Online Appendix for

"Addressing Strategic Uncertainty with Incentives and Information"

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A. Proofs for Section 2

Proof of Lemma 1. Let us first recall our rationalizability notion. Given an incentive scheme $\sigma = \langle q, \chi \rangle$ we define the sets $\{T_i^\sigma(\kappa)\}_{i \in N, \kappa \in \mathbb{Z}_+}$ as follows. Let $T_i^\sigma(0) := \emptyset$, and then, recursively for $\kappa \in \mathbb{N}$, let $T_i^\sigma(\kappa)$ be the set of all $t_i \in T_i^q$ such that every $\eta \in \Delta\left(2^{N \setminus \{i\}} \times T_{-i}^q \times \Omega\right)$ with $\operatorname{marg}_{T_{-i}^q \times \Omega} \eta = q_i(\cdot|t_i)$ and $\{j \in N \setminus \{i\} : \ t_j \in T_i^\sigma(\kappa - 1)\} \subseteq J$, $\forall (J, t_{-i}, \omega) \in \operatorname{supp}(\eta)$ has

$$\sum_{J \subseteq N \setminus \{i\}, \ t_{-i} \in T_{-i}^q, \ \omega \in \Omega} \eta(J, t_{-i}, \omega) \left[u_i\left(J \cup \{i\}, \chi_i(t_i), \omega\right) - u_i\left(J, \chi_i(t_i), \omega\right)\right] > 0.$$

By definition of interim correlated rationalizability (Dekel et al., 2007), incentive scheme σ is UIF if and only if $\bigcup_{\kappa=0}^{\infty} T_i^\sigma(\kappa) = T_i^q$ for every $i \in N$.

Now, in what follows, say a type profile t has no ties if $t_i^R \neq t_j^R$ for all distinct $i, j \in N$.

To prove the first assertion, suppose $\sigma = \langle q, \chi \rangle$ is a strict ranking scheme. Let us prove by induction on $\kappa \in \mathbb{Z}_+$ that, if $i \in N$ and $t_i \in T_i^q$ have $t_i^R = \kappa$, then $t_i \in T_i^\sigma(\kappa)$—from which it will follow directly that σ is UIF. The claim holds vacuously for $\kappa = 0$, so take $\kappa \in \mathbb{N}$ and $i \in N$, and assume the claim holds for all $i' \in N$ and all $\kappa' \in \{0, \ldots, \kappa - 1\}$. Next observe that $\chi_i(t_i) \in \mathcal{X}_i^*(\mu_i^q(\kappa))$ because σ is a strict ranking scheme; and the inductive hypothesis implies $t_{-i} \in T_{-i}^\sigma(\kappa - 1)$ for every $t_{-i} \in T_{-i}^q$ such that (t_i, t_{-i}) has no ties and $\pi_j(t) < \pi_i(t)$. Hence, by definition, $t_i \in T_i^\sigma(\kappa)$ as desired.

To prove the second assertion, suppose $\sigma = \langle q, \chi \rangle$ is an arbitrary UIF incentive scheme.

For each $i \in N$, define the map $k_i^\sigma : T_i^q \to \mathbb{N}$ by letting $k_i^\sigma(t_i) := \min\{\kappa \in \mathbb{N} :\ t_i \in T_i^\sigma(\kappa)\}$. It is easy to see some one-to-one function $\tilde{\lambda} : \bigcup_{i\in N} [\{i\} \times T_i^q] \to \mathbb{N}$ exists such that, for any $i, j \in N$ and $t_i \in T_i^q$, $t_j \in T_j^\sigma$ with $k_i^\sigma(t_i) > k_j^\sigma(t_j)$, we have $\tilde{\lambda}_i(t_i) > \tilde{\lambda}_j(t_j)$. Then, define $\lambda : \bigcup_{i\in N} [\{i\} \times T_i^q] \to \mathbb{N}^2$ by letting $\lambda_i(t_i) := (\tilde{\lambda}_i(t_i), 1)$.

Now, define the incentive scheme $\sigma^* := \langle q^*, \chi^* \rangle$ by letting

$$q^*(t^*, \omega) := q\left(\left(\lambda_i^{-1}(t_i^*)\right)_{i\in N},\ \omega\right)$$

for every $t^* \in (\mathbb{N}^2)^N$ and $\omega \in \Omega$, and letting $\chi_i^*(t_i^*) := \chi_i\left(\lambda_i^{-1}(t_i^*)\right)$ for every $i \in N$ and $t_i^* \in T_i^{q^*}$. That the modified scheme is UIF follows from the original scheme being UIF (Dekel et al., 2007, by Proposition 1) given that type t_i's hierarchy of beliefs over $X \times \Omega$ under σ are the same as type $\lambda_i(t_i)$'s under σ^*. Further, because σ^* generates the same distribution over $X \times \Omega$ as σ does, it follows directly that $V(\sigma^*) = V(\sigma)$. All that remains is to see σ^* is a strict ranking scheme. That q^* exhibits no ties is immediate from the construction. Moreover, given any $i, j \in N$, observe any $t_i^* \in T_i^{q^*}$ and $t_j^* \in T_j^{q^*}$ have $k_i^{\sigma^*}(t_i^*) > k_j^{\sigma^*}(t_j^*)$ if and only if $k_i^\sigma(\lambda_i^{-1}(t_i^*)) > k_j^\sigma(\lambda_j^{-1}(t_j^*))$, which in turn implies $t_i^{R*} > t_j^{R*}$. It therefore follows from $t_i^* \in T_i^{\sigma^*}\left(k_i^{\sigma^*}(t_i^*)\right)$ that $\chi_i^*(t_i^*) \in \mathcal{X}_i^*(\mu_i^{q^*}(t_i^*))$, and so σ^* is a strict ranking scheme. *Q.E.D.*

