Online Appendix for

“Addressing Strategic Uncertainty with Incentives and Information”

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A. Proofs for Section 2

Proof of Lemma 1. Let us first recall our rationalizability notion. Given an incentive scheme $\sigma = \langle q, \chi \rangle$ we define the sets $\{T^\sigma_i(\kappa)\}_{i \in N, \kappa \in \mathbb{Z}_+}$ as follows. Let $T^\sigma_i(0) := \emptyset$, and then, recursively for $\kappa \in \mathbb{N}$, let $T^\sigma_i(\kappa)$ be the set of all $t_i \in T^q_i$ such that every $\eta \in \Delta \left(2^{N\setminus\{i\}} \times T^q_{-i} \times \Omega\right)$ with $\text{marg}_{T^q_{-i}, \kappa} \eta = q_i(\cdot \mid t_i)$ and $\{ j \in N \setminus \{i\} : t_j \in T^\sigma_i(\kappa - 1) \} \subseteq J$, $\forall (J, t_{-i}, \omega) \in \text{supp}(\eta)$ has

$$\sum_{J \subseteq N \setminus \{i\}, t_{-i} \in T^q_{-i}, \omega \in \Omega} \eta(J, t_{-i}, \omega) \left[ u_i(J \cup \{i\}, \chi_i(t_i), \omega) - u_i(J, \chi_i(t_i), \omega) \right] > 0.$$ 

By definition of interim correlated rationalizability (Dekel et al., 2007), incentive scheme $\sigma$ is UIF if and only if $\bigcup_{\kappa=0}^\infty T^\sigma_i(\kappa) = T^q_i$ for every $i \in N$.

Now, in what follows, say a type profile $t$ has no ties if $t^R_i \neq t^R_j$ for all distinct $i, j \in N$.

To prove the first assertion, suppose $\sigma = \langle q, \chi \rangle$ is a strict ranking scheme. Let us prove by induction on $\kappa \in \mathbb{Z}_+$ that, if $i \in N$ and $t_i \in T^q_i$ have $t^R_i = \kappa$, then $t_i \in T^\sigma_i(\kappa)$—from which it will follow directly that $\sigma$ is UIF. The claim holds vacuously for $\kappa = 0$, so take $\kappa \in \mathbb{N}$ and $i \in N$, and assume the claim holds for all $i' \in N$ and all $\kappa' \in \{0, \ldots, \kappa - 1\}$. Next observe that $\chi_i(t_i) \in \mathcal{X}^*_i(\mu^q_i(\kappa))$ because $\sigma$ is a strict ranking scheme; and the inductive hypothesis implies $t_{-i} \in T^\sigma_{-i}(\kappa - 1)$ for every $t_{-i} \in T^q_{-i}$ such that $(t_i, t_{-i})$ has no ties and $\pi_j(t) < \pi_i(t)$. Hence, by definition, $t_i \in T^\sigma_i(\kappa)$ as desired.

To prove the second assertion, suppose $\sigma = \langle q, \chi \rangle$ is an arbitrary UIF incentive scheme.
For each $i \in N$, define the map $k^\sigma_i : T^q_i \to \mathbb{N}$ by letting $k^\sigma_i(t_i) := \min\{\kappa \in \mathbb{N} : t_i \in T^\sigma_i(\kappa)\}$.

It is easy to see some one-to-one function $\tilde{\lambda} : \bigcup_{i \in N} \{i\} \times T^q_i \to \mathbb{N}$ exists such that, for any $i, j \in N$ and $t_i \in T^q_i$, $t_j \in T^q_j$ with $k^\sigma_i(t_i) > k^\sigma_j(t_j)$, we have $\tilde{\lambda}_i(t_i) > \tilde{\lambda}_j(t_j)$. Then, define $\lambda : \bigcup_{i \in N} \{i\} \times T^q_i \to \mathbb{N}^2$ by letting $\lambda_i(t_i) := (\tilde{\lambda}_i(t_i), 1)$.

Now, define the incentive scheme $\sigma^* := \langle q^*, \chi^* \rangle$ by letting

$$q^*(t^*, \omega) := q\left((\lambda_i^{-1}(t_i))^i \in N, \omega\right)$$

for every $t^* \in (\mathbb{N}^2)^N$ and $\omega \in \Omega$, and letting $\chi^*_i(t^*_i) := \chi_i(\lambda_i^{-1}(t^*_i))$ for every $i \in N$ and $t^*_i \in T^q_i$. That the modified scheme is UIF follows from the original scheme being UIF (Dekel et al., 2007, by Proposition 1) given that type $t_i$’s hierarchy of beliefs over $X \times \Omega$ under $\sigma$ are the same as type $\lambda_i(t_i)$’s under $\sigma^*$. Further, because $\sigma^*$ generates the same distribution over $X \times \Omega$ as $\sigma$ does, it follows directly that $V(\sigma^*) = V(\sigma)$. All that remains is to see $\sigma^*$ is a strict ranking scheme. That $q^*$ exhibits no ties is immediate from the construction. Moreover, given any $i, j \in N$, observe any $t^*_i \in T^q_i$ and $t^*_j \in T^q_j$ have $k^\sigma_i(t^*_i) > k^\sigma_j(t^*_j)$ if and only if $k^\sigma_i(\lambda_i^{-1}(t^*_i)) > k^\sigma_j(\lambda_j^{-1}(t^*_j))$, which in turn implies $t^*_i R^* t^*_j$. It therefore follows from $t^*_i \in T^q_i \ (k^\sigma_i(t^*_i))$ that $\chi^*_i(t^*_i) \in X^*_i(\mu^q_i(t^*_i))$, and so $\sigma^*$ is a strict ranking scheme. Q.E.D.

