Appendix B Proofs and Additional Results

Proof of Proposition 2. By Shephard's lemma, changes in the price of good i are given by

$$d\log p_i = -d\log A_i + \sum_{j\in\mathcal{N}} \Omega_{ij} d\log p_j + \sum_{f\in\mathcal{G}} \Omega_{if} d\log w_f,$$

solving this system gives

$$d\log p_i = -\sum_{j\in\mathcal{N}} \Psi_{ij} d\log A_j + \sum_{f\in\mathcal{G}} \Psi_{if} d\log w_f.$$

Furthermore,

$$d\log w_f = d\log\lambda_f + d\log E - d\log L_f.$$

Hence, the change in real GDP is given by

$$\begin{aligned}
d\log Y &= d\log E - \sum_{j\in mathcalN} \Omega_{0j} d\log p_j,\\
&= d\log E + \sum_{j\in\mathcal{N}} \Psi_{0j} d\log A_j - \sum_{f\in\mathcal{G}} \Psi_{0f} d\log w_f,\\
&= d\log E + \sum_{j\in\mathcal{N}} \Psi_{0j} d\log A_j - \sum_{f\in\mathcal{G}} \Psi_{0f}\left(d\log\lambda_f + d\log E - d\log L_f\right),\\
&= d\log E + \sum_{j\in\mathcal{N}} \lambda_j d\log A_j - \sum_{f\in\mathcal{G}} \lambda_f\left(d\log\lambda_f + d\log E - d\log L_f\right),\\
&= \sum_{j\in\mathcal{N}} \lambda_j d\log A_j + \sum_{f\in\mathcal{G}} \lambda_f d\log L_f,
\end{aligned}$$

using the fact that $\Psi_{0i} = \lambda_i$ and $\sum_{f\in\mathcal{G}}\lambda_f = 1$. To complete the proof, note that

$$d\log L_f = \min\{d\log\bar{L}_f, d\log\lambda_f + d\log E - d\log\bar{L}_f\}.$$

□

Proof of Proposition 3. The proof is provided in text. □

Proof of Proposition 4. This is a special case of Proposition 8. □

Proof of Proposition 5. Combine (4.1) and (4.2) with Proposition 8 and let $\theta = 1$. □

Proof of Lemma 1. This is a special case of Lemma 2. □

Proof of Proposition 6. This is a special case of Proposition 9. □

Proof of Proposition 7. From Proposition 10 in Appendix D, we know that

$$d\log\lambda_k = Cov_{\Omega^{(0)}}\left(d\log\omega_0, \frac{\Psi_{(k)}}{\lambda_k}\right) + \sum_{j\in\mathcal{N}}\lambda_j(\theta-1)Cov_{\Omega^{(j)}}\left(\sum_{i\in\mathcal{N}}\Psi_{(i)}d\log A_i - \sum_{f\in\mathcal{G}}\Psi_{(f)}\left(d\log\lambda_f - d\log L_f\right), \frac{\Psi_{(k)}}{\lambda_k}\right).$$

with

$$d\log L_f = \begin{cases} d\log\bar{L}_f, & \text{for} \quad f\in\mathcal{K}, \\ \min\left\{d\log\lambda_f + d\log E, d\log\bar{L}_f\right\}, & \text{for} \quad f\in\mathcal{L}. \end{cases}$$

Now, use the identity

$$\sum \lambda_{j\in 1+\mathcal{N}} Cov_{\Omega^{(j)}}\left(\Psi_{(f)}, \Psi_{(k)}\right) = \lambda_k\lambda_f\left[\frac{\Psi_{fk}-\delta_{fk}}{\lambda_k} + \frac{\Psi_{kf}-\delta_{fk}}{\lambda_f} + \frac{\delta_{fk}}{\lambda_k} - 1\right],$$

where δ_{fk} is Kronecker delta, to get

$$d\log\lambda_k = Cov_{\Omega^{(0)}}\left(d\log\omega_0, \frac{\Psi_{(k)}}{\lambda_k}\right) - (\theta-1)\sum_{f\in\mathcal{G}}[\delta_{fk}-\lambda_f]\left(d\log\lambda_f - d\log L_f\right) + \frac{1}{\lambda_k}\sum_{f\in\mathcal{G}}(\theta-1)\left[\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(f)}\Psi_{(k)}\right) - \lambda_l\lambda_k\right]\left(d\log\lambda_f - d\log L_f\right),$$

where we use the fact that

$$Cov_{\Omega^{(0)}}\left(\sum_{f\in\mathcal{G}}\Psi_{(f)}\left(d\log\lambda_f - d\log L_f\right), \frac{\Psi_{(k)}}{\lambda_k}\right) = \frac{1}{\lambda_k}\sum_{f\in\mathcal{G}}\left[\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(f)}\Psi_{(k)}\right) - \lambda_l\lambda_k\right]\left(d\log\lambda_f - d\log L_f\right).$$

Rearrange this to get

$$d\log\lambda_k = \frac{1}{\theta}Cov_{\Omega^{(0)}}\left(d\log\omega_0, \frac{\Psi_{(k)}}{\lambda_k}\right) + \frac{\theta-1}{\theta}d\log L_k - \frac{1}{\lambda_k}\frac{(1-\theta)}{\theta}\sum_{f\in\mathcal{G}}\left[\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(f)}\Psi_{(k)}\right)\right]\left(d\log\lambda_f - d\log L_f\right).$$

□

Proof of Proposition 8. Suppose that there is only one capital factor, and conjecture an equilibrium where every labor factor becomes demand-constrained. For each demand-constrained factor, from Proposition 7, we have

$$
\begin{aligned}
\frac{1}{\theta} d\log\lambda_k =& \frac{1}{\theta} Cov_{\Omega^{(0)}}\left(d\log\omega_0, \frac{\Psi_{(k)}}{\lambda_k}\right) + d\log E - \frac{1}{\theta} d\log E \\
&+ \frac{1}{\lambda_k}\frac{(1-\theta)}{\theta}\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{D})}\Psi_{(k)}\right) d\log E \\
&- \frac{1}{\lambda_k}\frac{(1-\theta)}{\theta}\sum_{f\in\mathcal{S}}\left[\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(f)}\Psi_{(k)}\right)\right]\left(d\log\lambda_f - d\log \bar{L}_f\right) \\
\lambda_k d\log\lambda_k =& Cov_{\Omega^{(0)}}\left(d\log\omega_0, \Psi_{(k)}\right) - (1-\theta)\lambda_k d\log E \\
&+ (1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{D})}\Psi_{(k)}\right) d\log E \\
&- (1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(k)}\right)\left(d\log\lambda_{\mathcal{S}} - d\log\bar{L}_{\mathcal{S}}\right).
\end{aligned}
$$