Proof of Theorem 1. We first show that $\sup_{\sigma \text{ is UIF}} V(\sigma) \leq \sup_{\mu\in\mathcal{M}(p_0)} \sum_{i\in N} \widehat{v}_i^*(\mu)$. Given Lemma 1, it suffices to show that the principal's value for a strict ranking scheme $\langle q, \chi \rangle$ is no greater than $\sup_{\mu\in\mathcal{M}(p_0)} \sum_{i\in N} \widehat{v}_i^*(\mu)$. Bayesian updating implies that a given agent i's belief is, on average, equal to the true distribution over total states:

$$\sum_{t\in T^q} q(t)\mu_i^q(\cdot|t_i) = \sum_{t_i\in T_i^q} q_i(t_i)\mu_i^q(\cdot|t_i) = \mu^q \in \mathcal{M}(p_0).$$

Hence, the belief distribution $\tau_i \in \Delta\Delta(\Pi \times \Omega)$ given by $\sum_{t_i\in T_i^q} q_i(t_i)\delta_{\mu_i^q(\cdot|t_i)}$ is feasible in the program defining $\widehat{v}_i^*(\mu^q)$. It follows that

$$\sum_{t\in T^q,\ \omega\in\Omega} q(t,\omega)v_i^*(\mu_i^q(\cdot|t_i)) \le \widehat{v}_i^*\left(\mu^q\right),$$

and so summing over $i \in N$ yields $V(q) \le \sum_{i\in N} \widehat{v}_i^*(\mu^q)$.

To show $\sup_{\sigma \text{ is UIF}} V(\sigma) \ge \sup_{\mu\in\mathcal{M}(p_0)} \sum_{i\in N} \widehat{v}_i^*(\mu)$, consider an arbitrary $\mu \in \mathcal{M}(p_0)$ and $\varepsilon > 0$. We will construct a strict ranking scheme $\sigma = \langle q, \chi\rangle$ such that $V(\sigma) \ge \sum_{i\in N} [\widehat{v}_i^*(\mu) - 3\varepsilon]$. To do so, observe $\widehat{v}_i^*$ is bounded above by some constant $L_i \in \mathbb{R}$ for each $i \in N$ because v_i^* is. In what follows, let $m \in \mathbb{N}$ be large enough that $m \ge |N|$ and $\frac{2|N|}{m}\left[L_i - v_i^*(\overline{x}_i, \omega)\right] \le \varepsilon$ for each $i \in N$ and $\omega \in \Omega$.

Consider any $i \in N$. Some $\tau_i \in \Delta\Delta(\Pi \times \Omega)$ exists such that $\int \mu_i \, \mathrm{d}\tau_i(\mu_i) = \mu$ and $\int v_i^* \, \mathrm{d}\tau_i \ge \widehat{v}_i^*(\mu) - \varepsilon$. For each $\mu_i \in \operatorname{supp}(\tau_i)$, the definition of v_i^* implies some $x_i^{\mu_i} \in \mathcal{X}_i^*(\mu_i)$ exists such that $\sum_{\pi\in\Pi,\omega\in\Omega} \mu_i(\pi,\omega)\ v_i(x_i^{\mu_i},\omega) \ge v^*(\mu_i) - \varepsilon$. By the splitting lemma, some $\gamma_i : \Pi\times\Omega \to \Delta\mathbb{N}$ exists such that, when the prior distribution over $\Pi\times\Omega$ is μ and the results of Blackwell experiment γ_i are observed, the induced distribution of beliefs over $\Pi \times \Omega$ is τ_i. Letting $\overline{s}_i \in \mathbb{N}$ denote the number of positive-probability signals in $\mathbb{N}$ given prior μ and experiment γ_i, we can assume without loss that the positive-probability signals are exactly $\{1, \ldots, \overline{s}_i\}$. For each $s_i \in \{1, \ldots, \overline{s}_i\}$, let $x_i^{s_i}$ denote $x_i^{\mu_i}$, where μ_i is the belief induced by signal realization s_i from this experiment.

Now, we construct our incentive scheme $\sigma = \langle q, \chi\rangle$. Define the prior $q \in \Delta[(\mathbb{N}^2)^N \times \Omega]$ by letting, for each $t = (t_i^R, t_i^S)_{i\in N} \in (\mathbb{N}^2)^N$ and $\omega \in \Omega$,

$$q(t,\omega) := \begin{cases} \frac{1}{m}\mu(\pi,\omega)\prod_{i\in N}\gamma_i(t_i^S|\pi,\omega) & : \ \exists \ell \in \{0,\ldots,m-1\} \text{ with } t_i^R = \ell + \pi_i \text{ for all } i \in N, \\ 0 & : \ \text{otherwise;} \end{cases}$$

and the allocation rule $\chi = (\chi_i)_{i \in N}$ via

$$\chi_i(t_i^R, t_i^S) := \begin{cases} x_i^{t_i^S} & : \ t_i^S \le \overline{s}_i \text{ and } N \le t_i^R \le m, \\ \overline{x}_i & : \ \text{otherwise.} \end{cases}$$

