Online Appendix to the paper

Inference for Losers

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1. Proof of Proposition 1

Note that the event $\{\theta = \hat{\theta}_{(k)}\}$ is equivalent to the union of events

$$\bigcup_{(\tau^{\mathcal{L}},\tau^{\mathcal{U}})\in S_k(\theta)} \{M(\tau^{\mathcal{L}},\tau^{\mathcal{U}})X \leq 0\}, \tag{1}$$

where the $K \times K$ matrices $M(\tau^{\mathcal{L}},\tau^{\mathcal{U}})$ are defined as follows, supposing $\theta = \theta_i$:

1) for $j \neq i$ such that $\theta_j \in \tau^{\mathcal{U}}$, the j^{th} row of $M(\tau^{\mathcal{L}},\tau^{\mathcal{U}})$ is composed entirely of zeros except for a -1 in the j^{th} entry and a 1 in the i^{th} entry,
2) for $j \neq i$ such that $\theta_j \in \tau^{\mathcal{L}}$, the j^{th} row of $M(\tau^{\mathcal{L}},\tau^{\mathcal{U}})$ is composed entirely of zeros except for a 1 in the j^{th} entry and a -1 in the i^{th} entry, and
3) the i^{th} row of $M(\tau^{\mathcal{L}},\tau^{\mathcal{U}})$ is composed entirely of zeros.

For example, if $\tau^{\mathcal{L}} = \{\theta_j \in \Theta : j > i\}$ and $\tau^{\mathcal{U}} = \{\theta_j \in \Theta : j < i\}$, then

$$M(\tau^{\mathcal{L}},\tau^{\mathcal{U}}) = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & \dots & i & i+1 & i+2 & \dots & K \end{array} \\ \begin{pmatrix} -1 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & \dots & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & -1 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & 0 & 0 & \dots & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ i \\ i+1 \\ i+2 \\ \vdots \\ K. \end{array}$$

Define

$$A(\tau^{\mathcal{L}},\tau^{\mathcal{U}}) := \left[M(\tau^{\mathcal{L}},\tau^{\mathcal{U}}) \quad \mathbf{0}\right],$$

where $\mathbf{0}$ is a $K \times K$ matrix of zeros. Define

$$W := (X^T, Y^T)^T$$

and note that the event $\{M(\tau^{\mathcal{L}},\tau^{\mathcal{U}})X \leq 0\}$ is equivalent to the event $\{A(\tau^{\mathcal{L}},\tau^{\mathcal{U}})W \leq 0\}$. Define

$$\tilde{Z}^*_\theta := W - cY(\theta),$$

where $c = Cov(W, Y(\theta))/\Sigma_Y(\theta)$. By Lemma 5.1 of Lee et al. (2016), the event $\{A(\tau^{\mathcal{L}},\tau^{\mathcal{U}})W \leq 0\}$ is equivalent to the event

$$\left\{\mathcal{L}(\theta, \tilde{Z}^*_\theta, \tau^{\mathcal{L}},\tau^{\mathcal{U}}) \leq Y(\theta) \leq \mathcal{U}(\theta, \tilde{Z}^*_\theta, \tau^{\mathcal{L}},\tau^{\mathcal{U}}), 0 \leq \mathcal{V}(\theta, \tilde{Z}^*_\theta)\right\},$$

where

$$\mathcal{L}(\theta, z, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}) := \max_{j:(Ac)_j<0} \frac{-(Az)_j}{(Ac)_j},$$
$$\mathcal{U}(\theta, z, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}) := \min_{j:(Ac)_j>0} \frac{-(Az)_j}{(Ac)_j},$$
$$\mathcal{V}(\theta, z) := \min_{j:(Ac)_j=0} -(Az)_j.$$

Note that

$$(2) \qquad (A\tilde{Z}^*_{\theta_i})_j = \begin{cases} -(Z_\theta(\theta') - Z_\theta(\theta)) & \text{if } \theta' \in \tau^{\mathcal{U}} \\ Z_\theta(\theta') - Z_\theta(\theta) & \text{if } \theta' \in \tau^{\mathcal{L}} \\ 0 & \text{if } \theta' = \theta \end{cases}$$

and

$$(3) \qquad (Ac)_j = \begin{cases} -\frac{\Sigma_{XY}(\theta',\theta)-\Sigma_{XY}(\theta)}{\Sigma_Y(\theta)} & \text{if } \theta' \in \tau^{\mathcal{U}} \\ \frac{\Sigma_{XY}(\theta',\theta)-\Sigma_{XY}(\theta)}{\Sigma_Y(\theta)} & \text{if } \theta' \in \tau^{\mathcal{L}} \\ 0 & \text{if } \theta' = \theta. \end{cases}$$

