Online Appendix: Sectoral Heterogeneity and Monetary Policy

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A Extensions to Model without Inflation

A.1 Adjustment Costs and Habit Formation

The quantitative model presented in section III included adjustment costs. However, the analytical results in section I did not include adjustment costs. This section extends the analysis in section I to include adjustment costs and habit formation of various kinds.

Consider the model analyzed in section I, but with household utility:

$$\sum \beta^t \theta_t \left[u\left(\vec{c}_t, \vec{c}_{t-1}\right) - v\left(n_t\right) \right]$$

This specification nests various common forms of internal habit formation and consumption adjustment costs, as long as they affect the utility function rather than the budget constraint.[1] Household optimality conditions are just as in the baseline model except that the good j pricing equation is:

$$p_t^j = \frac{u_{c_t^j}}{\lambda_t} + \frac{\left(u_{t+1}\right)_{c_t^j}}{R_{t+1}\lambda_{t+1}} + \frac{1-\delta^j}{R_{t+1}} p_{t+1}^j$$

The only difference relative to the baseline case is the inclusion of the term $\left(u_{t+1}\right)_{c_t^j} / \left(R_{t+1}\lambda_{t+1}\right)$ in the good j pricing equation. This term, which will generally be negative, reflects the cost of higher past consumption in the following period. Note that the inclusion of adjustment costs might also affect the value of marginal utility of consumption. Firm optimality conditions and market clearing expressions are just as in the baseline model.

My first result is that the expressions that characterize optimal policy in the one-period fixed price case and the N-period fixed price case without commitment do not change.

[1] Thus it differs from the form of adjustment cost used in the quantitative model in section III.

Proposition 1 (Optimal Policy without Commitment and Adjustment Costs)**.** *With adjustment costs and N-period fixed prices, the optimal policy without commitment satisfies:*

$$\sum_j \varepsilon_R^j \gamma_t^j \tau_t^j = \sum_j \varepsilon_R^j \gamma_t^j \left(\frac{1-\delta^j}{R_{t+1}}\right) \tau_{t+1}^j$$

With $N = 1$, this becomes:

$$\sum_j \varepsilon_R^j \gamma_t^j \tau_t^j = 0$$

Proof. See appendix C. □

Intuitively, the only effect of adjustment costs is to change the magnitudes of the interest elasticities and the precise values of the labor wedges. The static sectoral tradeoff and the effect of durable overhang are fully summarized by the same expressions.

Matters change a bit with commitment. The best we can manage is the following:

Proposition 2 (Optimal Policy with Commitment and Adjustment Costs)**.** *With adjustment costs and N-period fixed prices, the optimal policy with commitment satisfies:*

$$\sum_{t=0}^{N-1} \beta^t \theta_t \lambda_t y_t \sum_j \gamma_t^j \varepsilon_{R_k}^{y_t^j} \chi_t^j = 0$$

where

$$\chi_t^j = \tau_t^j - \left(\frac{1-\delta^j}{R_{t+1}}\right) \tau_{t+1}^j$$

is the overhang-augmented labor wedge.

Proof. See appendix C. □

This expression is as in Proposition 5 except the sum is up to $t = N - 1$, rather than $t = k$. This is because, with the inclusion of lagged consumption in utility, it is no longer the case that past interest rates have no effect on future consumption. Moreover, we can no longer simplify the expression to apply to two periods only, since the effect of forward guidance is no longer the same for all interest rates after the current period. This is simply because the equations for period t demand depend on $\vec{c}_{t-1}$ and $\vec{c}_{t+1}$, and thus may depend in an arbitrary fashion on all past and future interest rates.

A.2 Uncertainty

The baseline model in section I assumed certainty. This was a convenient assumption to obtain simple optimal policy expressions, but raises the question whether the general results are sensitive to this assumption. This section extends the results to the stochastic case, and shows that the general character of optimal policy does not change.

Flexible Prices. Consider the model in section (I.A) with the inclusion of uncertainty. In particular, I suppose that the demand shock θ is stochastic, and also allow parameters of the production function to be stochastic.[2] I continue to assume that the real bond is safe, so that R_{t+1} is known at time t. Under these assumptions, the household budget constraint is unchanged, and the household objective function becomes:

$$U = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \theta_t [u(\vec{c}_t) - v(n_t)]$$

The household optimality expressions are:

$$\frac{v_{n_t}}{\lambda_t} = w_t \tag{1}$$

$$1 = \mathbb{E}_t \left[\frac{\beta\theta_{t+1}}{\theta_t} \frac{\lambda_{t+1}}{\lambda_t} R_{t+1} \right] \tag{2}$$

$$p_t^j = \frac{u_{c_t^j}}{\lambda_t} + (1-\delta^j)\mathbb{E}_t \left[\frac{\beta\theta_{t+1}}{\theta_t} \frac{\lambda_{t+1}}{\lambda_t} p_{t+1}^j \right] \tag{3}$$

To see what difference uncertainty makes in the demand equations, we combine the good j demand equation with the Euler equation to obtain:

$$p_t^j = \frac{u_{c_t^j}}{\lambda_t} + \left(\frac{1-\delta^j}{R_{t+1}} \right) \frac{\mathbb{E}_t \left[\theta_{t+1}\lambda_{t+1} p_{t+1}^j \right]}{\mathbb{E}_t \left[\theta_{t+1}\lambda_{t+1} \right]}$$

Under certainty, the term $\theta_{t+1}\lambda_{t+1}$ on the right-hand side cancel out, whereas with uncertainty they do not. Thus there is an additional effect due to covariance between future prices and the future marginal utility of consumption.

[2] Since the firm production functions given in (6) were time dependent, they already allowed for predictable change in production parameters such as productivity or the shape of the production function. We are now allowing these changeds to be stochastic.

One-period fixed prices. We now turn to optimal policy in the case of one-period fixed prices. The optimal policy expression turns out to be just as in the case with certainty:

Proposition 3 (One-period fixed prices with uncertainty)**.** *With one-period fixed prices and uncertainty, the optimal policy is to set the interest rate so that:*

$$\sum_j \varepsilon_R^j \gamma_t^j \tau_t^j = 0$$

Proof. See appendix C. □

Intuitively, the inclusion of uncertainty makes no difference to optimal policy under one-period fixed prices because optimal tradeoff between sectoral production in the current period is unaffected by uncertainty. Effects of future uncertainty are captured in the equilibrium prices, which are flexible and therefore respond optimally to changes in the covariance arising from changes in current production.

N-period fixed prices without commitment. We next suppose that prices are fixed for multiple periods, and the monetary authority lacks commitment. In this case the policy rule is quite similar to the case with certainty, with the only difference deriving from covariance between future wedges and future marginal utility. The following proposition gives the optimal policy expressions:

Proposition 4 (N-period fixed prices with uncertainty, no commitment)**.** *With N-period fixed prices, no commitment, and uncertainty, the optimal policy is to set the interest rate so that:*

$$\sum_j \varepsilon_R^j \gamma_t^j \tau_t^j = \sum_j \varepsilon_R^j \gamma_t^j \left(\frac{1-\delta^j}{R_{t+1}}\right) \frac{\mathbb{E}_t\left[\tau_{t+1}^j \theta_{t+1} \lambda_{t+1}\right]}{\mathbb{E}_t\left[\theta_{t+1}\lambda_{t+1}\right]}$$

Proof. See appendix C. □

Recall that when $\delta^1 = 1$, so that good 1 is a nondurable good, $\lambda_t = u_{c_t^1}$. This expression differs from the case with certainty by the inclusion of the covariance between the future labor wedge τ_{t+1}^j and the future marginal utility of consumption $\theta_{t+1}\lambda_{t+1}$. A positive covariance in sector j implies that production in this sector is relatively low when aggregate consumption is low, in other words the output gap in this sector is procyclical. This makes the effects of durable overhang greater.

Uncontingent commitment. We next consider cases where the central bank is able to commit to a future interest rates. With commitment, we must make a further distinction: that the central bank can commit to a future path of interest rates, or that it may commit to a future path of *state contingent* interest rates. I start with the former.

Suppose that the monetary authority can commit to a particular path of future interest rates, but cannot commit to a state-contingent rate. This may be because the state is partially unobservable to market participants, and thus only an announced future rate allows the monetary authority to maintain its reputation. Then it is unclear whether the monetary authority will prefer to commit or to retain the flexibility to respond to future shocks. Suppose that the monetary authority commits to a particular path of interest rates over the next N periods. Thus at time 0 the central bank chooses $\{R_{t+1}\}_{t=0}^{N-1}$. The following proposition gives the expression for the optimal choice of R_k:

Proposition 5 (Optimal Policy under Uncontingent Commitment)**.** *In the problem with uncertainty and N-period fixed prices, when the central bank must choose a path of interest rates at time 0, the optimal choice of R_k satisfies:*

$$\mathbb{E}_0 \sum_{t=0}^{k} \beta^t \theta_t \lambda_t y_t \sum_j \gamma_t^j \varepsilon_{R_{k+1}}^{y_t^j} \chi_t^j = 0$$

where

$$\chi_t^j = \tau_t^j - \left(\frac{1-\delta^j}{R_{t+1}}\right) \frac{\mathbb{E}_t\left[\theta_{t+1}\lambda_{t+1}\tau_{t+1}^j\right]}{\mathbb{E}_t\left[\theta_{t+1}\lambda_{t+1}\right]}$$

is the durable overhang-augmented labor wedge.

Proof. See appendix C. □

As in the certainty case, sectors are equally sensitive to all interest rates more than one period ahead.

Lemma 1 (Symmetric effects of forward guidance under uncertainty)**.** *In the model with uncertainty, $\varepsilon_{R_k}^{y_t^j} = \varepsilon_{R_\ell}^{y_t^j}$ for $k, \ell \in [t+2, N]$.*

Proof. See appendix C. □

Lemma 1 immediate allows us to obtain an analogous result to Proposition 6.

Corollary 1. *The optimal policy expression in Proposition 5 for $k > 0$ may be written:*

$$\mathbb{E}_0 \frac{y_k}{R_k} \sum_j \gamma_k^j \varepsilon_{R_{k+1}}^{y_k^j} \chi_k^j = \mathbb{E}_0 y_{k-1} \sum_j \gamma_{k-1}^j \left(\varepsilon_{R_k}^{y_{k-1}^j} - \varepsilon_{R_{k+1}}^{y_{k-1}^j} \right) \chi_{k-1}^j$$

Proof. Take the difference between the expressions for the optimal choices of R_{k+1} and R_k in Proposition 5, and apply Lemma 1. □

Compare this expression to the optimal policy expression without commitment, which we can write as:

$$\mathbb{E}_k \sum_j \gamma_k^j \varepsilon_{R_{k+1}}^{y_k^j} \chi_k^j = 0$$

The expression with commitment differs in two ways from the no commitment case in two respects. First, the right-hand side contains $\varepsilon_{R_k}^{y_{k-1}^j} - \varepsilon_{R_{k+1}}^{y_{k-1}^j}$. This captures the potential benefit of using forward guidance, which depends on the differential sensitivity of sectoral volatility to current and future interest rates. Second, the lefthand side contains $\mathbb{E}_0$ rather than $\mathbb{E}_k$. This captures that the monetary authority cannot make the choice of future interest rates state contingent, and thus must commit to future interest rates with the information set available at time 0, rather than at time t.

State-contingent Commitment. Now suppose the central bank can commit to a path of *state-contingent* future interest rates. Let the shock at time t be s_t, let the history of shocks from time 0 to time t by $s^t = \{s_0, \ldots, s_t\}$, and let the probability of this history be $q(s^t)$. Then at time 0, the central bank chooses:

$$\left\{ R_{t+1}\left(s^t\right) \right\}_{t=0}^{N-1}$$

Let $V_N\left(\vec{c}_{N-1}, s^N\right)$ be the flexible price value function at time N. Then the objective function of the central bank is:

$$\sum_{t=0}^{N-1} \sum_{S^t} q\left(s^t\right) \beta^t \theta_t \left[u\left(\vec{c}_t\right) - v\left(n_t\left(\vec{c}_t, \vec{c}_{t-1}\right)\right)\right] + \beta^T \sum_{S^T} q\left(s^T\right) \theta_T V\left(\vec{c}_{T-1}\right)$$

where demand $c_t^j\left(s^t\right)$ may depend on the entire path of interest rates $\left\{R_{t+1}\left(s^t\right)\right\}_{t=0}^{N-1}$. Note that Lemma 1 still applies, so that demand at time t does not depend on past choices of interest rates, since current prices are fixed. This demand $c_t^j(s^t)$ depends only on $\left\{R_{t+k}\left(s^{t+k}\right)\right\}_{k=1}^{N-t}$. The following proposition gives a condition for the optimal choice of $R_{k+1}\left(s_*^k\right)$.