Aggregating over all demand-constrained factors gives

$$
\begin{aligned}
\lambda_{\mathcal{D}} d\log\lambda_{\mathcal{D}} = \sum_{k\in\mathcal{D}}\lambda_k d\log\lambda_k =& Cov_{\Omega^{(0)}}\left(d\log\omega_0, \Psi_{(\mathcal{D})}\right) - (1-\theta)\left[\lambda_{\mathcal{D}} - \mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{D})}^2\right)\right] d\log E \\
&- (1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right)\left(d\log\lambda_{\mathcal{S}} - d\log\bar{L}_{\mathcal{S}}\right) \\
=& Cov_{\Omega^{(0)}}\left(d\log\omega_0, \Psi_{(\mathcal{D})}\right) - (1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{D})}\Psi_{(\mathcal{S})}\right) d\log E \\
&+ (1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right)\frac{1}{\lambda_{\mathcal{S}}}\lambda_{\mathcal{D}} d\log\lambda_{\mathcal{D}} \\
&+ (1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right) d\log\bar{L}_{\mathcal{S}} \\
=& \frac{Cov_{\Omega^{(0)}}\left(d\log\omega_0, \Psi_{(\mathcal{D})}\right)}{1-(1-\theta)\frac{1}{\lambda_{\mathcal{S}}}\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right)} \\
& \frac{-(1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{D})}\Psi_{(\mathcal{S})}\right) d\log E + (1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right) d\log\bar{L}_{\mathcal{S}}}{1-(1-\theta)\frac{1}{\lambda_{\mathcal{S}}}\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right)}.
\end{aligned}
$$

Where we use the fact that $\lambda_{\mathcal{D}} d\log\lambda_{\mathcal{D}} = -\lambda_{\mathcal{S}} d\log\lambda_{\mathcal{S}}$. Finally, combine the equation above with Proposition 2 to get,

$$
\begin{aligned}
d\log Y &= \lambda_{\mathcal{S}} d\log \bar{L}_{\mathcal{S}} + \lambda_{\mathcal{D}} d\log\lambda_{\mathcal{D}} + \lambda_{\mathcal{D}} d\log E \\
&= \frac{Cov_{\Omega^{(0)}}\left(d\log\omega_0, \Psi_{(\mathcal{D})}\right)}{1-(1-\theta)\frac{1}{\lambda_{\mathcal{S}}}\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right)} \\
&\quad \frac{-(1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{D})}\Psi_{(\mathcal{S})}\right)d\log E}{1-(1-\theta)\frac{1}{\lambda_{\mathcal{S}}}\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right)} + \lambda_{\mathcal{D}} d\log E \\
&\quad + \frac{(1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right)d\log \bar{L}_{\mathcal{S}}}{1-(1-\theta)\frac{1}{\lambda_{\mathcal{S}}}\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right)} + \lambda_{\mathcal{S}} d\log \bar{L}_{\mathcal{S}} \\
&= \frac{\lambda_{\mathcal{S}} Cov_{\Omega^{(0)}}\left(d\log\omega_0, \Psi_{(\mathcal{D})}\right)}{\lambda_{\mathcal{S}}-(1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right)} \\
&\quad + \frac{-\lambda_{\mathcal{S}}(1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{D})}\Psi_{(\mathcal{S})}\right)d\log E}{\lambda_{\mathcal{S}}-(1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right)} + \lambda_{\mathcal{D}} d\log E \\
&\quad + \lambda_{\mathcal{S}}\frac{(1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right)d\log \bar{L}_{\mathcal{S}}}{\lambda_{\mathcal{S}}-(1-\theta)\mathbb{E}_{\Omega^{(0)}}\left(\Psi_{(\mathcal{S})}\Psi_{(\mathcal{D})}\right)} + \lambda_{\mathcal{S}} d\log \bar{L}_{\mathcal{S}}
\end{aligned}
$$

□

Proof of Lemma 2. Define the function $\Phi(L^0) \mapsto L$ by

$$
\begin{aligned}
w_f L_f^0 &= \sum_{j\in\mathcal{N}} \Psi_{jf}\left(\frac{w_f^{1-\sigma}}{\sum_k \Psi_{jk} w_k^{1-\sigma}}\right) p_j c_j, \\
p_j c_j &= (\bar{\Omega}_{0i} - \kappa_i)\, E, \\
E &= \frac{(1-\beta)\sum_i (1-\kappa_i)}{\beta}\frac{\bar{E}_*}{1+i}\sum_h \bar{\lambda}_h^*\left(\frac{L_h^0}{L_h^*}(1-\phi_h) + \phi_h\right), \\
\tilde{w}_f &= \min\{\underline{w}_f, w_f\}\mathbf{1}(f\in\mathcal{L}) + w_f\mathbf{1}(f\in\mathcal{K}), \\
L_f &= \min\left\{\frac{1}{\tilde{w}_f}\sum_{j\in\mathcal{N}} \Psi_{jf}\left(\frac{\tilde{w}_f^{1-\sigma}}{\sum_k \Psi_{jk}\tilde{w}_k^{1-\sigma}}\right) p_j c_j, \bar{L}_f\right\}.
\end{aligned}
$$

An equilibrium is when $L^0 = L$. We show that Φ is an increasing function mapping $\prod_{f\in\mathcal{G}}[0, \bar{L}_f]$ into itself.

By Lemma 3, w_{-i} is increasing and w_i is decreasing in L_i^0. This means that $\tilde{w}_{-i}$ is increasing in L_i^0 and $\tilde{w}_i$ is decreasing in i if $w_i > \underline{w}_i$. Hence, L_{-i} is increasing in L_i^0,and L_i is increasing in L_i^0. This proves that Φ is a monotone function, and so we can apply Tarski (1955). ◻

Lemma 3. *For the following system of equations*

$$w_f L_f = \sum_{j \in \mathcal{N}} \Psi_{jf} \left(\frac{w_f^{1-\sigma}}{\sum_k \Psi_{jk} w_k^{1-\sigma}} \right) \Omega_{0j} E,$$

w_{-i} is increasing and w_i is decreasing in L_i.