**Proof of Theorem 1.** We first show that $\sup_{\sigma} V(\sigma) \leq \sup_{\mu \in \mathcal{M}(p_0)} \sum_{i \in N} \hat{v}_i^*(\mu)$. Given Lemma 1, it suffices to show that the principal’s value for a strict ranking scheme $\langle q, \chi \rangle$ is no greater than $\sup_{\mu \in \mathcal{M}(p_0)} \sum_{i \in N} \hat{v}_i^*(\mu)$. Bayesian updating implies that a given agent $i$’s belief is, on average, equal to the true distribution over total states:

$$\sum_{t_i \in T_i^q} q(t)\mu_i^q(\cdot|t_i) = \sum_{t_i \in T_i^q} q_i(t_i)\mu_i^q(\cdot|t_i) = \mu^q \in \mathcal{M}(p_0).$$

Hence, the belief distribution $\tau_i \in \Delta\Delta(\Pi \times \Omega)$ given by $\sum_{t_i \in T_i^q} q_i(t_i)\delta_{\mu_i^q(\cdot|t_i)}$ is feasible in the
program defining $\hat{v}_i^*(\mu^q)$. It follows that

$$\sum_{t \in T^i, \omega \in \Omega} q(t, \omega)v_i^*(\mu_i^q(\cdot|t_i)) \leq \hat{v}_i^*(\mu^q),$$

and so summing over $i \in N$ yields $V(q) \leq \sum_{i\in N} \hat{v}_i^*(\mu^q)$.

To show $\sup_{\sigma \text{ is } \text{UIF}} V(\sigma) \geq \sup_{\mu \in \mathcal{M}(p_0)} \sum_{i \in N} \hat{v}_i^*(\mu)$, consider an arbitrary $\mu \in \mathcal{M}(p_0)$ and $\varepsilon > 0$. We will construct a strict ranking scheme $\sigma = \langle q, \chi \rangle$ such that $V(\sigma) \geq \sum_{i \in N} [\hat{v}_i^*(\mu) - 3\varepsilon]$. To do so, observe $\hat{v}_i^*$ is bounded above by some constant $L_i \in \mathbb{R}$ for each $i \in N$ because $v_i^*$ is. In what follows, let $m \in \mathbb{N}$ be large enough that $m \geq |N|$ and $\frac{2|N|}{m} [L_i - v_i^*(\pi_i, \omega)] \leq \varepsilon$ for each $i \in N$ and $\omega \in \Omega$.

Consider any $i \in N$. Some $\tau_i \in \Delta \Delta(\Pi \times \Omega)$ exists such that $\int \mu_i \; d\tau_i(\mu_i) = \mu$ and $\int v_i^* \; d\tau_i \geq \hat{v}_i^*(\mu) - \varepsilon$. For each $\mu_i \in \text{supp}(\tau_i)$, the definition of $v_i^*$ implies some $x_i^{\mu_i} \in \mathcal{X}_i^*(\mu_i)$ exists such that $\sum_{\pi \in \Pi, \omega \in \Omega} \mu_i(\pi, \omega) \; v_i(x_i^{\mu_i}, \omega) \geq v^*(\mu_i) - \varepsilon$. By the splitting lemma, some $\gamma_i : \Pi \times \Omega \rightarrow \Delta \mathbb{N}$ exists such that, when the prior distribution over $\Pi \times \Omega$ is $\mu$ and the results of Blackwell experiment $\gamma_i$ are observed, the induced distribution of beliefs over $\Pi \times \Omega$ is $\tau_i$. Letting $\overline{s}_i \in \mathbb{N}$ denote the number of positive-probability signals in $\mathbb{N}$ given prior $\mu$ and experiment $\gamma_i$, we can assume without loss that the positive-probability signals are exactly $\{1, \ldots, \overline{s}_i\}$. For each $s_i \in \{1, \ldots, \overline{s}_i\}$, let $x_i^{s_i}$ denote $x_i^{\mu_i}$, where $\mu_i$ is the belief induced by signal realization $s_i$ from this experiment.

Now, we construct our incentive scheme $\sigma = \langle q, \chi \rangle$. Define the prior $q \in \Delta[(\mathbb{N}^2)^N \times \Omega]$ by letting, for each $t = (t_i^R, t_i^S)_{i \in N} \in (\mathbb{N}^2)^N$ and $\omega \in \Omega$,

$$q(t, \omega) := \begin{cases} \frac{1}{m} \mu(\pi, \omega) \prod_{i \in N} \gamma_i(t_i^S|\pi, \omega) & : \exists \ell \in \{0, \ldots, m - 1\} \text{ with } t_i^R = \ell + \pi_i \text{ for all } i \in N, \\ 0 & : \text{otherwise}; \end{cases}$$
and the allocation rule $\chi = (\chi_i)_{i \in N}$ via

$$
\chi_i(t^R_i, t^S_i) := \begin{cases} 
    t^S_i & : t^S_i \leq \bar{s}_i \text{ and } N \leq t^R_i \leq m, \\
    \bar{F}_i & : \text{otherwise}.
\end{cases}
$$