Proposition 6 (Optimal Policy under uncertainty with full commitment)**.** *The optimal choice of $R_{k+1}\left(s_*^k\right)$ in the N-period fixed price model with full commitment is:*

$$\sum_{t=0}^{k} q\left(s_*^t\right)\beta^t\theta_{t*}\lambda_{t*}y_{t*}\left\{\sum_j \varepsilon_{R_{k+1}\left(s_*^k\right)}^{y_t^j}\gamma_{t*}^j\chi_{t*}^j\right\}=0$$

where x_{t*} denotes $x(s_*^t)$, where s_*^t lies on the path defined by s_*^k, i.e. $s_*^k \subset s_*^t$.

Proof. See appendix C. □

To understand these expressions let's analyze the choices of particular interest rates. First the choice of R_1 yields the expression:

$$\sum_j \varepsilon_{R_1}^{y_0^j}\gamma_0^j\chi_0^j=0$$

which is just the same as in the case without commitment, since as before future demand does not depend on past interest rates.[3] Now consider the choice of $R_2\left(s_*^1\right)$. This yields:

$$\theta_0\lambda_0 y_0\left\{\sum_j \varepsilon_{R_2\left(s_*^1\right)}^{y_0^j}\gamma_0^j\chi_0^j\right\}+q\left(s_*^1\right)\beta\theta_{1*}\lambda_{1*}y_{1*}\left\{\sum_j \varepsilon_{R_2\left(s_*^1\right)}^{y_1^j}\gamma_{1*}^j\chi_{1*}^j\right\}=0$$

Rearranging and taking the difference, we obtain:

$$\sum_j \varepsilon_{R_2\left(s_*^1\right)}^{y_1^j}\gamma_{1*}^j\chi_{1*}^j=\frac{\theta_0\lambda_0 y_0}{\beta\theta_{1*}\lambda_{1*}y_{1*}}\left[\sum_j \gamma_0^j\chi_0^j\left(\varepsilon_{R_1}^{y_0^j}-\frac{\varepsilon_{R_2\left(s_*^1\right)}^{y_0^j}}{q\left(s_*^1\right)}\right)\right]$$

This captures the role of forward guidance. Note that since s_*^1 represents only a subset of states reachable from the initial period, the demand elasticity with respect to the interest rate that prevails in these states will generally be less than the demand elasticity with respect to an interest rate

[3]That is, Lemma 1 holds for the stochastic case.

that holds across all states. Dividing the elasticity by $q(s_*^t)$ corrects for this factor, so that the result captures the probability-adjusted interest elasticity of demand. Thus these elasticities should be similar in magnitude, even if the state is quite unlikely to occur.

With contingent interest rates, the symmetric effect of forward guidance at various time horizons no longer holds, and thus we cannot obtain an analogous simplified expression for all future interest rates.

A.3 Input-Output Linkages and Intermediate Goods

The model in section I assumes that all goods produced by firms in a sector are combined into a sector-specific final good, which is then sold directly to consumers. However, in reality a significant fraction of goods produced by a sector are sold to firms in other sectors to be used as intermediate inputs in production. The resulting network structure of production is a source of sectoral heterogeneity in itself, and also breaks the link between the interest elasticity of demand for final goods (arising from household demand), and the interest elasticity of demand for all goods from a given sector, which now includes demand from other sectors. In this section I extend the baseline model without inflation to include intersectoral input-output linkages through intermediate goods, and analyze how this affects the optimal monetary policy. I first present the model, and then give the optimal policy expression with one-period fixed prices.

Model with input-output linkages. Suppose that firms in sector j produce using technology:

$$y_t^j = f_t^j\left(n_t^j, \left\{x_t^{jk}\right\}\right) \tag{4}$$

where x_t^{jk} denotes the use of good k in the production of good j. I further assume that all intermediate goods depreciate fully upon use.[4] I assume that all sectoral production functions satisfy the Inada conditions with respect to each input, and feature nonincreasing returns to scale in all inputs.[5]

[4]This is generally the case for intermediate goods. If they did not fully depreciate, they would be capital goods, and an analysis of capital goods is beyond the scope of this paper.

[5]These assumptions are stronger than are necessary to derive the results, but keep things simple.

Firms seek to minimize cost of production for given output y_t^j. The cost function of a firm in sector j is:

$$C_t^j\left(y_t^j\right) = \min_{n_t^j, \left\{x_t^{jk}\right\}} \left\{ w_t n_t^j + \sum_k p_t^k x_t^{jk} \right\} \quad \text{s.t. } y_t^j \leq f_t^j\left(n_t^j, \left\{x_t^{jk}\right\}\right) \tag{5}$$

Optimal production satisfies conditions:

$$w_t = \phi_t^j f_{n_t}^j \tag{6}$$
$$p_t^k = \phi_t^j f_{x_t^{jk}}^j \tag{7}$$

where ϕ_t^j is the Lagrange multiplier on the constraint in (5), which equals the marginal cost of production. Under flexible prices, firms in sector j will set price equal to marginal cost, and therefore we have:

$$p_t^j = \phi_t^j \tag{8}$$

Market clearing for good j is:

$$y_t^j = c_t^j - \left(1 - \delta^j\right) c_{t-1}^j + \sum_k x_t^{kj} \tag{9}$$

The remaining conditions are as before: household behavior is determined by (2) – (5); firm profits are defined by (7); asset and labor market clearing conditions (9) and (11) hold. Together with equations (6) – (9) above, these define the flexible price equilibrium.

One-period fixed prices. Now suppose that, as in section I.B, prices in period t are fixed at given levels $p_t^j = \bar{p}_t^j$. Since firms can no longer adjust their prices, equation (8) no longer holds, and firms are obligated to meet the demand they face. Note that firms still optimize to minimize costs, conditional on demand for their goods. As before, the equilibrium in periods $t+1$ and later is the same as the flexible price equilibrium given consumption bundle $\left\{c_t^j\right\}$, and is therefore unique and optimal. Equilibrium is determined by the choice of the real interest rate R_{t+1} by the central bank. The optimal policy is given in the following proposition:

Proposition 7 (Optimal Policy with input-output linkages)**.** *The optimal choice of the interest rate R_{t+1} satisfies the condition:*

$$\sum_j \gamma_t^j \varepsilon_R^j \tau_t^j = 0$$

where

$$\varepsilon_R^j = -\frac{dy_t^j}{dR_t}\frac{R_t}{y_t^j}$$
$$\gamma_t^j = \frac{p_t^j y_t^j}{\sum_k p_t^k y_t^k}$$

Proof. See appendix C. □

Thus we find that optimal policy satisfies a condition that can be written just as in the model without intermediate goods. However, the definition of the terms changes in two respects relative to the model without intermediate goods: (1) ε_R^j is the interest elasticity of total demand for good j, including demand for intermediate goods from firms in other sectors; and (2) γ_t^j is now the share of sector j in total output, including intermediate goods, rather than the GDP share (which only includes final goods).

This result suggests that nothing fundamental changes with the introduction of input-output linkages. However, this is possibly misleading, as there are at least two effects operating below the surface, which one might expect to have opposite effects. First, input-output linkages tend to diffuse an increase in demand in one sector across many others, which is likely to attenuate differential interest sensitivity of demand, since increased demand in interest-sensitive sectors will be transmited to other sectors through network connections, excepting very particular network structures. Working the other way, sticky prices will prevent firms from optimally adjusting their mix of inputs in response to changes in relative prices, interfering with optimal adjustment to shocks and likely increasing sectoral labor wedges. Fully characterizing these effects, and quantitatively assessing their importance, is beyond the scope of this paper.

B Derivation of Calvo Pricing Model

Household optimality conditions are:

$$1 = \mathbb{E}_t \left[\frac{\beta\theta_{t+1}}{\theta_t} \frac{R_t}{\Pi^1_{t+1}} \left(\frac{c^1_t}{c^1_{t+1}} \right)^{\sigma_1} \right] \tag{10}$$

$$w_t = \psi_t n_t^{\zeta} \left(c^1_t\right)^{\sigma_1} \tag{11}$$

$$q^j_t = \chi^j_t \frac{\left(c^j_t\right)^{-\sigma_j}}{\left(c^1_t\right)^{-\sigma_1}} \tag{12}$$

where the interest rate R_t and the wage w_t pay in terms of good 1, the numeraire, and where $q^j_t = p^j_t / p^1_t$ denotes the relative price of good j (meaning that $q^1_t = 1$), and where $\Pi^j_t = p^j_t / p^j_{t-1}$ denotes the inflation rate in good j.

Final Good Aggregator. Suppose that in sector j there is a unit interval of intermediate good firms indexed by i. The output of intermediate good firms is combined by a Dixit-Stiglitz aggregator firm to produce final goods:

$$y^j_t = \left(\int_i \left(y^j_{it}\right)^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

This implies demand function for intermediate goods:

$$\frac{y^j_{it}}{y^j_t} = \left(\frac{p^j_{it}}{p^j_t} \right)^{-\varepsilon}$$

Combining these, we obtain an expression for the aggregate price level:

$$\left(p^j_t\right)^{1-\varepsilon} = \int_i \left(p^j_{it}\right)^{1-\varepsilon} \tag{13}$$

Intermediate Good Firms. Suppose that intermediate good firms produce using production functions:

$$y^j_{it} = z^j_t \left(n^j_{it}\right)^{1-\alpha}$$

Suppose further that firms discount real profits in time t at rate $Q_t = \beta^t \theta_t \left(c^1_t\right)^{-\sigma_1}$. Suppose that there is a per-unit production subsidy $1+\tau = \frac{\varepsilon}{\varepsilon-1}$, which is chosen to offset under production from firms' market power.

Suppose that every period a random fraction $1-\phi_j$ of firms can adjust their nominal prices, with all other firms leaving their prices unchanged. Then the price-setting problem of firm i that can adjust its price in period t is to choose p^j_{it} to maximize:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \phi_j^s Q_{t+s} \left[\left(p^1_{t+s}\right)^{-1} (1+\tau) \left(p^j_{it}\right)^{1-\varepsilon} \left(p^j_{t+s}\right)^{\varepsilon} y^j_{t+s} - w_{t+s} \left(\left(\frac{p^j_{it}}{p^j_{t+s}} \right)^{-\varepsilon} \frac{y^j_{t+s}}{z^j_{t+s}} \right)^{\frac{1}{1-\alpha_j}} \right]$$

Assuming $1+\tau = \frac{\varepsilon}{\varepsilon-1}$, some simplification yields optimality condition:

$$\left(\frac{p^{j*}_t}{p^j_{t-1}} \right)^{1+\frac{\varepsilon\alpha_j}{1-\alpha_j}} = \frac{1}{1-\alpha_j} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \phi_j^s Q_{t+s} w_{t+s} \left(\frac{p^j_{t+s}}{p^j_{t-1}} \right)^{\frac{\varepsilon}{1-\alpha_j}} \left(\frac{y^j_{t+s}}{z^j_{t+s}} \right)^{\frac{1}{1-\alpha_j}}}{\mathbb{E}_t \sum_{s=0}^{\infty} \phi_j^s Q_{t+s} q^j_{t+s} \left(\frac{p^j_{t+s}}{p^j_{t-1}} \right)^{\varepsilon-1} y^j_{t+s}}$$

We can express this recursively as:

$$\left(\Pi^{j*}_t\right)^{1+\frac{\varepsilon\alpha_j}{1-\alpha_j}} = \frac{1}{1-\alpha_j} \frac{X^j_t}{Z^j_t} \tag{14}$$

where $\Pi^{j*}_t = p^{j*}_t / p^j_{t-1}$, and where X^j_t and Z^j_t are defined recursively as

$$X^j_t = \left(\Pi^j_t\right)^{\frac{\varepsilon}{1-\alpha_j}} \left\{ w_t \left(y^j_t / z^j_t\right)^{\frac{1}{1-\alpha_j}} + \mathbb{E}_t \left[\frac{\beta\phi_j\theta_{t+1}}{\theta_t} \left(\frac{c^1_t}{c^1_{t+1}} \right)^{\sigma_1} X^j_{t+1} \right] \right\} \tag{15}$$

$$Z^j_t = \left(\Pi^j_t\right)^{\varepsilon-1} \left\{ q^j_t y^j_t + \mathbb{E}_t \left[\frac{\beta\phi_j\theta_{t+1}}{\theta_t} \left(\frac{c^1_t}{c^1_{t+1}} \right)^{\sigma_1} Z^j_{t+1} \right] \right\} \tag{16}$$

Aggregate price dynamics. Suppose that all firms follow the pricing rule above. Then at every point in time, we can distinguish between the aggregate price p^j_t, and the price of adjusting firms p^{j*}_t. From the expression for the price index p^j_t, the aggregate price level evolves according to:

$$\left(p^j_t\right)^{1-\varepsilon} = \phi_j \left(p^j_{t-1}\right)^{1-\varepsilon} + (1-\phi_j) \left(p^{j*}_t\right)^{1-\varepsilon}$$

We can write this is inflation terms as:

$$\left(\Pi^j_t\right)^{1-\varepsilon} = 1 + (1-\phi_j) \left[\left(\Pi^{j*}_t\right)^{1-\varepsilon} - 1 \right] \tag{17}$$

where $\Pi^j_t = p^j_t / p^j_{t-1}$ and $\Pi^{j*}_t = p^{j*}_t / p^j_{t-1}$.