By construction, this scheme has no ties: $t_i^R \neq t_j^R$ for all distinct $i, j \in N$ and any supported type profile $t \in T^q$. Moreover, for each $i \in N$, a direct computation shows every type $t_i \in T_i^q$ with $|N| \le t_i^R \le m$ has belief $\mu_i^q(\cdot|t_i) = \mu_i^{t_i^S}$ and thus has $\chi_i(t_i) = x_i^{t_i^S} \in \mathcal{X}_i^*(\mu_i^q(\cdot|t_i))$. Because every other $t_i \in T_i^q$ has $\chi_i(t_i) = \overline{x}_i \in \bigcap_{\mu_i \in \Delta(\Pi \times \Omega)} \mathcal{X}_i^*(\mu_i)$, it follows that σ is a strict ranking scheme. Finally, let us bound (from below) the value of this scheme to the principal. To do so, consider any agent $i \in N$ and $s_i \in \{1, \dots, \overline{s}_i\}$, and observe that σ generates belief $\mu_i^{s_i} \in \Delta(\Pi \times \Omega)$ for agent i with probability

$$\begin{aligned} \text{marg}_i q \left\{ t_i = (t_i^R, t_i^S) \in T_i^q : \ \mu_i^q(\cdot|t_i) = \mu_i^{t_i^S} \right\} \ &\ge \ \sum_{\pi \in \Pi, \ \omega \in \Omega} \sum_{\ell=0}^{m-1} \tfrac{1}{m} \mathbf{1}_{|N| \le \ell + \pi_i \le m} \mu(\pi, \omega) \gamma_i(s_i|\pi, \omega) \\ &\ge \ \left(1 - \tfrac{2|N|}{m}\right) \sum_{\pi \in \Pi, \ \omega \in \Omega} \mu(\pi, \omega) \gamma_i(s_i|\pi, \omega) \\ &\ge \ \left(1 - \tfrac{2|N|}{m}\right) \tau_i(\mu_i^{s_i}). \end{aligned}$$

Hence, the principal's payoff from this strict ranking scheme is

$$\begin{aligned} V(\sigma) \ &\ge \ \sum_{i \in N} \left\{ \tfrac{2|N|}{m} \left[\min_{\omega \in \Omega} v_i^*(\overline{x}_i) \right] + \left(1 - \tfrac{2|N|}{m}\right) \sum_{s_i=1}^{\overline{s}_i} \tau_i(\mu_i^{s_i}) \sum_{\omega \in \Omega} \text{marg}_\Omega \mu_i^{s_i}(\omega) \ v_i(x_i^{s_i}, \omega) \right\} \\ &\ge \ \sum_{i \in N} \left\{ \tfrac{2|N|}{m} L_i - \varepsilon + \left(1 - \tfrac{2|N|}{m}\right) \sum_{s_i=0}^{\overline{s}_i - 1} \tau_i(\mu_i^{s_i}) \left[v_i^*(\mu_i^{s_i}) - \varepsilon \right] \right\} \\ &\ge \ \sum_{i \in N} \left\{ \tfrac{2|N|}{m} \widehat{v}_i^*(\mu) - \varepsilon + \left(1 - \tfrac{2|N|}{m}\right) \left[\widehat{v}_i^*(\mu) - 2\varepsilon \right] \right\} \\ &\ge \ \sum_{i \in N} \left[\widehat{v}_i^*(\mu) - 3\varepsilon \right], \end{aligned}$$

as required. *Q.E.D.*

Proof of Fact 1. Let $\mathcal{P}$ denote the set of Borel probability measures on $\Delta(\Pi \times \Omega)$, a compact space when endowed with its weak* topology.

Take any $i \in N$. Because an upper semicontinuous function over a compact space attains a maximum, for any $\mu \in \Delta(\Pi \times \Omega)$, the program $\sup_{\tau_i \in \mathcal{P}:\ \int \mu_i \, \mathrm{d}\tau_i(\mu_i) = \mu} \int v_i^* \, \mathrm{d}\tau_i$—which relaxes the program defining $\widehat{v}_i^*(\mu)$ by allowing distributions with infinite support—admits an optimum. Moreover, by the upper semicontinuous version of Berge's theorem, this optimal value is an upper semicontinuous function of μ. Now, Carathéodory's theorem tells us some optimum to the aforementioned program has affinely independent (hence, of cardinality no more than $N! * |\Omega|$) support. It follows that the program defining $\widehat{v}_i^*(\mu)$ admits an optimum, and that $\widehat{v}_i^*$ is upper semicontinuous.

Finally, because $\sum_{i \in N} \widehat{v}_i^*$ is upper semicontinuous and $\mathcal{M}(p_0)$ is compact, the program $\sup_{\mu \in \mathcal{M}(p_0)} \sum_{i \in N} \widehat{v}_i^*(\mu)$ admits an optimum. *Q.E.D.*

B. Proofs for Section 3

Toward proving the results of Section 3, some preliminary claims will be useful.

Claim 1. *Suppose $i \in N$ and $\mu \in \Delta(\Pi \times \Omega)$. If τ_i is an optimal solution to*

$$\min_{\tau_i \in \Delta\Delta(\Pi \times \Omega)} \int \frac{c_i(\mu_i^\Omega)}{\iota_i(\mu_i^\Pi)} \, d\tau_i(\mu_i) \quad \text{subject to} \quad \int \mu_i \, d\tau_i(\mu_i) = \mu,$$

then no $\tilde{\omega}, \hat{\omega} \in \Omega$ with $c_i(\tilde{\omega}) = c_i(\hat{\omega})$ and distinct $\tilde{\beta}, \hat{\beta} \in \Delta\Pi$ have both $\tilde{\beta} \otimes \delta_{\tilde{\omega}}$ and $\hat{\beta} \otimes \delta_{\hat{\omega}}$ in the support of τ_i.