By construction, this scheme has no ties: $t^R_i \neq t^R_j$ for all distinct $i, j \in N$ and any supported type profile $t \in T^q$. Moreover, for each $i \in N$, a direct computation shows every type $t_i \in T^q_i$ with $|N| \leq t^R_i \leq m$ has belief $\mu_i^q(\cdot | t_i) = \mu_i^R$ and thus has $\chi_i(t_i) = x^t_i \in X_i^s(\mu_i^q(\cdot | t_i))$. Because every other $t_i \in T^q_i$ has $\chi_i(t_i) = \bar{F}_i \in \bigcap_{\mu_i \in \Delta(\Pi \times \Omega)} X_i^s(\mu_i)$, it follows that $\sigma$ is a strict ranking scheme. Finally, let us bound (from below) the value of this scheme to the principal. To do so, consider any agent $i \in N$ and $s_i \in \{1, \ldots, \bar{s}_i\}$, and observe that $\sigma$ generates belief $\mu_i^{s_i} \in \Delta(\Pi \times \Omega)$ for agent $i$ with probability

$$
marg q \left\{ t_i = (t_i^R, t_i^S) \in T^q_i : \mu_i^q(\cdot | t_i) = \mu_i^{s_i} \right\} \geq \sum_{\pi \in \Pi} \sum_{\omega \in \Omega} \frac{1}{m} \sum_{\ell=0}^{m-1} 1_{|N| \leq \ell + \pi_i \leq m} \mu(\pi, \omega) \gamma_i(s_i | \pi, \omega) \\
\geq \left(1 - \frac{2|N|}{m}\right) \sum_{\pi \in \Pi, \omega \in \Omega} \mu(\pi, \omega) \gamma_i(s_i | \pi, \omega) \\
\geq \left(1 - \frac{2|N|}{m}\right) \tau_i(\mu_i^{s_i}).
$$

Hence, the principal’s payoff from this strict ranking scheme is

$$
V(\sigma) \geq \sum_{i \in N} \left\{ \frac{2|N|}{m} \left[ \min_{\omega \in \Omega} v_i^*(x_i^s) \right] + \left(1 - \frac{2|N|}{m}\right) \sum_{s_i=1}^{\bar{s}_i} \tau_i(\mu_i^{s_i}) \sum_{\omega \in \Omega} \marg q(\mu_i^{s_i}(\omega)) v_i(x_i^{s_i}, \omega) \right\} \\
\geq \sum_{i \in N} \left\{ \frac{2|N|}{m} L_i - \varepsilon + \left(1 - \frac{2|N|}{m}\right) \sum_{s_i=0}^{\bar{s}_i-1} \tau_i(\mu_i^{s_i}) \left[ v_i^*(\mu_i^{s_i}) - \varepsilon \right] \right\} \\
\geq \sum_{i \in N} \left\{ \frac{2|N|}{m} \hat{v}_i^*(\mu) - \varepsilon + \left(1 - \frac{2|N|}{m}\right) \left[ \hat{v}_i^*(\mu) - 2\varepsilon \right] \right\} \\
\geq \sum_{i \in N} [\hat{v}_i^*(\mu) - 3\varepsilon],
$$

as required. \[Q.E.D.\]
Proof of Fact 1. Let $\mathcal{P}$ denote the set of Borel probability measures on $\Delta(\Pi \times \Omega)$, a compact space when endowed with its weak* topology.

Take any $i \in N$. Because an upper semicontinuous function over a compact space attains a maximum, for any $\mu \in \Delta(\Pi \times \Omega)$, the program $\sup_{\tau_i \in \mathcal{P}} \int \mu_i \ d\tau_i = \mu \int v_i^* \ d\tau_i$—which relaxes the program defining $\tilde{v}_i^*(\mu)$ by allowing distributions with infinite support—admits an optimum. Moreover, by the upper semicontinuous version of Berge’s theorem, this optimal value is an upper semicontinuous function of $\mu$. Now, Carathéodory’s theorem tells us some optimum to the aforementioned program has a finely independent (hence, of cardinality no more than $N! \times |\Omega|$) support. It follows that the program defining $\tilde{v}_i^*(\mu)$ admits an optimum, and that $\tilde{v}_i^*$ is upper semicontinuous.

Finally, because $\sum_{i \in N} \tilde{v}_i^*$ is upper semicontinuous and $\mathcal{M}(p_0)$ is compact, the program $\sup_{\mu \in \mathcal{M}(p_0)} \sum_{i \in N} \tilde{v}_i^*(\mu)$ admits an optimum. Q.E.D.

B. Proofs for Section 3

Toward proving the results of Section 3, some preliminary claims will be useful.

Claim 1. Suppose $i \in N$ and $\mu \in \Delta(\Pi \times \Omega)$. If $\tau_i$ is an optimal solution to

$$\min_{\tau_i \in \Delta(\Pi \times \Omega)} \int \frac{c_i(\mu_i^\Pi)}{\bar{v}_i(\mu_i^\Pi)} \ d\tau_i(\mu_i) \quad \text{subject to} \quad \int \mu_i \ d\tau_i(\mu_i) = \mu,$$

then no $\bar{\omega}, \hat{\omega} \in \Omega$ with $c_i(\bar{\omega}) = c_i(\hat{\omega})$ and distinct $\bar{\beta}, \hat{\beta} \in \Delta\Pi$ have both $\bar{\beta} \otimes \delta_{\bar{\omega}}$ and $\hat{\beta} \otimes \delta_{\hat{\omega}}$ in the support of $\tau_i$.