Price Dispersion. In addition to the aggregate price index, we also need to track price dispersion. We would like a measure price dispersion that relates aggregate sectoral labor demand n_t^j to aggregate sectoral output y_t^j. Aggregate sectoral labor demand satisfies:

$$n_t^j = \int_i n_{it}^j = \int_i \left(\frac{y_{it}^j}{z_t^j}\right)^{\frac{1}{1-\alpha}}$$

Using the intermediate good demand function, we can write this as:

$$n_t^j = \left(\frac{y_t^j}{z_t^j}\right)^{\frac{1}{1-\alpha}} \left[\int_i \left(\frac{p_{it}^j}{p_t^j}\right)^{-\frac{\varepsilon}{1-\alpha}}\right]$$

This allows us to define a notion of price dispersion

$$\Delta_t^j = \int_i \left(\frac{p_{it}^j}{p_t^j}\right)^{-\frac{\varepsilon}{1-\alpha}}$$

which satisfies:

$$y_t^j = z_t^j \left(\frac{n_t^j}{\Delta_t^j}\right)^{1-\alpha}$$

Dynamics of Price Dispersion. Price dispersion can be written as:

$$\Delta_t^j = \left(p_t^j\right)^{\frac{\varepsilon}{1-\alpha}} \cdot \int_i \left(p_{it}^j\right)^{-\frac{\varepsilon}{1-\alpha}} \tag{18}$$

Price dispersion evolves over time according to:

$$\Delta_t^j = \left(\Pi_t^j\right)^{\frac{\varepsilon}{1-\alpha}} \left[\phi_j \left(\Delta_{t-1}^j\right) + (1-\phi_j)\left(\Pi_t^{j*}\right)^{-\frac{\varepsilon}{1-\alpha}}\right]$$

New Keynesian Phillips Curve To derive the New Keynesian Phillips Curve, equation (16), we log-linearize equations (14) – (16) and (17). This yields:

$$\left(1 + \frac{\varepsilon\alpha_j}{1-\alpha_j}\right)\pi_t^{j*} = \log X_t^j - \log Z_t^j$$

$$\begin{aligned}
\log X_t^j &= \frac{\varepsilon}{1-\alpha_j}\pi_t^j + (1-\beta\phi_j)\left(\tilde{w}_t + \tilde{n}_t^j\right) + \beta\phi_j \mathbb{E}_t\left[\sigma_1\left(\tilde{c}_t^1 - \tilde{c}_{t+1}^1\right) + \log X_{t+1}^j\right] \\
\log Z_t^j &= (\varepsilon - 1)\,\pi_t^j + (1-\beta\phi_j)\left(\tilde{q}_t^j + \tilde{y}_t^j\right) + \beta\phi_j \mathbb{E}_t\left[\sigma_1\left(\tilde{c}_t^1 - \tilde{c}_{t+1}^1\right) + \log Z_{t+1}^j\right] \\
\pi_t^j &= (1-\phi_j)\,\pi_t^{j*}
\end{aligned}$$

Combining these and expression in terms of π_t^j only, we obtain:

$$\pi_t^j = \left(\frac{1-\alpha_j}{1-\alpha_j+\varepsilon\alpha_j}\right)\left(\frac{1-\beta\phi_j}{\phi_j}\right)\left(\tilde{w}_t + \tilde{n}_t^j - \tilde{q}_t^j - \tilde{y}_t^j\right) + \beta\mathbb{E}_t\left[\pi_{t+1}^j\right]$$

which is the NKPC.

Model with Durable Good in section III. The model in section III differs from the model above in only a few respects. Household optimality conditions (10) and (11) continue to hold, but now (12) is:

$$q_t = \chi_t \frac{d_t^{-\sigma_d}}{c_t^{-\sigma_c}} - \varphi q_t\left(d_t - d_{t-1}\right) + \beta\mathbb{E}_t\left[\frac{\theta_{t+1}}{\theta_t}\left(\frac{c_t}{c_{t+1}}\right)^{\sigma_c} q_{t+1}\left\{1-\delta+\varphi\left(d_{t+1}-d_t\right)\right\}\right] \tag{19}$$

The expressions that determine inflation are all the same except that equation (15) now includes a cost-push shock μ_t^j, which enters as follows:

$$X_t^j = \left(\Pi_t^j\right)^{\frac{\varepsilon}{1-\alpha_j}}\left\{\mu_t^j w_t\left(y_t^j/z_t^j\right)^{\frac{1}{1-\alpha_j}} + \mathbb{E}_t\left[\frac{\beta\phi_j\theta_{t+1}}{\theta_t}\left(\frac{c_t^1}{c_{t+1}^1}\right)^{\sigma_1} X_{t+1}^j\right]\right\} \tag{20}$$

Aside from these changes, all other equations from section II.A hold.

C Omitted Proofs

Proof of Proposition 1. The first-order condition of the optimal policy problem is:

$$\sum_j\left(u_{c_t^j} - \frac{v_{n_t}}{f_{n_t}^j} + \frac{\beta\theta_{t+1}}{\theta_t}V_{c_t^j}\right)\frac{dc_t^j}{dR} = 0$$

The envelope conditions are:

$$V_{c_t^j} = \left(1-\delta^j\right)\frac{v_{n_t}}{f_{n_t}^j}$$

Combining these we obtain:

$$\sum_j\left(u_{c_t^j} - \frac{v_{n_t}}{f_{n_t}^j} + \frac{\beta\theta_{t+1}}{\theta_t}\left(1-\delta^j\right)\frac{v_{n_{t+1}}}{f_{n_{t+1}}^j}\right)\frac{dc_t^j}{dR} = 0$$

Next we use that in the next period, since prices are flexible, we have:

$$w_{t+1} = \frac{v_{n_{t+1}}}{u_{c^1_{t+1}}} = p^j_{t+1} f^j_{n_{t+1}}$$

and therefore $\frac{v_{n_{t+1}}}{f^j_{n_{t+1}}} = p^j_{t+1} u_{c^1_{t+1}}$. This may not hold in period t, where instead we have:

$$\frac{v_{n_t}}{f^j_{n_t}} = \left(1 - \tau^j_t\right) p^j_t u_{c^1_t}$$

where τ^j_t is the labor wedge. Therefore optimal policy becomes (after dividing through by $u_{c^1_t}$):

$$\sum_j \left(\frac{u_{c^j_t}}{u_{c^1_t}} + \frac{\beta \theta_{t+1}}{\theta_t} \left(1 - \delta^j\right) p^j_{t+1} \frac{u_{c^1_{t+1}}}{u_{c^1_t}} - \left(1 - \tau^j_t\right) p^j_t \right) \frac{dc^j_t}{dR} = 0$$

Now we substitute in the sector j asset pricing equation to obtain:

$$\sum_j \tau^j_t p^j_t \frac{dc^j_t}{dR} = 0$$

We now write this in terms of interest elasticities of demand and GDP shares. First since $y^j_t = c^j_t - \left(1 - \delta^j\right) c^j_{t-1}$, it follows that $\frac{dy^j_t}{dR} = \frac{dc^j_t}{dR}$. Then we multiply and divide each term of the sum by y^j_t to put things in terms of production, multiply the entire expression through by $-R$, and then divide through by GDP, which is $y_t = \sum_j p^j_t y^j_t$. Then the expression can be written as:

$$\sum_j \varepsilon^j_R \gamma^j_t \tau^j_t = 0$$

where $\varepsilon^j_R = -\frac{dy^j_t}{dR_{t+1}} \frac{R_{t+1}}{y^j_t}$, $\gamma^j_t = \frac{p^j_t y^j_t}{\sum_j p^j_t y^j_t}$, and $\tau^j_t = 1 - \frac{w_t}{p^j_t f^j_{n_{t+1}}}$. $\square$

Proof of Proposition 2. The flexible price equilibrium is the unique optimum of the economy. Optimality requires $\tau^j_t = 0$, i.e. $p^j_t = w_t / f^j_{n_t}$. This implies that for every i, j , the relative price between sectors satisfies:

$$\frac{p^j_t}{p^i_t} = \frac{f^j_{n_t}}{f^i_{n_t}}$$

Since $f^j_{n_t}/f^i_{n_t}$ is pinned down by production, this implies that the relative price p^j_t/p^i_t is also pinned down. Due to the normalization $p^1_t = 1$, this requires that $p^j_t = p^{j,flex}_t$ for all j. This is only feasible if $\bar{p}^j = p^{j,flex}_t$.

If there is linear production in each sector, then optimality requires $p^j_t = z^j_t/z^i_t$. Since we start out at an optimum, we know this holds for $\bar{p}^j$. Thus, in the absence of idiosyncratic sectoral demand shocks, this expression continues to hold. □

Proof of Proposition 3. Let ε and τ be random variables which take on values (ε^j, τ^j) in state j, which occurs with probability γ^j. Then $\varepsilon^y = E[\varepsilon]$, $\tau^y = E[\tau]$, and $\sum_j \gamma^j \left(\varepsilon^j_R - \varepsilon^y_R\right)(\tau^j - \tau^y) = \mathrm{Cov}(\varepsilon, \tau) = E[\varepsilon\tau] - E[\varepsilon]E[\tau]$. The static policy rule is then $E[\varepsilon\tau] = \sum_j \gamma^j \varepsilon^j_R \tau^j = 0$, and therefore under optimal policy we have $E[\varepsilon]E[\tau] + \mathrm{Cov}(\varepsilon, \tau) = 0$. Dividing through by ε^y_R then yields the result. □

Proof of Proposition 4. The optimality expression is just as in the case with one-period fixed prices:

$$\sum_j \left(u_{c^j_t} - \frac{v_{n_t}}{f^j_{n_t}} + \frac{\beta\theta_{t+1}}{\theta_t}\left(1 - \delta^j\right)\frac{v_{n_{t+1}}}{f^j_{n_{t+1}}}\right)\frac{dc^j_t}{dR} = 0$$

But now it may be the case that $\tau^j_{t+1} \neq 0$. Therefore we must use the expression:

$$\frac{v_{n_t}}{f^j_{n_t}} = \left(1 - \tau^j_t\right) p^j_t u_{c^1_t}$$

in both period t and $t+1$. Using this, the expression above becomes:

$$\sum_j \left(\frac{u_{c^j_t}}{u_{c^1_t}} + \left(\frac{1 - \delta^j}{R_{t+1}}\right)\left(1 - \tau^j_{t+1}\right) p^j_{t+1} - \left(1 - \tau^j_t\right) p^j_t\right)\frac{dc^j_t}{dR} = 0$$

Now we use the expression for p^j_t, together with the fact that $p^j_t = p^j_{t+1} = \bar{p}^j$, to obtain:

$$\sum_j \bar{p}^j \left(\tau^j_t - \left(\frac{1 - \delta^j}{R_{t+1}}\right)\tau^j_{t+1}\right)\frac{dc^j_t}{dR} = 0$$