Proof. Suppose $\tilde{\omega}, \hat{\omega} \in \Omega$ with $c_i(\tilde{\omega}) = c_i(\hat{\omega}) =: \bar{c}_i$ and distinct $\tilde{\beta}, \hat{\beta} \in \Delta\Pi$ have both $\tilde{\beta} \otimes \delta_{\tilde{\omega}}$ and $\hat{\beta} \otimes \delta_{\hat{\omega}}$ in the support of τ_i. Then, some $\varepsilon \in (0, 1]$ and $\check{\tau}_i \in \Delta\Delta(\Pi \times \Omega)$ exists such that

$$\tau_i = (1 - \varepsilon)\check{\tau}_i + \tfrac{\varepsilon}{2}\delta_{\tilde{\beta} \otimes \delta_{\tilde{\omega}}} + \tfrac{\varepsilon}{2}\delta_{\hat{\beta} \otimes \delta_{\hat{\omega}}}.$$

The alternative belief distribution

$$\tau_i' = (1-\varepsilon)\check{\tau}_i + \varepsilon\delta_{\frac{1}{2}\left(\tilde{\beta}\otimes\delta_{\tilde{\omega}}+\hat{\beta}\otimes\delta_{\hat{\omega}}\right)}$$

is then feasible in the given program. Moreover, by strict convexity of $\frac{\bar{c}_i}{\iota_i(\beta)}$ in $\beta \in \Delta\Pi$, the latter attains a strictly lower loss, so that τ_i is not optimal. *Q.E.D.*

Claim 2. *Suppose $i \in N$ and $\beta_0 \in \Delta\Pi$. If τ_i is an optimal solution to the program*

$$\min_{\tau_i\in\Delta\Delta(\Pi\times\Omega)} \int \frac{c_i(\mu_i^\Omega)}{\iota_i(\mu_i^\Pi)}\ d\tau_i(\mu_i) \quad \textit{subject to} \quad \int (\mu_i^\Pi, \mu_i^\Omega)\ d\tau_i(\mu_i) = (\beta_0, p_0), \tag{4}$$

then some alternative optimal $\tilde{\tau}_i$ exists such that

- *Each $\omega \in \Omega$ admits a unique $\tilde{\beta}_\omega \in \Delta\Pi$ such that $\tilde{\tau}_i(\tilde{\beta}_\omega \otimes \delta_\omega) = p_0(\omega)$;*
- *Any μ_i in the support of τ_i and any $\omega, \hat{\omega} \in \Omega$ in the support of μ_i^Ω have $\tilde{\beta}_\omega = \tilde{\beta}_{\hat{\omega}}$.*

Proof. Let $\tilde{\tau}_i := \int\int \delta_{\mu_i^\Pi\otimes\delta_\omega}\ \mathrm{d}\mu_i^\Omega(\omega)\ \mathrm{d}\tau_i(\mu_i) \in \Delta\Delta(\Pi\times\Omega)$.

Various features are immediate from the construction. First, the average marginal distributions under $\tilde{\tau}_i$ are the same as those under τ_i, making $\tilde{\tau}_i$ feasible in the program. Second, because the fraction $\frac{c_i(\mu_i^\Omega)}{\iota_i(\mu_i^\Pi)}$ is affine in μ_i^Ω when holding μ_i^Π fixed, we know $\tilde{\tau}_i$ yields the same value in program (4) as τ_i does, and so is optimal too. Third, every $\tilde{\mu}_i$ in the support of $\tilde{\tau}_i$ admits some $\tilde{\beta} \in \Delta\Pi$ and $\omega \in \Omega$ for which $\tilde{\mu}_i = \tilde{\beta} \otimes \delta_\omega$. Fourth, for any μ_i in the support of τ_i and any $\omega, \hat{\omega} \in \Omega$ in the support of μ_i^Ω, some $\tilde{\beta} \in \Delta\Pi$ has both $\tilde{\beta}\otimes\delta_\omega$ and $\tilde{\beta}\otimes\delta_{\hat{\omega}}$ in the support of $\tilde{\tau}_i$—indeed, $\tilde{\beta} = \mu_i^\Pi$ has this property.

The claim will then follow if we know that no $\omega \in \Omega$ and distinct $\tilde{\beta}, \hat{\beta} \in \Delta\Pi$ have both $\tilde{\beta}\otimes\delta_\omega$ and $\hat{\beta}\otimes\delta_\omega$ in the support of $\tilde{\tau}_i$. And indeed, this fact follows directly from Claim 1. *Q.E.D.*

Claim 3. *For any $c_H \geq c_L > 0$, the program*

$$\min_{(\beta^H,\beta^L)\in[0,1]^2} \left\{\tfrac{c_H}{(1-\beta^H)(P_1-P_0)+\beta^H(P_2-P_1)} + \tfrac{c_L}{(1-\beta^L)(P_1-P_0)+\beta^L(P_2-P_1)}\right\} \ \textit{subject to} \ \ \beta^H + \beta^L = 1$$

has a unique optimal solution (β^H, β^L)*. It has*

$$\beta^H = \begin{cases} \frac{\sqrt{c_H} - \varphi\sqrt{c_L}}{(1-\varphi)(\sqrt{c_H}+\sqrt{c_L})} & : \ \varphi\sqrt{c_H} < \sqrt{c_L} \\ 1 & : \ \textit{otherwise.} \end{cases}$$

Moreover, if $c_H > c_L$*, then* $\beta^H > \frac{1}{2}$*.*

Proof. Substituting in $\beta^L = 1 - \beta^H$, we can view the program as an optimization over $\beta^H \in [0, 1]$. The loss is continuous in β^H so that an optimum exists, and it is strictly convex in β^H so that this optimum is unique. Direct computation shows that the given form of β^H satisfies the first-order condition, and hence is the optimum.

