Two-Stage Contests with Private Information

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Online Appendix

Uniqueness of equilibrium with convex cost

In this section we show that Theorem 1 holds for more general cost of effort. We denote the cost function of effort as $c(e)$, which is the same for high and low ability contestants. The cost function is assumed to be twice differentiable on the non-negative reals, strictly increasing and weakly convex, with the cost of zero effort being zero.

Lemma A.1. *In any equilibrium and for any history* η_2, $BR^{\ell}_{i2}(\sigma^*, \eta_2) \cup BR^{h}_{i2}(\sigma^*, \eta_2) = [0, x^*]$ *for* $i = s, w$ *and* $x^* > 0$*. Equilibrium output distributions,* $H^*_{2s}(x|\eta_2), L^*_{2s}(x|\eta_2), H^*_{2w}(x|\eta_2)$*, and* $L^*_{2w}(x|\eta_2)$*, are continuous on* $(0, x^*]$ *and for* $x \in BR^{\ell}_{i2}(\sigma^*, \eta_2)$ *and* $x' \in BR^{h}_{i2}(\sigma^*, \eta_2)$*, then* $x \leq x'$*.*

Proof. The proof follows in three steps:

(1) We first show there is no output at which both contestants have an atom and if a contestant has an atom, it is at zero. It follows that $F^*_{i2}(x|\eta_2)$ are continuous on $(0, \infty)$ for $i = s, w$ and any η_2.

Assume both contestants produce x with positive probability. Because the cost of effort is continuous, either contestant can improve payoffs by producing output slightly above this atom. Then x is not a best response of that contestant, a contradiction.

Assume that contestant i produces $x > 0$ with positive probability. Then by the continuity of the cost function in output, there is a $\delta > 0$ such that for all $\hat{x} \in (x - \delta, x)$, $\hat{x} \notin BR^{\theta}_{-i2}(\sigma^*, \eta_2)$ for $\theta = \ell, h$. This implies, that contestant i would do better by playing $x - \delta/2$, and therefore $x \notin BR^{\theta}_{i2}(\sigma^*, \eta_2)$, a contradiction.

(2) Next, if $\hat{x} > 0$ is not a best response for any ability of one of the contestants, then for all $x > \hat{x}$, x is not a best response for either type of either contestant.

Step (1) implies that equilibrium payoffs are continuous over positive outputs. Given $x' \notin BR^{\ell}_{i2}(\sigma^*, \eta_2) \cup BR^{h}_{i2}(\sigma^*, \eta_2)$, for some $i = s, w$, $\exists \tilde{x}^h, \tilde{x}^\ell$ for which $\mathbb{E}[\pi_{i2}(\tilde{x}^h)|a^h, \sigma^*, \eta_2] > \mathbb{E}[\pi_{i2}(x')|a^h, \sigma^*, \eta_2] + \varepsilon$ and $\mathbb{E}[\pi_{i2}(\tilde{x}^\ell)|a^\ell, \sigma^*, \eta_2] > \mathbb{E}[\pi_{i2}(x')|a^\ell, \sigma^*, \eta_2] + \varepsilon$. Then, every output in the neighborhood of x' in this neighborhood cannot be a best response of either type of contestant i, and therefore also cannot be a best response for any type of contestant $-i$, who could improve expected payoffs by lowering output.

Define $X^* = \{x | x > \hat{x}$ and $x \in BR^{\ell}_{i2}(\sigma^*, \eta_2) \cup BR^{h}_{i2}(\sigma^*, \eta_2)\}$. Let $x_* = \inf\{X^*\}$. Then, there is a neighborhood below x_* for which all outputs are not best responses for any ability type of either contestant. By continuity of payoffs, this would imply that there is an $x \in X^*$ that gives lower expected payoffs than $\hat{x}$ for both types of each contestant, a contradiction. Therefore, x_* does not exist and X^* is empty. This implies that $\sup\{BR^{\ell}_{s2}(\sigma^*, \eta_2) \cup BR^{h}_{s2}(\sigma^*, \eta_2)\} = \sup\{BR^{\ell}_{w2}(\sigma^*, \eta_2) \cup BR^{h}_{w2}(\sigma^*, \eta_2)\} \equiv x^*$ and the combined best response sets of each contestant is $[0, x^*]$.

(3) In any equilibrium, for $x \in BR^{\ell}_{i2}(\sigma^*, \eta_2)$ and $x' \in BR^{h}_{i2}(\sigma^*, \eta_2)$, then $x \leq x'$.
Assume otherwise, $\exists x \in BR^{\ell}_{i2}(\sigma^*, \eta_2)$ and $x' \in BR^{h}_{i2}(\sigma^*, \eta_2)$, with $x > x'$. Then

$$\mathbb{E}[\pi_{i2}(x) - \pi_{i2}(x')|a^{\ell}, \sigma^*, \eta_2] = p_2(F^*_{i2}(x|\eta_2) - F^*_{i2}(x'|\eta_2)) - (c(x) - c(x')) \geq 0.$$

Because the cost function is increasing and weakly convex, and $x > x'$, then

$$\mathbb{E}[\pi_{i2}(x) - \pi_{i2}(x')|a^{h}, \sigma^*, \eta_2] = p_2(F^*_{i2}(x|\eta_2) - F^*_{i2}(x'|\eta_2)) - (c(x/a^h) - c(x'/a^h)) > 0.$$

This contradicts $x' \in BR^{h}_{i2}(\sigma^*, \eta_2)$.

□

Proof of Proposition 7

From Lemma A.1, there are x^*_s and x^*_w such that $x^*_i = \sup\{BR^{\ell}_{i2}(\sigma^*, \eta_2)\} = \inf\{BR^{h}_{i2}(\sigma^*, \eta_2)\}$, for $i = s, w$, and $x^* \equiv \sup\{BR^{h}_{i2}(\sigma^*, \eta_2)\}$ which is the same for $i = s, w$.

Each contestant must be indifferent between all $x \in (x^*_i, x^*)$ when they have high ability. Each high ability contestant has the same marginal cost of output so indifference implies that the expected output density must also be the same: $f^*_{s2}(x) = f^*_{w2}(x)$ for $x \in (\max\{x^*_s, x^*_w\}, x^*)$. Since $f^*_{i2}(x) = \mu_i h^*_{i2}(x)$ for all $x \in (x^*_i, x^*)$, then $h^*_{s2}(x) \leq h^*_{w2}(x)$ for all $x \in (\max\{x^*_s, x^*_w\}, x^*)$. Then $H_{i2}(x^*_i) = 0$, requires that $x^*_s \leq x^*_w$.

Therefore, for the remainder of the construction, there are three intervals to consider: the best response set of the low types of both the stronger and the weaker contestants, $0 \leq x \leq x^*_s$, the best response set of the low type of the weaker contestant and the high type of the strong contestant, $x^*_s \leq x \leq x^*_w$, and best response set of the high types of each contestant, $x^*_w \leq x \leq x^*$.

Within their best response sets, contestants must be indifferent between all output levels. For example, the strong contestant with high ability must be indifferent to picking all outputs between x^*_s and x^*. This puts a condition on $F_{w2}(x)$, the output distribution of the weak contestant, on the interval $[x^*_s, x^*]$:

$$p_2 F^*_{w2}(x') - c\left(\frac{x}{a^h}\right) = p_2 F^*_{w2}(x') - c\left(\frac{x'}{a^h}\right).$$

Rearranging and taking the limit as $x \to x'$, $\lim_{x \to x'} \frac{F^*_{w2}(x) - F^*_{w2}(x')}{c(\frac{x}{a^h}) - c(\frac{x'}{a^h})} = \frac{1}{p_2}$. Then the output density of the strong contestant is

$$f^*_{w2}(x') = \lim_{x \to x'} \frac{F^*_{w2}(x) - F^*_{w2}(x')}{c\left(\frac{x}{a^h}\right) - c\left(\frac{x'}{a^h}\right)} \frac{c\left(\frac{x}{a^h}\right) - c\left(\frac{x'}{a^h}\right)}{a^h\left(\frac{1}{a^h}(x - x')\right)} = \frac{c'\left(\frac{x'}{a^h}\right)}{p_2 a^h}.$$

A similar calculation on each interval for each contestant allows us to characterize the densities of the output on each of the intervals below.

- $x^*_w \leq x \leq x^*$: $h^*_{s2}(x) = \frac{c'(x/a^h)}{p_2 a^h \mu_s}$, $h^*_{w2}(x) = \frac{c'(x/a^h)}{p_2 a^h \mu_w}$, $f^*_{s2}(x) = f^*_{w2}(x) = \frac{c'(x/a^h)}{p_2 a^h}$.

- $x_s^* \le x \le x_w^*$: $h_{s2}^*(x) = \frac{c'(x)}{p_2\mu_s}$, $\ell_{w2}^*(x) = \frac{c'(x/a^h)}{p_2 a^h(1-\mu_w)}$, $f_{s2}^*(x) = \frac{c'(x)}{p_2}$, $f_{w2}^*(x) = \frac{c'(x/a^h)}{p_2 a^h}$.
- $0 \le x \le x_s^*$: $\ell_{s2}^*(x) = \frac{c'(x)}{p_2(1-\mu_s)}$, $\ell_{w2}^*(x) = \frac{c'(x)}{p_2(1-\mu_w)}$, $f_{s2}^*(x) = f_{w2}^*(x) = \frac{c'(x)}{p_2}$.