Now, as before, we note that $\frac{dy^j_t}{dR} = \frac{dc^j_t}{dR}$, we multiply and divide each term of the sum by y^j_t, multiply the entire expression by $-R$, and then divide through by GDP $y_t = \sum_j p_t^j y_t^j$, to obtain:

$$\sum_j \varepsilon_R^j \gamma_t^j \tau_t^j = \sum_j \varepsilon_R^j \gamma_t^j \left(\frac{1-\delta^j}{R_{t+1}}\right) \tau_{t+1}^j$$

$\square$

Proof of Lemma 1. We show this by backward induction. First consider the expressions in period $T = N - 1$, i.e. the last period with fixed prices. Here the expressions as:

$$\begin{aligned} \frac{u_{c_T^j}\left(\vec{c}_T\right)}{u_{c_T^1}\left(\vec{c}_T\right)} &= \bar{p}^j - \frac{1-\delta^j}{R_{T+1}} p_{T+1}^j\left(\vec{c}_T\right) \\ \frac{u_{c_T^1}\left(\vec{c}_T\right)}{u_{c_{T+1}^1}\left(\vec{c}_T\right)} &= \frac{\beta\theta_{T+1}}{\theta_T} R_{T+1} \end{aligned}$$

where current marginal utility is a function of current consumption only from the assumption of time-separability of utility, and where future prices and consumption depend only on current consumption because these are defined by the flexible price equilibrium for given initial state, which here is just $\vec{c}_T$. Note that we have N_j equations in N_j unknowns, given entirely in terms of $(\vec{c}_T, R_{T+1}, \bar{p}^j)$. Thus these expressions implicitly define current demand as a function of current fixed prices and the current interest rate only:

$$\vec{c}_T\left(R_{T+1}\bar{p}^j\right)$$

Now we show by induction that demand equations for earlier periods depend only on future interest rates. Suppose this is true for all future periods up to period T. Then in period t we have:

$$\begin{aligned} \frac{u_{c_t^j}\left(\vec{c}_t\right)}{u_{c_t^1}\left(\vec{c}_t\right)} &= \bar{p}^j\left(1 - \frac{1-\delta^j}{R_{t+1}}\right) \\ \frac{u_{c_t^1}\left(\vec{c}_t\right)}{u_{c_{t+1}^1}\left(\left\{R_s\right\}_{s\geq t+2}\right)} &= \frac{\beta\theta_{t+1}}{\theta_t} R_{t+1} \end{aligned}$$

These are again N_j equations in N_j unknowns, only in terms of $\left(\vec{c}_t, \vec{\bar{p}}, \left\{R_s\right\}_{s\geq t+1}\right)$. Therefore these expressions implicitly define $\vec{c}_t\left(\vec{\bar{p}}, \left\{R_s\right\}_{s\geq t+1}\right)$. $\square$

Proof of Proposition 5. Consider the choice of R_k for $k \in \{1, \ldots, N\}$. The optimality condition is:

$$\sum_j \left[\sum_{t=0}^{N-2} \frac{\beta^t \theta_t}{\theta_0} \left(\frac{\partial U_t}{\partial c_t^j} + \frac{\beta \theta_{t+1}}{\theta_t} \frac{\partial U_{t+1}}{\partial c_t^j} \right) \frac{dc_t^j}{dR_k} \right] = 0$$
$$+ \sum_j \left[\frac{\beta^{N-1} \theta_{N-1}}{\theta_0} \left(\frac{\partial U_{N-1}}{\partial c_{N-1}^j} + \frac{\beta \theta_N}{\theta_{N-1}} \frac{dV_N}{dc_{N-1}^j} \right) \frac{dc_{N-1}^j}{R_k} \right] = 0$$

Now we use the fact that:

$$U_{c_t^j} = u_{c_t^j} - \frac{v_{n_t}}{f_{n_t}^j} = u_{c_t^1} \left[\frac{u_{c_t^j}}{u_{c_t^1}} - \left(1 - \tau_t^j\right) p_t^j \right]$$
$$U_{c_{t-1}^j} = \left(1 - \tau_t^j\right) \left(1 - \delta^j\right) u_{c_t^1} p_t^j$$

Then the optimality condition becomes:

$$\sum_j \left[\sum_{t=0}^{k-1} \beta^t \lambda_t \bar{p}^j \tau_t^j \left(1 - \left(\frac{1 - \delta^j}{R_{t+1}} \right) \frac{\tau_{t+1}^j}{\tau_t^j} \right) \frac{dc_t^j}{dR_k} \right] = 0$$

Note that this holds for every $R_k \in \{R_1, \ldots, R_N\}$. We can then write this as:

$$\sum_{t=0}^{k-1} \beta^t \lambda_t y_t \left[\sum_j \left(\gamma_t^j \tau_t^j - \gamma_t^j \tau_{t+1}^j \left(\frac{1 - \delta^j}{R_{t+1}} \right) \right) \varepsilon_{R_k}^{y_t^j} \right] = 0$$

□

Proof of Lemma 2. Demand for c_t^j for $j \neq 1$ is defined by the set of equations:

$$u_{c_t^j} = \left(1 - \frac{1 - \delta^j}{R_{t+1}} \right) \bar{p}^j u_{c_t^1}$$
$$u_{c_t^1} = \frac{\beta \theta_{t+1}}{\theta_t} R_{t+1} u_{c_{t+1}^1}$$

Iterating the Euler equation forward in time, we obtain:

$$u_{c_t^1} = \beta^{N-t} \frac{\theta_N}{\theta_t} \left(\prod_{s=t+1}^{N} R_s \right) u_{c_N^1}$$

Substituting this into the system of equations above yields:

$$u_{c_t^j} = \left(1 - \frac{1-\delta^j}{R_{t+1}}\right)\bar{p}^j u_{c_t^1}$$
$$u_{c_t^1} = \beta^{N-t}\frac{\theta_N}{\theta_t}\left(\prod_{s=t+1}^{N} R_s\right) u_{c_N^1}$$

Since there are N_j of these equations and N_j unknowns, this set of equations determines $\vec{c}_t\left(\{R_s\}_{s=t+1}^N, \bar{p}^j, \theta_t, \theta_N, u_{c_N^1}\right)$. Now consider R_k for $k \in [t+2, N]$. By Lemma 1, $u_{c_N^1}$ is not a function of R_k. Then if we differentiate the system of equations above by R_k, we obtain:

$$\sum_i u_{c_t^j c_t^i}\frac{dc_t^i}{dR_k} = \left(1 - \frac{1-\delta^j}{R_{t+1}}\right)\bar{p}^j\left(\sum_i u_{c_t^1 c_t^i}\frac{dc_t^i}{dR_k}\right)$$
$$\sum_i u_{c_t^1 c_t^i}\frac{dc_t^i}{dR_k} = \frac{u_{c_t^1}}{R_k}$$

We can write this in terms of consumption demand interest elasticities as:

$$\sum_i u_{c_t^j c_t^i} c_t^i \varepsilon_{R_k}^{c_t^i} = \left(1 - \frac{1-\delta^j}{R_{t+1}}\right)\bar{p}^j\left(\sum_i u_{c_t^1 c_t^i} c_t^i \varepsilon_{R_k}^{c_t^i}\right)$$
$$\sum_i u_{c_t^1 c_t^i} c_t^i \varepsilon_{R_k}^{c_t^i} = -u_{c_t^1}$$

This yields N_j equations in N_j unknowns, namely the interest elasticities $\varepsilon_{R_k}^{c_t^i}$. But note that these expressions are the same for any $k \in [t+2, N]$, and thus the elasticities are the same. This proves the proposition. $\square$

Proof of Proposition 6. We can write the optimality expressions as:

$$\beta^{k-1}\lambda_{k-1}y_{k-1}\left[\sum_j\left(\gamma_{k-1}^j\tau_{k-1}^j - \gamma_{k-1}^j\tau_k^j\left(\frac{1-\delta^j}{R_{k1}}\right)\right)\varepsilon_{R_k}^{y_{k-1}^j}\right] =$$
$$-\sum_{t=0}^{k-2}\beta^t\lambda_t y_t\left[\sum_j\left(\gamma_t^j\tau_t^j - \gamma_t^j\tau_{t+1}^j\left(\frac{1-\delta^j}{R_{t+1}}\right)\right)\varepsilon_{R_k}^{y_t^j}\right]$$

Take the difference:

$$\beta^{k-1}\lambda_{k-1}y_{k-1}\left[\sum_j\left(\gamma^j_{k-1}\tau^j_{k-1}-\gamma^j_{k-1}\tau^j_k\left(\frac{1-\delta^j}{R_{k1}}\right)\right)\varepsilon^{y^j_{k-1}}_{R_k}\right]=$$
$$\sum_{t=0}^{k-2}\beta^t\lambda_t y_t\left[\sum_j\left(\gamma^j_t\tau^j_t-\gamma^j_t\tau^j_{t+1}\left(\frac{1-\delta^j}{R_{t+1}}\right)\right)\left(\varepsilon^{y^j_t}_{R_{k-1}}-\varepsilon^{y^j_t}_{R_k}\right)\right]$$

The right-hand terms cancel for $t<k-2$. Then we obtain:

$$\sum_j\gamma^j_{k-1}\left(\tau^j_{k-1}-\tau^j_k\left(\frac{1-\delta^j}{R_{k1}}\right)\right)\varepsilon^{y^j_{k-1}}_{R_k}=$$
$$\frac{R_{k-1}y_{k-2}}{y_{k-1}}\left[\sum_j\gamma^j_{k-2}\left(\tau^j_{k-2}-\tau^j_{k-1}\left(\frac{1-\delta^j}{R_{k-1}}\right)\right)\left(\varepsilon^{y^j_{k-2}}_{R_{k-1}}-\varepsilon^{y^j_{k-2}}_{R_k}\right)\right]$$

□

Proof of Proposition 7. The demand equations can be written:

$$u_{c^j_t}=\left(1-\frac{1-\delta^j}{R_1}\right)\bar{p}^j u_{c^1_t}$$
$$u_{c^1_t}=\beta^2\frac{\theta_{t+2}}{\theta_t}R_{t+1}R_{t+2}u_{c^1_{t+2}}$$

where the second equation is the Euler equation iterated forward an extra period. Combining these, we obtain:

$$u_{c^j_t}=\left(1-\frac{1-\delta^j}{R_1}\right)\bar{p}^j\beta^2\frac{\theta_{t+2}}{\theta_t}R_{t+1}R_{t+2}u_{c^1_{t+2}}$$
$$u_{c^1_t}=\beta^2\frac{\theta_{t+2}}{\theta_t}R_{t+1}R_{t+2}u_{c^1_{t+2}}$$

Since (by Lemma 1) c^j_{t+2} does not depend on R_{t+1} or R_{t+2}, we can immediately compute the following:

$$u_{c^j_t c^j_t}\frac{dc^j_t}{dR_{t+2}}=\frac{u_{c^j_t}}{R_{t+2}}$$
$$u_{c^j_t c^j_t}\frac{dc^j_0}{dR_{t+1}}=\frac{u_{c^j_t}}{r_{t+1}+\delta^j}$$

We can express this as:

$$\sigma_t^j \varepsilon_{R_{t+2}}^{y_t^j} \frac{y_t^j}{c_t^j} = 1$$
$$\sigma_t^j \varepsilon_{R_{t+1}}^{y_0^j} \frac{y_t^j}{c_t^j} = \frac{1+r_{t+1}}{r_{t+1}+\delta^j}$$

where $\sigma_t^j = -\frac{u_{c_t^j c_t^j} c_t^j}{u_{c_t^j}}$. Or in other words:

$$\varepsilon_{R_{t+2}}^{y_t^j} = \left(\frac{r_{t+1}+\delta^j}{1+r_{t+1}}\right) \varepsilon_{R_{t+1}}^{y_t^j}$$

Taking the difference, this implies:

$$\varepsilon_{R_{t+1}}^{y_t^j} - \varepsilon_{R_{t+2}}^{y_t^j} = \left(\frac{1-\delta^j}{R_{t+1}}\right) \varepsilon_{R_{t+1}}^{y_t^j}$$

Plugging this into the optimal policy expression found in 5 yields the last expression. □