Thus,

$$\{\theta' : (Ac)_j < 0\} = \Big\{\theta' : \big(\theta' \in \tau^{\mathcal{L}} \text{ and } \Sigma_{XY}(\theta) > \Sigma_{XY}(\theta',\theta)\big) \text{ or } \big(\theta' \in \tau^{\mathcal{U}} \text{ and } \Sigma_{XY}(\theta) < \Sigma_{XY}(\theta',\theta)\big)\Big\}$$

and

$$\{\theta' : (Ac)_j > 0\} = \Big\{\theta' : \big(\theta' \in \tau^{\mathcal{L}} \text{ and } \Sigma_{XY}(\theta) < \Sigma_{XY}(\theta',\theta)\big) \text{ or } \big(\theta' \in \tau^{\mathcal{U}} \text{ and } \Sigma_{XY}(\theta) > \Sigma_{XY}(\theta',\theta)\big)\Big\}.$$

Define

$$S^{\mathcal{L}}_\theta(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) := \{\theta' : (Ac)_j > 0\},$$
$$S^{\mathcal{U}}_\theta(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) := \{\theta' : (Ac)_j < 0\},$$
$$S^{\mathcal{V}}_\theta := \{\theta' : (Ac)_j) = 0\}$$

Equations (2) and (3) imply

$$\frac{-(A\tilde{Z}^*_{\theta_i})_j}{(Ac)_j} = Q_\theta(\theta').$$

This allows us to rewrite

$$\mathcal{L}(\theta, \tilde{Z}^*_{\theta_i}, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}) := \max_{\theta' \in S^{\mathcal{U}}_\theta(\tau^{\mathcal{L}}, \tau^{\mathcal{U}})} Q_\theta(\theta'),$$

$$\mathcal{U}(\theta, \tilde{Z}^*_{\theta_i}, \tau^{\mathcal{L}}, \tau^{\mathcal{U}}) := \min_{\theta' \in S^{\mathcal{L}}_\theta(\tau^{\mathcal{L}}, \tau^{\mathcal{U}})} Q_\theta(\theta')$$

and

$$\mathcal{V}(\theta, \tilde{Z}^*_{\theta_i}) := \min_{j \in S^{\mathcal{V}}_\theta} -\Big(Z_\theta(\theta') - Z_\theta(\theta)\Big).$$

Thus by (1), we have established that the event $\{\theta = \hat{\theta}_{(k)}\}$ is equivalent to the union of events

$$\bigcup_{(\tau^{\mathcal{L}},\tau^{\mathcal{U}})\in S_k(\theta)} \left\{\mathcal{L}(\theta,\tilde{Z}^*_\theta,\tau^{\mathcal{L}},\tau^{\mathcal{U}}) \le Y(\theta) \le \mathcal{U}(\theta,\tilde{Z}^*_\theta,\tau^{\mathcal{L}},\tau^{\mathcal{U}}), 0 \le \mathcal{V}(\theta,\tilde{Z}^*_\theta)\right\}.$$

Note that the event $\{\theta \in \widehat{R}\}$ is equivalent to the union of events $\cup_{k\in R}\{\theta = \hat{\theta}_{(k)}\}$ so that when $\mathcal{V}(\theta,z) \ge 0$,

$$\mathcal{Y}_R(\theta,z) = \bigcup_{(\tau^{\mathcal{L}},\tau^{\mathcal{U}})\in \mathcal{S}'_P(\theta)} \left[\mathcal{L}(\theta,z,\tau^{\mathcal{L}},\tau^{\mathcal{U}}),\mathcal{U}(\theta,z,\tau^{\mathcal{L}},\tau^{\mathcal{U}})\right],$$

where

$$\mathcal{S}'_P(\theta) := \bigcup_{k\in R} \mathcal{S}_k(\theta).$$

Finally, note that

$$\begin{aligned}
&\bigcup_{(\tau^{\mathcal{L}},\tau^{\mathcal{U}})\in \mathcal{S}'_P(\theta)} \left[\mathcal{L}(\theta,z,\tau^{\mathcal{L}},\tau^{\mathcal{U}}),\mathcal{U}(\theta,z,\tau^{\mathcal{L}},\tau^{\mathcal{U}})\right] \\
&= \bigcup_{(\tau^{\mathcal{L}},\tau^{\mathcal{U}})\in \mathcal{S}_P(\theta)} \left[\mathcal{L}(\theta,z,\tau^{\mathcal{L}},\tau^{\mathcal{U}}),\mathcal{U}(\theta,z,\tau^{\mathcal{L}},\tau^{\mathcal{U}})\right],
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}_P(\theta) := \{(\tau^{\mathcal{L}},\tau^{\mathcal{U}}) \in \bigcup_{k\in R} \mathcal{S}_k(\theta) : Q_\theta(\theta') \le Q_\theta(\theta'')\ \forall \theta' \in S^{\mathcal{L}}_\theta(\tau^{\mathcal{L}},\tau^{\mathcal{U}}), \\
\theta'' \in S^{\mathcal{U}}_\theta(\tau^{\mathcal{L}},\tau^{\mathcal{U}})\},
\end{aligned}$$

since if $Q_\theta(\theta') > Q_\theta(\theta'')$ for some $\theta' \in S^{\mathcal{L}}_\theta(\tau^{\mathcal{L}},\tau^{\mathcal{U}}), \theta'' \in S^{\mathcal{U}}_\theta(\tau^{\mathcal{L}},\tau^{\mathcal{U}})$, then

$$\mathcal{L}(\theta,z,\tau^{\mathcal{L}},\tau^{\mathcal{U}}) \ge Q_\theta(\theta') > Q_\theta(\theta'') \ge \mathcal{U}(\theta,z,\tau^{\mathcal{L}},\tau^{\mathcal{U}}).$$

