that $h^k(\widetilde{\mathbf{B}}) = \beta^k u_c(c^*(\widetilde{b}), n^*(\widetilde{b}))$ under the deviation. In order to satisfy the resource constraint and implementability condition, let the government deviate today to a consumption and labor allocation $\{\widetilde{c}, \widetilde{n}\}$ which satisfies

$$\widetilde{c} + g = \widetilde{n} \tag{B.11}$$

and

$$\begin{aligned} u_c(\widetilde{c}, \widetilde{n})\widetilde{c} + u_n(\widetilde{c}, \widetilde{n})\widetilde{n} - (u_c(\widetilde{c}, \widetilde{n}) - u_c(c^*(\widetilde{b}), n^*(\widetilde{b})))b_1 = \\ u_c(c^*(\widetilde{b}), n^*(\widetilde{b}))\left(\widehat{b} + \frac{\beta}{1-\beta}(\widehat{b} - \widetilde{b})\right) \end{aligned} \tag{B.12}$$

where we have appealed to the definition of $\widehat{b}$ in (21). For such a deviation to be weakly dominated, it must be that

$$V(\mathbf{B}) \geq u(\widetilde{c}, \widetilde{n}) + \beta W\left(\{b_k\}_{k=1}^{\infty}\right)|_{b_k = \widetilde{b}\ \forall k}. \tag{B.13}$$

Clearly, the value of the right hand side of $(B.13)$ equals $V(\mathbf{B})$ if $\widetilde{b} = \widehat{b}$. Therefore, it must be that $\widetilde{b} = \widehat{b}$ with $\{\widetilde{c}, \widetilde{n}\} = \{c^*(\widehat{b}), n^*(\widehat{b})\}$ maximizes the right hand side of $(B.13)$ subject to $(B.11)$, and $(B.12)$. More specifically, we can consider the solution to the following program

$$\max_{\widetilde{c}, \widetilde{n}, \widetilde{b}} u(\widetilde{c}, \widetilde{n}) + \beta W\left(\{b_k\}_{k=1}^{\infty}\right)|_{b_k = \widetilde{b}\ \forall k} \text{ s.t. } (B.11) \text{ and } (B.12). \tag{B.14}$$

For the deviation to not strictly increase welfare, $\widetilde{b}=\widehat{b}$ must be a solution to $(B.14)$. By Assumption 1, $W\left(\{b_k\}_{k=1}^{\infty}\right)|_{b_k=\widetilde{b}\ \forall k}=u(c^*,n^*)/(1-\beta)$ where $\{c^*,n^*\}=\{c^*(\widetilde{b}),n^*(\widetilde{b})\}$ are the unique levels of consumption and labor which maximize welfare given $\widetilde{b}$ and are defined in (18) and (19). Letting μ_1 represent the Lagrange multiplier on the implementability condition for the program defining $W\left(\{b_k\}_{k=1}^{\infty}\right)|_{b_k=\widetilde{b}\ \forall k}$ in (15) – (17), it follows from first order conditions that

$$\text{(B.15)}\qquad \begin{array}{c} u_c(c^*,n^*)+u_n(c^*,n^*)+ \\ \mu_1\left(\begin{array}{c} u_c(c^*,n^*)+u_n(c^*,n^*) \\ +u_{cc}(c^*,b^*)(c^*-\widetilde{b})+u_{cn}(c^*,n^*)(c^*-\widetilde{b}+n^*)+u_{nn}(c^*,n^*)n^* \end{array}\right)=0. \end{array}$$

Since $\{c^*,n^*\}\neq\{c^{fb},n^{fb}\}$ by the statement of the lemma, $(B.15)$ implies that $\mu_1\neq 0$. Using this observation, implicit differentiation of (18) and (19) taking $(B.15)$ into account implies

$$\text{(B.16)}\qquad c^{*\prime}(\widetilde{b})=n^{*\prime}(\widetilde{b})=-\mu_1\frac{u_c(c^*,n^*)}{u_c(c^*,n^*)+u_n(c^*,n^*)}.$$

Finally, by the Envelope condition,

$$\text{(B.17)}\qquad \frac{dW\left(\{b_k\}_{k=1}^{\infty}\right)|_{b_k=\widetilde{b}\ \forall k}}{d\widetilde{b}}=-\mu_1\frac{u_c\left(c^*,n^*\right)}{1-\beta}.$$