It remains to characterize the cutoff points, x_w^*, x_s^* and x^*, and $L_{w2}^*(0)$. In equilibrium, the distribution of output for each contestant must satisfy

$$L_{i2}^*(x_i^*) = 1, \quad H_{i2}^*(x_i^*) = 0, \quad F_{i2}^*(x_i^*) = 1 - \mu_i, \text{ and } \quad F_{i2}^*(x^*) = 1.$$

Additionally, the strong contestant chooses no effort with zero probability, so $L_{s2}^*(0) = 0$. Using $L_{s2}^*(x_s^*) = 1$ and the definition of $\ell_{s2}^*(x)$ on $[0, x_s^*]$, we calculate x_s^*.

$$\int_0^{x_s^*} \ell_{s2}^*(x)dx = L_{s2}^*(x_s^*) - L_{s2}^*(0) = \frac{c(x_s^*)}{p_2(1-\mu_s)} = 1$$

Then $c(x_s^*) = p_2(1-\mu_s)$, so that $x_s^* = c^{-1}(p_2(1-\mu_s))$. Similarly, $x_w^* = c^{-1}(p_2(1-\mu_w))$. From these endpoints we can calculate x^*.

$$\int_{x_s^*}^{x_w^*} h_{s2}^*(x)dx = \frac{c(x_w^*) - c(x_s^*)}{\mu_s} = \frac{(1-\mu_w) - (1-\mu_s)}{\mu_s} = \frac{\mu_s - \mu_w}{\mu_s}$$

$$\int_{x_w^*}^{x^*} h_{s2}^*(x_s)dx = 1 - \frac{\mu_s - \mu_w}{\mu_s} = \frac{\mu_w}{\mu_s}$$

$$\int_{x_w^*}^{x^*} f_{s2}^*(x_s)dx = \frac{1}{p_2}\left(c\left(\frac{x^*}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right)\right) = \mu_w$$

$$x^* = a^h c^{-1}\left(p_2\mu_w + c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right)\right)$$

Lastly, we pin down the probability that the weaker contestant exerts no effort.

$$\int_{x_s^*}^{x_w^*} \ell_{w2}^*(x)dx = \frac{1}{p_2(1-\mu_w)}\left[c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_s))}{a^h}\right)\right]$$

$$\int_0^{x_s^*} \ell_{w2}^*(x)dx = \frac{c(c^{-1}(p_2(1-\mu_s)))}{p_2(1-\mu_w)} - 0 = \frac{1-\mu_s}{1-\mu_w}$$

$$L_{w2}^*(0) = \mu_s - \mu_w - \frac{1}{p_2(1-\mu_w)}\left[c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_s))}{a^h}\right)\right]$$

Lemma A.2. *Given history $\eta_2 = (x_{s1}, x_{w1})$ with associated beliefs, $\mu_s \ge \mu_w$, the second contest continuation value of each contestant conditional on their ability are*

$$\begin{aligned} v_s^h(\mu_s,\mu_w) &= v_w^h(\mu_w,\mu_s) = p_2(1-\mu_w) - c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right), \\ v_s^\ell(\mu_s,\mu_w) &= p_2(\mu_s-\mu_w) - \left[c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_s))}{a^h}\right)\right], \\ &= v_w^\ell(\mu_w,\mu_s) = 0. \end{aligned}$$

Proof. The expected payoffs of a high ability contestant are equal to the value of winning less the cost of producing output x^*, as producing x^* guarantees a win.

$$v_s^h(\mu_s, \mu_w) = v_w^h(\mu_w, \mu_s) = p_2 - c(x^*/a^h) = p_2(1 - \mu_w) - c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right)$$

The expected payoffs of low ability contestants is equal to the probability they win given they exert no effort. This is the probability the other contestant puts in no effort.[1]

$$\begin{aligned} v_s^\ell(\mu_s, \mu_w) &= p_2(1-\mu_w)L_{w2}^*(0) \\ &= p_2(\mu_s - \mu_w) - \left[c\left(\frac{c^{-1}(p_2(1-\mu_w))}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_s))}{a^h}\right)\right] \\ v_w^\ell(\mu_w, \mu_s) &= p_2(1-\mu_s)L_{s2}^*(0) = 0 \end{aligned}$$

□

Proposition A.1. *Let $F_{\mu_{-i}}(M) = \Pr(\mu_{-i} \leq M)$ be the belief distribution of contestant $-i$'s ability resulting from the first contest, and let $\underline{M} = \sup\{M | F_{\mu_{-i}}(M) = 0\}$ and $\overline{M} = \inf\{M | F_{\mu_{-i}}(M) = 1\}$. For all $\mu_i \in (\underline{M}, \overline{M})$, expected payoffs in the second contest decrease for high ability players as μ_i increases, $\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i^h(\mu_i, \mu_{-i})] < 0$, and increase with μ_i for low ability players, $\frac{\partial}{\partial \mu_i}\mathbb{E}[v_i^\ell(\mu_i, \mu_{-i})] > 0$.*

Proof. In the second contest, for a given pair of beliefs, contestants will expect the following payoffs:

$$v_i^h(\mu_i, \mu_{-i}) = p_2(1 - \min\{\mu_i, \mu_{-i}\}) - c\left(\frac{c^{-1}(p_2(1-\min\{\mu_i, \mu_{-i}\}))}{a^h}\right)$$

$$v_i^\ell(\mu_i, \mu_{-i}) = \begin{cases} p_2(\mu_i - \mu_{-i}) - \left[c\left(\frac{c^{-1}(p_2(1-\mu_{-i}))}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a^h}\right)\right], & \text{if } \mu_i \geq \mu_{-1} \\ 0, & \text{otherwise} \end{cases}.$$

For a high ability contestant believed to be high ability with probability μ_i and with opponent's belief distribution, $F_{\mu_{-i}}$, the expected payoff in the second contest is

$$\begin{aligned} &\mathbb{E}[v_i^h(\mu_i, \mu_{-i})] \\ &= \int_0^1 \left(p_2(1-\min\{\mu_i, \mu_{-i}\}) - c\left(\frac{c^{-1}(p_2(1-\min\{\mu_i, \mu_{-i}\}))}{a^h}\right)\right) dF_{\mu_{-i}}(\mu_{-i}). \end{aligned}$$

As the opponent believes the contestant is stronger, the change in expected payoff is

$$\frac{\partial}{\partial \mu_i}\mathbb{E}_{\mu_{-i}}[v_i^h(\mu_i, \mu_{-i})] = \left(p_2 + \frac{\partial}{\partial \mu_i} c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a^h}\right)\right)(F_{\mu_{-i}}(\mu_i) - 1).$$

[1] Here we technically are assuming the contestant wins all ties at zero, but if the agent exerts a tiny amount of effort and we let that effort shrink to zero, then this is the payoff of the agent in the limit. Since the payoffs are continuous, these limits must be the payoffs of the low ability contestants.

For a low ability contestant, the expected payoff is

$$\mathbb{E}[v_i^\ell(\mu_i,\mu_{-i})] \\ = \int_0^{\mu_i} \left(p_2(\mu_i-\mu_{-i}) + c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a^h}\right) - c\left(\frac{c^{-1}(p_2(1-\mu_{-i}))}{a^h}\right)\right) dF_{\mu_{-i}}(\mu_{-i}),$$

with change in expected payoff

$$\frac{\partial}{\partial\mu_i}\mathbb{E}[v_i^\ell(\mu_i,\mu_{-i})] = \left(p_2 + \frac{\partial}{\partial\mu_i} c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a^h}\right)\right) F_{\mu_{-i}}(\mu_i).$$

Given the assumptions on the cost of effort, $c'(e)>0$ and $c''(e)\geq 0$,

$$\frac{\partial}{\partial\mu_i} c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a^h}\right) = -\frac{1}{a^h} c'\left(\frac{c^{-1}(p_2(1-\mu_i))}{a^h}\right)\frac{1}{c'(c^{-1}(p_2(1-\mu_i)))} \in \left[-\frac{p_2}{a^h}, 0\right).$$

Define $d(\mu_i) \equiv \left[p_2 + \frac{\partial}{\partial\mu_i} c\left(\frac{c^{-1}(p_2(1-\mu_i))}{a_h}\right)\right]$. For all μ_i, $d(\mu_i) \in \left[\frac{p_2(a^h-1)}{a^h}, p_2\right)$. It follows that

$$\frac{\partial}{\partial\mu_i} E[v_i^h(\mu_i,\mu_{-i})] = d(\mu_i)(F_{\mu_{-i}}(\mu_i)-1) \text{ and } \frac{\partial}{\partial\mu_i} E[v_i^\ell(\mu_i,\mu_{-i})] = d(\mu_i)F_{\mu_{-i}}(\mu_i),$$

where the former derivative is strictly negative and the later is strictly positive when $\mu_i \in (\underline{M}_{-i}, \overline{M}_{-i})$. □

Lemma A.3. *In every SPBE, $\mu^*(x)$ is weakly increasing in x for all $x \in X_{i1} \equiv X_{i1}^h \cup X_{i1}^\ell$.*

Proof. Let $x, x' \in X_{i1}$ such that $x < x'$ and $\mu(x) > \mu(x')$. Then $0 \leq \mu(x') < \mu(x) \leq 1$ which implies $x \in X_{i1}^h \subseteq BR_{i1}^h$ and $x' \in X_{i1}^\ell \subseteq BR_{i1}^\ell$. Best responses require

$$\begin{aligned} p_1\mathbb{E}[w_i(x',x_{-i1})] - c(x') + \mathbb{E}[v_i^\ell(\mu(x'),\mu(x_{-i1}))] \\ \geq p_1\mathbb{E}[w_i(x,x_{-i1})] - c(x) + \mathbb{E}[v_i^\ell(\mu(x),\mu(x_{-i1}))], \text{ and} \\ p_1\mathbb{E}[w_i(x',x_{-i1})] - c(x'/a^h) + \mathbb{E}[v_i^h(\mu(x'),\mu(x_{-i1}))] \\ \leq p_1\mathbb{E}[w_i(x,x_{-i1})] - c(x/a^h) + \mathbb{E}[v_i^h(\mu(x),\mu(x_{-i1}))]. \end{aligned}$$