This yields the statement of the proposition. ■

2. *Proof of Proposition 2*

Note that for

$$CI^\beta_{P,\theta} = \left[Y(\theta) - c_\beta\sqrt{\Sigma_Y(\theta)}, Y(\theta) + c_\beta\sqrt{\Sigma_Y(\theta)}\right]$$

the level β projection interval for $\mu_Y(\theta)$,

$$\begin{aligned}
&\sum_{k\in R} Pr_\mu\left\{\hat{\mu}^H_{\alpha,k} \ge \mu_Y\left(\hat{\theta}_{(k)}\right)\right\} \\
&= \sum_{k\in R}\sum_{\theta\in\Theta} Pr_\mu\left\{\theta = \hat{\theta}_{(k)}, \hat{\mu}^H_{\alpha,k} \ge \mu_Y(\theta)\right\} \\
&= \sum_{\theta\in\Theta} E_\mu\left[\sum_{k\in R} 1\left\{\theta = \hat{\theta}_{(k)}, \hat{\mu}^H_{\alpha,k} \ge \mu_Y(\theta)\right\}\right] \\
&= \sum_{\theta\in\Theta} E_\mu\left[1\left\{\theta \in \widehat{R}, \hat{\mu}^H_{\alpha,\theta} \ge \mu_Y(\theta)\right\}\right] \\
&= \sum_{\theta\in\Theta} Pr_\mu\left\{\theta \in \widehat{R}, \hat{\mu}^H_{\alpha,\theta} \ge \mu_Y(\theta)\right\}
\end{aligned}$$

$$= \begin{array}{c} \sum_{\theta\in\Theta} Pr_\mu \left\{ \hat{\mu}^H_{\alpha,\theta} \geq \mu_Y(\theta) | \theta \in \widehat{R}, \mu_Y(\theta) \in CI^\beta_{P,\theta} \right\} Pr_\mu \left\{ \theta \in \widehat{R}, \mu_Y(\theta) \in CI^\beta_{P,\theta} \right\} + \\ \sum_{\theta\in\Theta} Pr_\mu \left\{ \hat{\mu}^H_{\alpha,\theta} \geq \mu_Y(\theta) | \theta \in \widehat{R}, \mu_Y(\theta) \notin CI^\beta_{P,\theta} \right\} Pr_\mu \left\{ \theta \in \widehat{R}, \mu_Y(\theta) \notin CI^\beta_{P,\theta} \right\}, \end{array}$$

where we write $\hat{\mu}^H_{\alpha,\theta}$ to denote the hybrid estimator of $\mu_Y(\theta)$ for $\theta \in \widehat{R}$,

$$\hat{\mu}^H_{\alpha,\theta} = \sum_{k\in R} 1\left\{ \hat{\theta}_{(k)} = \theta \right\} \hat{\mu}^H_{\alpha,k}.$$

Proposition 7 of AKM implies, however, that

$$Pr_\mu \left\{ \hat{\mu}^H_{\alpha,\theta} \geq \mu_Y(\theta) | \theta \in \widehat{R}, \mu_Y(\theta) \in CI^\beta_{P,\theta} \right\} = \alpha,$$

so

$$\left| \sum_{k\in R} \left(Pr_\mu \left\{ \hat{\mu}^H_{\alpha,k} \geq \mu_Y\left(\hat{\theta}_{(k)}\right) \right\} - \alpha \right) \right|$$

$$= \left| \begin{array}{c} \sum_{\theta\in\Theta} \left(Pr_\mu \left\{ \hat{\mu}^H_{\alpha,\theta} \geq \mu_Y(\theta) | \theta \in \widehat{R}, \mu_Y(\theta) \in CI^\beta_{P,\theta} \right\} - \alpha \right) Pr_\mu \left\{ \theta \in \widehat{R}, \mu_Y(\theta) \in CI^\beta_{P,\theta} \right\} \\ + \sum_{\theta\in\Theta} \left(Pr_\mu \left\{ \hat{\mu}^H_{\alpha,\theta} \geq \mu_Y(\theta) | \theta \in \widehat{R}, \mu_Y(\theta) \notin CI^\beta_{P,\theta} \right\} - \alpha \right) Pr_\mu \left\{ \theta \in \widehat{R}, \mu_Y(\theta) \notin CI^\beta_{P,\theta} \right\} \end{array} \right|$$

$$= \left| \sum_{\theta\in\Theta} \left(Pr_\mu \left\{ \hat{\mu}^H_{\alpha,\theta} \geq \mu_Y(\theta) | \theta \in \widehat{R}, \mu_Y(\theta) \notin CI^\beta_{P,\theta} \right\} - \alpha \right) Pr_\mu \left\{ \theta \in \widehat{R}, \mu_Y(\theta) \notin CI^\beta_{P,\theta} \right\} \right|$$