Proof. Suppose $\bar{\omega}, \hat{\omega} \in \Omega$ with $c_i(\bar{\omega}) = c_i(\hat{\omega}) =: \bar{c}_i$ and distinct $\bar{\beta}, \hat{\beta} \in \Delta\Pi$ have both $\bar{\beta} \otimes \delta_{\bar{\omega}}$ and $\hat{\beta} \otimes \delta_{\hat{\omega}}$ in the support of $\tau_i$. Then, some $\varepsilon \in (0, 1]$ and $\bar{\tau}_i \in \Delta\Delta(\Pi \times \Omega)$ exists such that

$$\tau_i = (1 - \varepsilon)\bar{\tau}_i + \frac{\varepsilon}{2} \delta_{\bar{\beta} \otimes \delta_{\bar{\omega}}} + \frac{\varepsilon}{2} \delta_{\hat{\beta} \otimes \delta_{\hat{\omega}}}.$$
The alternative belief distribution
\[ \tau'_i = (1 - \varepsilon)\tau_i + \varepsilon \delta_{1/2}(\tilde{\beta} \otimes \delta_\omega + \hat{\beta} \otimes \delta_\omega) \]
is then feasible in the given program. Moreover, by strict convexity of \( \frac{\tau}{i(\beta)} \) in \( \beta \in \Delta \Pi \), the latter attains a strictly lower loss, so that \( \tau_i \) is not optimal. \( Q.E.D. \)

**Claim 2.** Suppose \( i \in N \) and \( \beta_0 \in \Delta \Pi \). If \( \tau_i \) is an optimal solution to the program
\[
\min_{\tau_i \in \Delta \Pi(\Pi \times \Omega)} \int \frac{c_i(\mu^\Omega_i)}{i(\mu^\Pi_i)} \, d\tau_i(\mu_i) \quad \text{subject to} \quad \int (\mu^\Pi_i, \mu^\Omega_i) \, d\tilde{\tau}_i(\mu_i) = (\beta_0, p_0),
\]
then some alternative optimal \( \tilde{\tau}_i \) exists such that

- Each \( \omega \in \Omega \) admits a unique \( \tilde{\beta}_\omega \in \Delta \Pi \) such that \( \tilde{\tau}_i(\tilde{\beta}_\omega \otimes \delta_\omega) = p_0(\omega) \);
- Any \( \mu_i \) in the support of \( \tau_i \) and any \( \omega, \hat{\omega} \in \Omega \) in the support of \( \mu^\Omega_i \) have \( \tilde{\beta}_\omega = \tilde{\beta}_{\hat{\omega}} \).

**Proof.** Let \( \tilde{\tau}_i := \int \int \delta_{\mu^\Omega_i \otimes \delta_\omega} \, d\mu^\Omega_i(\omega) \, d\tau_i(\mu_i) \in \Delta \Delta(\Pi \times \Omega) \).

Various features are immediate from the construction. First, the average marginal distributions under \( \tilde{\tau}_i \) are the same as those under \( \tau_i \), making \( \tilde{\tau}_i \) feasible in the program. Second, because the fraction \( \frac{c_i(\mu^\Omega_i)}{i(\mu^\Pi_i)} \) is affine in \( \mu^\Omega_i \) when holding \( \mu^\Pi_i \) fixed, we know \( \tilde{\tau}_i \) yields the same value in program (4) as \( \tau_i \) does, and so is optimal too. Third, every \( \tilde{\mu}_i \in \text{the support of } \tilde{\tau}_i \) admits some \( \tilde{\beta} \in \Delta \Pi \) and \( \omega \in \Omega \) for which \( \tilde{\mu}_i = \tilde{\beta} \otimes \delta_\omega \). Fourth, for any \( \mu_i \) in the support of \( \tau_i \) and any \( \omega, \hat{\omega} \in \Omega \) in the support of \( \mu^\Omega_i \), some \( \tilde{\beta} \in \Delta \Pi \) has both \( \tilde{\beta} \otimes \delta_\omega \) and \( \hat{\beta} \otimes \delta_{\hat{\omega}} \) in the support of \( \tilde{\tau}_i \)—indeed, \( \tilde{\beta} = \mu^\Pi_i \) has this property.

The claim will then follow if we know that no \( \omega \in \Omega \) and distinct \( \tilde{\beta}, \hat{\beta} \in \Delta \Pi \) have both \( \tilde{\beta} \otimes \delta_\omega \) and \( \hat{\beta} \otimes \delta_{\hat{\omega}} \) in the support of \( \tilde{\tau}_i \). And indeed, this fact follows directly from Claim 1. \( Q.E.D. \)

**Claim 3.** For any \( c_H \geq c_L > 0 \), the program
\[
\min_{(\beta^H, \beta^L) \in [0,1]^2} \left\{ \frac{c_H}{(1-\beta^H)(P_1-P_0) + \beta^H(P_2-P_1)} + \frac{c_L}{(1-\beta^L)(P_1-P_0) + \beta^L(P_2-P_1)} \right\} \quad \text{subject to} \quad \beta^H + \beta^L = 1
\]
has a unique optimal solution \((\beta^H, \beta^L)\). It has

\[
\beta^H = \begin{cases} 
\frac{\sqrt{c_H} - \phi \sqrt{c_L}}{(1 - \phi)(\sqrt{c_H} + \sqrt{c_L})} & : \phi \sqrt{c_H} < \sqrt{c_L} \\
1 & : otherwise.
\end{cases}
\]

Moreover, if \(c_H > c_L\), then \(\beta^H > \frac{1}{2}\).