Proof. Start by noting that

$$Cov_{\Omega^{(0)}}(\Psi_{(f)}, \Psi_{(k)}) = \sum_l \Omega_{0l} \Psi_{lf} \left[\Psi_{lk} - \lambda_k \right],$$

Using this fact, and Proposition 10, we can simplify

$$\begin{aligned}
\lambda_k d\log\lambda_k &= -(\theta-1) \sum_{f \in \mathcal{G}} \left[-\lambda_f \lambda_k - Cov_{\Omega^{(0)}}(\Psi_{(f)}, \Psi_{(k)}) + \mathbf{1}(f=k)\lambda_k \right] \left(d\log\lambda_f - d\log L_f \right) \\
&= \sum_{f \in \mathcal{G}} \left[(1-\theta)\mathbf{1}(f=k)\lambda_k - (1-\theta)\lambda_f\lambda_k - (1-\theta) Cov_{\Omega^{(0)}}(\Psi_{(f)}, \Psi_{(k)}) \right] \left(d\log\lambda_f - d\log L_f \right) \\
&= \sum_{f \in \mathcal{G}} \left[(1-\theta)\mathbf{1}(f=k)\lambda_k - (1-\theta)\lambda_f\lambda_k - (1-\theta) \sum_l \Omega_{0l}\Psi_{lf} \left(\Psi_{kf} - \lambda_k \right) \right] \left(d\log\lambda_f - d\log L_f \right) \\
&= \sum_{f \in \mathcal{G}} \left[(1-\theta)\mathbf{1}(f=k)\lambda_k - (1-\theta) \sum_l \Omega_{0l}\Psi_{lf}\Psi_{kf} \right] \left(d\log\lambda_f - d\log L_f \right) \\
&= \sum_{f \in \mathcal{G}} \left[(1-\theta)\mathbf{1}(f=k)\lambda_k - (1-\theta) \left[\mathbb{E}_{\Omega^{(0)}}(\Psi_{(f)}\Psi_{(k)}) \right] \right] \left(d\log\lambda_f - d\log L_f \right) \\
&= \sum_{f \in \mathcal{G}} \left[(1-\theta)\mathbf{1}(f=k)\lambda_k - (1-\theta) \left[\mathbb{E}_{\Omega^{(0)}}(\Psi_{(f)}\Psi_{(k)}) \right] \right] \left(d\log\lambda_f - d\log L_f \right) \\
&= -\sum_{f \in \mathcal{G}} \left[(1-\theta) \left[\mathbb{E}_{\Omega^{(0)}}(\Psi_{(f)}\Psi_{(k)}) \right] \right] \left(d\log\lambda_f - d\log L_f \right) + \left[(1-\theta)\lambda_k \right] \left(d\log\lambda_k - d\log L_k \right) \\
&= -(1-\theta) \sum_{f \in \mathcal{G}} \left[\mathbb{E}_{\Omega^{(0)}}(\Psi_{(f)}\Psi_{(k)}) \right] \left(d\log\lambda_f - d\log L_f \right) + \left[(1-\theta)\lambda_k \right] \left(d\log\lambda_k - d\log L_k \right)
\end{aligned}$$

Let

$$B_{kf} = \left[\mathbb{E}_{\Omega^{(0)}}(\Psi_{(f)} \frac{\Psi_{(k)}}{\lambda_k}) \right]$$

We know that

$$\sum_f B_{kf} = 1.$$

Hence, letting $x = d\log\lambda/d\log L_i$ be a column vector and e_i the ith basis vector, we can write

$$\begin{aligned}\theta x &= -(1-\theta)Bx - (I-(1-\theta)B)\,e_i\\ x &= -\left(\theta I + (1-\theta)B\right)^{-1}(I-(1-\theta)B)\,e_i = -A\,(I-(1-\theta)B)\,e_i.\end{aligned}$$

By Lemma 4, $A = (\theta I + (1-\theta)B)^{-1}$ is an M-matrix, hence by A_5 of Theorem 6.2.3 of Berman and Plemmons (1979), $-A\,(I-(1-\theta)B)\,e_i$ has the same signs as $-(I-(1-\theta)B)$. Since $-(I-(1-\theta)B)$ has negative diagonal and positive off-diagonal elements, this means that x_i is negative and x_{-i} is positive, as needed. □

Lemma 4. *The matrix defined in Lemma 4*

$$A = (\theta I + (1-\theta)B)^{-1}$$

is an M-matrix.

Proof. By Theorem 6.2.3 of Berman and Plemmons (1979), it suffices to prove that A^{-1} has all positive elements and that A is a Z-matrix. The fact that A^{-1} has all positive elements is immediate from its definition. To show that A is a Z-matrix, note that we can write

$$\begin{aligned}A &= (\theta I + (1-\theta)B)^{-1},\\ &= (I-(1-\theta)(I-B))^{-1},\\ &= \sum_{n=0}^{\infty}(1-\theta)^n(I-B)^n.\end{aligned}$$

Hence, since $I-B$ is an M-matrix, $(1-\theta)(I-B)X$ does not change the sign of the columns of X for any X. Hence, by induction, and the fact that M-matrices are closed under addition, we have that A has the same sign as the elements of $(I-B)$,and hence A is a Z-matrix. □

Proof of Proposition 9. To prove the statements regarding $\Delta\log L$, we use Theorem 3 from Milgrom and Roberts (1994). Since $\Delta\log Y$ is a monotone function of $\Delta\log L$, this also establishes the results about real GDP. It remains to prove the claims regarding inflation. To prove that labor supply shocks (on their own) are inflationary, we need to show that the price level p^Y is decreasing in by $\bar{L}$. To do so, consider some negative labor shocks then

$$\begin{aligned}
\Delta \log p^Y &= \Delta \log E - \Delta \log Y \\
&\geq \Delta \log E - \sum_f \bar{\lambda}_f \Delta \log L_f \\
&= \Delta \log \left(\sum_h \bar{\lambda}^*_h \left(\frac{L_h}{L^*_h}(1-\phi_h) + \phi_h \right) \right) - \sum_f \bar{\lambda}_f \Delta \log L_f \\
&\geq \Delta \log \left(\sum_h \bar{\lambda}^*_h \frac{L_h}{L^*_h} \right) - \sum_f \bar{\lambda}_f \Delta \log L_f \\
&= \Delta \log \left(\sum_h \bar{\lambda}^*_h \exp(\log L_h / L^*_h) \right) - \sum_f \bar{\lambda}_f \Delta \log L_f \\
&\geq \sum_h \bar{\lambda}^*_h \Delta \log L_h - \sum_f \bar{\lambda}_f \Delta \log L_f = 0,
\end{aligned}$$

as long as $\bar{\lambda}^* = \bar{\lambda}$. The second line follows from the fact that $\log Y$ is log-concave (see Baqaee and Farhi, 2019b).