Now consider the solution to $(B.14)$. Let μ_0 correspond to the Lagrange multiplier on $(B.12)$. First order conditions with respect to $\widetilde{c}$ and $\widetilde{n}$ imply

$$\text{(B.18)}\qquad \begin{array}{c} u_c(\widetilde{c},\widetilde{n})+u_n(\widetilde{c},\widetilde{n})+ \\ \mu_0\left(\begin{array}{c} u_c(\widetilde{c},\widetilde{n})+u_n(\widetilde{c},\widetilde{n}) \\ +u_{cc}(\widetilde{c},\widetilde{n})(\widetilde{c}-b_1)+u_{cn}(\widetilde{c},\widetilde{n})(\widetilde{c}-b_1+\widetilde{n})+u_{nn}(\widetilde{c},\widetilde{n})\widetilde{n} \end{array}\right)=0. \end{array}$$

Since $\{\widetilde{c},\widetilde{n}\}\neq\{c^{fb},n^{fb}\}$ by the statement of the lemma, $(B.18)$ implies that $\mu_0\neq 0$. Since the solution admits $\widetilde{b}=\widehat{b}\in\left(\underline{b},\overline{b}\right)$, then we can ignore the bounds on government debt, and first order conditions with respect to $\widetilde{b}$ taking into account $(B.16)$ and $(B.17)$ yields

$$\text{(B.19)}\qquad \mu_0\mu_1\frac{u_{cc}(c^*,n^*)+u_{cn}(c^*,n^*)}{u_c(c^*,n^*)+u_n(c^*,n^*)}\left(\widehat{b}-b_1+\frac{\beta}{1-\beta}(\widehat{b}-\widetilde{b})\right)+\frac{\beta}{1-\beta}(\mu_0-\mu_1)=0.$$

Note that $(B.15)$ and $(B.18)$ imply that

$$\text{(B.20)}\qquad \frac{\beta}{1-\beta}(\mu_0-\mu_1)=\frac{\beta}{1-\beta}\mu_0\mu_1\frac{u_{cc}(c^*,n^*)+u_{cn}(c^*,n^*)}{u_c(c^*,n^*)+u_n(c^*,n^*)}(\widetilde{b}-b_1)$$

Now consider if $\widetilde{b} = \widehat{b}$ so that $\{\widetilde{c}, \widetilde{n}\} = \{c^*, n^*\}$. In that case, use $(B.20)$ to substitute into $(B.19)$ to achieve:

$$\text{(B.21)} \qquad \mu_0 \mu_1 \frac{u_{cc}(c^*, n^*) + u_{cn}(c^*, n^*)}{u_c(c^*, n^*) + u_n(c^*, n^*)} (\widehat{b} - b_1) = 0.$$

If it were the case that $\widehat{b} \neq b_1$, then $(B.21)$ would require that $u_{cc}(c^*, n^*) + u_{cn}(c^*, n^*) = 0$, which contradicts the fact that $u_{cc}(c^*, n^*) + u_{cn}(c^*, n^*) < 0$. Therefore, $\widehat{b} = b_1$.

Case 2. Suppose that $\widehat{b} = b^{laffer}$. In this case, consider an analogous perturbation as in case 1 which reduces $\widehat{b}$ locally. For such a perturbation to be weakly dominated, the analog of $(B.21)$ requires

$$\text{(B.22)} \qquad \mu_0 \mu_1 \frac{u_{cc}(c^*, n^*) + u_{cn}(c^*, n^*)}{u_c(c^*, n^*) + u_n(c^*, n^*)} (\widehat{b} - b_1) \geq 0$$

It follows from $(B.17)$ that $\mu_1 > 0$ since any reduction in inherited debt can facilitate higher consumption and higher welfare. Since $u_{cc}(c^*, n^*) + u_{cn}(c^*, n^*) < 0$, satisfaction of $(B.22)$ requires

$$\text{(B.23)} \qquad \mu_0 (\widehat{b} - b_1) \leq 0.$$

Given that $\{\widetilde{c}, \widetilde{n}\} = \{c^*, n^*\} = \{c^{laffer}, c^{laffer} + g\}$, it can be verified from $(B.9)$ that if $b_1 < (>)\ \widehat{b} = b^{laffer}$, then $(B.18)$ implies that $\mu_0 > (<)\ 0$. This follows from the fact that $c^{laffer} < c^{fb}$ and the term in parentheses multiplying μ_0 in equation $(B.18)$ is equal to 0 if $b_1 = b^{laffer}$ and is increasing in b_1. Therefore, $(B.23)$ cannot hold unless $\widehat{b} = b_1$.■