Proof of Proposition 8. First we take a second-order approximation of U_t around the natural level of output. Because the natural level includes all shocks, we don't need to calculate with respect to the shocks. We write:

$$U_t \approx U_t^n + \theta_t \sum_j \chi_t^j \left[u_c^j \left(c_t^j - c_t^{j,n}\right) + \frac{1}{2} u_{cc}^j \left(c_t^j - c_t^{j,n}\right)^2\right]$$
$$- \theta_t \psi_t \left[v_n \sum_j \left(n_t^j - n_t^{j,n}\right) + \frac{1}{2} v_{nn} \left(\sum_j \left(n_t^j - n_t^{j,n}\right)\right)^2\right]$$

Next we use the fact that for variable x, to second-order, $x_t/x_t^n - 1 = \tilde{x}_t + \frac{1}{2}\tilde{x}_t^2$, where x^n is the natural (flexible price) level of x, and where $\tilde{x} = \log(x/x^n)$. After dropping higher order terms, this yields:

$$U_t \approx U_t^n + \theta_t \sum_j \chi_t^j \left[u_c^j c_t^{j,n} \left(\tilde{c}_t^j + \frac{1}{2}\left(\tilde{c}_t^j\right)^2\right) + \frac{1}{2} u_{cc}^j \left(c_t^{j,n}\right)^2 \left(\tilde{c}_t^j\right)^2\right]$$
$$- \theta_t \psi_t \left[v_n \sum_j n_t^{j,n} \left(\tilde{n}_t^j + \frac{1}{2}\left(\tilde{n}_t^j\right)^2\right) + \frac{1}{2} v_{nn} \left(\sum_j n_t^{j,n} \tilde{n}_t^j\right)^2\right]$$

Now we look at what happens to the first-order terms. These are:

$$U_t^1 = \theta_t \sum_j \left[\chi_t^j u_c^j c_t^{j,n} \tilde{c}_t^j - \psi_t v_n n_t^{j,n} \tilde{n}_t^j\right]$$

At the natural level of output, we know that:

$$\psi_t v_n = \chi_t^j u_c^j \left(1 - \alpha_j\right) \frac{c_t^{j,n}}{n_t^{j,n}}$$

from optimal production and demand. Substituting this into the expression above yields:

$$U_t^1 = \theta_t \sum_j u_c^j c_t^{j,n} \chi_t^j \left[\tilde{c}_t^j - \left(1 - \alpha_j\right) \tilde{n}_t^j\right]$$

Next we observe that, from the production function together with $c_t^j = y_t^j$, we have:

$$\tilde{c}_t^j = \left(1 - \alpha_j\right) \tilde{n}_t^j - \left(1 - \alpha_j\right) \log\left(\Delta_t^j\right)$$

Thus we are left with:

$$U_t^1 = -\theta_t \sum_j u_c^j c_t^{j,n} \chi_t^j \left(1 - \alpha_j\right) \log\left(\Delta_t^j\right)$$

Next we simplify the price dispersion term. We adopt the following conventions: p_{it}^j denotes the log price of firm i in sector j, p_t^j denotes the overall price index of sector j, and $\bar{p}_t^j = \int_i p_{it}^j$ denotes the average log price in sector j. By taking the second-order taylor expansion of the definition of the price index defined in equation (13) (though note that there lower case p denotes level prices rather than logs), we obtain:

$$p_t^j \approx \bar{p}_t^j + \frac{(1 - \varepsilon)}{2} \mathrm{var}_i \left(p_{it}^j\right)$$

Then, by taking a second-order taylor approximation of the definition of price dispersion from equation (18), using the previous equation, and dropping higher order terms, we obtain:

$$\left(1 - \alpha_j\right) \log \Delta_t^j \approx \frac{\varepsilon}{2} \left(1 + \frac{\alpha_j \varepsilon}{1 - \alpha_j}\right) \mathrm{var}_i \left(p_{it}^j\right)$$

The welfare loss can therefore be written as:

$$
\begin{aligned}
U_t \approx U_t^n &+ \theta_t \frac{1}{2} \sum_j \chi_t^j \left[u_c^j c_t^{j,n} \left(\tilde{c}_t^j\right)^2 + u_{cc}^j \left(c_t^{j,n}\right)^2 \left(\tilde{c}_t^j\right)^2 \right] \\
&- \theta_t \psi_t \frac{1}{2} \left[v_n \sum_j n_t^{j,n} \left(\tilde{n}_t^j\right)^2 + v_{nn} \left(\sum_j n_t^{j,n} \tilde{n}_t^j \right)^2 \right] \\
&- \theta_t \sum_j u_c^j c_t^{j,n} \chi_t^j \frac{\varepsilon}{2} \left(1 + \frac{\alpha_j \varepsilon}{1 - \alpha_j}\right) \mathrm{var}_i \left(p_{it}^j\right)
\end{aligned}
$$

Now we simplify this further. Note that all terms are now second order. Note that all coefficients can be expanded around the steady state like so:

$$
u_c^j \left(c_t^{j,n}\right) \approx u_c^j \left(\bar{c}^j\right) + u_{cc}^j \left(\bar{c}^j\right) \cdot \left(c_t^{j,n} - \bar{c}^j\right)
$$

and the same with shocks. However, since we have dropped all first-order terms, the extra terms will all drop, and we are left with terms in steady state terms:

$$
\begin{aligned}
U_t \approx U_t^n &+ \sum_j \frac{\chi^j}{2} \left[\bar{u}_c^j \bar{c}^j \left(\tilde{c}_t^j\right)^2 + \bar{u}_{cc}^j \left(\bar{c}^j\right)^2 \left(\tilde{c}_t^j\right)^2 \right] - \frac{\psi}{2} \left[\bar{v}_n \sum_j \bar{n}^j \left(\tilde{n}_t^j\right)^2 + \bar{v}_{nn} \tilde{n}_t^2 \right] \\
&- \sum_j \bar{u}_c^j \bar{c}^j \chi^j \frac{\varepsilon}{2} \left(1 + \frac{\alpha_j \varepsilon}{1 - \alpha_j}\right) \mathrm{var}_i \left(p_{it}^j\right)
\end{aligned}
$$

The final step is to explain the determination of the variance of log prices. This satisfies:

$$
\mathrm{var}_i \left(p_{it}^j\right) = \int_i \left(p_{it}^j - p_{t-1}^j\right)^2 - \left(p_t^j - \bar{p}_{t-1}^j\right)^2
$$

where we have used the fact that, to first order, $p_t^j = \bar{p}_t^j$. Now by the Calvo adjustment rule:

$$
\int_i \left(p_{it}^j - p_{t-1}^j\right)^2 = \phi_j \int_i \left(p_{i,t-1}^j - p_{t-1}^j\right)^2 + (1 - \phi_j) \left(p_t^{j*} - p_{t-1}^j\right)^2
$$

Finally, from equation (17), to first-order we have:

$$
p_t^j - p_{t-1}^j = (1 - \phi_j) \left(p_t^{j*} - p_{t-1}^j\right)
$$

This leaves us with:

$$\text{var}_i\left(p_{it}^j\right) = \phi_j \text{var}_i\left(p_{it-1}^j\right) + \left(\frac{\phi_j}{1-\phi_j}\right)\left(\pi_t^j\right)^2$$

Iterating backward in time yields:

$$\text{var}_i\left(p_{it}^j\right) = \sum_{s=0}^{t}\left(\frac{\phi_j}{1-\phi_j}\right)\phi_j^s\left(\pi_{t-s}^j\right)^2$$

Now consider the overall welfare term. This is:

$$W = \sum_{t=0}^{\infty}\beta^t U_t$$

Consider the component relating to inflation. This is:

$$\begin{aligned}
W_\pi &= -\sum_{t=0}^{\infty}\beta^t \sum_j \bar{u}_c^j \bar{c}^j \chi^j \frac{\varepsilon}{2}\left(1+\frac{\alpha_j\varepsilon}{1-\alpha_j}\right)\text{var}_i\left(p_{it}^j\right) \\
&= -\sum_{t=0}^{\infty}\beta^t \sum_j \bar{u}_c^j \bar{c}^j \chi^j \frac{\varepsilon}{2}\left(1+\frac{\alpha_j\varepsilon}{1-\alpha_j}\right)\sum_{s=0}^{t}\left(\frac{\phi_j}{1-\phi_j}\right)\phi_j^s\left(\pi_{t-s}^j\right)^2 \\
&= -\sum_j \bar{u}_c^j \bar{c}^j \chi^j \frac{\varepsilon}{2\Phi_j}\sum_{t=0}^{\infty}\beta^t\left(\pi_t^j\right)^2
\end{aligned}$$

where

$$\Phi_j = \frac{1-\phi_j}{\phi_j}\left(\frac{1-\phi_j\beta}{1+\frac{\varepsilon_j\alpha_j}{1-\alpha_j}}\right)$$

and therefore lifetime welfare can be written as:

$$W \approx W_t^n + \sum_{t=0}^{\infty}\beta^t \tilde{U}_t$$

where now:

$$\tilde{U}_t = \frac{1}{2}\sum_j \chi^j\left(\bar{c}^j\right)^{1-\sigma_j}\left\{(1-\sigma_j)\left(\tilde{c}_t^j\right)^2 - \frac{\varepsilon}{\Phi_j}\left(\pi_t^j\right)^2\right\} - \frac{\psi}{2}\left(\bar{n}\right)^{1+\zeta}\left\{\sum_j \frac{\bar{n}^j}{\bar{n}}\left(\tilde{n}_t^j\right)^2 + \zeta\tilde{n}_t^2\right\}$$

where we have used $\sigma_j = -u_{cc}^j c^j / u_c^j$ and $\zeta = v_{nn} n / v_n$. □

Proof of proposition 9. I first establish an important lemma.

Lemma 2 (Marginal Change in Welfare from Policy Change)**.** *Suppose there is a marginal change in policy variable Q. Then the resulting change in $\tilde{U}_t$ is:*

$$\frac{d\tilde{U}_t}{dQ} = (\bar{c}_1)^{-\sigma_1} \sum_j \bar{\gamma}_j \left[\frac{d\tilde{c}_t^j}{dQ} \tau_t^j - \frac{\varepsilon}{\Phi_j} \pi_t^j \frac{d\pi_t^j}{dQ} \right]$$

where τ_t^j and Φ_j are as defined previously, and where

$$\bar{\gamma}_j = \frac{\bar{q}_j \bar{c}_j}{\sum_k \bar{q}_k \bar{c}_k}$$

is the steady state GDP share of sector j. This can also be written in terms of inflation only as:

$$\frac{d\tilde{U}_t}{dQ} = (\bar{c}_1)^{-\sigma_1} \sum_j \frac{\bar{\gamma}_j}{\Phi_j} \left[\beta \frac{d\tilde{c}_t^j}{dQ} \mathbb{E}_t \pi_{t+1}^j - \left(\varepsilon \frac{d\pi_t^j}{dQ} + \frac{d\tilde{c}_t^j}{dQ} \right) \pi_t^j \right]$$

Proof. Suppose that some policy instrument Q is adjusted which has some effect on period t variables. Then the marginal effect on U_t is, to second-order:

$$\begin{aligned}
\frac{d}{dQ}\left(\tilde{U}_t - U_t^n\right) \approx & \sum_j \chi^j \bar{u}_c^j \bar{C}^j (1-\sigma_j)\, \tilde{c}_t^j \frac{d\tilde{c}_t^j}{dQ} \\
& - \psi \bar{v}_n \bar{N} \left[\sum_j \frac{\bar{N}^j}{\bar{N}} \left(\frac{1}{1-\alpha_j} \right)^2 \tilde{c}_t^j \frac{d\tilde{c}_t^j}{dQ} + \zeta \tilde{n}_t \sum_j \frac{N^{j,n}}{\bar{N}} \left(\frac{1}{1-\alpha_j} \right) \frac{d\tilde{c}_t^j}{dQ} \right] \\
& - \sum_j \bar{u}_c^j \bar{C}^j \chi^j \frac{\varepsilon}{\Phi_j} \pi_t^j \frac{d\pi_t^j}{dQ}
\end{aligned}$$

where we have made use of $\tilde{c}_t^j = (1-\alpha_j)\tilde{n}_t^j$ to first-order. Now we simplify a bit. Note that we can write the first portion as:

$$\sum_j \chi^j \bar{u}_c^j \bar{C}^j (1-\sigma_j)\, \tilde{c}_t^j \frac{d\tilde{c}_t^j}{dQ} - \psi \bar{v}_n \bar{N} \sum_j \frac{\bar{N}^j}{\bar{N}} \left(\frac{1}{1-\alpha_j}\right)^2 \tilde{c}_t^j \frac{d\tilde{c}_t^j}{dQ}$$
$$= \sum_j \left[\chi^j \bar{u}_c^j \bar{C}^j (1-\sigma_j) - \psi \bar{v}_n \bar{N}^j \left(\frac{1}{1-\alpha_j}\right)^2\right] \tilde{c}_t^j \frac{d\tilde{c}_t^j}{dQ}$$
$$= \sum_j \chi^j \bar{u}_c^j \bar{C}^j \left[1-\sigma_j - \frac{1}{1-\alpha_j}\right] \tilde{c}_t^j \frac{d\tilde{c}_t^j}{dQ}$$

And the term:

$$-\psi \bar{v}_n \bar{N} \left[\zeta \left(\sum_k \frac{N^k}{\bar{N}} \left(\frac{1}{1-\alpha_k}\right) \tilde{c}_t^k\right) \frac{\bar{N}^j}{\bar{N}} \left(\frac{1}{1-\alpha_j}\right) \frac{d\tilde{c}_t^j}{dQ}\right] = \sum_j \left[\left(\frac{\alpha_j}{1-\alpha_j} + \sigma_j\right) \tilde{c}_t^j + \tau_t^j\right] \chi^j \bar{u}_c^j \bar{C}^j \frac{d\tilde{c}_t^j}{dQ}$$

Therefore the marginal welfare term simplifies to:

$$\frac{d}{dQ}\left(\tilde{U}_t - U_t^n\right) \approx \bar{u}_c^1 \sum_j \bar{\gamma}_j \left[\frac{d\tilde{c}_t^j}{dQ} \tau_t^j - \frac{\varepsilon}{\Phi_j} \pi_t^j \frac{d\pi_t^j}{dQ}\right]$$

where Φ_j is as defined previously, and where $\bar{\gamma}_j$ is the steady state GDP share of sector j.