To prove that aggregate demand shocks, like forward guidance shocks, (on their own) are deflationary, we need to show that the price level p^Y is increasing $E_*/(1+i)$. To do so, consider some shock then

$$\begin{aligned}
\Delta \log p^Y &= \Delta \log E - \Delta \log Y \\
&\geq \Delta \log E - \sum_f \bar{\lambda}_f \Delta \log L_f \\
&= \Delta \log \left(\frac{(1-\beta)\sum_i (1-\kappa_i)}{\beta} \frac{\bar{E}_*}{1+i} \sum_h \bar{\lambda}^*_h \left(\frac{L_h}{L^*_h}(1-\phi_h) + \phi_h \right) \right) - \sum_f \bar{\lambda}_f \Delta \log L_f \\
&\geq \Delta \log \left(\frac{\bar{E}_*}{1+i} \right) - \sum_f \bar{\lambda}_f \Delta \log L_f \\
&\geq \Delta \log \left(\frac{\bar{E}_*}{1+i} \right) - \sum_f \bar{\lambda}_f \Delta \log \left(\frac{\bar{E}_*}{1+i} \right) \\
&\geq 0.
\end{aligned}$$

□

Proof of Proposition 10. This follows from an application of Proposition 9 in Baqaee and Farhi (2019a). □

Appendix C A Microfoundation for Demand Shocks

When household preferences are Cobb-Douglas, there is a simple microfoundation for the demand shocks motivated by health concerns. To see this, consider households with log preferences

$$(1-\beta)\left[\sum_{i\in\mathcal{N}}\bar{\Omega}_{0i}\log c_i - H\left(\{c_i\}_{i\in\mathcal{N}}\right)\right]+\beta\sum_i \bar{\Omega}_{0i}\log c_i^*,$$

where $\beta\in[0,1]$ captures households' time-preferences, and c_i and c_i^* are current and future consumption of good i. The function $H\left(\{c_i\}_{i\in\mathcal{N}}\right)$ is a homothetic aggregator that captures health concerns of the household associated with consumption today. We assume there are no health concerns in the future. We let the disutility of consumption due to health concerns be

$$H\left(\{c_i\}_{i\in\mathcal{N}}\right)=\sum_i \kappa_i \log c_i,$$

where $\kappa_i \geq 0$ captures the riskiness of consuming good i. As κ_i increases, households choose to spend a smaller fraction of their permanent income on purchasing i. We call an increase in κ_i an individual negative demand shock for sector i (in contrast to aggregate demand shocks which affect spending on all goods produced this period).

The health-risk parameters κ then map into shocks to both the intersectoral composition of demand

$$\Delta\log\omega_{0i}=\Delta\log\frac{\bar{\Omega}_{0i}-\kappa_i}{(1-\sum_{j\in\mathcal{N}}\kappa_j)\bar{\Omega}_{0i}},$$

and shocks to aggregate demand

$$\Delta\log\zeta=-\Delta\log(1+i)-\Delta\log\frac{\beta}{1-\beta}+\Delta\log\bar{E}_*+\Delta\log(1-\sum_{j\in\mathcal{N}}\kappa_j).$$

For future reference, when we refer to an aggregate demand shock, we mean a change in $\Delta\log\zeta$ that keeps the intersectoral composition of final demand $\Delta\log\omega_0=0$ constant.

Appendix D Extension: Generalizing to Arbitrary CES Economies

This appendix shows how Proposition 7 generalizes to arbitrary nested-CES production networks.[28] To do this, suppose that each good $i\in\mathcal{N}$ is produced with the production

[28] Our results can easily be extended beyond the nested-CES case along the lines of Section 5 in Baqaee and Farhi (2019a).

function

$$\frac{y_i}{\overline{y}_i} = \frac{A_i}{\overline{A}_i} \left(\sum_{j \in \mathcal{N}+\mathcal{G}} \bar{\omega}_{ij} \left(\frac{x_{ij}}{\overline{x}_{ij}} \right)^{\frac{\theta_i - 1}{\theta_i}} \right)^{\frac{\theta_i}{\theta_i - 1}},$$

where we now allow the elasticity of substitution θ_i to vary across producers. The households' consumption function is

$$\frac{y_0}{\overline{y}_0} = \left(\sum_{j \in \mathcal{N}+\mathcal{G}} \bar{\omega}_{0j} \frac{\omega_{0j}}{\bar{\omega}_{0j}} \left(\frac{x_{0j}}{\overline{x}_{0j}} \right)^{\frac{\theta_0 - 1}{\theta_0}} \right)^{\frac{\theta_0}{\theta_0 - 1}},$$

where ω_{0j} are sectoral demand shocks with $\sum_j \omega_{0j} = 1$. where x_{ij} are intermediate inputs from j used by i. In these equations, variables with over-lines are normalizing constants.To simplify the notation below, we think of ω_0 as a $1 \times (1 + \mathcal{N} + \mathcal{G})$ vector with k-th element ω_{0k}.

We now show how changes in factor income shares $d \log \lambda_f$ are determined, which along with Propositions 1 and 2 pins down output, employment, and inflation. Recall that for a matrix M, we denote by $M_{(i)}$ its i-th row by $M^{(j)}$ its j-th column. We write $Cov_{\Omega^{(j)}}(\cdot, \cdot)$ to denote the covariance of two vectors of size $1 + \mathcal{N} + \mathcal{G}$ using the j-the row of the input-ouput matrix $\Omega^{(j)}$ as a probability distribution.

Proposition 10 (Propagation)**.** *Changes in sales and factor shares are given by*

$$\begin{aligned} d \log \lambda_k = {} & \theta_0 Cov_{\Omega^{(0)}} \left(d \log \omega_{(0)}, \frac{\Psi_{(k)}}{\lambda_k} \right) \\ & + \sum_{j \in 1+\mathcal{N}} \lambda_j (\theta_j - 1) Cov_{\Omega^{(j)}} \left(\sum_{i \in \mathcal{N}} \Psi_{(i)} d \log A_i - \sum_{f \in \mathcal{G}} \Psi_{(f)} \left(d \log \lambda_f - d \log L_f \right), \frac{\Psi_{(k)}}{\lambda_k} \right) \end{aligned}$$

almost everywhere, where changes in factor employments are given by

$$d \log L_f = \begin{cases} d \log \bar{L}_f, & \text{for} \quad f \in \mathcal{K}, \\ \min \left\{ d \log \lambda_f + d \log E, d \log \bar{L}_f \right\}, & \text{for} \quad f \in \mathcal{L}. \end{cases}$$