$$\leq \sum_{\theta\in\Theta} \left| Pr_\mu \left\{ \hat{\mu}^H_{\alpha,\theta} \geq \mu_Y(\theta) | \theta \in \widehat{R}, \mu_Y(\theta) \notin CI^\beta_{P,\theta} \right\} - \alpha \right| Pr_\mu \left\{ \theta \in \widehat{R}, \mu_Y(\theta) \notin CI^\beta_{P,\theta} \right\}$$

$$\leq |R| \cdot \max\{\alpha, 1-\alpha\} \cdot \beta.$$

A parallel argument establishes the coverage statement. ■

3. *Efficient Computation of the Truncation Set*

The number of parameter arrangements that characterize the truncation set in Proposition 1 is $|\mathcal{S}_P(\theta)| = \sum_{k\in R} C_{k-1}(K-1)$ where C_k is the "choose k element" operator. Even assuming we can compute the truncation set for each arrangement $(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) \in \mathcal{S}_P(\theta)$ in constant time, exhaustive search will find the truncation set $\mathcal{Y}_R(\theta, z)$ in exponential time. This is infeasible when considering a large parameter space.

Here we describe an algorithm to compute the truncation set in $O(K \log K)$ time. We begin with the following observation.

OBSERVATION 1: *Suppose $\Sigma_{XY}(\theta', \theta) \neq \Sigma_{XY}(\theta)$ $\forall \theta' \in \Theta_{-\theta}$. Then for all $\tau^{\mathcal{L}}, \tau^{\mathcal{U}}$ and $\tilde{\tau}^{\mathcal{L}}, \tilde{\tau}^{\mathcal{U}}$*

$$S^{\mathcal{L}}_\theta(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) = S^{\mathcal{L}}_\theta(\tilde{\tau}^{\mathcal{L}}, \tilde{\tau}^{\mathcal{U}}) \iff (\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) = (\tilde{\tau}^{\mathcal{L}}, \tilde{\tau}^{\mathcal{U}})$$

and

$$S^{\mathcal{U}}_\theta(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) = S^{\mathcal{U}}_\theta(\tilde{\tau}^{\mathcal{L}}, \tilde{\tau}^{\mathcal{U}}) \iff (\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) = (\tilde{\tau}^{\mathcal{L}}, \tilde{\tau}^{\mathcal{U}}).$$

That is, we can construct a unique $\tau^{\mathcal{L}}$ and $\tau^{\mathcal{U}}$ given $S^{\mathcal{L}}_\theta(\tau^{\mathcal{L}}, \tau^{\mathcal{U}})$ and $S^{\mathcal{U}}_\theta(\tau^{\mathcal{L}}, \tau^{\mathcal{U}})$. For example, if $\theta' \in S^{\mathcal{L}}_\theta(\tau^{\mathcal{L}}, \tau^{\mathcal{U}})$ and $\Sigma_{XY}(\theta) > \Sigma_{XY}(\theta', \theta)$, then $\theta' \in \tau^{\mathcal{L}}$. Moreover, we can construct $(\tau^{\mathcal{L}}, \tau^{\mathcal{U}})$ in $O(K)$ (i.e., linear) time.

This observation suggests a $O(K \log K)$ algorithm for computing the truncation set as follows:

1) Begin by setting $\mathcal{Y}_R(\theta, z)$ to $\emptyset$ and sorting $\Theta_{-\theta} := \{\tilde{\theta}_1, \ldots, \tilde{\theta}_{K-1}\}$ such that $Q_\theta(\tilde{\theta}_1) \geq \ldots \geq Q_\theta(\tilde{\theta}_{K-1})$. Sorting takes $O(K \log K)$ time.

2) Assume $S_\theta^{\mathcal{L}}(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) = \Theta_{-\theta}$ and $S_\theta^{\mathcal{U}}(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) = \emptyset$ and construct $\tau^{\mathcal{L}}$ and $\tau^{\mathcal{U}}$ to satisfy this assumption. By Observation 1, these $\tau^{\mathcal{L}}$ and $\tau^{\mathcal{U}}$ are unique and can be constructed in linear time.

3) Step through $\tilde{\theta}_1, \ldots, \tilde{\theta}_{K-1}$, removing each parameter $\tilde{\theta}'$ from $S_\theta^{\mathcal{L}}(\tau^{\mathcal{L}}, \tau^{\mathcal{U}})$ and adding it to $S_\theta^{\mathcal{U}}(\tau^{\mathcal{L}}, \tau^{\mathcal{U}})$. Observation 1 implies that this step also switches the parameter from $\tau^{\mathcal{L}}$ to $\tau^{\mathcal{U}}$ or vice versa. This guarantees we consider all $S_\theta^{\mathcal{L}}(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) \subseteq \Theta_{-\theta}$ and $S_\theta^{\mathcal{U}}(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) \subseteq \Theta_{-\theta}$ satisfying

$$Q_\theta(\theta') \leq Q_\theta(\theta'') \;\forall \theta' \in S_\theta^{\mathcal{L}}(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}), \theta'' \in S_\theta^{\mathcal{U}}(\tau^{\mathcal{L}}, \tau^{\mathcal{U}})$$

and

$$S_\theta^{\mathcal{L}}(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) \cap S_\theta^{\mathcal{U}}(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) = \emptyset.$$

Thus, $\mathcal{S}_P(\theta)$ is contained in the set of arrangements we consider in this stepwise procedure.