Finally, supposing $c_H > c_L$, let us show $\beta^H > \frac{1}{2}$. Indeed, in this case,

$$2(\sqrt{c_H} - \varphi\sqrt{c_L}) - (1-\varphi)(\sqrt{c_H} + \sqrt{c_L}) = (1+\varphi)(\sqrt{c_H} - \sqrt{c_L}) > 0,$$

so that $\beta^H \geq \min\left\{1, \frac{\sqrt{c_H}-\varphi\sqrt{c_L}}{(1-\varphi)(\sqrt{c_H}+\sqrt{c_L})}\right\} > \frac{1}{2}$. *Q.E.D.*

B.1. Toward Proposition 1

Proof of Proposition 1. Some optimal solution to program (3) exists by Fact 1. Moreover, by Claim 1, any optimal solution (μ, τ_1, τ_2) has $\tau_1^\Pi\left(\mu^\Pi\right) = \tau_2^\Pi\left(\mu^\Pi\right) = 1$.

Hence, all that remains to see is that the program

$$\min_{\beta\in\Delta\Pi} \sum_{i\in N} \frac{c_i}{\iota_i(\beta)}$$

is uniquely solved by setting

$$\beta(\pi^1) = \begin{cases} \frac{\sqrt{c_H} - \varphi\sqrt{c_L}}{(1-\varphi)(\sqrt{c_H}+\sqrt{c_L})} & : \ \varphi\sqrt{c_H} < \sqrt{c_L} \\ 1 & : \ \text{otherwise,} \end{cases}$$

which follows directly from Claim 3 (with $\beta(\pi^1)$ corresponding to β^H in that claim). *Q.E.D.*

B.2. Toward Proposition 2

Claim 4. *Suppose $c_1(1) = c_2(2) > c_2(1) = c_1(2)$. Let $i \in N$, let $\beta_0 \in \Delta\Pi$ be uniform, and suppose τ_i is a feasible solution to the program (4) from Claim 2's statement. Then, some feasible solution to program (3) exists that generates loss $2\int \frac{c_i(\mu_i^\Omega)}{\iota_i(\mu_i^\Pi)}\, d\tau_i(\mu_i)$.*

Proof. Let $\psi : \Pi \times \Omega \to \Pi \times \Omega$ be the involution that changes every coordinate.[8] Define $\Psi : \Delta(\Pi \times \Omega) \to \Delta(\Pi \times \Omega)$ by letting $\Psi(\tilde{\mu}) := \tilde{\mu} \circ \psi^{-1}$ for every $\tilde{\mu} \in \Delta(\Pi \times \Omega)$. Let j be such that $N = \{i, j\}$, and define $\tau_j := \tau_i \circ \Psi^{-1}$. It follows from $v_1^* = v_2^* \circ \Psi$ that

$$\sum_{k \in N} \int \frac{c_k(\mu_k^\Omega)}{\iota_k(\mu_j^\Pi)}\, \mathrm{d}\tau_k(\mu_k) = 2\int \frac{c_i(\mu_i^\Omega)}{\iota_i(\mu_i^\Pi)}\, \mathrm{d}\tau_i(\mu_i).$$

If some $\mu \in \Delta(\Pi \times \Omega)$ is such that (μ, τ_1, τ_2) is feasible in program (3), we will have a feasible triple with the desired property. To that end, define $\mu := \int \mu_i\, \mathrm{d}\tau_i(\mu_i)$, and note that $\int \mu_j\, \mathrm{d}\tau_j(\mu_j) = \Psi(\mu)$ by construction. It then suffices to observe that $\mu = \Psi(\mu)$. But this property follows from both marginals μ^Π, μ^Ω being uniform on their respective domains.[9] *Q.E.D.*

Claim 5. *Suppose $c_1(1) = c_2(2) =: c_H > c_L := c_2(1) = c_1(2)$. Let $i \in N$, let $\beta_0 \in \Delta\Pi$ be uniform, and suppose τ_i is an optimal solution to the program (4) from Claim 2's statement. If $\tau_i\{\mu_i \in \Delta(\Pi \times \Omega) :\ \mu_i^\Omega(\omega) = 1$ for some $\omega \in \Omega\} = 1$, then $\tau_i(\beta_1^* \otimes \delta_1) = \tau_i(\beta_2^* \otimes \delta_2) = \frac{1}{2}$, where*

$$\begin{aligned} \beta_1^*(\pi^1) = \beta_2^*(\pi^2) &= \begin{cases} \frac{\sqrt{c_H} - \varphi\sqrt{c_L}}{(1-\varphi)(\sqrt{c_H}+\sqrt{c_L})} & : \ \varphi\sqrt{c_H} < \sqrt{c_L} \\ 1 & : \ \textit{otherwise} \end{cases} \\ &> \tfrac{1}{2}. \end{aligned}$$

[8] So, if $N = \{i, j\} = \{i', j'\}$, then $\psi(\pi^i, i') = (\pi^j, j')$.