The intuition for Proposition 10 is similar to that of Proposition 7. Changes in factor shares depend on changes in the composition of final demand and on relative prices:

$$d \log \lambda_k = \theta_0 Cov_{\Omega^{(0)}} \left(d \log \omega_0, \Psi_{(k)} / \lambda_k \right) + \sum_{j \in 1+\mathcal{N}} \lambda_j (\theta_j - 1) Cov_{\Omega^{(j)}} \left(-d \log p, \Psi_{(k)} / \lambda_k \right).$$

The first term on the right-hand side $\theta_0 Cov_{\Omega^{(0)}}(d\log\omega_0, \Psi_{(k)}/\lambda_k)$ on the right-hand side is the direct effect of shocks to the sectoral composition of final demand on the sales of k.The second term $\sum_{j\in 1+\mathcal{N}} \lambda_j(\theta_j - 1)Cov_{\Omega^{(j)}}(-d\log p, \Psi_{(k)}/\lambda_k)$ on the right-hand side captures the changes in the sales of k from substitutions by producers j downstream from k. If producer j has an elasticity of substitution θ_j below one so that its inputs are complements, then it shifts its expenditure towards those inputs with higher price increases, and this increases the demand for k if those inputs also use k intensively (measured by Ψ_{lk}/λ_k).The result follows from noticing that changes in relative prices are, in turn, given by changes in factor shares

$$d\log p_k - d\log E = -\sum_{i\in\mathcal{N}} \Psi_{ki} d\log A_i + \sum_{f\in\mathcal{G}} \Psi_{kf} d\log(d\log\lambda_f - d\log L_f),$$

where we use the fact that $d\log w_f - d\log E = d\log\lambda_f + d\log E - d\log L_f$.

Appendix E Extension: Semi-Flexible Wages

In practice, we might imagine that wages can fall albeit not by enough to clear the market. For each factor $f \in \mathcal{L}$, suppose the following conditions hold

$$\left(\frac{w_f}{\bar{w}_f} - \left(\frac{L_f}{\bar{L}_f}\right)^{\frac{1}{\gamma_f}}\right)\left(L - \bar{L}_f\right) = 0, \quad L_f \le \bar{L}_f, \quad \left(\frac{L_f}{\bar{L}_f}\right)^{\frac{1}{\gamma_f}} \le \frac{w_f}{\bar{w}_f}.$$

The parameter γ_f controls the degree of downward wage flexibility. If $\gamma_f = \infty$, then the wage is perfectly rigid downwards. If $\gamma_f = 0$, then the wage is fully flexible, and we recover the neoclassical case.

E.1 Generalizing the Results

Collectively, Propositions 1 and 2, as well as Proposition 10 in Appendix D, pin down all equilibrium outcomes. So, we discuss how each can be generalized. Proposition 1 remains exactly the same as before, so we do not restate it. The only change to Proposition

2 is that we now have

$$
\begin{aligned}
d\log Y &= \sum_{i\in\mathcal{N}} \lambda_i d\log A_i + \sum_{f\in\mathcal{G}} \lambda_f d\log L_f, \\
&= \sum_{i\in\mathcal{N}} \lambda_i d\log A_i + \sum_{f\in\mathcal{G}} \lambda_f d\log \bar{L}_f + \sum_{f\in\mathcal{L}} \frac{\gamma_f}{1+\gamma_f}\lambda_f \min\left\{d\log\lambda_f + d\log E - d\log\bar{L}_f, 0\right\},
\end{aligned}
$$

In particular, this implies that Corollary 1 about the behavior of inflation remains unchanged:

$$
d\log p^Y = \frac{1}{\rho} d\log\zeta - \frac{1}{\rho}\sum_{i\in\mathcal{N}} \lambda_i d\log A_i - \frac{1}{\rho}\sum_{f\in\mathcal{G}} \phi_f \lambda_f d\log L_f.
$$

Hence, reductions in employment are still inflationary in the absence of exogenous negative demand shocks.

The only endogenous objects left to be determined are the factor shares. Proposition 10 can be generalized to pin down factor shares. In particular, changes in factor shares solve the following linear system:

$$
\begin{aligned}
d\log\lambda_k = \theta_0 Cov_{\Omega^{(0)}}\left(d\log\omega_{(0)}, \frac{\Psi_{(k)}}{\lambda_k}\right) + \sum_{j\in 1+\mathcal{N}} \lambda_j(\theta_j - 1) Cov_{\Omega^{(j)}}\left(\sum_{i\in\mathcal{N}} \Psi_{(i)} d\log A_i\right. \\
\left. - \sum_{f\in\mathcal{G}} \Psi_{(f)}\left(d\log\lambda_f - d\log L_f + d\log E\right), \frac{\Psi_{(k)}}{\lambda_k}\right)
\end{aligned}
$$

where

$$
d\log L_f = \begin{cases} \frac{\gamma_f}{1+\gamma_f}\left(d\log\lambda_f + d\log E\right) + \frac{1}{1+\gamma_f} d\log\bar{L}_f & \text{if} \quad f\in\mathcal{D} \\ d\log\bar{L}_f & \text{if} \quad f\in\mathcal{S}. \end{cases}
$$

Appendix F Extension: Investment

To model investment, we add intertemporal production functions into the model. An investment function transforms goods and factors in the present period into goods that can be used in the future. In this case, instead of breaking the problem into an intertemporal and intratemporal problem, we must treat both problems at once. In this section, we first discuss the general local comparative statics with investment, extending the results in Section 3, then we discuss a special case with simple sufficient statistics and global comparative statics, extending the results in Section 5.2.

In the body of the paper, we assumed that prices in the future p_*^Y were fixed, which meant that nominal expenditures in the future were also fixed $p_*^Y Y_* = E_*$. In the version of the model with investment, output in the future Y_* is not exogenous, so assuming p_*^Y is no longer equivalent to assuming E_* is fixed. Therefore, we consider both situations.

F.1 General local comparative statics

When we add investment to the model, we can still use Proposition 2 without change. However, we can no longer use the Euler equation to pin down nominal expenditures today, since nominal GDP today includes investment expenditures and output tomorrow can no longer taken to be exogenous. Instead, to determine $d\log E$, we must use a version of Proposition 10. For this subsection, we assume that nominal expenditures in the future period are fixed and we denote the future period by $*$.