Because we step through $\tilde{\theta}_1, \ldots, \tilde{\theta}_{K-1}$, this step takes linear time.

4) After each step in (3), check if $\exists k \in R$ such that $K - |\tau^{\mathcal{L}}| \geq k \geq |\tau^{\mathcal{U}}| + 1$, implying $(\tau^{\mathcal{L}}, \tau^{\mathcal{U}}) \in \mathcal{S}_P(\theta)$. If so, add $[Q(\tilde{\theta}_{j+1}), Q(\tilde{\theta}_j)]$ to $\mathcal{Y}_R(\theta, z)$.

Assuming ties are rare (i.e., $|S_\theta^{\mathcal{V}}|$ is small), we can check if $k \in R$ in constant time by first creating an array $A = [\mathbf{1}\{1 \in R\}, \ldots, \mathbf{1}\{K-1 \in R\}]^T$ in linear time then using $k \in R \iff A[k]$.

After the initial sorting in $O(K \log K)$ time, the rest of the algorithm takes linear time. Therefore, the entire algorithm takes $O(K \log K)$ time.

We formalize the algorithm as follows.

Algorithm 1: Efficient computation of the truncation set

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Result: The truncation set 𝒴_R(θ, z)
𝒴_R(θ, z) ← ∅;
Sort Θ_{-θ} such that Q(θ̃_1) ≥ ... ≥ Q(θ̃_{K-1});
Construct τ^ℒ, τ^𝒰 such that S_θ^ℒ(τ^ℒ, τ^𝒰) = Θ_{-θ} and S_θ^𝒰(τ^ℒ, τ^𝒰) = ∅;
if ∃k ∈ R : |τ^ℒ| = K − k, |τ^𝒰| = k − 1 then
    𝒴_R(θ, z) ← 𝒴_R(θ, z) ∪ [Q_θ(θ̃_1), ∞);
end
for θ̃_j ∈ Θ_{-θ} do
    if θ̃_j ∈ τ^ℒ then
        τ^ℒ ← τ^ℒ \ {θ̃_j};
        τ^𝒰 ← τ^𝒰 ∪ {θ̃_j};
    else
        τ^ℒ ← τ^ℒ ∪ {θ̃_j};
        τ^𝒰 ← τ^𝒰 \ {θ̃_j};
    end
    if ∃k ∈ R : |τ^ℒ| = K − k, |τ^𝒰| = k − 1 then
        if j < K − 1 then
            𝒴_R(θ, z) ← 𝒴_R(θ, z) ∪ [Q_θ(θ̃_{j+1}), Q_θ(θ̃_j)];
        else
            𝒴_R(θ, z) ← 𝒴_R(θ, z) ∪ (−∞, Q_θ(θ̃_{K-1})];
        end
    end
end
```

4. Additional Simulation Results

Figure 1 reports simulation results for conventional estimators and CIs, along with our corrected procedures, in the simulations calibrated to Chetty and Hendren (2018) and Chetty et al. (2018), based on 5,000 simulation draws.

5. Full Empirical Results

Tables 1 and 2 report naive and corrected estimates and CIs for all 50 CZs.

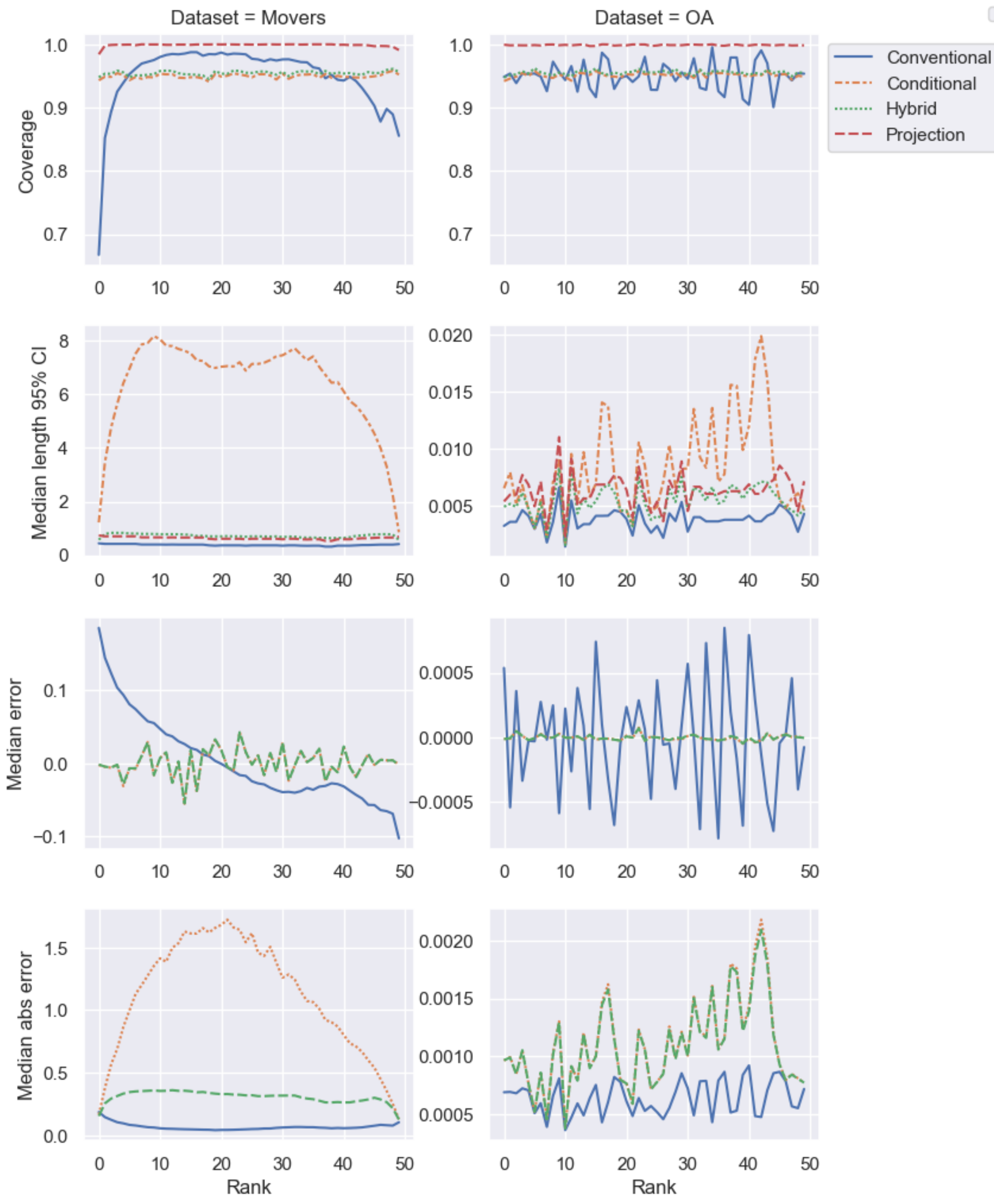

FIGURE 1. SIMULATED PERFORMANCE OF ESTIMATORS AND CIS

Note: Estimator and CI performance for simulations calibrated to Chetty and Hendren, 2018 (first column) and Chetty et al., 2018 (second column). The first row shows the simulated coverage of nominal 95% CIs, while the second shows the median estimation error. The third row reports the median length of nominal 95% CIs, while the fourth row plots the median absolute estimation error of the estimators considered.