[9] Consider the 2×2 matrix whose (i', j') entry is $\mu(\pi^{i'}, j') - \frac{1}{4}$ for each $i', j' \in N$. Every row and every column of this matrix sums to zero, and so it is proportional to $\pm \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

Proof. Assume τ_i has the hypothesized properties. First, observe no $\omega \in \Omega$ and distinct $\tilde{\beta}, \hat{\beta} \in \Delta\Pi$ have both $\tilde{\beta} \otimes \delta_\omega$ and $\hat{\beta} \otimes \delta_\omega$ in the support of τ_i, by Claim 1. Hence, some $\beta_1, \beta_2 \in \Delta\Pi$ exist such that $\tau_i\{\beta_1 \otimes \delta_1, \beta_2 \otimes \delta_2\} = 1$. Optimality of τ_i for program (4) then tells us $(\beta_i(\pi^i), \beta_i(\pi^j))$ is an optimal solution to

$$\min_{(\beta^H,\beta^L)\in[0,1]^2} \left\{ \frac{c^H}{(1-\beta^H)(P_1-P_0)+\beta^H(P_2-P_1)} + \frac{c^L}{(1-\beta^L)(P_1-P_0)+\beta^L(P_2-P_1)} \right\} \text{ subject to } \beta^H + \beta^L = 1.$$

The claim then follows directly from Claim 3. *Q.E.D.*

Now, we prove Proposition 2.

Proof of Proposition 2. Let (μ, τ_1, τ_2) be any optimal solution to (3) (which exists by Fact 1).

Our first step is to construct an alternative optimum that satisfies a symmetry property. To construct such an optimum, recall the map $\Psi : \Delta(\Pi \times \Omega) \to \Delta(\Pi \times \Omega)$ defined in the proof of Claim 4. Symmetry of p_0 implies $\Psi(\mu) \in \mathcal{M}(p_0)$ because $\mu \in \mathcal{M}(p_0)$; because $\mathcal{M}(p_0)$ is convex, it therefore also contains $\hat{\mu} := \frac{1}{2}\left[\mu + \Psi(\mu)\right]$. For each $\{i, j\} = N$, define $\hat{\tau}_i := \frac{1}{2}\left[\tau_i + \tau_j \circ \Psi^{-1}\right]$.

Some properties of $(\hat{\mu}, \hat{\tau}_1, \hat{\tau}_2)$ are immediate from the construction. First, the mean of $\hat{\tau}_i$ is $\hat{\mu}$ for each $i \in N$, so that $(\hat{\mu}, \hat{\tau}_1, \hat{\tau}_2)$ is feasible in program (3). Second, $\hat{\tau}_1 = \hat{\tau}_2 \circ \Psi^{-1}$. Third, that $v_1^* = v_2^* \circ \Psi$ implies $(\hat{\mu}, \hat{\tau}_1, \hat{\tau}_2)$ attains the same value as (μ, τ_1, τ_2) does in program (3), and so is optimal too.

Now, let $\beta_! \in \Delta\Pi$ be the uniform distribution and $i \in N$. Let us show, for $\beta_0 = \beta_!$ and $i \in N$, that $\hat{\tau}_i$ solves the program (4) defined in Claim 2's statement. Assume otherwise for a contradiction. So some $\check{\tau}_i \in \Delta\Delta(\Pi \times \Omega)$ has $\int (\mu_i^\Pi, \mu_i^\Omega)\, \mathrm{d}\check{\tau}_i(\mu_i) = (\beta_!, p_0)$ and $\int \frac{c_i(\mu_i^\Omega)}{\iota_i(\mu_i^\Pi)}\, \mathrm{d}\check{\tau}_i(\mu_i) < \int \frac{c_i(\mu_i^\Omega)}{\iota_i(\mu_i^\Pi)}\, \mathrm{d}\hat{\tau}_i(\mu_i)$. By Claim 4, some feasible solution to program (3) generates loss $2\int \frac{c_i(\mu_i^\Omega)}{\iota_i(\mu_i^\Pi)}\, \mathrm{d}\check{\tau}_i(\mu_i)$, contradicting the (previously established) optimality of $(\hat{\mu}, \hat{\tau}_1, \hat{\tau}_2)$ in program (3).

Having established $\hat{\tau}_i$ is optimal in program (4), for $\beta_0 = \beta_!$ and $i \in N$, let $\tilde{\tau}_i$ be as delivered by Claim 2. So $\tilde{\tau}_i$ is optimal in program (4), and

- Each $\omega \in \Omega$ admits a unique $\tilde{\beta}^i_\omega \in \Delta\Pi$ such that $\tilde{\tau}_i(\tilde{\beta}^i_\omega \otimes \delta_\omega) = p_0(\omega)$;
- Any μ_i in the support of $\hat{\tau}_i$ and any $\omega, \hat{\omega} \in \Omega$ in the support of μ_i^Ω have $\tilde{\beta}^i_\omega = \tilde{\beta}^i_{\hat{\omega}}$.

We can then apply Claim 5 to $\tilde{\tau}_i$, to learn $\tilde{\tau}_i$ is the uniform distribution over $\{\beta_1^* \otimes \delta_1,\ \beta_2^* \otimes \delta_2\}$. That $\beta_1^* \neq \beta_2^*$ (which holds because $\beta_1^*(\pi^1) = \beta_2^*(\pi^2) > \frac{1}{2}$) then implies (by the second bullet above) no μ_i in the support of $\hat{\tau}_i$ has μ_i^Ω putting positive probability on both values for the fundamental state.