In particular, let λ_i^I denote the intertemporal sales share — expenditures on quantity i as a share of the net present value of household income. Furthermore, let $\bar{\Omega}^I$ represent the intertemporal input-output matrix, which includes the capital accumulation equations. Then, letting intertemporal consumption be the zero-th good, and abstracting from shocks to the sectoral composition of demand for simplicity, we can write

$$d\log\lambda_k^I = \sum_j \lambda_j^I(\theta_j - 1) Cov_{\Omega^{I,(j)}}\left(\sum_{i\in\mathcal{N}} \Psi_{(i)}^I d\log A_i - \sum_{f\in\mathcal{G}} \Psi_{(f)}^I \left(d\log\lambda_f^I - d\log L_f\right), \frac{\Psi_{(k)}^I}{\lambda_k^I}\right)$$

almost everywhere, where changes in factor employments are given by

$$d\log L_f = \begin{cases} d\log\bar{L}_f, & \text{for} \quad f\in\mathcal{K}, \\ \min\left\{d\log\lambda_f^I - d\log\lambda_*^I, d\log\bar{L}_f\right\}, & \text{for} \quad f\in\mathcal{L}. \end{cases}$$

This follows from the fact that nominal expenditures on each factor f is given by $d\log\lambda_f^I + d\log E^I$, where E^I is the net-present value of household income. However, since nominal expenditures in the future are fixed, we have $d\log E_* = d\log\lambda_*^I + d\log E^I = 0$. This allows us to write nominal expenditures on each factor as $d\log\lambda_f^I - d\log\lambda_*^I$.

F.2 Global Comparative Statics

We can extend the results in Section 5.2 to the model with investment. To do so, we assume that the intertemporal elasticity of substitution ρ is the same as the intersectoral elasticities of substitution $\rho = \theta_j = \theta$ for every $j \in \mathcal{N}$. In this case, the initial factor shares are, once again, a sufficient statistic for the production network. Furthermore, we can also prove that the set of equilibria form a lattice under some additional assumptions.

Proposition 11. *Suppose that the intertemporal elasticity of substitution, the elasticities of substitution in production and in final demand are all the same θ. Suppose that there are only shocks to potential factor supplies $\Delta \log \bar{L}$. If future nominal expenditure is fixed, then assuming in addition that $\theta < 1$, there is a unique best and worst equilibrium: for any other equilibrium, $\Delta \log L$ are lower than at the best and higher than at the worst. Furthermore, both in the best and in the worst equilibrium, $\Delta \log L$ are increasing in $\Delta \log \bar{L}$.*

Intuitively, a negative shock to potential factor supply today potentially reduces output tomorrow by reducing resources available for consumption tomorrow. Since nominal expenditures tomorrow are fixed, this raises the price level tomorrow. If the elasticity of substitution θ is less than one, then the increase in the price level tomorrow reduces expenditures on non-shocked factor markets and potentially causes them to become slack.

In Proposition 11, we assume that nominal expenditures in the final period are fixed. If instead we assume that the nominal price level in the future is fixed, rather than nominal expenditures, then Proposition 11 applies regardless of the value of the elasticity of substitution θ.

Appendix G Extension: Bankruptcies

The paper abstracts from capital market frictions and bankruptcies. In this appendix, we briefly discuss how our results can be extended to the case with these frictions. We begin by generalizing our comparative statics to a case with firm exits. We then make three observations: (1) in a production network, the negative effects of demand shocks are amplified if there are exits because of an intermediate-input multiplier; (2) exits, by acting as endogenous negative supply shocks, can change the flow of spending and cause Keynesian spillovers outside of the Cobb-Douglas case; and, (3) firm failures, by potentially destroying intangible firm-specific capital, can reduce output in the future, and by lowering output in the future, reduce aggregate demand today through the Euler equation.

G.1 Local Comparative Statics with Bankruptcies

To capture firm failures, we modify the general structure described in Section 2 as follows. We assume that output in sector $i \in \mathcal{N}$ is a CES aggregate of identical producers j each with constant returns production functions $y_{ik} = A_i f_i(x_{ij}^k)$, where x_{ij}^k is the quantity of industry

j's output used by producer k in industry i. Assuming all firms within an industry use the same mix of inputs, sectoral output is

$$y_i = \left(\int y_{ik}^{\frac{\sigma_i - 1}{\sigma_i}} dk \right)^{\frac{\sigma_i}{\sigma_i - 1}} = M_i^{\frac{1}{\sigma_i - 1}} A_i f_i(x_{ij}), \tag{G.1}$$

where x_{ij} is the quantity of input j used by industry i, M_i is the mass of producers in industry i, $\sigma_i > 1$ is the elasticity of substitution across producers, and A_i is an exogenous productivity shifter. From this equation, we see that a change in the mass of operating firms acts like a productivity shock and changes the industry-level price. Therefore, if shocks outside sector i trigger a wave of exits, then this will set in motion endogenous negative productivity shock $(1/(\sigma_i - 1))\Delta \log M_i$ in sector i.

Suppose that each firm must maintain a minimum level of revenue in order to continue operation.[29, 30] We are focused on a short-run application, so we do not allow new entry, but of course, this would be important for long-run analyses.[31]

The mass of firms that operate in equilibrium is therefore given by

$$M_i = \min \left\{ \frac{\lambda_i E}{\bar{\lambda}_i \bar{E}} \bar{M}_i, \bar{M}_i \right\},$$

where $\bar{M}_i$ is the exogenous initial mass of varieties, $\lambda_i E$ is nominal revenue earned by sector i and $\bar{\lambda}_i \bar{E}$ is the initial nominal revenue earned by i. If nominal revenues fall relative to the baseline, then the mass of producers declines to ensure that sales per producer remain constant. In order to capture government-mandated shutdowns of certain firms, we allow for shocks that reduce the exogenous initial mass of producers $\Delta \log \bar{M}_i \leq 0$.

We can generalize Propositions 2 and 10 to this context. The only difference is that we must replace $d \log A_i$ by $d \log A_i + (1/(\sigma_i - 1)) d \log M_i$, where

$$d \log M_i = d \log \bar{M}_i + \min\{d \log \lambda_i + d \log E - d \log \bar{M}_i, 0\}.$$

[29]One possible micro-foundation is each producer must pay its inputs in advance by securing within-period loans and that these loans have indivisibilities: only loans of size greater than some minimum level can be secured. This minimum size is assumed to coincide with the initial costs $\bar{\lambda}_i \bar{E}/\bar{M}_i$ of the producer.