We further know that:

$$\tau_t^j = \frac{1}{\Phi_j}\left(\beta \mathbb{E}_t \pi_{t+1}^j - \pi_t^j\right)$$

And therefore:

$$\frac{d}{dQ}\left(\tilde{U}_t - U_t^n\right) \approx \sum_j \frac{\bar{\gamma}_j}{\Phi_j} \left[\frac{d\tilde{c}_t^j}{dQ}\left(\beta \mathbb{E}_t \pi_{t+1}^j - \pi_t^j\right) - \varepsilon \pi_t^j \frac{d\pi_t^j}{dQ}\right]$$

The form with inflation only follows directly from substituting (16) into this expression. □

Lemma 2 gives the effect on welfare from a marginal change in an unspecified policy instrument Q. In practice, we are interested in the effect of interest rates. Given the lack of durable goods in the model, it is clear that, while future interest rates may affect current variables, past interest rates have no effects. I put this in a lemma as well:

Lemma 3 (Zero Effect of Past Interest Rates)**.** *In the log-linear model, the for any $s > 0$:*

$$\frac{d\tilde{c}_t^j}{dR_{t+s}} = 0 \tag{21}$$

$$\frac{d\tilde{\pi}_t^j}{dR_{t+s}} = 0 \tag{22}$$

Proof. Since there are no durable goods, all equilibrium variables in period t and after can be defined in terms of interest rates R_{t+s}, for $s > 0$, and shocks realized at time t only. Since none of these are affected by past interest rates, the result immediately follows. □

Now we are in position to prove the proposition. The problem is to choose R_t to maximize $\tilde{U}_t$ subject to the equilibrium conditions. The optimality condition is therefore $d\tilde{U}_t/dR_t = 0$. We then use the expressions for marginal period welfare from lemma 2 to obtain the result.

□

Proof of Proposition 10. The problem is to choose R_T to maximize $W = \sum_{t=0}^{\infty} \beta^t \tilde{U}_t$. The optimality expression is therefore

$$\sum_{t=0}^{\infty} \beta^t \frac{d\tilde{U}_t}{dR_T} = 0$$

Now we substitute in the expression for $d\tilde{U}_t/dQ$ from Lemma 2, with $Q = R_T$, and set $d\tilde{U}_t/dR_T = 0$ for $t > T$ from Lemma 3. Letting $\mathbb{E}_t \pi_{t+1} = \pi_{t+1}$ due to certainty, and rearranging the sum slightly, yields the given expression. □

Proof of Proposition 1. Consider the N-period fixed price case without commitment. We write the problem of the policymaker as:

$$V_t\left(\vec{c}_{t-1}\right) = \max_{R_{t+1}} \left\{ u\left(\vec{c}_t, \vec{c}_{t-1}\right) - v\left(n_t\left(\vec{c}_t, \vec{c}_{t-1}\right)\right) + \frac{\beta\theta_{t+1}}{\theta_t} V_{t+1}\left(\vec{c}_t\right) \right\} = \max_{R_{t+1}} \left\{U_t\right\}$$

where V_{t+1} is defined recursively, with V_{t+N} being the flexible price value function. We again have:

$$U_{c_t^j} = u_{c_t^j} - \frac{v_{n_t}}{f_{n_t}^j} + \frac{\beta\theta_{t+1}}{\theta_t} V_{c_t^j}$$

The envelope condition is:

$$V_{c_{t-1}^j} = (u_t)_{c_{t-1}^j} + \left(1-\delta^j\right) \frac{v_{n_t}}{f_{n_t}^j} = (u_t)_{c_{t-1}^j} + \left(1-\delta^j\right) p_t^j \lambda_t \left(1-\tau_t^j\right)$$

Thus we obtain:

$$\begin{aligned} U_{c_t^j} &= u_{c_t^j} - \frac{v_{n_t}}{f_{n_t}^j} + \frac{\beta\theta_{t+1}}{\theta_t}\left[(u_{t+1})_{c_t^j} + \left(1-\delta^j\right) p_{t+1}^j \lambda_{t+1}\left(1-\tau_{t+1}^j\right)\right] \\ &= p_t^j \lambda_t \tau_t^j - \lambda_t \left(\frac{1-\delta^j}{R_{t+1}}\right) p_{t+1}^j \tau_{t+1}^j \end{aligned}$$

Now the optimality condition is just as before (after observing that $p_{t+1}^j = p_t^j$):

$$\sum_j \varepsilon_R^j \gamma_t^j \left(\tau_t^j - \left(\frac{1-\delta^j}{R_{t+1}}\right)\tau_{t+1}^j\right) = 0$$

which can also be written:

$$\sum_j \varepsilon_R^j \gamma_t^j \tau_t^j = \sum_j \varepsilon_R^j \gamma_t^j \left(\frac{1-\delta^j}{R_{t+1}}\right)\tau_{t+1}^j$$

The expression for $N = 1$ is implied, after we note that in this case $\tau_{t+1}^j = 0$. □

Proof of Proposition 2. With commitment the central bank chooses the path of interest rates $\{R_k\}_{k=1}^N$ to maximize objective function:

$$\sum_{t=0}^{N-1} \beta^t \theta_t \left[u\left(\vec{c}_t, \vec{c}_{t-1}\right) - v\left(n_t\left(\vec{c}_t, \vec{c}_{t-1}\right)\right)\right] + \beta^N \theta_N V_N\left(\vec{c}_{N-1}\right)$$

where $V_N\left(\vec{c}_{N-1}\right)$ is the flexible price value function entering period N. The optimal choice of R_k satisfies:

$$\sum_{t=0}^{N-1} \beta^t \theta_t \sum_j \left(u_{c_t^j} - \frac{v_{n_t}}{f_{n_t}^j}\right)\left(c_t^j\right)_{R_k} + \sum_{t=1}^{N} \beta^t \theta_t \sum_j \left[\left((u_t)_{c_{t-1}^j} + \left(1-\delta^j\right)\frac{v_{n_t}}{f_{n_t}^j}\right)\left(c_{t-1}^j\right)_{R_k}\right] = 0$$

Adjusting the time indices yields:

$$\sum_{t=0}^{N-1} \beta^t \theta_t \sum_j \left\{\left(u_{c_t^j} + \beta\frac{\theta_{t+1}}{\theta_t}(u_{t+1})_{c_t^j} - \frac{v_{n_t}}{f_{n_t}^j} + \beta\frac{\theta_{t+1}}{\theta_t}\left(1-\delta^j\right)\frac{v_{n_{t+1}}}{f_{n_{t+1}}^j}\right)\left(c_t^j\right)_{R_k}\right\} = 0$$

Using $v_{n_t}/f^j_{n_t} = \lambda_t p^j_t \left(1 - \tau^j_t\right)$ and $u_{c^j_t} + \beta \frac{\theta_{t+1}}{\theta_t} \left(u_{t+1}\right)_{c^j_t} = p^j_t \lambda_t - \frac{1-\delta^j}{R_{t+1}} p^j_{t+1} \lambda_t$, we can write this as:

$$\sum_{t=0}^{N-1} \beta^t \theta_t \lambda_t y_t \sum_j \gamma^j_t \varepsilon^{y^j_t}_{R_k} \chi^j_t = 0$$

where

$$\chi^j_t = \tau^j_t - \left(\frac{1-\delta^j}{R_{t+1}}\right) \tau^j_{t+1}$$

is the overhang-augmented labor wedge. □

Proof of Proposition 3. As before, labor can be expressed as a function of demand as:

$$n_t\left(\vec{c}_t\right) = \sum_j \left(f^j_t\right)^{-1} \left(c^j_t - \left(1 - \delta^j\right) c^j_{t-1}\right)$$

Then the optimal policy problem is to choose R_{t+1} to maximize:

$$V_t\left(\vec{c}_{t-1}\right) = \max_R \left\{u\left(\vec{c}_t\left(R\right)\right) - v\left(n_t\left(\vec{c}_t\left(R\right)\right)\right) + \mathbb{E}_t\left[\frac{\beta\theta_{t+1}}{\theta_t} V_{t+1}\left(\vec{c}_t\left(R\right)\right)\right]\right\}$$

The optimality expression is:

$$\sum_j \left(u_{c^j_t} + \mathbb{E}_t\left[\frac{\beta\theta_{t+1}}{\theta_t} V_{c^j_t}\right] - \frac{v_{n_t}}{f^j_{n_t}}\right) \frac{dc^j_t}{dR} = 0$$

From the envelope condition, we have:

$$V_{c^j_{t-1}} = \left(1 - \delta^j\right) \frac{v_{n_t}}{f^j_{n_t}}$$

But since the next period has flexible prices, this implies:

$$V_{c^j_t} = \left(1 - \delta^j\right) p^j_{t+1} \lambda_{t+1}$$

Substituting this into the expression above, we obtain:

$$\sum_j \left(u_{c^j_t} + \mathbb{E}_t\left[\frac{\beta\theta_{t+1}}{\theta_t} \left(1 - \delta^j\right) p^j_{t+1} \lambda_{t+1}\right] - \frac{v_{n_t}}{f^j_{n_t}}\right) \frac{dc^j_t}{dR} = 0$$

This we may write this as:

$$\sum_j \lambda_t p^j_t \left(1 - \frac{w_t}{p^j_t f^j_{n_t}}\right) \frac{dc^j_t}{dR} = 0$$

which, as before, we may write as:

$$\sum_j \varepsilon_R^j \gamma_t^j \tau_t^j = 0$$

$\square$

Proof of Proposition 4. We again write the optimal policy problem as:

$$V_t\left(\vec{c}_{t-1}\right) = \max_R \left\{ u\left(\vec{c}_t\left(R\right)\right) - v\left(n_t\left(\vec{c}_t\left(R\right)\right)\right) + \mathbb{E}_t\left[\frac{\beta\theta_{t+1}}{\theta_t} V_{t+1}\left(\vec{c}_t\left(R\right)\right)\right]\right\}$$

The optimality expression is again:

$$\sum_j \left(u_{c_t^j} + \mathbb{E}_t\left[\frac{\beta\theta_{t+1}}{\theta_t} V_{c_t^j}\right] - \frac{v_{n_t}}{f_{n_t}^j}\right)\frac{dc_t^j}{dR} = 0$$

But now it may be the case that $\tau_{t+1}^j \neq 0$. Thus we must use the expression:

$$V_{c_t^j} = \left(1-\delta^j\right)\frac{v_{n_{t+1}}}{f_{n_{t+1}}^j} = \left(1-\delta^j\right)\left(1-\tau_{t+1}^j\right)p_{t+1}^j\lambda_{t+1}$$