Rank	CZ	Naive Estimate	SE	$\hat{\mu}_{\frac{1}{2}}$	CI, L	CI, U	$\hat{\mu}^H_{\frac{1}{2}}$	CI^H, L	CI^H, U
1	Seattle	0.229	0.082	0.189	-0.163	0.385	0.189	-0.076	0.389
2	Washington DC	0.163	0.077	0.108	-0.407	0.502	0.108	-0.130	0.447
3	Cleveland	0.124	0.107	-2.528	-7.229	0.666	-0.284	-0.290	0.533
4	Fort Worth	0.121	0.090	1.175	-3.884	5.305	0.464	-0.230	0.470
5	Minneapolis	0.116	0.120	1.821	-3.341	5.677	0.571	-0.359	0.582
6	Portland	0.102	0.114	-1.979	-6.182	3.066	-0.332	-0.340	0.554
7	Dayton	0.098	0.152	3.340	-6.863	9.898	0.678	-0.496	0.686
8	Las Vegas	0.088	0.071	-0.623	-2.859	1.804	-0.180	-0.189	0.371
9	Buffalo	0.084	0.112	2.159	-2.180	6.154	0.510	-0.364	0.519
10	Kansas City	0.067	0.126	-2.575	-6.633	2.909	-0.413	-0.421	0.568
11	Salt Lake City	0.063	0.129	2.253	-5.547	6.363	0.554	-0.442	0.563
12	San Diego	0.054	0.080	-0.793	-3.029	2.510	-0.249	-0.258	0.371
13	Jacksonville	0.050	0.100	0.548	-4.874	4.825	0.430	-0.339	0.438
14	Milwaukee	0.045	0.135	-2.343	-11.619	7.979	-0.471	-0.477	0.569
15	Denver	0.042	0.101	0.884	-4.158	4.242	0.427	-0.350	0.433
16	Boston	0.038	0.089	0.988	-3.315	3.838	0.375	-0.312	0.384
17	Cincinnati	0.030	0.130	-0.669	-5.722	6.061	-0.463	-0.476	0.538
18	Grand Rapids	0.024	0.155	-5.905	-7.176	7.224	-0.570	-0.574	0.626
19	Philadelphia	0.022	0.071	1.436	-3.016	4.679	0.293	-0.256	0.297
20	San Francisco	0.017	0.083	0.612	-2.708	3.630	0.330	-0.311	0.341
21	Newark	0.008	0.064	-0.049	-1.872	1.671	-0.049	-0.246	0.263
22	Phoenix	0.000	0.065	0.142	-1.396	1.947	0.142	-0.261	0.258
23	San Jose	-0.011	0.101	-3.891	-7.711	1.863	-0.398	-0.400	0.388
24	Sacramento	-0.012	0.081	3.098	-3.216	3.588	0.298	-0.327	0.300
25	Columbus	-0.016	0.122	0.725	-4.816	4.784	0.449	-0.490	0.457
26	Miami	-0.021	0.063	-0.218	-3.041	2.401	-0.218	-0.267	0.226
27	St. Louis	-0.025	0.120	-3.504	-7.525	7.166	-0.480	-0.488	0.440
28	Dallas	-0.027	0.073	1.796	-1.248	4.517	0.251	-0.320	0.256
29	Houston	-0.039	0.069	-1.676	-3.939	1.044	-0.302	-0.306	0.238
30	Nashville	-0.041	0.129	4.901	-3.845	12.259	0.453	-0.549	0.457
31	Providence	-0.053	0.130	-3.767	-5.353	4.298	-0.550	-0.556	0.459
32	Pittsburgh	-0.056	0.125	-0.056	-5.656	5.544	-0.056	-0.540	0.428
33	Indianapolis	-0.059	0.129	3.315	-5.538	5.841	0.434	-0.564	0.440
34	Baltimore	-0.068	0.100	0.574	-1.486	3.778	0.303	-0.476	0.325
35	Detroit	-0.092	0.081	-3.197	-9.292	0.155	-0.402	-0.404	0.202
36	Atlanta	-0.093	0.055	1.998	-0.865	5.777	0.117	-0.313	0.119
37	Manchester	-0.102	0.139	0.550	-5.445	6.337	0.421	-0.649	0.441
38	Bridgeport	-0.115	0.093	0.164	-1.462	2.342	0.164	-0.496	0.256
39	Tampa	-0.138	0.068	-0.383	-1.832	0.561	-0.377	-0.409	0.146