[30]Another possible micro-foundation is as follows. Producers within a sector charge a CES markup $\mu_i = \sigma_i/(\sigma_i - 1)$ over marginal cost. These markups are assumed to be offset by corresponding production subsidies. Producers have present nominal debt obligations corresponding to their initial profits $(1 - 1/\mu_i)\bar{\lambda}_i \bar{E}/\bar{M}_i$. The same is true in the future. If present profits $(1 - 1/\mu_i)\lambda_i E/M_i$ fall short of the required nominal debt payment $(1 - 1/\mu_i)\bar{\lambda}_i \bar{E}/\bar{M}_i$, then the firm goes bankrupt and exits. Alternatively, we can imagine that there is no future debt obligation but that firms cannot borrow.

[31]See Baqaee (2018) and Baqaee and Farhi (2020a) for production networks with both entry and exit.

This backs up the claim that the $d\log M_i$'s act like endogenous negative productivity shocks. They provide a mechanism whereby a negative demand shock, say in the composition of demand or in aggregate demand $d\log\zeta$, triggers exits that are isomorphic to negative supply shocks.

As in the other examples, the general lesson is that the output response, to a first-order, is again given by an application of Hulten's theorem along with an amplification effect which depends on how the network redistributes demand and triggers Keynesian unemployment in some factors and firm failures in some sectors.

Having generalized the local comparative statics, we now make three observations about the way bankruptcies can propagate and affect aggregates. In order to simplify the exposition, we abstract away from HtM households for the rest of the section.

G.2 Intermediate Multiplier of Bankruptcies

If there are increasing returns, then firm failures can also affect supply today directly. As the economy scales down, marginal cost goes up. Our formulation of industry-level production functions (G.1) have this property due to the love-of-variety effect. Hence, firm exits act like negative TFP shocks, and if there are intermediate inputs, then these endogenous negative TFP shocks are amplified.

To see this, consider a Cobb-Douglas model where $\rho = \theta_0 = \theta_j = 1$ and negative demand shocks. In this case, since there are no HtM households, the effect on output is given by

$$d\log Y = \sum_{i\in\mathcal{N}} \lambda_i \frac{1}{\sigma_i - 1} d\log M_i = \sum_{i\in\mathcal{N}} \lambda_i \frac{1}{\sigma_i - 1}(d\log\lambda_i + d\log E).$$

Using the fact that $d\log\lambda_i + d\log E = -\sum_{j\in\mathcal{N}} \Psi_{ji} d\kappa_j/\lambda_i$, we can write

$$d\log Y = -\sum_{i\in\mathcal{N}} \frac{1}{\sigma_i - 1} \sum_{j\in\mathcal{N}} \Psi_{ji} d\kappa_j.$$

Hence, the higher is the use of intermediate inputs, the larger are the elements of the Leontief inverse Ψ, and the larger is the negative effect on output. Intuitively, a reduction in demand causes exits at every step in the supply chain, and so the longer the supply chains, the more costly the exits.

G.3 Bankruptcies and Expenditure Switching

In the previous example, we deliberately chose a Cobb-Douglas economy since the expenditure shares do not respond to relative prices. If the elasticities of substitution are not all equal to one, the endogenous TFP shocks associated with exits, by changing expenditure shares and the flow of spending, can trigger additional cascades of unemployment and failure.

To make this concrete, consider a simple example economy without intermediate inputs where each sector uses only its own labor. Assume that there are no shocks to aggregate demand ($d\log\zeta = 0$). Set the intertemporal elasticity of substitution $\rho = 1$ and share of HtM households $1-\phi = 0$ to ensure that nominal expenditure is constant ($d\log E = 0$). We also assume that there are no exogenous shocks to productivities ($d\log A_i = 0$), no shocks to potential labor ($d\log\bar{L}_f = 0$), and no shocks to the sectoral composition of demand ($d\log\omega_{0j} = 0$). Finally, we assume that all sectors have the same within-sector elasticity of substitution $\sigma_i = \sigma > 1$.

We focus on exogenous shocks $d\log\bar{M}_i \leq 0$ capturing government-mandated shutdowns. We show how endogenous failures can amplify these negative supply shocks. The insights are more general and also apply to shocks to potential labor. Similarly, failures can be triggered by negative aggregate demand shocks, and the resulting endogenous negative supply shocks can result in stagflation with simultaneous reductions in output and increases in inflation.

We start by analyzing the case where sectors are complements, and then consider the case where they are substitutes. For brevity, we jump directly to the final result and leave the derivations in a different appendix — Appendix H.

Shut-down shock with complements. Assume that sectors are complements ($\theta < 1$) and consider the government-mandated shutdown of some firms in only one sector i. The change in output is given by

$$d\log Y = \lambda_i \frac{1}{\sigma-1} d\log\bar{M}_i + \frac{(1-\theta)(1-\lambda_i)\frac{\sigma}{\sigma-1}}{1-(1-\theta)(1-\lambda_i)\left(1-\frac{1}{\sigma-1}\frac{\lambda_i}{1-\lambda_i}\right)}\lambda_i \frac{1}{\sigma-1} d\log\bar{M}_i.$$

The first term on the right-hand side is the direct reduction in output from the shut-down in sector i. The second term capture the further indirect equilibrium reduction in output due to firm failures and Keynesian unemployment in the other sectors. Intuitively, the shutdown in sector i raises the relative price of i, and because of complementarities, demand in the rest of the sectors falls. This reduction in nominal spending causes unemployment and additional exits in the other sectors.

Shut-down shock with substitutes. Consider the same experiment as above but assume now that sectors are substitutes ($\theta > 1$). Shut-downs in i raise the price of i relative to other sectors, and cause substitution away from i. As long as the elasticity of substitution within the sector $\sigma > 1$ is large enough and the elasticity of substitution across sectors $\theta > 1$ is not too large, the shut-down in sector i causes unemployment in sector i, but no additional firm failures in sector i. Furthermore, the other sectors maintain full employment and experiences no failures. In this case the response of output is given by

$$d\log Y = \lambda_i \frac{1}{\sigma - 1} d\log \bar{M}_i + \frac{(\theta_0 - 1)(1 - \lambda_i)}{1 - (\theta_0 - 1)\lambda_i} \lambda_i \frac{1}{\sigma - 1} d\log \bar{M}_i,$$

where the first term on the right-hand side is the direct effect of the shutdown and the second term is the amplification from the indirect effect of the shutdown which results in Keynesian unemployment in i.

G.4 Scarring Effect of Bankruptcies

One of the primary concerns about firm failures is that it results in the destruction of irreversible investments. This lowers output in the future, and through the Euler equation, depresses spending today.[32] In other words, the destruction of irreversible investments can act like an endogenous negative aggregate demand shock. To see this, for simplicity, assume there are no HtM agents and suppose that when firms exit in the first period $d\log M$, they do not return in the next period.