Using this, the expression above becomes:

$$\sum_j \left(u_{c_t^j} + \left(1-\delta^j\right)\mathbb{E}_t\left[\frac{\beta\theta_{t+1}}{\theta_t}\left(1-\tau_{t+1}^j\right)p_{t+1}^j\lambda_{t+1}\right] - \frac{v_{n_t}}{f_{n_t}^j}\right)\frac{dc_t^j}{dR} = 0$$

Now we use the expression for p_t^j, together with the fact that $p_t^j = p_{t+1}^j = \bar{p}^j$, and the definition of R_{t+1} to obtain:

$$\sum_j \bar{p}^j \left(\tau_t^j - \left(\frac{1-\delta^j}{R_{t+1}}\right)\frac{\mathbb{E}_t\left[\theta_{t+1}\lambda_{t+1}\tau_{t+1}^j\right]}{\mathbb{E}_t\left[\theta_{t+1}\lambda_{t+1}\right]}\right)\frac{dc_t^j}{dR} = 0$$

Now, as before, we note that $\frac{dy_t^j}{dR} = \frac{dc_t^j}{dR}$, we multiply and divide each term of the sum by y_t^j, multiply the entire expression by $-R$, and then divide through by GDP $y_t = \sum_j p_t^j y_t^j$, to obtain:

$$\sum_j \varepsilon_R^j \gamma_t^j \tau_t^j = \sum_j \varepsilon_R^j \gamma_t^j \left(\frac{1-\delta^j}{R_{t+1}}\right)\frac{\mathbb{E}_t\left[\theta_{t+1}\tau_{t+1}^j\lambda_{t+1}\right]}{\mathbb{E}_t\left[\theta_{t+1}\lambda_{t+1}\right]}$$

$\square$

Proof of Proposition 5. The problem of the monetary authority is:

$$\max_{\{R_{t+1}\}_{t=0}^{N-1}} \sum_{t=0}^{N-1} \sum_{s^t} q\left(s^t\right) \beta^t \theta_t \left[u\left(\vec{c}_t\right) - v\left(n_t\left(\vec{c}_t, \vec{c}_{t-1}\right)\right)\right] + \beta^T \sum_{S^T} q\left(s^T\right) \theta_T V\left(\vec{c}_{T-1}\right)$$

As before, past choices of interest rates do not affect future equilibrium variables. Thus the optimality condition for the choice of R_{k+1} is:

$$\sum_{t=0}^{k} \sum_{s^t} q\left(s^t\right) \beta^t \theta_t \sum_j \left[\left(u_{c_t^j} - \frac{v_{n_t}}{f_{n_t^j}}\right) \left(c_t^j\right)_{R_{k+1}}\right] + \sum_{t=1}^{k+1} \sum_{s^t} q\left(s^t\right) \beta^t \theta_t \sum_j \left(1 - \delta^j\right) \frac{v_{n_t}}{f_{n_t^j}} \left(c_{t-1}^j\right)_{R_{k+1}} = 0$$

We reindex to write these as:

$$\sum_{t=0}^{k} \sum_{s^t} q\left(s^t\right) \beta^t \theta_t \sum_j \left[u_{c_t^j} - \frac{v_{n_t}}{f_{n_t^j}} + \beta\left(1 - \delta^j\right) \sum_{s^{t+1} \subset s^t} \frac{q\left(s^{t+1}\right)}{q\left(s^t\right)} \frac{\theta_{t+1}}{\theta_t} \frac{v_{n_{t+1}}}{f_{n_{t+1}^j}}\right] \left(c_t^j\right)_{R_{k+1}} = 0$$

Using $v_{n_t}/f_{n_t^j} = \left(1 - \tau_t^j\right) \lambda_t p_t^j$ and $u_{c_t^j} = p_t^j \lambda_t - \beta\left(1 - \delta^j\right) \mathbb{E}_t \frac{\theta_{t+1}}{\theta_t} \lambda_{t+1} p_{t+1}^j$, we can write this as:

$$\sum_{t=0}^{k} \sum_{s^t} q\left(s^t\right) \beta^t \theta_t \sum_j \left[\tau_t^j - \left(\frac{1 - \delta^j}{R_{t+1}}\right) \frac{\sum_{s^{t+1} \subset s^t} q\left(s^{t+1}\right) \theta_{t+1} \lambda_{t+1} \tau_{t+1}^j}{\sum_{s^{t+1} \subset s^t} q\left(s^{t+1}\right) \theta_{t+1} \lambda_{t+1}}\right] \lambda_t p^j \left(c_t^j\right)_{R_{k+1}} = 0$$

or just:

$$\mathbb{E}_0 \sum_{t=0}^{k} \beta^t \theta_t \lambda_t y_t \sum_j \gamma_t^j \varepsilon_{R_{k+1}}^{y_t^j} \chi_t^j = 0$$

where

$$\chi_t^j = \tau_t^j - \left(\frac{1 - \delta^j}{R_{t+1}}\right) \frac{\mathbb{E}_t\left[\theta_{t+1} \lambda_{t+1} \tau_{t+1}^j\right]}{\mathbb{E}_t\left[\theta_{t+1} \lambda_{t+1}\right]}$$

is the durable overhang-augmented labor wedge. □

Proof of Lemma 1 . As in the certainty case, we can iterate the Euler equation forward to show that the effect of the interest more than 1 period ahead is the same. First observe that the good j demand equation at time $t < N-1$ can be written as:

$$u_{c_t^j} = \left(1 - \frac{1 - \delta^j}{R_{t+1}}\right) p^j \lambda_t$$

Since p^j is fixed, the only moving parts here are R_{t+1} and λ_t. By iterating the Euler equation forward, we obtain:

$$\theta_t \lambda_t = \beta^{N-t} \left(\prod_{k=t}^{N-1} R_{k+1} \right) \mathbb{E}_t \left[\theta_N \lambda_N \right]$$

Note that R_{k+1} for $k \in [t+1, N-1]$ has exactly the same effect on demand.

□

Proof of Proposition 6. Start with the objective function

$$\sum_{t=0}^{N-1} \sum_{S^t} q\left(s^t\right) \beta^t \theta_t \left[u\left(\vec{c}_t\right) - v\left(n_t\left(\vec{c}_t, \vec{c}_{t-1}\right)\right) \right] + \beta^T \sum_{S^T} q\left(s^T\right) \theta_T V\left(\vec{c}_{T-1}\right)$$

The first-order condition with respect to $R_{k+1}\left(s_*^k\right)$ satisfies:

$$\sum_{t=0}^{k} q\left(s_*^t\right) \beta^t \theta_t \sum_j \left(u_{c_t^j} - \frac{v_{n_t}}{f_{n_t}^j} \right) \frac{dc_t^j}{dR_{k+1}\left(s_*^k\right)} + \sum_{t=0}^{k} \sum_{s^{t+1} \subset s_*^t} q\left(s^{t+1}\right) \beta^{t+1} \theta_{t+1} \sum_j \left[\left(1 - \delta^j\right) \frac{v_{n_{t+1}}}{f_{n_{t+1}}^j} \frac{dc_t^j}{dR_{k+1}\left(s_*^k\right)} \right] = 0$$

where the states s_*^t are taken to lie on the path defined by s_*^k, that is $s_*^k \subset s_*^t$. Note that since the choice of $R_{k+1}\left(s_*^k\right)$ may depend on the entire history of past states, its effect on c_t^j only occurs in the state s_*^t, i.e. the state that yields directly to s_*^k. However, this increase in $c_t^j\left(s_*^t\right)$ will lead to an increase in the initial stock of goods $c_t^j\left(s^{t+1}\right)$ in all state s^{t+1} reachable from s_*^t, including ones not leading to s_*^k. Now we use the fact that $u_{c_t^j} = \bar{p}^j \lambda_t \left(1 - \frac{1-\delta^j}{R_{t+1}}\right)$, $\frac{v_n}{f_n^j} = \left(1 - \tau_t^j\right) \bar{p}^j \lambda_t$, and

$$\frac{1}{R_{t+1}\left(s_*^t\right)} = \mathbb{E}_{t*} \left[\beta \frac{\theta_{t+1}}{\theta_t} \frac{\lambda_{t+1}}{\lambda_t} \right] = \sum_{s^{t+1} \subset s_*^t} \left[\frac{q\left(s^{t+1}\right)}{q\left(s_*^t\right)} \beta \frac{\theta_{t+1}}{\theta_t} \lambda_{t+1} \right]$$

to obtain:

$$\sum_{t=0}^{k} q\left(s_*^t\right) \beta^t \theta_t \lambda_t y_t \left\{ \sum_j \varepsilon^{y_t^j}_{R_{k+1}\left(s_*^k\right)} \gamma_t^j \left(\tau_t^j - \left(\frac{1-\delta^j}{R_{t+1}\left(s_*^t\right)} \right) \frac{\mathbb{E}_{t*}\left[\theta_{t+1} \lambda_{t+1} \tau_{t+1}^j\right]}{\mathbb{E}_{t*}\left[\lambda_{t+1} \theta_{t+1}\right]} \right) \right\} = 0$$

□

Proof of Proposition 7. Let the flexible price value function of the economy entering period $t+1$ be $V_{t+1}\left(\left\{c_t^j\right\}\right)$. Now we need to determine equilibrium variables in period t conditional on the interest rate. As before, household demand satisfies (12), which (together with the Euler equation) defines demand functions $c_t^j(R_{t+1})$ just as described in section I.B. Next we observe that, given demand $\vec{c}_t$, we can implicitly define $\left(n_t, \left\{n_t^j, y_t^j, \left\{x_t^{jk}\right\}\right\}\right)$ by:

$$y_t^j = f_t^j\left(n_t^j, \left\{x_t^{jk}\right\}\right) \tag{23}$$

$$y_t^j = c_t^j - \left(1-\delta^j\right) c_{t-1}^j + \sum_k x_t^{kj} \tag{24}$$

$$\frac{v_{n_t}}{u_{c_t^1}} = \bar{p}_t^k \frac{f_{n_t}^j}{f_{x_t^{jk}}^j} \tag{25}$$

$$n_t = \sum_j n_t^j \tag{26}$$

Now consider the optimal choice of R_{t+1}. This satisfies:

$$\max_{R_{t+1}} \left\{U\left(\left\{c_t^j(R_{t+1})\right\}\right) - v\left(n_t(R_{t+1})\right) + \beta V\left(\left\{c_t^j(R_{t+1})\right\}\right)\right\} \tag{27}$$

This problem yields optimality condition:

$$\sum_j \left(U_{c_t^j} + \beta V_{c_t^j}\right) \frac{dc_t^j}{dR_{t+1}} = v_{n_t} \frac{dn_t}{dR_{t+1}} \tag{28}$$

As before, we have $V_{c_t^j} = \left(1-\delta^j\right)\lambda_{t+1} p_{t+1}^j$, and therefore (28) becomes:

$$\sum_j p_t^j \frac{dc_t^j}{dR_{t+1}} = w_t \frac{dn_t}{dR_{t+1}}$$

where we have used the fact from the household problem that:

$$p_t^j = \frac{U_{c_t^j}}{\lambda_t} + \beta\left(1-\delta^j\right)\frac{\lambda_{t+1}}{\lambda_t} p_{t+1}^j$$

and that $v_{n_t} = w_t \lambda_t$.