This implies that

$$\begin{aligned} p_1(\mathbb{E}[w_i(x',x_{-i1})] - \mathbb{E}[w_i(x,x_{-i1})]) + \mathbb{E}[v_i^\ell(\mu(x'),\mu(x_{-i1}))] - \mathbb{E}[v_i^\ell(\mu(x),\mu(x_{-i1}))] \\ \geq c(x') - c(x), \text{ and} \\ p_1(\mathbb{E}[w_i(x',x_{-i1})] - \mathbb{E}[w_i(x,x_{-i1})]) + \mathbb{E}[v_i^h(\mu(x'),\mu(x_{-i1}))] - \mathbb{E}[v_i^h(\mu(x),\mu(x_{-i1}))] \\ \leq c(x'/a^h) - c(x/a^h). \end{aligned}$$

From Proposition A.1, $\mu(x) > \mu(x')$ implies

$$\begin{aligned} \mathbb{E}[v_i^\ell(\mu(x'),\mu(x_{-i1}))] - \mathbb{E}[v_i^\ell(\mu(x),\mu(x_{-i1}))] \leq 0, \\ \text{and } \mathbb{E}[v_i^h(\mu(x'),\mu(x_{-i1}))] - \mathbb{E}[v_i^h(\mu(x),\mu(x_{-i1}))] \geq 0. \end{aligned}$$

Combining the previous inequalities,

$$c(x') - c(x) \leq p_1(\mathbb{E}[w_i(x', x_{-i1})] - \mathbb{E}[w_i(x, x_{-i1})]) \leq c(x'/a^h) - c(x/a^h),$$

which cannot be true given $a^h > 1$, $c''(x) \geq 0$ and $c'(x) > 0$. $\square$

Proposition A.2. *Given $p_1 = 0$ and $p_2 > 0$, there is a unique SPBE where $X_{i1} = \{0\}$.*

Proof. Equilibrium conditions are satisfied when $H_1^*(x) = L_1^*(x) = 1$ for $x \geq 0$ and 0 otherwise (i.e. the output densities of both high and low ability contestants consist of a single mass point at $x = 0$), $\mu^*(x) = \hat{\mu}$ for $x \geq 0$, and second period distribution functions are as characterized in the proof of Proposition 7.

To show that there can be no equilibrium where $\tilde{x} \in X_{i1}$, such that $\tilde{x} > 0$, assume that there is. Then $\tilde{x} \in BR_{i1}^{\ell}(\sigma_{-i}) \cup BR_{i1}^{h}(\sigma_{-i})$. If $\tilde{x} \in BR_{i1}^{\ell}(\sigma_{-i})$ then $\mathbb{E}[v_i^{\ell}(\mu(\tilde{x}), \mu(x_{-i1}))] - \mathbb{E}[v_i^{\ell}(\mu(0), \mu(x_{-i1}))] \geq c(\tilde{x}) > 0$ which implies that $\mu(\tilde{x}) > \mu(0) \geq 0$. Because $\mu(\tilde{x}) > 0$, equilibrium conditions on the belief function require that $\tilde{x} \in X_{i1}^h \subset BR_{i1}^h$ and therefore $\mathbb{E}[v_i^h(\mu(\tilde{x}), \mu(x_{-i1}))] - \mathbb{E}[v_i^h(\mu(0), \mu(x_{-i1}))] \geq c\left(\frac{\tilde{x}}{a^h}\right) > 0$, which cannot be true when $\mu(\tilde{x}) > \mu(0)$, a contradiction.

If $\tilde{x} \in BR_{i1}^h(\sigma_{-i}) \setminus BR_{i1}^{\ell}(\sigma_{-i})$, then $\mu(\tilde{x}) < \mu(0)$ which implies by Lemma A.3 that $0 \notin X_{i1}$. This however would require the existence of a positive output in $BR_{i1}^{\ell}(\sigma_{-i})$, which we just ruled out. $\square$

Lemma A.4. *Let $p_1 > 0$. For any SPBE, first contest output distributions are continuous and therefore $\mathbb{E}[w_i(x_{i1}, x_{-i1})|x_{i1}] = F_1^*(x_{i1}) = \hat{\mu}H_1^*(x_{i1}) + (1 - \hat{\mu})L_1^*(x_{i1})$ is continuous.*

Proof. In a symmetric equilibrium, if an output is played with positive probability by one type of contestant, then it must be played with positive probability by both contestants of this type. Let $\tilde{x} \in \{X_{i1}^{\ell} \cup X_{i1}^h\}$ be played with probability $q > 0$. Then

$$\mathbb{E}[w_i(\tilde{x}, x_{-i1})] + \frac{q}{2} \leq \mathbb{E}[w_i(x, x_{-i1})] \text{ for all } x > \tilde{x}.$$

Since, $\tilde{x} \in BR_{i1}^{\theta}(\sigma_{-i})$, for some θ, then for all $x \geq 0$,

$$\begin{aligned} p_1\mathbb{E}[w_i(\tilde{x}, x_{-i1})] - c(\tilde{x}/a^{\theta}) + \mathbb{E}[v_i^{\theta}(\mu_i(\tilde{x}), \mu_{-i})] \\ \geq p_1\mathbb{E}[w_i(x, x_{-i1})] - c(x/a^{\theta}) + \mathbb{E}[v_i^{\theta}(\mu_i(x), \mu_{-i})]. \end{aligned}$$

Combining the above inequalities,

$$p_1\frac{q}{2} \leq \mathbb{E}[v_i^{\theta}(\mu_i(\tilde{x}), \mu_{-i})] - \mathbb{E}[v_i^{\theta}(\mu_i(x), \mu_{-i})] + c(x/a^{\theta}) - c(\tilde{x}/a^{\theta}).$$

By continuity of the cost function, $\exists\varepsilon > 0$ such that for all $x \in (\tilde{x}, \tilde{x} + \varepsilon)$, we have $c\left(\frac{\tilde{x}+\varepsilon}{a^{\theta}}\right) - c\left(\frac{\tilde{x}}{a^{\theta}}\right) < p_1\frac{q}{2}$. Then for each x in this range

$$\mathbb{E}[v_i^{\theta}(\mu_i(\tilde{x}), \mu_{-i})] - \mathbb{E}[v_i^{\theta}(\mu_i(x), \mu_{-i})] > 0. \tag{1}$$

From Proposition A.1, if $\theta = \ell$, then $\mu_i(\tilde{x}) > \mu_i(x)$ and $\tilde{x} \in \{X_{i1}^{\ell} \cap X_{i1}^h\}$. Similarly, if $\theta = h$, then $\mu_i(\tilde{x}) < \mu_i(x)$ and $\tilde{x} \in \{X_{i1}^{\ell} \cap X_{i1}^h\}$. In either case, $\tilde{x} \in \{BR_{i1}^{\ell}(\sigma_{-i}) \cap BR_{i1}^h(\sigma_{-i})\}$. However, (1) cannot hold for both $\theta = \ell$ and $\theta = h$, a contradiction. $\square$

Lemma A.5. *Let $p_1, p_2 > 0$. Define $x_{\ell *} = \inf X^\ell_{i1}$, $x^*_\ell = \sup X^\ell_{i1}$, $x_{h*} = \inf X^h_{i1}$, and $x^*_h = \sup X^h_{i1}$. In any SPBE, the best response sets of low and high ability contestants in the first contest are intervals with $BR^\ell_{i1}(\sigma^*) = [0, x^*_\ell]$, $BR^h_{i1}(\sigma^*) = [x_{h*}, x^*_h]$ and $0 = x_{\ell *} \le x_{h*} < x^*_\ell \le x^*_h$.*

Proof. From Lemma A.4 we now can use the fact that $L^*_1(x)$ and $H^*_1(x)$, and therefore $F^*_1(x)$, are continuous in x and we have that in equilibrium $\mathbb{E}[w_i(x, x_{-i1})] = \Pr(x_{-i1} < x|\sigma^*_{-i}) = \Pr(x_{-i1} \le x|\sigma^*_{-i}) = F^*_1(x)$. Combined with Lemma A.3, we have $\Pr(\mu^*(x_{-i1}) < \mu^*(x)|\sigma^*_{-i}) \le \mathbb{E}[w_i(x, x_{-i1})] = F^*_1(x) \le \Pr(\mu^*(x_{-i1}) \le \mu^*(x)|\sigma^*_{-i}) = F_{\mu_{-i}}(\mu^*(x))$. The proof follows in four steps.

(1) We first show that $x_{\ell *} = 0$. We do this by first showing that $x_{\ell *} \le x_{h*}$, and then showing that $x_{\ell *}$ cannot be larger than zero.

Let $x_{h*} < x_{\ell *}$. Since $x_{h*} = \inf X^h_{i1}$, $\forall \varepsilon > 0$, $\exists x_\varepsilon$ such that $x_{h*} \le x_\varepsilon < x_{h*} + \varepsilon$ and $x_\varepsilon \in X^h_{i1}$. In particular, this holds for $\varepsilon^* = x_{\ell *} - x_{h*}$. Then $x_{\varepsilon^*} \in \{X^h_{i1} \setminus X^\ell_{i1}\}$ and $\mu^*(x_{\varepsilon^*}) = 1$. However, from Lemma A.3 we would have $\mu^*(x) = 1$ for all $x \in X^\ell_{i1}$, which cannot hold. Therefore $x_{h*} \ge x_{\ell *}$.