**Proof.** Substituting in \(\beta^L = 1 - \beta^H\), we can view the program as an optimization over \(\beta^H \in [0, 1]\). The loss is continuous in \(\beta^H\) so that an optimum exists, and it is strictly convex in \(\beta^H\) so that this optimum is unique. Direct computation shows that the given form of \(\beta^H\) satisfies the first-order condition, and hence is the optimum.

Finally, supposing \(c_H > c_L\), let us show \(\beta^H > \frac{1}{2}\). Indeed, in this case,

\[
2(\sqrt{c_H} - \phi \sqrt{c_L}) - (1 - \phi)(\sqrt{c_H} + \sqrt{c_L}) = (1 + \phi)(\sqrt{c_H} - \sqrt{c_L}) > 0,
\]

so that \(\beta^H \geq \min \left\{ 1, \frac{\sqrt{c_H} - \phi \sqrt{c_L}}{(1 - \phi)(\sqrt{c_H} + \sqrt{c_L})} \right\} > \frac{1}{2}.

Q.E.D.

**B.1. Toward Proposition 1**

**Proof of Proposition 1.** Some optimal solution to program (3) exists by Fact 1. Moreover, by Claim 1, any optimal solution \((\mu, \tau_1, \tau_2)\) has \(\tau_1^\Pi (\mu^\Pi) = \tau_2^\Pi (\mu^\Pi) = 1\).

Hence, all that remains to see is that the program

\[
\min_{\beta \in \Delta_1} \sum_{i \in N} \frac{c_i}{k_i(\beta)}
\]

is uniquely solved by setting

\[
\beta(\pi^1) = \begin{cases} 
\frac{\sqrt{c_H} - \phi \sqrt{c_L}}{(1 - \phi)(\sqrt{c_H} + \sqrt{c_L})} & : \phi \sqrt{c_H} < \sqrt{c_L} \\
1 & : otherwise,
\end{cases}
\]
which follows directly from Claim 3 (with $\beta(\pi^1)$ corresponding to $\beta^H$ in that claim). Q.E.D.

B.2. Toward Proposition 2

Claim 4. Suppose $c_1(1) = c_2(1) = c_1(2)$. Let $i \in N$, let $\beta_0 \in \Delta \Pi$ be uniform, and suppose $\tau_i$ is a feasible solution to the program (4) from Claim 2’s statement. Then, some feasible solution to program (3) exists that generates loss $2\int \frac{c_i(\mu^0_j)}{\mu_j} d\tau_i(\mu_i)$.

Proof. Let $\psi : \Pi \times \Omega \rightarrow \Pi \times \Omega$ be the involution that changes every coordinate. Define $\Psi : \Delta(\Pi \times \Omega) \rightarrow \Delta(\Pi \times \Omega)$ by letting $\Psi(\tilde{\mu}) := \tilde{\mu} \circ \psi^{-1}$ for every $\tilde{\mu} \in \Delta(\Pi \times \Omega)$. Let $j$ be such that $N = \{i, j\}$, and define $\tau_j := \tau_i \circ \Psi^{-1}$. It follows from $v_i^* = v_2^* \circ \Psi$ that

$$\sum_{k \in N} \int \frac{c_k(\mu^0_j)}{\mu_j} d\tau_k(\mu_k) = 2 \int \frac{c_i(\mu^0_j)}{\mu_j} d\tau_i(\mu_i).$$

If some $\mu \in \Delta(\Pi \times \Omega)$ is such that $(\mu, \tau_1, \tau_2)$ is feasible in program (3), we will have a feasible triple with the desired property. To that end, define $\mu := \int \mu_i d\tau_i(\mu_i)$, and note that $\int \mu_j d\tau_j(\mu_j) = \Psi(\mu)$ by construction. It then suffices to observe that $\mu = \Psi(\mu)$. But this property follows from both marginals $\mu^\Pi, \mu^\Omega$ being uniform on their respective domains. Q.E.D.

Claim 5. Suppose $c_1(1) = c_2(2) =: c_H > c_L := c_2(1) = c_1(2)$. Let $i \in N$, let $\beta_0 \in \Delta \Pi$ be uniform, and suppose $\tau_i$ is an optimal solution to the program (4) from Claim 2’s statement. If $\tau_i \{\mu_i \in \Delta(\Pi \times \Omega) : \mu_i^\Omega(\omega) = 1 \text{ for some } \omega \in \Omega\} = 1$, then $\tau_i(\beta_1^* \otimes \delta_1) = \tau_i(\beta_2^* \otimes \delta_2) = \frac{1}{2}$, where

$$\beta_1^*(\pi^1) = \beta_2^*(\pi^2) = \begin{cases} \frac{\sqrt{c_H - \varphi_c c_L}}{(1-\varphi_c)(\sqrt{c_H} + \sqrt{c_L})} & : \varphi_c \sqrt{c_H} < \sqrt{c_L} \\ 1 & : \text{otherwise} \end{cases}$$

$8$So, if $N = \{i, j\} = \{i', j'\}$, then $\psi(i', j') = (i', j')$.