40	New York	-0.148	0.047	-0.036	-0.502	0.669	-0.036	-0.339	0.043
41	Los Angeles	-0.170	0.040	-1.282	-3.215	-0.089	-0.323	-0.325	-0.076
42	Austin	-0.171	0.115	4.240	-3.533	16.929	0.270	-0.622	0.272
43	Chicago	-0.180	0.059	-0.180	-1.599	1.239	-0.180	-0.416	0.056
44	Orlando	-0.189	0.069	0.050	-1.186	1.764	0.050	-0.471	0.085
45	San Antonio	-0.206	0.098	0.104	-0.987	1.882	0.104	-0.625	0.188
46	Charlotte	-0.248	0.096	-0.607	-2.499	0.498	-0.581	-0.632	0.162
47	Port St. Lucie	-0.263	0.090	-0.263	-2.245	1.719	-0.263	-0.624	0.098
48	Raleigh	-0.278	0.105	0.216	-0.605	2.436	0.093	-0.620	0.141
49	Fresno	-0.377	0.100	-0.858	-3.056	-0.085	-0.732	-0.775	-0.071
50	New Orleans	-0.391	0.111	0.206	-0.524	2.859	0.009	-0.532	0.050

Table 1—Estimates based on Chetty and Hendren (2018) data

Note: Naive and corrected estimates based on movers data. The column $\hat{\mu}_{\frac{1}{2}}$ reports the conditionally median-unbiased estimate for each CZ, conditioning on the CZ's rank, while columns CI, L and CI, U report the lower and upper endpoints for the conditional CI, respectively. Columns $\hat{\mu}^H_{\frac{1}{2}}$, CI^H, L, and CI^H, U similarly report estimates and CI endpoints based on the hybrid approach.

Rank	CZ	Naive Estimate	SE	$\hat{\mu}_{\frac{1}{2}}$	CI, L	CI, U	$\hat{\mu}^H_{\frac{1}{2}}$	CI^H, L	CI^H, U
1	San Francisco	0.456617	0.000816	0.444	0.422	0.457	0.454	0.453	0.457
2	Salt Lake City	0.456580	0.001297	0.488	0.457	0.527	0.462	0.457	0.462
3	Boston	0.453016	0.000907	0.453	0.450	0.455	0.453	0.450	0.455
4	Minneapolis	0.451732	0.001171	0.452	0.449	0.456	0.452	0.449	0.456
5	San Jose	0.448515	0.001040	0.449	0.446	0.451	0.449	0.446	0.451
6	Newark	0.445783	0.000750	0.446	0.444	0.447	0.446	0.444	0.447
7	Pittsburgh	0.441542	0.001082	0.441	0.438	0.444	0.441	0.438	0.444
8	New York	0.440292	0.000445	0.440	0.439	0.441	0.440	0.439	0.441
9	Seattle	0.438663	0.000903	0.438	0.435	0.441	0.438	0.436	0.441
10	Manchester	0.437685	0.001678	0.439	0.435	0.449	0.439	0.434	0.444
11	Los Angeles	0.430816	0.000352	0.431	0.430	0.432	0.431	0.430	0.432
12	Providence	0.430099	0.001387	0.432	0.427	0.440	0.432	0.427	0.436
13	Washington DC	0.427107	0.000758	0.427	0.425	0.429	0.427	0.425	0.429
14	Sacramento	0.426089	0.000925	0.424	0.416	0.429	0.424	0.422	0.429
15	San Diego	0.425785	0.000856	0.427	0.424	0.435	0.427	0.424	0.429
16	Houston	0.421016	0.000605	0.416	0.398	0.422	0.419	0.419	0.422
17	Denver	0.420967	0.001042	0.436	0.416	0.464	0.425	0.417	0.425
18	Bridgeport	0.420453	0.001045	0.421	0.417	0.428	0.421	0.417	0.424
19	Portland	0.419309	0.001166	0.420	0.417	0.424	0.420	0.417	0.424