If $0 < x_{\ell *} < x_{h*}$, then by Lemma A.4, $\exists \delta$ with $0 < \delta < x_{h*} - x_{\ell *}$ such that $\forall x \in (x_{\ell *}, x_{\ell *} + \delta)$ we have $|p_1(F^*_1(x) - F^*_1(0))| = |p_1(F^*_1(x) - F^*_1(x_{\ell *}))| < c(x_{\ell *})$. Let $x_\delta \in X^\ell_{i1} \cap (x_{\ell *}, x_{\ell *} + \delta)$. Then $\mu(x_\delta) = 0$ and $p_1(F^*_1(x_\delta) - F^*_1(0)) < c(x_\delta)$. However this implies

$$p_1 F^*_1(0) + \mathbb{E}[v^\ell_i(\mu(0), \mu_{-i})] > p_1 F^*_1(x_\delta) + \mathbb{E}[v^\ell_i(\mu(x_\delta), \mu_{-i})] - c(x_\delta),$$

and therefore $x_\delta \notin BR^\ell_{i1}(\sigma_{-i})$, a contradiction.

If $0 < x_{\ell *} = x_{h*}$, then $\exists x_\ell, x_h$ such that $x_\ell \le x_h$, $x_\ell \in X^\ell_{i1}$, $x_h \in X^h_{i1}$, and $p_1(F^*_1(x_\ell) - F^*_1(x_{\ell *})) = p_1 F^*_1(x_\ell) < c(x_{\ell *}) < c(x_\ell)$ and $p_1(F^*_1(x_h) - F^*_1(x_{h*})) = p_1 F^*_1(x_h) < c(x_{h*}/a_h) < c(x_h/a_h)$, by the continuity of $F^*_1(x)$. It follows that $x_\ell \in X^\ell_{i1}$ implies

$$p_1 F^*_1(x_\ell) - c(x_\ell) + E[v^\ell_i(\mu(x_\ell), \mu_{-i})] \ge p_1 F^*_1(0) - c(0) + \mathbb{E}[v^\ell_i(\mu(0), \mu_{-i})]$$

and $\mathbb{E}[v^\ell_i(\mu(x_\ell), \mu_{-i})] > \mathbb{E}[v^\ell_i(\mu(0), \mu_{-i})]$, which requires $\mu(x_\ell) > \mu(0)$.

Similarly, $x_h \in X^h_{i1}$ implies that $\mu(x_h) < \mu(0)$. Combining these two inequalities leads to $\mu(x_h) < \mu(x_\ell)$. This contradicts Lemma A.3. Therefore we must have $0 = x_{\ell *} \le x_{h*}$.

(2) We next show that $x_{h*} \le x^*_\ell$.

If $x^*_\ell < x_{h*}$, then $\forall x \in (x^*_\ell, x_{h*})$, $x \notin \{X^\ell_{i1} \cup X^h_{i1}\}$. Let $\tilde{x} = \frac{x^*_\ell + x_{h*}}{2}$ and $\varepsilon = c(x_{h*}/a^h) - c(\tilde{x}/a^h)$. There is a $\delta > 0$ such that $\forall x \in (x_{h*}, x_{h*} + \delta)$, $p_1(F^*_1(x) - F^*_1(x_{h*})) < \varepsilon$. Pick an x_δ such that $x_\delta \in (x_{h*}, x_{h*} + \delta)$ and $x_\delta \in X^h_{i1}$. Then $p_1(F_1(x_\delta) - F_1(x_{h*})) = p_1(F_1(x_\delta) - F_1(x')) < \varepsilon$, $c(x_\delta/a^h) - c(\tilde{x}/a^h) > \varepsilon$, and $\mathbb{E}[v^h_i(\mu(x_\delta), \mu_{-i})] \le \mathbb{E}[v^h_i(\mu(\tilde{x}), \mu_{-i})]$. Then

$$p_1 F^*_1(\tilde{x}) + \mathbb{E}[v^h_i(\mu(\tilde{x}), \mu_{-i})] - c\left(\frac{\tilde{x}}{a^h}\right) > p_1 F^*_1(x_\delta) + \mathbb{E}[v^h_i(\mu(x_\delta), \mu_{-i})] - c\left(\frac{x_\delta}{a^h}\right),$$

a contradiction. So we can conclude that $x^*_\ell \le x_{h*}$.

Also $x^*_\ell \le x^*_h$. If we assume otherwise, then we can find $x \in \{X^\ell_{i1} \setminus X^h_{i1}\}$ where $x > x^*_h$ and $\mu(x) = 0$. Lemma A.3 rules out this possibility.

We have shown so far that $0 = x_{\ell *} \leq x_{h*} \leq x^*_\ell \leq x^*_h$.

(3) For all $x \in (x_{\ell *}, x_{h*})$, $x \in BR^\ell_{i1}(\sigma_{-i})$ and for all $x \in (x^*_\ell, x^*_h)$, $x \in BR^h_{i1}(\sigma_{-i})$.

Given $x_{\ell *} < x_{h*}$, let $X^\ell_c = \{x | x \in (x_{\ell *}, x_{h*}) \setminus BR^\ell_{i1}(\sigma_{-i})\}$. If $x \in X^\ell_c$, then $\exists \varepsilon > 0$ such that for all $x' \in (x_{\ell,*}, x_{h,*}) \cap X^\ell_{i1}$,

$$p_1 F^*_1(x) + \mathbb{E}[v^\ell_i(\mu(x), \mu_{-i})] - c(x) < p_1 F^*_1(x') + \mathbb{E}[v^\ell_i(\mu(x'), \mu_{-i})] - c(x') - \varepsilon,$$

where $\mathbb{E}[v^\ell_i(\mu(x), \mu_{-i})] \geq \mathbb{E}[v^\ell_i(\mu(x'), \mu_{-i})]$ as $\mu(x') = 0$. Therefore $p_1 F^*_1(x) - c(x) < p_1 F^*_1(x') - c(x') - \varepsilon$, and for all $x' > x$, $p_1(F^*_1(x') - F^*_1(x)) > c(x') - c(x) - \varepsilon$.

Since $F^*_1(x)$ and $c(x)$ are continuous, then there is a $\delta_\varepsilon > 0$ such that for all $x' \in X^\ell_{i1}$, $|x' - x| \geq \delta_\varepsilon$. This implies that x is contained in an interval which is a subset of X^ℓ_c. Let a and b be the infimum and supremem of this interval respectively.

- If $b < x_{h*}$, then $\exists x' < x_{h*}$, $x' \in X^\ell_{i1}$ where $|x' - b| < \delta, \forall \delta > 0$. Then, by the continuity of $F^*_1(x)$, $\exists x' \in X^\ell_{i1}$ and $p_1(F^*_1(x') - F^*_1(b)) < c(b) - c(\frac{a+b}{2})$. Then we know that

  $$p_1 F^*_1(x') - p_1 F^*_1\left(\frac{a+b}{2}\right) < c(b) - c\left(\frac{a+b}{2}\right) \text{ and}$$
  $$\mathbb{E}[v^\ell_i(\mu(x'), \mu_{-i})] \leq \mathbb{E}\left[v^\ell_i\left(\mu\left(\frac{a+b}{2}\right), \mu_{-i}\right)\right],$$

  which contradicts $x' \in BR^\ell_{i1}(\sigma_{-i})$.

- If $b = x_{h*}$, then $\exists x' \in X^h_{i1}$, where $|x' - x_{h*}| < \delta$, $\forall \delta > 0$. We can take $x' \in X^h_{i1}$ such that $p_1(F^*_1(x') - F^*_1(x_{h*})) < c(\frac{b}{a^h}) - c(\frac{a+x_{h*}}{2a^h})$.
  - If $x' \notin X^\ell_{i1}$ then $\mu(x') = 1$, but since $\mathbb{E}[v^h_i(\mu(x'), \mu_{-i})] \leq \mathbb{E}\left[v^h_i\left(\mu\left(\frac{a+x_{h*}}{2}\right), \mu_{-i}\right)\right]$, then this contradicts $x' \in BR^h_{i1}(\sigma_{-i})$.
  - If $x' \in X^\ell_1$, then $\mu(x') \in [0, 1]$. If $\mu(x') \leq \mu(\frac{a+x_{h*}}{2})$, then this contradicts $x' \in BR^\ell_{i1}(\sigma_{-i})$, but if $\mu(x') \geq \mu(\frac{a+x_{h*}}{2})$, this contradicts $x' \in BR^h_{i1}(\sigma_{-i})$.

Therefore X^ℓ_c must be empty.

Similarly, define $X^h_c = \{x | x \in (x^*_\ell, x^*_h) \setminus BR^h_{i1}(\sigma_{-i})\}$ and let $x \in X^h_c$. Then $\exists \delta_\varepsilon > 0$ such that for all $x' \in X^h_{i1}$, $|x' - x| \geq \delta_\varepsilon > 0$. Take a and b to be the infimum and supremum respectively of the interval of X^h_c containing x noting that $b < x^*_h$.

There is an $x' \in X^h_{i1}$ where $|x' - b| < \delta$ for all $\delta > 0$. Then we can take $x' \in BR^h_{i1}(\sigma_{-i})$ such that $p_1(F^*_1(x') - F^*_1(b)) < c(\frac{b}{a^h}) - c(\frac{b+a}{2a^h})$. This implies $p_1(F^*_1(x') - F^*_1(\frac{b+a}{2})) < c(\frac{x'}{a^h}) - c(\frac{b+a}{2a^h})$ and

$$p_1 F^*_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a^h}\right) + \mathbb{E}\left[v^h_i\left(\mu\left(\frac{b+a}{2}\right), \mu_{-i}\right)\right]$$
$$> p_1 F^*_1(x') - c\left(\frac{x'}{a^h}\right) + \mathbb{E}[v^h_i(\mu(x'), \mu_{-i})].$$

This contradicts $x' \in BR^h_{i1}(\sigma_{-i})$, and therefore X^h_c must be empty.