Given the previous observation, for each $i \in N$, we can now apply Claim 5 to $\hat{\tau}_i$, to learn $\hat{\tau}_i$ is the uniform distribution over $\{\beta_1^* \otimes \delta_1,\ \beta_2^* \otimes \delta_2\}$ too. But then, by construction of $\hat{\tau}_i$, it would follow that $\tau_i \in \Delta\{\beta_1^* \otimes \delta_1,\ \beta_2^* \otimes \delta_2\}$ too. Finally, because $\int \mu_i^\Omega \,\mathrm{d}\tau_i(\mu_i) = p_0$, the only possibility for τ_i is that it is uniform as well. Because the pair (τ_1, τ_2) determines the total state distribution, the proposition follows. *Q.E.D.*

B.3. Toward Proposition 3

Claim 6. *Suppose c_2 is constant. If (μ, τ_1, τ_2) is optimal in program (3), then some alternative optimal $(\tilde{\mu}, \tilde{\tau}_1, \tilde{\tau}_2)$ exists such that*

- *The distribution $\tilde{\tau}_2$ is degenerate;*
- *Each $\omega \in \Omega$ admits a unique $\tilde{\beta}_\omega \in \Delta\Pi$ such that $\tilde{\tau}_1(\tilde{\beta}_\omega \otimes \delta_\omega) = p_0(\omega)$;*
- *Any μ_1 in the support of τ_1 and any $\omega, \hat{\omega} \in \Omega$ in the support of μ_1^Ω have $\tilde{\beta}_\omega = \tilde{\beta}_{\hat{\omega}}$.*

Proof. Let $\tilde{\tau}_1$ be as delivered by Claim 2 for $i = 1$ and $\beta_0 := \mu^\Pi$. Then, let $\tilde{\tau}_1 := \int \mu_1 \,\mathrm{d}\tilde{\tau}_1(\mu_1)$ and $\tilde{\tau}_2 := \delta_{\tilde{\mu}}$. By construction, $(\tilde{\mu}, \tilde{\tau}_1, \tilde{\tau}_2)$ is feasible in program (3), so all that remains is to see $(\tilde{\mu}, \tilde{\tau}_1, \tilde{\tau}_2)$ attains a weakly lower loss than (μ, τ_1, τ_2) does.

Let us observe $\int \frac{c_i(\mu_i^\Omega)}{\iota_i(\mu_i^\Pi)} \,\mathrm{d}\tilde{\tau}_i(\mu_i) \leq \int \frac{c_i(\mu_i^\Omega)}{\iota_i(\mu_i^\Pi)} \,\mathrm{d}\tau_i(\mu_i)$ for each agent $i \in N$. For $i = 1$, the inequality follows from optimality of $\tilde{\tau}_1$ in program (4) from Claim 2's statement. For $i = 2$, the inequality follows from $\tilde{\tau}^\Pi$ being degenerate, the identity $\tilde{\mu}^\Pi = \mu^\Pi$, and the integrand $\frac{c_2(\mu_2^\Omega)}{\iota_2(\mu_2^\Pi)} = \frac{c_2}{\iota_2(\mu_2^\Pi)}$ being a convex function of the marginal μ_2^Π. *Q.E.D.*

Claim 7. *Suppose c_2 is constant and a unique $\vec{\beta} \in (\Delta\Pi)^\Omega$ minimizes*

$$\int \frac{c_1(\omega)}{\iota_1(\beta_\omega)}\ dp_0(\omega) + \frac{c_2}{\iota_2\big(\int \beta_\omega\ dp_0(\omega)\big)},$$

and $\beta_\omega \neq \beta_{\hat{\omega}}$ for all distinct $\omega, \hat{\omega} \in \Omega$, then every optimal solution (μ, τ_1, τ_2) to program (3) has

- $\tau_1(\beta_\omega \otimes \delta_\omega) = p_0(\omega)$ *for every* $\omega \in \Omega$*;*
- $\tau_2^\Pi\left(\int \beta_\omega\ dp_0(\omega)\right) = 1$*;*
- τ_1^Π *is a strict mean-preserving spread of* τ_2^Π*, and* τ_1^Ω *is a strict mean-preserving spread of* τ_2^Ω*.*

Proof. The third point follows immediately from the first two given that the entries of $\vec{\beta}$ are distinct: the first point implies τ_1^Ω is maximally informative and τ_1^Π is strictly informative, while the second point implies τ_2^Π is uninformative and τ_2^Ω is not maximally informative. Moreover, the second point follows directly from the first because the entries of $\vec{\beta}$ are all distinct, given Claim 1. So we turn to showing every optimal (μ, τ_1, τ_2) for program (3) satisfies the first point.