In particular, by the Envelope theorem, output in the future falls by

$$d\log Y_* = \sum_{i \in \mathcal{N}} \frac{\lambda_i^*}{\sigma_i - 1} d\log M_i = \sum_{i \in \mathcal{S}} \frac{\lambda_i^*}{\sigma_i - 1} d\log \bar{M}_i + \sum_{i \in \mathcal{D}} \frac{\lambda_i^*}{\sigma_i - 1} \left(d\log \lambda_i + d\log E \right).$$

The endogenous changes in $d\log Y_*$ then mean that the previously exogenous aggregate demand shock $d\log \zeta$, defined by (3.3) now contains an endogenous term

$$d\log \zeta = -\rho \Big(d\log(1+i) + \frac{1}{1-\beta} d\log \beta - d\log \bar{p}_*^Y \Big) + d\log Y_*.$$

However, the rest of the model remains the same. We can combine the Euler equation in

[32] This mechanism is the same as the one emphasized by Benigno and Fornaro (2018), except here it corresponds to the destruction of irreversible investment instead of reduced investment in innovation.

(3.4), with the aggregation and propagation equations in Propositions 2 and 10 (remembering that $d\log A_i$ should be replaced by $d\log A_i + d\log M_i/(\sigma_i - 1)$).

Appendix H Detailed Derivations for Example with Failures

Preliminaries. Changes in the sales of i are given by

$$d\log\lambda_i = (1-\theta_0)(1-\lambda_i)\Big(d\log p_i - \sum_{j\in\mathcal{N}}\lambda_j d\log p_j\Big), \tag{H.1}$$

where changes in the price of i depend on changes in the wage in i and on the endogenous reduction in the productivity of i driven by firm failures

$$d\log p_i = d\log w_i - \frac{1}{\sigma-1}d\log M_i. \tag{H.2}$$

The change in wages in i are given by

$$d\log w_i = \max\{d\log\lambda_i - d\log\bar{L}_i, 0\}, \tag{H.3}$$

and changes in the mass of producers in i are given by

$$d\log M_i = \min\{d\log\lambda_i, d\log\bar{M}_i\}. \tag{H.4}$$

We consider the effect of shutdown shocks $d\log\bar{M}_i$ starting with the case where sectors are complements and then the case where they are substitutes. The effect of negative labor shocks $d\log\bar{L}_i$ is similar.

Shut-down shock with complements. Assume that sectors are complements ($\theta < 1$) and consider the government-mandated shutdown of some firms in only one sector i. We can aggregate the non-shocked sectors into a single representative sector indexed by $-i$. We therefore have $d\log\bar{M}_i < 0 = d\log\bar{M}_{-i}$.

The closures of firms in i raise its price ($d\log p_i > 0$), which because of complementarities, increases its share ($d\log\lambda_i > 0$). It therefore does not trigger any further endogenous exit in this shocked sector ($d\log M_i = d\log\bar{M}_i$). In addition, the wages of its workers increases ($d\log w_i > 0$). The shock reduces expenditure on the other sectors ($d\log\lambda_{-i} < 0$), and this reduction in demand triggers endogenous exits ($d\log M_{-i} < 0$), pushes wages against their downward rigidity constraint ($d\log w_{-i} = 0$) and creates unemployment ($d\log L_{-i} < 0$), both of which endogenously amplify the reduction in output through failures and Keynesian effects.

Using equations (H.1), (H.2), (H.3), and (H.4), we find

$$d\log\lambda_i = -\frac{(1-\theta)(1-\lambda_i)}{1-(1-\theta)(1-\lambda_i)\left(1-\frac{1}{\sigma-1}\frac{\lambda_i}{1-\lambda_i}\right)}\frac{1}{\sigma-1}d\log\bar{M}_i > 0,$$

$$d\log M_{-i} = d\log L_{-i} = -\frac{\lambda_i}{1-\lambda_i}d\log\lambda_i < 0,$$

and finally

$$d\log Y = \lambda_i\frac{1}{\sigma-1}d\log\bar{M}_i + \frac{(1-\theta)(1-\lambda_i)\frac{\sigma}{\sigma-1}}{1-(1-\theta)(1-\lambda_i)\left(1-\frac{1}{\sigma-1}\frac{\lambda_i}{1-\lambda_i}\right)}\lambda_i\frac{1}{\sigma-1}d\log\bar{M}_i.$$

The first term on the right-hand side is the direct reduction in output from the shut-down in sector i. The second term capture the further indirect equilibrium reduction in output via firm failures and Keynesian unemployment in the other sectors.

Shut-down shock with substitutes. Consider the same experiment as above but assume now that sectors are substitutes ($\theta > 1$). We conjecture an equilibrium where sales in sector i do not fall more quickly than the initial shock $d\log\lambda_i - d\log\bar{M}_i > 0$. Sector i loses demand following the exogenous shutdown of some of its firms, and this results in unemployment in in the sector ($d\log L_i < 0$) but no endogenous firm failures ($d\log M_i = d\log\bar{M}_i$) . On the other hand, sector $-i$ maintains full employment and experiences no failures.

To verify that this configuration is indeed an equilibrium, we compute

$$d\log\lambda_i = \frac{(\theta-1)(1-\lambda_i)}{1-(\theta-1)\lambda_i}\frac{1}{\sigma-1}d\log\bar{M}_i.$$

We must verify that

$$0 > d\log\lambda_i > d\log\bar{M}_i.$$

The first inequality is verified as long as $\theta > 1$ is not too large. The second inequality is verified if $\sigma > 1$ is large enough and $\theta > 1$ is not too large.

If these conditions are violated, then we can get a jump in the equilibrium outcome. Intuitively, in those cases, the shutdown triggers substitution away from i, and that substitution is so dramatic than it causes more firms to shutdown, and the process feeds on itself ad infinitum. Any level of $d\log L_i < 0$ and $d\log M_i < d\log \bar{M}_i$ can then be supported as equilibria. Although we do not focus on it, this possibility illustrates how allowing for firm failures with increasing returns to scale can dramatically alter the model's behavior.

Assuming the regularity conditions above are satisfied, the response of output is given by

$$d\log Y = \lambda_i \frac{1}{\sigma - 1} d\log \bar{M}_i + \frac{(\theta_0 - 1)(1 - \lambda_i)}{1 - (\theta_0 - 1)\lambda_i} \lambda_i \frac{1}{\sigma - 1} d\log \bar{M}_i,$$

where the first term on the right-hand side is the direct effect of the shutdown and the second term is the amplification from the indirect effect of the shutdown which results in Keynesian unemployment in i.