(4) Lastly, we show that $x_{h*} < x^*_\ell$, and for all $x \in (x_{h*}, x^*_\ell)$, $x \in BR^\ell_{i1}(\sigma_{-i}) \cap BR^h_{i1}(\sigma_{-i})$.

If $x^*_\ell = x_{h*}$, then $\forall \delta > 0$, there is $x_\ell \in X^\ell_{i1}$ and $x_h \in X^h_{i1}$ where $|x_h - x_\ell| < \delta$. By the continuity of $F^*_1(x)$ and $c(x)$, there is x_h and x_ℓ for which

$$p_1 F^*_1(x_h) - c\left(\frac{x_h}{a^h}\right) - \left(p_1 F^*_1(x_\ell) - c\left(\frac{x_\ell}{a^h}\right)\right) < \mathbb{E}[v^h_i(\mu(x_\ell), \mu_{-i})] - \mathbb{E}[v^h_i(\mu(x_h), \mu_{-i})]$$

since $\mu(x_\ell) = 0$, $\mu(x_h) = 1$, and $\mathbb{E}[v^h_i(0, \mu_{-i})] - \mathbb{E}[v^h_i(1, \mu_{-i})] > 0$. Then

$$p_1 F^*_1(x_\ell) - c\left(\frac{x_\ell}{a^h}\right) + \mathbb{E}[v^h_i(\mu(x_\ell), \mu_{-i})] > p_1 F^*_1(x_h) - c\left(\frac{x_h}{a^h}\right) + \mathbb{E}[v^h_i(\mu(x_h), \mu_{-i})],$$

which contradicts $x_h \in BR^h_{i1}(\sigma_{-i})$.

Define $X_c = \{x | x \in (x_{h*}, x^*_\ell) \setminus (BR^\ell_{i1}(\sigma_{-i}) \cup BR^h_{i1}(\sigma_{-i}))\}$. From Lemma A.3, we know that for all $x' \in \{(x_{h*}, x^*_\ell) \cap (X^\ell_{i1} \cup X^h_{i1})\}$, $\mu(x') \in (0,1)$ as $\mu(x') = 1$, implies $x^*_\ell \leq x'$ and $\mu(x') = 0$ implies $x_{h*} \geq x'$. Therefore $x' \in X^\ell_{i1} \cap X^h_{i1}$.

Let $x \in X_c$ be given. Then for all $x', x'' \in \{(x_{h*}, x^*_\ell) \cap (X^\ell_{i1} \cap X^h_{i1})\}$ such that $x' < x < x''$ we must by Lemma A.3 have $\mu(x') \leq \mu(x'')$. Let $\mu^* \in [\sup\{\mu(x')\}, \inf\{\mu(x'')\}]$. These are well-defined as there is at least one such x' and x''.

If $\mu(x) \geq \mu^*$ then $\mathbb{E}[v^\ell_i(\mu(x), \mu_{-i})] \geq \mathbb{E}[v^\ell_i(\mu(x'), \mu_{-i})]$ for all x' and

$$\begin{aligned} p_1 F^*_1(x') - c(x') + \mathbb{E}[v^\ell_i(\mu(x'), \mu_{-i})] - \varepsilon_1 &> p_1 F^*_1(x) - c(x) + \mathbb{E}[v^\ell_i(\mu(x), \mu_{-i})] \\ \Rightarrow p_1 F^*_1(x') - c(x') - \varepsilon_1 &> p_1 F^*_1(x) - c(x). \end{aligned}$$

By continuity of $F^*_1(x)$ and $c(x)$, $\exists \delta_1 > 0$ such that $[x - \delta_1, x] \subset X_c$.

Similarly, if $\mu(x) < \mu^*$, then $\mathbb{E}[v^h_i(\mu(x), \mu_{-i})] \geq \mathbb{E}[v^h_i(\mu(x''), \mu_{-i})]$ for all x'' and

$$p_1 F^*_1(x'') - c\left(\frac{x''}{a^h}\right) - \varepsilon_2 > p_1 F^*_1(x) - c\left(\frac{x}{a^h}\right).$$

By continuity, $\exists \delta_2 > 0$ such that $[x, x + \delta_2] \subset X_c$. In either case, if $x \in X_c$, then there is an interval with some supremum b and infimum a such that $x \in (a, b) \subset X_c$.

If $b < x^*_\ell$, then there is an $\tilde{x} \in \{(x_{h*}, x^*_\ell) \cap X^\ell_{i1} \cap X^h_{i1}\}$ where $|\tilde{x} - b| < \delta$ for all $\delta > 0$, and therefore there is an $\tilde{x}$ where $p_1(F^*_1(\tilde{x}) - F^*_1(b)) < c(b/a^h) - c(\frac{b+a}{2a^h})$. It follows that $p_1(F^*_1(\tilde{x}) - F^*_1(\frac{b+a}{2})) < c(\tilde{x}/a^h) - c(\frac{b+a}{2a^h})$ and $p_1(F^*_1(\tilde{x}) - F^*_1(\frac{b+a}{2})) < c(\tilde{x}) - c(\frac{b+a}{2})$.

If $\mu((b+a)/2) < \mu(\tilde{x})$ then

$$\begin{aligned} p_1 F^*_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a^h}\right) + \mathbb{E}\left[v^h_i\left(\mu\left(\frac{b+a}{2}\right), \mu_{-i}\right)\right] & \\ > p_1 F^*_1(\tilde{x}) - c\left(\frac{\tilde{x}}{a^h}\right) + \mathbb{E}[v^h_i(\mu(\tilde{x}), \mu_{-i})]&. \end{aligned}$$

If $\mu((b+a)/2) \geq \mu(\tilde{x})$ then

$$p_1 F^*_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2}\right) + \mathbb{E}\left[v^\ell_i\left(\mu\left(\frac{b+a}{2}\right), \mu_{-i}\right)\right] > p_1 F^*_1(\tilde{x}) - c(\tilde{x}) + \mathbb{E}[v^\ell_i(\mu(\tilde{x}), \mu_{-i})].$$

In either case, this contradicts $\tilde{x} \in X^{\ell}_{i1} \cap X^h_{i1}$.

If $b = x^*_\ell$, then there is an $\tilde{x} \in X^h_{i1}$, such that $|\tilde{x} - b| < \delta$, and $\mu(\tilde{x}) = 1$. This implies that $p_1(F^*_1(\tilde{x}) - F^*_1(\frac{b+a}{2})) < c(\tilde{x}/a^h) - c(\frac{b+a}{2a^h})$, and

$$p_1 F^*_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a^h}\right) + \mathbb{E}\left[v^h_i\left(\mu\left(\frac{b+a}{2}\right), \mu_{-i}\right)\right] > p_1 F^*_1(\tilde{x}) - c\left(\frac{\tilde{x}}{a^h}\right) + \mathbb{E}[v^h_i(\mu(\tilde{x}), \mu_{-i})].$$

This contradicts $\tilde{x} \in X^h_{i1}$. Therefore X_c must be empty and for all $x \in (x_{h*}, x^*_\ell)$, we must have $x \in BR^\ell_{i1}(\sigma_{-i}) \cap BR^h_{i1}(\sigma_{-i})$. □

Lemma A.6. *In any SPBE, the belief function is continuous in output on $[0, x^*_h]$, is weakly increasing on (x_{h*}, x^*_ℓ), takes a value of zero for all $x \in [0, x_{h*}]$ when $x_{h*} > 0$, and takes a value of one for all $x \in [x^*_\ell, x^*_h]$ when $x^*_h > x^*_\ell$.*

Proof. To show that $\mu^*(x)$ is continuous on $(0, x^*_h)$, note that equilibrium expected payoffs of a low ability contestant are constant for all $x \in BR^\ell_{i1}(\sigma^*)$ and likewise for high ability contestants for all $x \in BR^h_{i1}(\sigma^*)$. Since $F^*_1(x)$ and $c(x)$ are continuous on $(0, \infty)$ and $\mathbb{E}[v^\ell_i(\mu^*(x), \mu_{-i})] = c(x) - p_1 F^*_1(x) + K^\ell(p_1, p_2)$ on $[0, x^*_\ell]$, then $\mathbb{E}[v^\ell_i(\mu^*(x), \mu_{-i})]$ must be continuous on this interval. Similarly, $\mathbb{E}[v^h_i(\mu^*(x), \mu_{-i})]$ is continuous on $[x_{h*}, x^*_h]$. Since $\mathbb{E}[v^h_i(\mu^*(x), \mu_{-i})]$ is strictly decreasing in $\mu^*(x)$, and $\mathbb{E}[v^\ell_i(\mu^*(x), \mu_{-i})]$ is strictly increasing in $\mu^*(x)$, then $\mu^*(x)$ must also be continuous on $BR^\ell_{i1}(\sigma^*) \cup BR^h_{i1}(\sigma^*) = [0, x^*_h]$.

To show the remaining properties of the equilibrium belief function, we first show that the set $[0, x^*_h] \setminus X_{i1}$ has no interior, i.e. there can be no interval $[a, b] \subset [0, x^*_h]$ where for all $x \in [a, b]$, $x \notin X_{i1}$. This implies that X_{i1} is dense in $[0, x^*_h]$.

If we let $[\tilde{a}, \tilde{b}] \subset [0, x^*_h] \setminus X_{i1}$ be given, then define a and b to be the infimum and supremum respectively of the interval in $[0, x^*_h] \setminus X_{i1}$ which contains $[\tilde{a}, \tilde{b}]$. Neither x_{h*} nor x^*_ℓ can be contained in the interval as they are the limit point of a subset of X_{i1}. Then the interval $[a, b]$ must be contained within either $[0, x_{h*}]$, $[x_{h*}, x^*_\ell]$, or $[x^*_\ell, x^*_h]$.