$9$Consider the $2 \times 2$ matrix whose $(i', j')$ entry is $\mu(\pi_{i'}, j') - \frac{1}{4}$ for each $i', j' \in N$. Every row and every column of this matrix sums to zero, and so it is proportional to $\pm \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Q.E.D.
Proof. Assume $\tau_i$ has the hypothesized properties. First, observe no $\omega \in \Omega$ and distinct $\hat{\beta}, \hat{\beta} \in \Delta \Pi$ have both $\hat{\beta} \otimes \delta_\omega$ and $\hat{\beta} \otimes \delta_\omega$ in the support of $\tau_i$, by Claim 1. Hence, some $\beta_1, \beta_2 \in \Delta \Pi$ exist such that $\tau_i \{ \beta_1 \otimes \delta_1, \beta_2 \otimes \delta_2 \} = 1$. Optimality of $\tau_i$ for program (4) then tells us $(\beta_i(\pi^i), \beta_i(\pi^j))$ is an optimal solution to

$$
\min_{(\beta^H, \beta^L) \in [0,1]^2} \left\{ \frac{c^H}{(1-\beta^H)(P_1-P_2)} + \beta^H(P_2-P_3) + \frac{c^L}{(1-\beta^L)(P_1-P_3)} + \beta^L(P_3-P_2) \right\} \text{ subject to } \beta^H + \beta^L = 1.
$$

The claim then follows directly from Claim 3. \textit{Q.E.D.}

Now, we prove Proposition 2.

Proof of Proposition 2. Let $(\mu, \tau_1, \tau_2)$ be any optimal solution to (3) (which exists by Fact 1).

Our first step is to construct an alternative optimum that satisfies a symmetry property. To construct such an optimum, recall the map $\Psi : \Delta(\Pi \times \Omega) \rightarrow \Delta(\Pi \times \Omega)$ defined in the proof of Claim 4. Symmetry of $p_0$ implies $\Psi(\mu) \in \mathcal{M}(p_0)$ because $\mu \in \mathcal{M}(p_0)$; because $\mathcal{M}(p_0)$ is convex, it therefore also contains $\hat{\mu} := \frac{1}{2}[\mu + \Psi(\mu)]$. For each $\{i, j\} = N$, define $\hat{\tau}_i := \frac{1}{2}[\tau_i + \tau_j \circ \Psi^{-1}]$.

Some properties of $(\hat{\mu}, \hat{\tau}_1, \hat{\tau}_2)$ are immediate from the construction. First, the mean of $\hat{\tau}_i$ is $\hat{\mu}$ for each $i \in N$, so that $\hat{\mu}, \hat{\tau}_1, \hat{\tau}_2$ is feasible in program (3). Second, $\hat{\tau}_1 = \hat{\tau}_2 \circ \Psi^{-1}$. Third, that $v^*_1 = v^*_2 \circ \Psi$ implies $(\hat{\mu}, \hat{\tau}_1, \hat{\tau}_2)$ attains the same value as $(\mu, \tau_1, \tau_2)$ does in program (3), and so is optimal too.

Now, let $\beta_i \in \Delta \Pi$ be the uniform distribution and $i \in N$. Let us show, for $\beta_0 = \beta_i$ and $i \in N$, that $\hat{\tau}_i$ solves the program (4) defined in Claim 2’s statement. Assume otherwise for a contradiction. So some $\hat{\tau}_i \in \Delta \Delta(\Pi \times \Omega)$ has $\int (\mu^i, \mu^i) \, d\hat{\tau}_i(\mu_i) = (\hat{\beta}_i, p_0)$ and $\int \frac{c_i(\mu^i_i)}{\mu^i(\mu^i_i)} \, d\hat{\tau}_i(\mu_i) < \int \frac{c_i(\mu^i_i)}{\mu^i(\mu^i_i)} \, d\hat{\tau}_i(\mu_i)$. By Claim 4, some feasible solution to program (3) generates loss $2 \int \frac{c_i(\mu^i_i)}{\mu^i(\mu^i_i)} \, d\hat{\tau}_i(\mu_i)$, contradicting the (previously established) optimality of $(\hat{\mu}, \hat{\tau}_1, \hat{\tau}_2)$ in program (3).
Having established \( \hat{\tau}_i \) is optimal in program (4), for \( \beta_0 = \beta_1 \) and \( i \in N \), let \( \hat{\tau}_i \) be as delivered by Claim 2. So \( \hat{\tau}_i \) is optimal in program (4), and

- Each \( \omega \in \Omega \) admits a unique \( \tilde{\beta}_\omega \in \Delta \Pi \) such that \( \hat{\tau}_i(\tilde{\beta}_\omega \otimes \delta_\omega) = p_0(\omega) \);

- Any \( \mu_i \) in the support of \( \hat{\tau}_i \) and any \( \omega, \tilde{\omega} \in \Omega \) in the support of \( \mu_i^\Omega \) have \( \tilde{\beta}_\omega = \tilde{\beta}_{\tilde{\omega}} \).

We can then apply Claim 5 to \( \hat{\tau}_i \), to learn \( \hat{\tau}_i \) is the uniform distribution over \{\( \beta_1^* \otimes \delta_1, \beta_2^* \otimes \delta_2 \)\}. That \( \beta_1^* \neq \beta_2^* \) (which holds because \( \beta_1^*(\pi^1) = \beta_2^*(\pi^2) > \frac{1}{2} \)) then implies (by the second bullet above) no \( \mu_i \) in the support of \( \hat{\tau}_i \) has \( \mu_i^\Omega \) putting positive probability on both values for the fundamental state.