Proof of Lemma 5

Suppose that $b_l = \widehat{b}\ \forall l \leq m$. Given $\mathbf{B}$, let $\widehat{\mathbf{B}}(1)$ represent the portfolio which sets $\widehat{b}_k = b_{k+1}$ so that no retrading takes place. Note that in such a portfolio, $\widehat{b}_1 = b_2$. Define $\widehat{\mathbf{B}}(2)$ analogously as the portfolio involving no retrading at the next date, so that $\widehat{b}_k = b_{k+2}$ under $\widehat{\mathbf{B}}(2)$, and define $\widehat{\mathbf{B}}(l)\ \forall l \leq m$ analogously. In any MPCE for which $b_1 = \widehat{b}$, a possible deviation sets $\{c, n\} = \{c^*(\widehat{b}), n^*(\widehat{b})\}$ and $b'_k = b_{k+1}$ so that no retrading takes place, where this deviation satisfies the resource constraint and implementability condition given $(18) - (19)$. For such a deviation to be weakly dominated, it is necessary that:

$$\text{(B.24)} \qquad V(\mathbf{B}) \geq u(c^*(\widehat{b}), n^*(\widehat{b})) + \beta V(\widehat{\mathbf{B}}(1)).$$

Forward induction on this argument implies that

$$V(\mathbf{B}) \geq \sum_{l=0}^{m-1} \beta^l u(c^*(\widehat{b}), n^*(\widehat{b})) + \beta^m V(\widehat{\mathbf{B}}(m)). \tag{B.25}$$

Combining $(B.10)$ with $(B.25)$, we achieve

$$V(\mathbf{B}) \geq V(\widehat{\mathbf{B}}(m)). \tag{B.26}$$

Now consider optimal policy starting from $\widehat{\mathbf{B}}(m)$. Note that since $b_l = \widehat{b}\ \forall l \leq m$, then following the same arguments as in the proof of Lemma 3, a feasible strategy starting from $\widehat{\mathbf{B}}(m)$ is to issue a flat debt maturity with all bonds equal to $\widehat{b}$. Such a strategy ensures a constant consumption and labor allocation forever equal to $\{c^*(\widehat{b}), n^*(\widehat{b})\}$. As such, it follows that $(B.26)$ holds with equality and that choosing a flat maturity distribution going forward is optimal.

Now we prove by contradiction that $b_{m+1} = \widehat{b}$. Suppose it were the case that $b_{m+1} \neq \widehat{b}$. This means that starting from $\widehat{\mathbf{B}}(m)$, the immediate debt which is owed by the government does not equal $\widehat{b}$. If this is the case, then the same arguments as those in the proof of Lemma 4 imply that there exists a deviation from the government's equilibrium strategy at $\widehat{\mathbf{B}}(m)$ which can strictly increase the government's welfare. However, if this is the case, $(B.26)$ which holds with equality is violated. Therefore, it must be that $b_{m+1} = \widehat{b}$.■

Proof of Proposition 1 and Corollaries 1 and 2

The proof of Proposition 1 follows directly by induction after appealing to Lemmas 4 and 5.

To prove the first corollary, note that for the statement of Proposition 1 to be false, it is necessary that $\{c, n\} = \{c^{fb}, n^{fb}\}$. However, if this is the case, then $(B.4)$ implies that

$$c^{fb} + \frac{u_n(c^{fb}, n^{fb})}{u_c(c^{fb}, n^{fb})} n^{fb} = -g = \sum_{k=1}^{\infty} \beta^{k-1} (1-\beta)\, b_k \geq \underline{b} \tag{B.27}$$

which contradicts $b_k > -g$.

To prove the second corollary, note that from Lemma 2, it is necessary that the continuation equilibrium starting from a flat government debt maturity entail consumption and labor equal to $\{c^*(b), n^*(b)\}$ forever. The arguments in the proof of Lemmas 4 and 5 imply that if the government were to choose a non-flat maturity distribution going forward, future governments would not choose $\{c^*(b), n^*(b)\}$ forever. Therefore, all solutions to $(12)-(14)$ admit $b'_k = b\ \forall k$.■