1. If $[a, b] \subset [0, x_{h*}]$, then for all $x \in [a, b]$, $x \in BR^\ell_{i1}(\sigma^*)$ and $F^*_1(x) = F^*_1(a)$. Therefore, $\mathbb{E}[v^\ell_i(\mu^*(b), \mu_{-i})] - c(b) = \mathbb{E}[v^\ell_i(\mu^*(a), \mu_{-i})] - c(a)$, which implies that $\mu(b) > \mu(a)$. Since $\mu(x)$ is continuous, then for all $\delta > 0$, there is an $x \in X^h_{i1}$ such that $|x - b| < \delta$ and $\mu(x) > 0$. If $x \in X^h_{i1} \setminus X^\ell_{i1}$, then $\mu^*(x)$=1, and $x \notin BR^h_{i1}(\sigma^*)$, a contradiction. If $x \in X^h_{i1} \cap X^\ell_{i1}$ then depending on the value of $\mu^*((a + b)/2)$, it must be that either $x \notin BR^h_{i1}(\sigma^*)$ or $x \notin BR^\ell_{i1}(\sigma^*)$, again a contradiction.

2. If $[a, b] \subset [x_{h*}, x^*_\ell]$, then for all $x \in [a, b]$, $x \in \{BR^\ell_{i1}(\sigma^*) \cap BR^h_{i1}(\sigma^*)\}$ which implies

$$\mathbb{E}[v^\ell_i(\mu^*(b), \mu_{-i})] - c(b) = \mathbb{E}[v^\ell_i(\mu^*(a), \mu_{-i})] - c(a),$$
$$\mathbb{E}[v^h_i(\mu^*(b), \mu_{-i})] - c(b/a^h) = \mathbb{E}[v^h_i(\mu^*(a), \mu_{-i})] - c(a/a^h).$$

However, rearranging these equations, it is clear they cannot hold at the same time as the right hand sides are both strictly positive which contradicts Proposition A.1.

$$\mathbb{E}[v_i^\ell(\mu^*(b), \mu_{-i})] - \mathbb{E}[v_i^\ell(\mu^*(a), \mu_{-i})] = c(b) - c(a)$$
$$\mathbb{E}[v_i^h(\mu^*(b), \mu_{-i})] - \mathbb{E}[v_i^h(\mu^*(a), \mu_{-i})] = c(b/a^h) - c(a/a^h)$$

3. If $[a,b] \subset [x_\ell^*, x_h^*]$, then for all $x \in [a,b]$, $x \in BR_{i1}^h(\sigma^*)$ and therefore,
$$\mathbb{E}[v_i^h(\mu^*(b), \mu_{-i})] - c(b/a^h) = \mathbb{E}[v_i^h(\mu^*(a), \mu_{-i})] - c(a/a^h),$$
and $\mu^*(b) < \mu^*(a) \leq 1$. Then for all $\delta > 0$, there is an $x \in X_{i1}^h$ such that $|x - b| < \delta$ and $\mu^*(x) = 1$. However, this contradicts the continuity of $\mu^*(x)$.

Now, if $x \in [0, x_{h*})$ and $\mu^*(x) = \varepsilon > 0$, then by the continuity of $\mu^*(x)$, $\exists \delta > 0$ where $\forall x', |x' - x| < \delta$, $\mu^*(x) > \varepsilon/2$. However for all $\delta > 0$ there is an $x' \in X_{i1}^\ell \setminus X_{i1}^h$ for which $\mu^*(x') = 0$, a contradiction. Therefore $\mu^*(x) = 0$ for all $x \in [0, x_{h*})$. Note that $\mu^*(x_{h*}) = 0$ when $x_{h*} > 0$, which follows from continuity from the left. Similarly, $\mu^*(x) = 1$ for all $x \in [x_\ell^*, x_h^*]$ when $x_\ell^* < x_h^*$. To show that $\mu^*(x)$ is weakly increasing on $[x_{h*}, x_\ell^*]$, let $x, y \in [x_{h*}, x_\ell^*]$ be such that, $\mu^*(x) > \mu^*(y)$ and $x < y$. Then there is an x' and y' arbitrarily close to x and y respectively, where $x', y' \in X_{i1}$ and therefore $\mu^*(x') \leq \mu^*(y')$. This is not consistent with $\mu^*(x)$ being continuous on $[0, x_h^*]$, a contradiction. $\square$

Proof of Theorem 2

There are up to three distinct intervals in each equilibrium. We will show that the endpoints of these intervals and the distribution functions on the intervals are completely determined by the first order conditions of the contestants.

Conditions for x being in $BR_{i1}^h(\sigma^*)$ and $BR_{i1}^\ell(\sigma^*)$ are

$$BR_{i1}^h(\sigma^*): \; p_1 F_1^*(x) + \mathbb{E}[v_i^h(\mu^*(x), \mu_{-i})] - c\left(\frac{x}{a^h}\right) = K^h(p_1, p_2),$$
$$BR_{i1}^\ell(\sigma^*): \; p_1 F_1^*(x) + \mathbb{E}[v_i^\ell(\mu^*(x), \mu_{-i})] - c(x) = K^\ell(p_1, p_2) = 0.$$

For all values of $p_1 > 0$ and $p_2 > 0$, Lemma A.5 shows that $x_{h*} < x_\ell^*$, and therefore the interval $[x_{h*}, x_\ell^*]$ is non-trivial. On this interval, $x \in X_{i1}^\ell \cup X_{i1}^h$ implies $x \in X_{i1}^\ell \cap X_{i1}^h \subset BR_{i1}^\ell(\sigma^*) \cap BR_{i1}^h(\sigma^*)$. Subtracting the condition for $BR_{i1}^\ell(\sigma^*)$ from the condition for $BR_{i1}^h(\sigma^*)$

$$\mathbb{E}[v_i^h(\mu^*(x), \mu_{-i})] - \mathbb{E}[v_i^\ell(\mu^*(x), \mu_{-i})] = c\left(\frac{x}{a^h}\right) - c(x) + K^h(p_1, p_2).$$

Taking the derivative of each side with respect to output gives (16):

$$\frac{d\mu^*(x)}{dx} d(\mu^*(x)) = c'(x) - \frac{1}{a^h} c'\left(\frac{x}{a^h}\right).$$

Note that on this interval, $\frac{d\mu^*(x)}{dx} > 0$ and therefore, $F_\mu^*(\mu^*(x)) = F_1^*(x)$.

Taking the derivative of the condition for X_1^ℓ and combining (16) we recover (17):

$$p_1 f_1^*(x) + \frac{d\mu^*(x)}{dx} d(\mu^*(x)) F_1^*(x) = c'(x)$$
$$\Rightarrow p_1 f_1^*(x) = c'(x)(1 - F_1^*(x)) + \frac{1}{a^h} c'\left(\frac{x}{a^h}\right) F_1^*(x).$$

From continuity of $F_1^*(x)$, $p_1 F_1^*(x_{h*}) = c(x_{h*})$. For a given x_{h*}, using the Picard - Lindelöf Theorem[2], we know that there is a unique solution for $f_1^*(x)$ on $[x_{h*}, x_\ell^*]$, and therefore $F_1^*(x)$ is determined on this interval.

To see why only one such x_{h*} can lead to an equilibrium, consider a different initial condition, $p_1 \tilde{F}^*(\tilde{x}_{h*}) = c(\tilde{x}_{h*})$ where $\tilde{x}_{h*} > x_{h*}$ and the associated $\tilde{f}_1^*(x)$ on $[\tilde{x}_{h*}, \tilde{x}_\ell^*]$. Then both $\tilde{F}_1^*(\tilde{x}_{h*}) > F_1^*(\tilde{x}_{h*})$ and $\tilde{\mu}^*(\tilde{x}_{h*}) < \mu^*(\tilde{x}_{h*})$, and for all $x \in [\tilde{x}_{h*}, x_\ell^*]$, $\tilde{F}_1^*(x) > F_1^*(x)$, $\tilde{f}_1^*(x) < f_1^*(x)$, and $\mu^*(x) > \tilde{\mu}^*(x)$. This implies that $\tilde{H}_1^*(x_\ell^*) = \int_0^{x_\ell^*} \tilde{\mu}^*(x)\tilde{f}_1^*(x)dx < \int_0^{x_\ell^*} \mu^*(x) f_1^*(x)dx = H_1^*(x_\ell^*)$ and therefore $\tilde{L}_1^*(x_\ell^*) > L_1^*(x_\ell^*) = 1$, a contradiction. Similarly, there cannot be an additional equilibrium where $\tilde{x}_{h*} < x_{h*}$.

The belief function on this interval is determined up to a constant by equation (16). The constant is determined by $\mu^*(x_{h,*})$ which is 0 when $x_{h*} > 0$, and needs to be characterized in equilibrium when $x_{h*} = 0$. Given this constant, the equilibrium strategies of high ability and low ability contestants can be constructed on this interval.

For small values of p_1 relative to p_2, this is the only non-trivial interval: $x_{h*} = 0$ and $x_\ell^* = x_h^*$. In this case, $\mu^*(x_{h*}) \in [0, \hat{\mu}]$ and $\mu^*(x_h^*) \in [\hat{\mu}, 1]$ both need to be determined in equilibrium along with x_h^*. By an argument similar to that for showing x_{h*} is unique, if $x_{h*} = 0$ then $\mu^*(x_{h*})$ is also uniquely determined. Then $\mu^*(x)$ and $F_1^*(x)$ are uniquely determined on this interval, and therefore x_h^* and $\mu^*(x_h^*)$ are also uniquely determined.