Given the previous observation, for each \( i \in N \), we can now apply Claim 5 to \( \hat{\tau}_i \), to learn \( \hat{\tau}_i \) is the uniform distribution over \{\( \beta_1^* \otimes \delta_1, \beta_2^* \otimes \delta_2 \)\} too. But then, by construction of \( \hat{\tau}_i \), it would follow that \( \tau_i \in \Delta \{\beta_1^* \otimes \delta_1, \beta_2^* \otimes \delta_2 \} \) too. Finally, because \( \int \mu_i^\Omega \, d\tau_i(\mu_i) = p_0 \), the only possibility for \( \tau_i \) is that it is uniform as well. Because the pair \( (\tau_1, \tau_2) \) determines the total state distribution, the proposition follows.

\[ Q.E.D. \]

B.3. Toward Proposition 3

Claim 6. Suppose \( c_2 \) is constant. If \( (\mu, \tau_1, \tau_2) \) is optimal in program (3), then some alternative optimal \( (\bar{\mu}, \bar{\tau}_1, \bar{\tau}_2) \) exists such that

- The distribution \( \bar{\tau}_2 \) is degenerate;

- Each \( \omega \in \Omega \) admits a unique \( \tilde{\beta}_\omega \in \Delta \Pi \) such that \( \bar{\tau}_1(\tilde{\beta}_\omega \otimes \delta_\omega) = p_0(\omega) \);

- Any \( \mu_1 \) in the support of \( \tau_1 \) and any \( \omega, \tilde{\omega} \in \Omega \) in the support of \( \mu_1^\Omega \) have \( \tilde{\beta}_\omega = \tilde{\beta}_{\tilde{\omega}} \).

Proof. Let \( \hat{\tau}_1 \) be as delivered by Claim 2 for \( i = 1 \) and \( \beta_0 := \mu^\Pi \). Then, let \( \bar{\tau}_1 := \int \mu_1 \, d\tilde{\tau}_1(\mu_1) \) and \( \bar{\tau}_2 := \delta_\bar{\mu} \). By construction, \( (\bar{\mu}, \bar{\tau}_1, \bar{\tau}_2) \) is feasible in program (3), so all that remains is to see \( (\bar{\mu}, \bar{\tau}_1, \bar{\tau}_2) \) attains a weakly lower loss than \( (\mu, \tau_1, \tau_2) \) does.

Let us observe \( \int c_i(\mu_i) \, d\tilde{\tau}_i(\mu_i) \leq \int c_i(\mu_i^\Omega) \, d\tilde{\tau}_i(\mu_i) \) for each agent \( i \in N \). For \( i = 1 \), the inequality follows from optimality of \( \hat{\tau}_1 \) in program (4) from Claim 2’s statement. For \( i = 2 \),
the inequality follows from $\tilde{\tau}^\Pi$ being degenerate, the identity $\tilde{\mu}^\Pi = \mu^\Pi$, and the integrand $\frac{c_2(\mu_2^\Omega)}{c_2(\mu_2^\Omega)}$ being a convex function of the marginal $\mu_2^\Pi$. \[ Q.E.D. \]

**Claim 7.** Suppose $c_2$ is constant and a unique $\tilde{\beta} \in (\Delta \Pi)^\Omega$ minimizes

$$\int \frac{c_1(\omega)}{c_1(\tilde{\beta}_\omega)} \, dp_0(\omega) + \frac{c_2}{c_2(\int \beta_\omega \, dp_0(\omega))},$$

and $\beta_\omega \neq \beta_\hat{\omega}$ for all distinct $\omega, \hat{\omega} \in \Omega$, then every optimal solution $(\mu, \tau_1, \tau_2)$ to program (3) has

- $\tau_1(\beta_\omega \otimes \delta_\omega) = p_0(\omega)$ for every $\omega \in \Omega$;
- $\tau_2^\Pi \left( \int \beta_\omega \, dp_0(\omega) \right) = 1$;
- $\tau_1^\Pi$ is a strict mean-preserving spread of $\tau_2^\Pi$, and $\tau_1^\Omega$ is a strict mean-preserving spread of $\tau_2^\Omega$.

**Proof.** The third point follows immediately from the first two given that the entries of $\tilde{\beta}$ are distinct: the first point implies $\tau_1^\Omega$ is maximally informative and $\tau_1^\Pi$ is strictly informative, while the second point implies $\tau_2^\Pi$ is uninformative and $\tau_2^\Omega$ is not maximally informative. Moreover, the second point follows directly from the first because the entries of $\tilde{\beta}$ are all distinct, given Claim 1. So we turn to showing every optimal $(\mu, \tau_1, \tau_2)$ for program (3) satisfies the first point.