20	Buffalo	0.415368	0.001131	0.415	0.413	0.418	0.415	0.413	0.418
21	Fort Worth	0.406522	0.000972	0.406	0.404	0.408	0.406	0.404	0.408
22	Miami	0.405049	0.000599	0.405	0.404	0.406	0.405	0.404	0.406
23	Austin	0.403423	0.001282	0.403	0.398	0.408	0.403	0.399	0.408
24	San Antonio	0.402263	0.000873	0.402	0.397	0.405	0.402	0.399	0.405
25	Philadelphia	0.401706	0.000661	0.402	0.400	0.405	0.402	0.400	0.404
26	Phoenix	0.397362	0.000814	0.397	0.392	0.399	0.397	0.394	0.399
27	Chicago	0.396876	0.000549	0.397	0.396	0.399	0.397	0.396	0.399
28	Kansas City	0.395848	0.001097	0.396	0.391	0.400	0.396	0.392	0.400
29	Fresno	0.394903	0.000925	0.395	0.393	0.398	0.395	0.393	0.398
30	Grand Rapids	0.392111	0.001353	0.392	0.389	0.395	0.392	0.389	0.395
31	Dallas	0.389610	0.000687	0.389	0.387	0.391	0.389	0.387	0.391
32	Milwaukee	0.388879	0.001242	0.364	0.321	0.392	0.384	0.384	0.392
33	Las Vegas	0.388836	0.001012	0.405	0.389	0.435	0.393	0.388	0.393
34	Orlando	0.385193	0.000903	0.379	0.353	0.386	0.382	0.382	0.386
35	Cleveland	0.385097	0.000917	0.382	0.350	0.417	0.382	0.381	0.389
36	Port St. Lucie	0.385023	0.001035	0.395	0.384	0.430	0.389	0.384	0.389
37	St. Louis	0.378417	0.000943	0.357	0.331	0.378	0.375	0.375	0.378
38	Nashville	0.378389	0.001202	0.413	0.378	0.473	0.383	0.377	0.383
39	New Orleans	0.377470	0.000954	0.374	0.356	0.381	0.374	0.374	0.381
40	Tampa	0.377310	0.000804	0.380	0.376	0.392	0.380	0.376	0.381
41	Dayton	0.374576	0.001244	0.370	0.350	0.377	0.371	0.370	0.377
42	Detroit	0.374336	0.000708	0.375	0.368	0.382	0.375	0.371	0.377
43	Cincinnati	0.374056	0.001046	0.375	0.365	0.389	0.375	0.370	0.378
44	Baltimore	0.373670	0.000918	0.375	0.371	0.382	0.375	0.371	0.378
45	Columbus	0.372527	0.001104	0.373	0.370	0.377	0.373	0.370	0.376
46	Raleigh	0.368923	0.001296	0.369	0.366	0.372	0.369	0.366	0.372
47	Indianapolis	0.363872	0.001201	0.364	0.362	0.366	0.364	0.361	0.366
48	Jacksonville	0.358469	0.001047	0.354	0.332	0.360	0.355	0.354	0.360
49	Atlanta	0.358312	0.000684	0.360	0.357	0.369	0.360	0.357	0.361
50	Charlotte	0.355478	0.001087	0.355	0.353	0.358	0.355	0.353	0.358

Table 2—Estimates based on Chetty et al. (2018) data

Note: Naive and corrected estimates based on OA data. The column $\hat{\mu}_{\frac{1}{2}}$ reports the conditionally median-unbiased estimate for each CZ, conditioning on the CZ's rank, while columns CI, L and CI, U report the lower and upper endpoints for the conditional CI, respectively. Columns $\hat{\mu}^H_{\frac{1}{2}}$, CI^H, L, and CI^H, U similarly report estimates and CI endpoints based on the hybrid approach.