For larger p_1, $x_{h*} > 0$ and/or $x_h^* > x_\ell^*$. When the intervals are non-trivial, then the belief functions on these intervals were characterized in Lemma A.6. Characterization of the output distributions directly follow. For $x \in [0, x_{h*})$, $\mathbb{E}[v_i^\ell(\mu^*(x), \mu_{-i})] = 0$ as $\mu^*(x) = 0$, and therefore $p_1 F_1^*(x) = c(x)$. For all $x \in [x_\ell^*, x_h^*]$, $\mathbb{E}[v_i^h(\mu^*(x), \mu_{-i})] = \mathbb{E}[v_i^h(1, \mu_{-i})]$ and $p_1 F_1^*(x) + \mathbb{E}[v_i^h(1, \mu_{-i})] = c(x/a^h) + K^h(p_1, p_2)$.

Given $F_1^*(x)$ and $\mu^*(x)$ on $[0, x^*]$, the output distribution of both the low and high ability contestants can be determined. Therefore $F_1^*(x), L_1^*(x)$ and $H_1^*(x)$ are uniquely characterized on X_{i1} where $\overline{X}_{i1} = [0, x_h^*]$. These distributions along with the second period output distributions $L_{i2}^*(x|\eta_2)$ and $H_{i2}^*(x|\eta_2)$ form the unique SPBE.

Proposition A.3. *Let $F_\mu^*(M)$ be the equilibrium belief distribution associated with prize ratio p_1/p_2 and $\tilde{F}_\mu^*(M)$ be associated with $\tilde{p}_1/\tilde{p}_2$. Then $p_1/p_2 > \tilde{p}_1/\tilde{p}_2$ implies $F_\mu^*(M) <_{SOSD} \tilde{F}_\mu^*(M)$.*

Proof. Belief distributions that arise after the first contest for different prize structures must be equal at least at one point. If the distributions do not cross then one distribution FOSD the other and the distributions cannot have the same expected value. However, the expectation of the probability that a contestant is high ability is $\hat{\mu}$ in either case.

Let $\tilde{\mu}(\hat{\tilde{x}}) = \mu(\hat{x}) = \hat{M}$ be a point of intersection for belief distributions $\tilde{F}_\mu(M)$ and $F_\mu(M)$. Note that

$$f_\mu(\hat{M}) = \frac{\partial}{\partial \mu} F_1(\mu^{-1}(\hat{M})) = \frac{f_1(\mu^{-1}(\hat{M}))}{\mu'(\mu^{-1}(\hat{M}))} = \frac{f_1(\hat{x})}{\mu'(\hat{x})}.$$

[2]The right hand side of equation (17) is continuous in x and uniformly Lipshitz continuous in $F_1^*(x)$ on the interval of $[x_{h*}, x_\ell^*]$. Also, due to the properties of the cost function, the distribution function is bounded between 0 and 1.

From equations (16) and (17),

$$\frac{f_1(\hat{x})}{\mu'(\hat{x})} = \frac{d(\mu(\hat{x}))\left(c'(x) - F_1(\hat{x})\left(c'(x) - \frac{1}{a_h}c'\left(\frac{\hat{x}}{a_h}\right)\right)\right)}{p_1\left(c'(x) - \frac{1}{a_h}c'\left(\frac{\hat{x}}{a_h}\right)\right)} = \frac{d(\mu(\hat{x}))}{p_1}\left(\frac{a_h^\alpha}{a_h^\alpha - 1} - F_1(\hat{x})\right)$$
$$= \frac{p_2}{p_1}\left(\frac{a_h^\alpha(1 - F_1(\hat{x})) + F_1(\hat{x})}{a_h^\alpha}\right)$$

Because $\tilde{F}_1(\tilde{\mu}^{-1}(\hat{M})) = F_1(\mu^{-1}(\hat{M}))$, then $\tilde{f}_\mu(\hat{M}) \lesseqgtr f_\mu(\hat{M})$ when $\frac{\tilde{p}_2}{\tilde{p}_1} \lesseqgtr \frac{p_2}{p_1}$. Given that $\tilde{f}_\mu(\hat{M}) < f_\mu(\hat{M})$, as in the case when first contest prize is increased for a fixed second contest prize, this implies that $\tilde{F}_\mu(M)$ crosses $F_\mu(M)$ exactly once from above and $\tilde{F}_\mu(M) <_{SOSD} F_\mu(M)$. An increase the second contest prize for fixed first contest prize implies $\tilde{F}_\mu(M)$ crosses $F_\mu(M)$ exactly once from below and $\tilde{F}_\mu(M) >_{SOSD} F_\mu(M)$. $\square$

Equilibrium construction

Assume the cost function takes the form, $c(x) = kx^\alpha$, with $\alpha \geq 1$ and $k > 0$ and let $\hat{\mu} = 1/2$. For the equilibrium of the first contest we find the ex-ante expected distribution of each contestant over each of the potential three ranges of output which depend on the values of p_1 and p_2. Let $A = \frac{a^{h^\alpha}}{a^{h^\alpha}-1}$.

For any values of p_1 and p_2, $x_{h*} < x_\ell^*$. For $x \in [x_{h*}, x_\ell^*]$ the expected output distribution satisfies equation (17). The family of solutions is

$$F_1^*(x) = Be^{(c(x/a^h) - c(x))/p_1} + A,$$

with boundary condition $F_1^*(x_{h*}) = \frac{1}{p_1}kx_{h*}^\alpha$. The solution is

$$F_1^*(x) = A - \left(A - \frac{1}{p_1}kx_{h*}^\alpha\right)e^{-\frac{1}{Ap_1}(kx^\alpha - kx_{h*}^\alpha)}.$$

The belief function satisfies the condition in equation (16) which simplifies under this parameterization to $p_2\mu'(x) = c'(x)$. The belief function is $\mu^*(x) = \frac{1}{p_2}\left(kx^\alpha + C\right)$, where $C = -c(x_{h*})$ if $x_{h*} > 0$ and $C = p_2\mu^*(x_{h*})$ if $x_{h*} = 0$. Therefore

$$F_1^*(x) = A - \left(A - \frac{1}{p_1}kx_{h*}^\alpha\right)e^{-\frac{p_2}{Ap_1}(\mu^*(x) - \mu^*(x_{h*}))},$$

where $\mu^*(x_{h*}) = 0$ when $x_{h*} > 0$.

If $x_{h*} > 0$, then $F_1^*(x) = \frac{1}{p_1}kx^\alpha$ and $\mu^*(x) = 0$ for $x \in [0, x_{h*}]$. If $x_\ell^* < x_h^*$, then $F_1^*(x) = \frac{1}{p_1}\left(\frac{1}{a^{h^\alpha}}kx^\alpha + K^h(p_1, p_2) - \mathbb{E}[v_i^h(1, \mu_{-i})]\right)$ and $\mu^*(x) = 1$ for $x \in [x_\ell^*, x_h^*]$.

Given $F_1^*(x)$ and $\mu^*(x)$, the output distribution of the both the high and low ability contestants comes from using $2F_1^*(x) = L_1^*(x) + H_1^*(x)$ and $\mu^*(x) = \frac{h_1^*(x)}{\ell_1^*(x) + h_1^*(x)}$.

Over the range $x \in [x_{h,*}, x_\ell^*]$ these distributions are

$$H_1^*(x) = 2(A - \frac{1}{p_1}kx_{h,*}^\alpha)\left(\frac{Ap_1}{p_2} - (\mu^*(x) + \frac{Ap_1}{p_2})e^{-\frac{p_2}{Ap_1}(\mu^*(x)-\mu^*(x_{h,*}))}\right) + 2A\mu^*(x_{h,*}) \text{ and}$$

$$L_1^*(x) = 2A(1-\mu^*(x_{h,*})) + 2(A - \frac{1}{p_1}kx_{h,*}^\alpha)\left((\mu^*(x) + \frac{Ap_1}{p_2} - 1)e^{-\frac{p_2}{Ap_1}(\mu^*(x)-\mu^*(x_{h,*}))} - \frac{Ap_1}{p_2}\right).$$

Small prize in first contest

Given p_2, for p_1 close enough to 0 (specifically for $p_1 < \frac{p_2}{2}((A^2 - A)\log(a^{h^\alpha}) - A)^{-1})$, both $x_{h*} = 0$ and $x_\ell^* = x_h^*$. The expected output distribution becomes

$$F_1^*(x) = A\left(1 - e^{-\frac{p_2}{Ap_1}(\mu^*(x)-\mu^*(x_{h*}))}\right), \text{ for } 0 \leq x \leq x_h^*.$$

Given $H_1^*(x_{h*}) = F_1^*(x_{h*}) = F_1^*(0) = 0$, the output distribution of the high ability contestant is

$$H_1^*(x) = \int_0^x \mu^*(t)f_1^*(t)dt = 2F_1^*(x)\left(\mu^*(x) + \frac{Ap_1}{p_2}\right) - 2A(\mu^*(x) - \mu^*(x_{h*})).$$

Combining $F_1^*(x_h^*) = 1$ and $H_1(x_h^*) = 1$ gives

$$\mu^*(x_h^*) - \mu^*(x_{h*}) = \frac{p_1}{p_2} + \frac{2\mu^*(x_h^*) - 1}{2A}.$$

Plugging back into $F_1^*(x_h^*) = 1$, we can solve the belief function at each end point:

$$\mu^*(x_h^*) = \frac{1}{2} + \frac{p_1}{p_2}(A^2\log(a^{h^\alpha}) - A) \text{ and } \mu^*(x_{h*}) = \frac{1}{2} + \frac{p_1}{p_2}((A^2 - A)\log(a^{h^\alpha}) - A).$$

Therefore $\mu^*(x_h^*) - \mu^*(x_{h*}) = \frac{p_1}{p_2}A\log(a^{h^\alpha})$ and $k{x_h^*}^\alpha = p_1 A\log(a^{h^\alpha})$.