Next we differentiate the market clearing constraint for good j with respect to R_{t+1} to obtain:

$$f^j_{n_t}\frac{dn^j_t}{dR_{t+1}} + \sum_k f^j_{x^{jk}_t}\frac{dx^{jk}_t}{dR_{t+1}} = \frac{dc^j_t}{dR_{t+1}} + \sum_k \frac{dx^{kj}_t}{dR_{t+1}}$$

Next we use the fact that $f^j_n/w_t = f_{x^{jk}}/p^k$ to obtain:

$$f^j_{n_t}\frac{dn^j_t}{dR_{t+1}} + \sum_k \frac{p^k_t}{w_t} f^j_{n_t}\frac{dx^{jk}_t}{dR_{t+1}} = \frac{dc^j_t}{dR_{t+1}} + \sum_k \frac{dx^{kj}_t}{dR_{t+1}}$$

Finally, we use the fact that:

$$\begin{aligned}
\frac{dn_t}{dR_{t+1}} &= \sum_j \frac{dn^j_t}{dR_{t+1}} \\
&= \sum_j \left[\left(1-\tau^j_t\right)\frac{p^j_t}{w_t}\frac{dc^j_t}{dR_{t+1}} + \left(1-\tau^j_t\right)\frac{p^j_t}{w_t}\sum_k \frac{dx^{kj}_t}{dR_{t+1}} - \sum_k \frac{p^k_t}{w_t}\frac{dx^{jk}_t}{dR_{t+1}}\right] \\
&= \sum_j \left(1-\tau^j_t\right)\frac{p^j_t}{w_t}\frac{dc^j_t}{dR_{t+1}} - \sum_j \tau^j_t \frac{p^j_t}{w_t}\sum_k \frac{dx^{kj}_t}{dR_{t+1}}
\end{aligned}$$

where we define τ^j_t implicitly by:

$$w_t = \left(1-\tau^j_t\right) p^j_t f^j_{n_t}$$

Substituting this into the original optimality condition, we obtain:

$$\sum_j \tau^j_t p^j_t \left(\frac{dc^j_t}{dR_{t+1}} + \sum_k \frac{dx^{kj}_t}{dR_{t+1}}\right) = 0$$

Next we observe that the term inside the parenthesis is just dy^j_t/dR_{t+1}. Therefore we can write this as:

$$\sum_j \tau^j_t \gamma^j_t \varepsilon^{y^j_t}_R = 0$$

where

$$\varepsilon^{y^j_t}_R = -\frac{dy^j_t}{dR_{t+1}}\frac{R_{t+1}}{y^j_t}$$

□

D Additional Figures

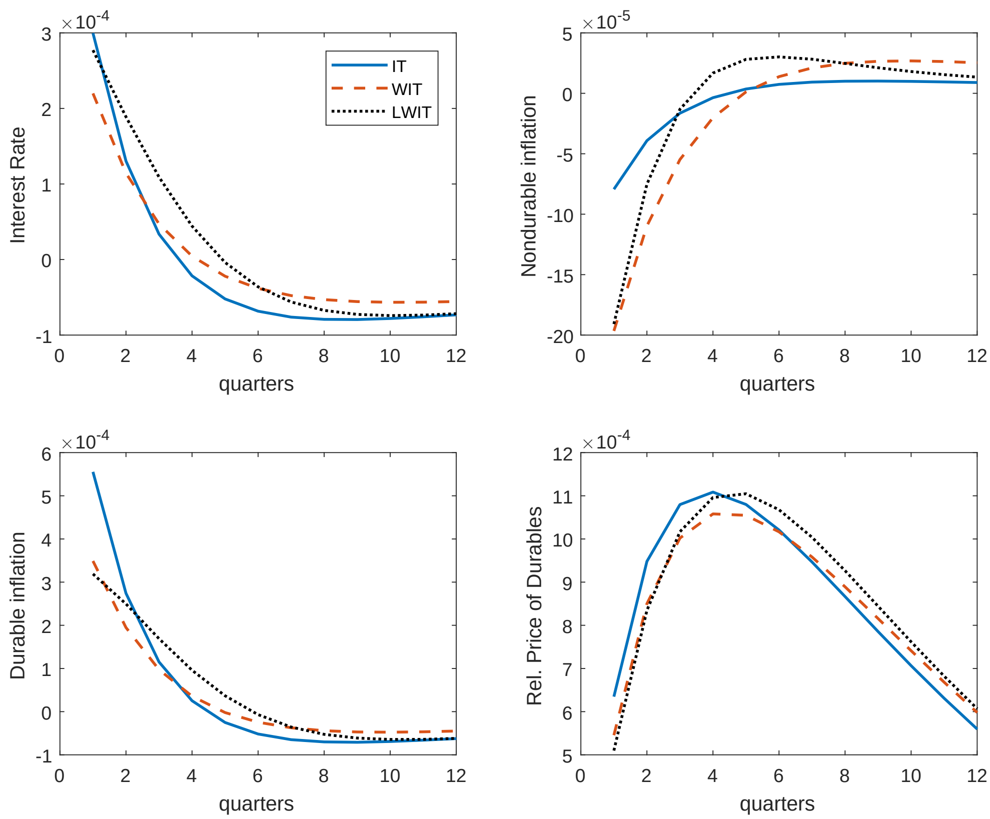

Figure 1: Impulse responses following 1-sd shock to aggregate productivity z.

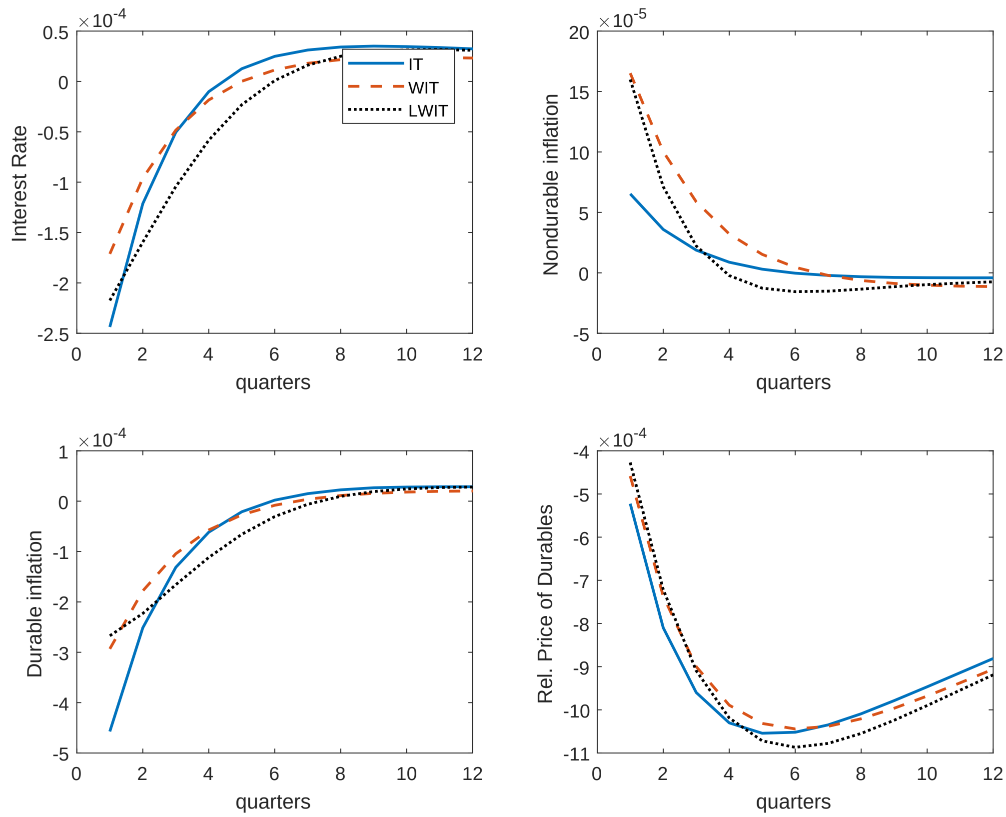

Figure 2: Impulse responses following 1-sd shock to nondurable markup μ_c.

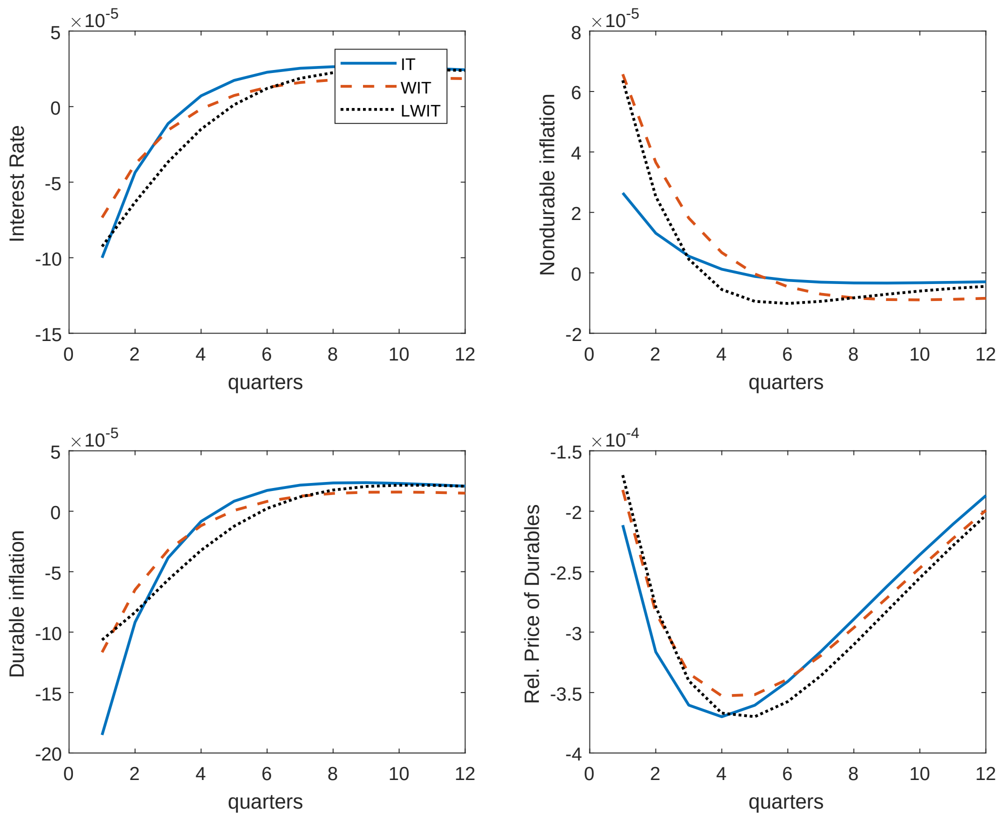

Figure 3: Impulse responses following 1-sd shock to aggregate markup μ.

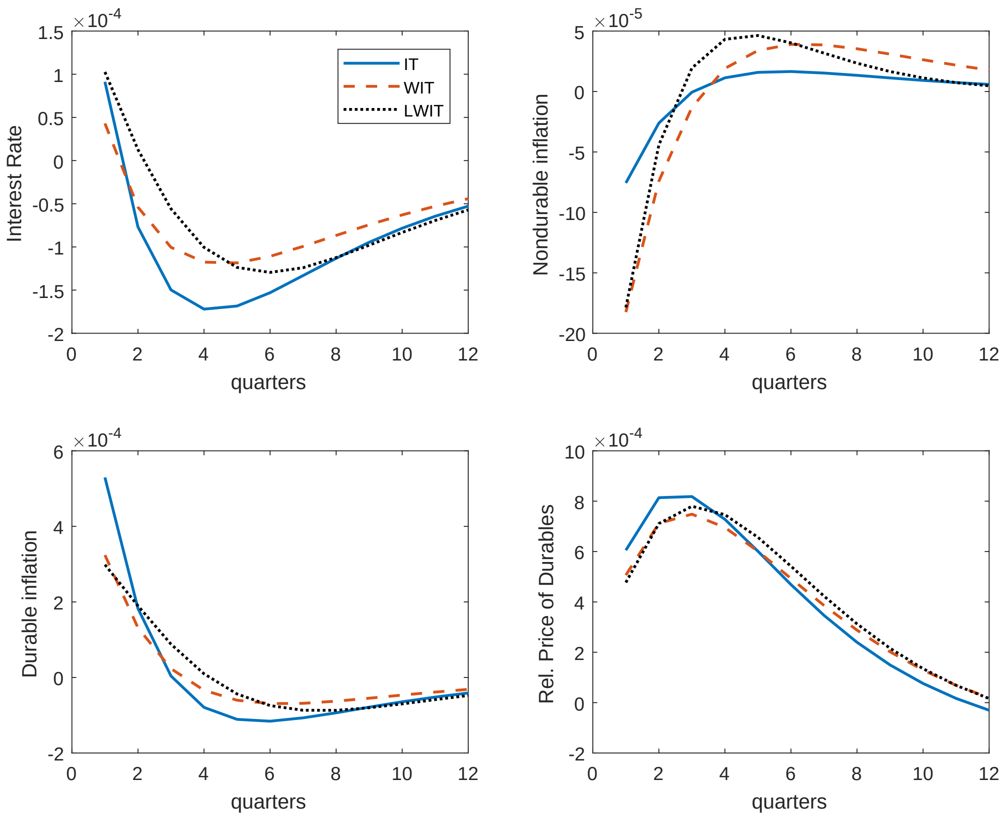

Figure 4: Impulse responses following 1-sd shock to discount factor θ.