Consider first any optimal $(\hat{\mu}, \hat{\tau}_1, \hat{\tau}_2)$ for program (3) with the property that $\hat{\tau}_1$ reveals the fundamental state—that is, such that every belief in the support of $\hat{\tau}_1$ takes the form $\hat{\beta} \otimes \delta_\hat{\omega}$ for some $\hat{\beta} \in \Delta \Pi$ and $\hat{\omega} \in \Omega$. By Claim 1, no $\omega \in \Omega$ and distinct $\beta, \tilde{\beta} \in \Delta \Pi$ can exist such that $\beta \otimes \delta_\omega$ and $\tilde{\beta} \otimes \delta_\hat{\omega}$ are both in the support of $\hat{\tau}_1$. Said differently, every $\omega \in \Omega$ admits a unique $\hat{\mu}_1$ in the support of $\hat{\tau}_1$ with $\hat{\mu}_1^\Omega(\hat{\omega}) > 0$. The uniqueness property of $\tilde{\beta}$ then directly implies that $\hat{\tau}_1(\beta_\omega \otimes \delta_\omega) = p_0(\omega)$ for every $\omega \in \Omega$.

In light of the above paragraph, it suffices to show, for any optimal $(\mu, \tau_1, \tau_2)$ for program (3), that $\tau_1$ reveals the fundamental state. To that end, apply Claim 6: some optimal solution
\((\mu_1, \bar{\tau}_1, \bar{\tau}_2)\) to program (3) exists such that:

- The distribution \(\bar{\tau}_2\) is degenerate;
- Each \(\omega \in \Omega\) admits a unique \(\bar{\beta}_\omega \in \Delta \Pi\) such that \(\bar{\tau}_1(\bar{\beta}_\omega \otimes \delta_\omega) = p_0(\omega)\);
- Any \(\mu_1\) in the support of \(\tau_1\) and any \(\omega, \hat{\omega} \in \Omega\) in the support of \(\mu_1^{\Omega}\) have \(\bar{\beta}_\omega = \bar{\beta}_{\hat{\omega}}\).

Now, the uniqueness property of \(\bar{\beta}\), together with optimality of \((\mu, \bar{\tau}_1, \bar{\tau}_2)\), implies \((\bar{\beta}_\omega)_{\omega \in \Omega} = \bar{\beta}\). Hence, because the entries of \(\bar{\beta}\) are distinct, it follows that every \(\mu_1\) in the support of \(\tau_1\) admits some \(\omega \in \Omega\) such that \(\mu_1^{\Omega}(\omega) = 1\). Said differently, \(\tau_1\) reveals the fundamental state, as required.  

\textit{Q.E.D.}

**Claim 8.** Take \(c_1(1) = c_H > c_L := c_2(1) = c_2(2) = c_1(2)\). The program

\[
\min_{\beta \in (\Delta \Pi)^{\Omega}} \int_{\Omega} c_1(\omega) \frac{d\rho_0(\omega)}{l_1(\beta_\omega)} + \frac{c_2}{l_2(\int \beta_\omega d\rho_0(\omega))}
\]

has a unique optimal solution \((\beta_1^{**}, \beta_2^{**})\). It has

\[
(\beta_1^{**}(\pi^1), \beta_2^{**}(\pi^1)) = \begin{cases} 
(2+\varphi)\sqrt{c_H}-3\varphi\sqrt{c_L}, & \frac{2-\varphi}{\sqrt{c_L}+\sqrt{c_H}} : \frac{\sqrt{c_H}}{\sqrt{c_L}} \leq \frac{3}{1+2\varphi} \\
(1, 1/3) : otherwise.
\end{cases}
\]

In particular, \(\beta_1^{**} \neq \beta_2^{**}\).

\textbf{Proof}. Substituting in \(\beta_\omega(\pi^2) = 1 - \beta_\omega(\pi^1)\) for each \(\omega \in \Omega\), we can view the program as an optimization over \((\beta_1(\pi^1), \beta_2(\pi^1)) \in [0, 1]^2\). The loss is continuous so that an optimum exists, and it is strictly convex so that this optimum is unique. Direct computation shows that the given form of \((\beta_1^{**}(\pi^1), \beta_2^{**}(\pi^1))\) satisfies the first-order condition, and hence is the optimum.

Finally, let us verify that \(\beta_1^{**} \neq \beta_2^{**}\). Given the form of the solution, we need only check that the numerators differ in the case that \(\frac{\sqrt{c_H}}{\sqrt{c_L}} \leq \frac{3}{1+2\varphi}\). And indeed,

\[
[(2 + \varphi)\sqrt{c_H} - 3\varphi\sqrt{c_L}] - [(2 - \varphi)\sqrt{c_L} - \varphi\sqrt{c_H}] = 2(1 + \varphi) (\sqrt{c_H} - \sqrt{c_L}) > 0.
\]
Now, we prove Proposition 3.

**Proof of Proposition 3.** Some optimal solution to program (3) exists by Fact 1. Moreover, any two triples that satisfy the conditions of the proposition’s statement—which yield the same total state distribution, provide the same information to agent 1 about the total state, and provide the same information to agent 2 about the ranking state—generate the exact same loss (and so are either both optimal or both suboptimal). Hence, given Claim 7, we need only see that \((\beta^*_{\omega})_{\omega \in \Omega}\) is the unique solution to the program

\[
\min_{\hat{\theta} \in (\Delta \Pi)^{\alpha}} \int \frac{\epsilon_1(\omega)}{\epsilon_1(\hat{\theta}_\omega)} \, dp_0(\omega) + \frac{\epsilon_2}{\epsilon_2 \left( \int \hat{\theta}_\omega \, dp_0(\omega) \right)}
\]

and that \(\beta_1^* \neq \beta_2^*\)—exactly what Claim 8 proves. \(Q.E.D.\)