Example of parameters that fall in this category: $c(e) = e^2$, $a^h = 2$ and $p_1 = .5$ and $p_2 = 1$.

Intermediate prize in first contest

For larger p_1 compared to p_2, (specifically $p_1 > \frac{p_2}{2}((A^2 - A)\log(a^{h^\alpha}) - A)^{-1})$, then $H_1^*(x_\ell^*) < 1$ and $x_\ell^* < x_h^*$. The expected output distribution is

$$F_1^*(x) = \begin{cases} A\left(1 - e^{\frac{-p_2}{Ap_1}(\mu^*(x)-\mu^*(x_{h*}))}\right) & 0 \leq x \leq x_\ell^* \\ \frac{1}{p_1}(\frac{k}{a^{h^\alpha}}x^\alpha + K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]) & x_\ell^* \leq x \leq x_h^* \end{cases}.$$

The output distributions of the high and low ability contestants are

$$H_1^*(x) = 2F_1^*(x)\left(\mu^*(x) + \frac{Ap_1}{p_2}\right) - 2A(\mu^*(x) - \mu^*(x_{h*})) \text{ and}$$

$$L_1^*(x) = 2F_1^*(x)\left(1 - \mu^*(x) - \frac{Ap_1}{p_2}\right) + 2A(\mu^*(x) - \mu^*(x_{h*})).$$

To characterize the equilibrium we need to solve for $\mu^*(0)$, x_ℓ^*, x_h^*, and $K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]$.

1. Continuity of the belief function: $\mu^*(x_\ell^*) = 1$ implies that $kx_\ell^{*\alpha} = p_2(1-\mu^*(x_{h*}))$:
2. $L_1^*(x_\ell^*) = 1$ gives an implicit equation for $\mu^*(x_{h*})$.
$$1 = 2A\left((1-\mu^*(x_{h*})) - \frac{Ap_1}{p_2}\left(1 - e^{-\frac{p_2}{Ap_1}(1-\mu^*(x_{h*}))}\right)\right)$$
3. By continuity of $F_1^*(x)$ at x_ℓ^*, we can find $K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]$:
$$A - Ae^{-\frac{p_2}{Ap_1}(1-\mu^*(x_{h*}))} = \frac{1}{p_1}\left(\frac{k}{a^{h^\alpha}}x_\ell^{*\alpha} + K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]\right).$$
Using the equation that determines $\mu^*(x_{h*})$ and the belief equations, this simplifies to
$$K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})] = \frac{p_2}{A}\left(\frac{1}{2} - \mu(x_{h*})\right).$$
4. From $F_1^*(x_h^*) = 1$ we can find x_h^*:
$$kx_h^{*\alpha} = p_1 a^{h^\alpha} - p_2(a^{h^\alpha} - 1)(1/2 - \mu^*(x_{h*})).$$

Example of parameters that fall in this category: $c(e) = e^2$, $a^h = 2$ and $p_1 = 0.8$ and $p_2 = 1$.

Large prize in first contest

For large enough p_1, all three intervals are non-trivial, $\mu^*(x_{h*}) = 0$ and $\mu^*(x_\ell^*) = 1$. The distribution functions are

$$F_1^*(x) = \begin{cases} \frac{1}{p_1}kx^\alpha & 0 \le x \le x_{h*} \\ A - (A - \frac{1}{p_1}kx_{h*}^\alpha)e^{\frac{-p_2}{Ap_1}\mu^*(x)} & x_{h*} \le x \le x_\ell^* \\ \frac{1}{p_1}(\frac{k}{a^{h^\alpha}}x^\alpha + K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]) & x_\ell^* \le x \le x_h^* \end{cases},$$

$$L_1^*(x) = \begin{cases} \frac{2}{p_1}kx^\alpha, & 0 \le x \le x_{h*} \\ 2A + 2(A - \frac{1}{p_1}kx_{h*}^\alpha)\left((\mu^*(x) + \frac{Ap_1}{p_2} - 1)e^{-\frac{p_2}{Ap_1}\mu^*(x)} - \frac{Ap_1}{p_2}\right), & x_{h*} \le x \le x_\ell^* \\ 1, & x_\ell^* \le x \le x_h^* \end{cases}, \text{ and}$$

$$H_1^*(x) = \begin{cases} 0, & 0 \le x \le x_{h*} \\ 2(A - \frac{1}{p_1}kx_{h*}^\alpha)\left(\frac{Ap_1}{p_2} - (\mu^*(x) + \frac{Ap_1}{p_2})e^{-\frac{p_2}{Ap_1}\mu^*(x)}\right), & x_{h*} \le x \le x_\ell^* \\ \frac{2}{p_1}(\frac{k}{a_h^\alpha}x^\alpha + K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]) - 1 & x_\ell^* \le x \le x_h^* \end{cases}.$$

Using $\mu^*(x_\ell^*) = 1$ and $L_1^*(x_\ell^*) = 1$ identifies the endpoints of the middle interval:

$$kx_{h*}^\alpha = p_1\left(A - \frac{(2A-1)p_2}{2Ap_1(1-e^{-\frac{p_2}{Ap_1}})}\right) \text{ and } kx_\ell^{*\alpha} = p_2 + p_1\left(A - \frac{(2A-1)p_2}{2Ap_1(1-e^{-\frac{p_2}{Ap_1}})}\right).$$

Continuity of the expected output distribution at x_ℓ^* gives

$$\frac{1}{p_1}\left(\frac{1}{a^{h^\alpha}}k(x_\ell^*)^\alpha + K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})]\right) = A - \left(A - \frac{1}{p_1}kx_{h*}^\alpha\right)e^{-\frac{p_2}{Ap_1}}.$$

Then the constant associated with the third interval is

$$K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})] = p_1 - \frac{p_2}{a_h^\alpha}\left(1 + \frac{2A-1}{2A}\frac{a^{h^\alpha}e^{-\frac{p_2}{Ap_1}} - 1}{1-e^{-\frac{p_2}{Ap_1}}}\right).$$

Using $F_1^*(x_h^*) = 1$ the endpoint of the upper interval is characterized by

$$kx_h^{*\alpha} = a^{h^\alpha}\left(p_1 - (K^h(p_1,p_2) - \mathbb{E}[v_i^h(1,\mu_{-i})])\right) = p_2\left(1 + \left(\frac{2A-1}{2A}\right)\frac{a^{h^\alpha}e^{-\frac{p_2}{Ap_1}} - 1}{1-e^{-\frac{p_2}{Ap_1}}}\right).$$

Example of parameters that fall in this category: $c(e) = e^2$, $a^h = 2$ and $p_1 = 1$ and $p_2 = 1$.

Second Stage Contest

As derived in the proof of Proposition 7, for given beliefs μ_w and μ_s, the expected output distribution of the weak and strong contestants are

$$L_{s2}^*(x) = \begin{cases} \frac{kx^\alpha}{p_2(1-\mu_s)}, & 0 \le x \le x_s^* \\ 1, & x_s^* \le x \le x^* \end{cases}, \quad H_{s2}^*(x) = \begin{cases} 0, & 0 \le x \le x_s^* \\ \frac{kx^\alpha - kx_s^{*\alpha}}{p_2\mu_s}, & x_s^* \le x \le x_w^* \\ 1 - \frac{kx^{*\alpha} - kx^\alpha}{a^{h^\alpha}p_2\mu_w}, & x_w^* \le x \le x^* \end{cases},$$

$$L_{w2}^*(x) = \begin{cases} \frac{kx^\alpha}{p_2(1-\mu_w)} + \frac{\mu_s - \mu_w}{1-\mu_w}\left(\frac{a^{h^\alpha}-1}{a^{h^\alpha}}\right), & 0 \le x \le x_s^* \\ 1 - \frac{kx_w^{*\alpha} - kx^\alpha}{a^{h^\alpha}p_2(1-\mu_w)}, & x_s^* \le x \le x_w^* \\ 1, & x_w^* \le x \le x^* \end{cases}, \text{ and}$$

$$H_{w2}^*(x) = \begin{cases} 0, & 0 \le x \le x_w^* \\ 1 - \frac{kx^{*\alpha} - kx^\alpha}{a^{h^\alpha}p_2\mu_w}, & x_w^* \le x \le x^* \end{cases}.$$

The expected output distributions are characterized by

$$F_{s2}^*(x) = \begin{cases} \frac{k}{p_2}x^\alpha, & 0 \le x \le x_w^* \\ 1 - \frac{kx^{*\alpha} - kx^\alpha}{a^{h^\alpha}p_2}, & x_w^* \le x \le x^* \end{cases} \quad \text{and}$$

$$F_{w2}^*(x) = \begin{cases} \frac{k}{p_2}x^\alpha + \left(\frac{a^{h^\alpha}-1}{a^{h^\alpha}}\right)(\mu_s - \mu_w), & 0 \le x \le x_s^* \\ 1 - \frac{kx^{*\alpha} - kx^\alpha}{a^{h^\alpha}p_2}, & x_s^* \le x \le x^* \end{cases},$$

where

$$
\begin{aligned}
k{x_w^*}^{\alpha} &= p_2(1-\mu_w), \\
k{x_s^*}^{\alpha} &= p_2(1-\mu_s), \text{ and} \\
k{x^*}^{\alpha} &= p_2(1-\mu_w) + p_2\mu_w {a^h}^{\alpha}.
\end{aligned}
$$