Consider first any optimal $(\hat{\mu}, \hat{\tau}_1, \hat{\tau}_2)$ for program (3) with the property that $\hat{\tau}_1$ reveals the fundamental state—that is, such that every belief in the support of $\hat{\tau}_1$ takes the form $\hat{\beta} \otimes \delta_{\hat{\omega}}$ for some $\hat{\beta} \in \Delta\Pi$ and $\hat{\omega} \in \Omega$. By Claim 1, no $\hat{\omega} \in \Omega$ and distinct $\beta, \hat{\beta} \in \Delta\Pi$ can exist such that $\beta \otimes \delta_{\hat{\omega}}$ and $\hat{\beta} \otimes \delta_{\hat{\omega}}$ are both in the support of $\hat{\tau}_1$. Said differently, every $\hat{\omega} \in \Omega$ admits a unique $\hat{\mu}_1$ in the support of $\hat{\tau}_1$ with $\hat{\mu}_1^\Omega(\hat{\omega}) > 0$. The uniqueness property of $\vec{\beta}$ then directly implies that $\hat{\tau}_1(\beta_{\hat{\omega}} \otimes \delta_{\hat{\omega}}) = p_0(\hat{\omega})$ for every $\hat{\omega} \in \Omega$.

In light of the above paragraph, it suffices to show, for any optimal (μ, τ_1, τ_2) for program (3), that τ_1 reveals the fundamental state. To that end, apply Claim 6: some optimal solution

$(\tilde{\mu}_1, \tilde{\tau}_1, \tilde{\tau}_2)$ to program (3) exists such that:

- The distribution $\tilde{\tau}_2^{\Pi}$ is degenerate;
- Each $\omega \in \Omega$ admits a unique $\tilde{\beta}_\omega \in \Delta\Pi$ such that $\tilde{\tau}_1(\tilde{\beta}_\omega \otimes \delta_\omega) = p_0(\omega)$;
- Any μ_1 in the support of τ_1 and any $\omega, \hat{\omega} \in \Omega$ in the support of μ_1^{Ω} have $\tilde{\beta}_\omega = \tilde{\beta}_{\hat{\omega}}$.

Now, the uniqueness property of $\vec{\beta}$, together with optimality of $(\tilde{\mu}, \tilde{\tau}_1, \tilde{\tau}_2)$, implies $(\tilde{\beta}_\omega)_{\omega\in\Omega} = \vec{\beta}$. Hence, because the entries of $\vec{\beta}$ are distinct, it follows that every μ_1 in the support of τ_1 admits some $\omega \in \Omega$ such that $\mu_1^{\Omega}(\omega) = 1$. Said differently, τ_1 reveals the fundamental state, as required. *Q.E.D.*

Claim 8. *Take $c_1(1) =: c_H > c_L := c_2(1) = c_2(2) = c_1(2)$. The program*

$$\min_{\vec{\beta}\in(\Delta\Pi)^{\Omega}} \int \frac{c_1(\omega)}{\iota_1(\beta_\omega)} \; dp_0(\omega) + \frac{c_2}{\iota_2\left(\int \beta_\omega \; dp_0(\omega)\right)}$$

has a unique optimal solution $(\beta_1^{**}, \beta_2^{**})$. It has

$$\left(\beta_1^{**}(\pi^1), \beta_2^{**}(\pi^1)\right) = \begin{cases} \left(\frac{(2+\varphi)\sqrt{c_H}-3\varphi\sqrt{c_L}}{(1-\varphi)(3\sqrt{c_L}+\sqrt{c_H})}, \frac{(2-\varphi)\sqrt{c_L}-\varphi\sqrt{c_H}}{(1-\varphi)(3\sqrt{c_L}+\sqrt{c_H})}\right) & : \frac{\sqrt{c_H}}{\sqrt{c_L}} \leq \frac{3}{1+2\varphi} \\ (1, 1/3) & : \textit{otherwise.} \end{cases}$$

In particular, $\beta_1^{**} \neq \beta_2^{**}$.

Proof. Substituting in $\beta_\omega(\pi^2) = 1 - \beta_\omega(\pi^1)$ for each $\omega \in \Omega$, we can view the program as an optimization over $(\beta_1(\pi^1), \beta_2(\pi^1)) \in [0,1]^2$. The loss is continuous so that an optimum exists, and it is strictly convex so that this optimum is unique. Direct computation shows that the given form of $(\beta_1^{**}(\pi^1), \beta_2^{**}(\pi^1))$ satisfies the first-order condition, and hence is the optimum.

Finally, let us verify that $\beta_1^{**} \neq \beta_2^{**}$. Given the form of the solution, we need only check that the numerators differ in the case that $\frac{\sqrt{c_H}}{\sqrt{c_L}} \leq \frac{3}{1+2\varphi}$. And indeed,

$$[(2+\varphi)\sqrt{c_H} - 3\varphi\sqrt{c_L}\,] - [(2-\varphi)\sqrt{c_L} - \varphi\sqrt{c_H}\,] = 2(1+\varphi)\left(\sqrt{c_H} - \sqrt{c_L}\right) > 0.$$

Q.E.D.

Now, we prove Proposition 3.

Proof of Proposition 3. Some optimal solution to program (3) exists by Fact 1. Moreover, any two triples that satisfy the conditions of the proposition's statement—which yield the same total state distribution, provide the same information to agent 1 about the total state, and provide the same information to agent 2 about the ranking state—generate the exact same loss (and so are either both optimal or both suboptimal). Hence, given Claim 7, we need only see that $(\beta_\omega^{**})_{\omega\in\Omega}$ is the unique solution to the program

$$\min_{\vec{\beta}\in(\Delta\Pi)^\Omega} \int \tfrac{c_1(\omega)}{\iota_1(\beta_\omega)}\ \mathrm{d}p_0(\omega) + \tfrac{c_2}{\iota_2\left(\int \beta_\omega\ \mathrm{d}p_0(\omega)\right)},$$

and that $\beta_1^{**} \neq \beta_2^{**}$—exactly what Claim 8 proves. *Q.E.D.*