Online appendix for **Capital Controls for Crisis Management in a Global Economy**

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A   Appendix

In this appendix we first present the full list of the 35 variables and equations in the model. We then solve the planner's problem for the optimal τ.

A.1   Full list of model variables and equations

There are 35 variables in the model: $c^b_{2,X}$, $c^b_{1,X}$, $c^b_{1,Y}$, $c^s_{2,X}$, $c^s_{1,X}$, $c^s_{1,Y}$, $c^{b*}_{2,X}$, $c^{b*}_{1,X}$, $c^{b*}_{1,Y}$, $c^{s*}_{2,X}$, $c^{s*}_{1,X}$, $c^{s*}_{1,Y}$, R, R^*, B^b_1, B^s_1, B^{b*}_1, B^{s*}_1, F_1, F^*_1, λ^b_1, λ^b_2, λ^s_1, λ^s_2, λ^{b*}_1, λ^{b*}_2, λ^{s*}_1, λ^{s*}_2, μ, μ^*, p, p^*, T, T^*, R^W_1

The 35 model equations are given by:

$$c^b_{2,X} : \left(c^b_{2,X}\right)^{-\sigma} = \lambda^b_2$$

$$c^b_{1,X} : \left(\frac{1}{\alpha^\alpha \left(1-\alpha\right)^{1-\alpha}} \left(c^b_{1,X}\right)^\alpha \left(c^b_{1,Y}\right)^{1-\alpha}\right)^{1-\sigma} \alpha \left(c^b_{1,X}\right)^{-1} = \lambda^b_1$$

$$c^b_{1,Y} : \left(\frac{1}{\alpha^\alpha \left(1-\alpha\right)^{1-\alpha}} \left(c^b_{1,X}\right)^\alpha \left(c^b_{1,Y}\right)^{1-\alpha}\right)^{1-\sigma} \left(1-\alpha\right) \left(c^b_{1,Y}\right)^{-1} = p\lambda^b_1$$

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$$c_{2,X}^s : \left(c_{2,X}^s\right)^{-\sigma} = \lambda_2^s$$

$$c_{1,X}^s : \left(\frac{1}{\alpha^\alpha \left(1-\alpha\right)^{1-\alpha}} \left(c_{1,X}^s\right)^\alpha \left(c_{1,Y}^s\right)^{1-\alpha}\right)^{1-\sigma} \alpha \left(c_{1,X}^s\right)^{-1} = \lambda_1^s$$

$$c_{1,Y}^s : \left(\frac{1}{\alpha^\alpha \left(1-\alpha\right)^{1-\alpha}} \left(c_{1,X}^s\right)^\alpha \left(c_{1,Y}^s\right)^{1-\alpha}\right)^{1-\sigma} \left(1-\alpha\right) \left(c_{1,Y}^s\right)^{-1} = p\lambda_1^s$$

$$c_{2,X}^{b*} : \left(c_{2,X}^{b*}\right)^{-\sigma} = \lambda_2^{b*}$$

$$c_{1,X}^{b*} : \left(\frac{1}{\alpha^\alpha \left(1-\alpha\right)^{1-\alpha}} \left(c_{1,X}^{b*}\right)^\alpha \left(c_{1,Y}^{b*}\right)^{1-\alpha}\right)^{1-\sigma} \alpha \left(c_{1,X}^{b*}\right)^{-1} = \lambda_1^{b*}$$

$$c_{1,Y}^{b*} : \left(\frac{1}{\alpha^\alpha \left(1-\alpha\right)^{1-\alpha}} \left(c_{1,X}^{b*}\right)^\alpha \left(c_{1,Y}^{b*}\right)^{1-\alpha}\right)^{1-\sigma} \left(1-\alpha\right) \left(c_{1,Y}^{b*}\right)^{-1} = p^*\lambda_1^{b*}$$

$$c_{2,X}^{s*} : \left(c_{2,X}^{s*}\right)^{-\sigma} = \lambda_2^{s*}$$

$$c_{1,X}^{s*} : \left(\frac{1}{\alpha^\alpha \left(1-\alpha\right)^{1-\alpha}} \left(c_{1,X}^{s*}\right)^\alpha \left(c_{1,Y}^{s*}\right)^{1-\alpha}\right)^{1-\sigma} \alpha \left(c_{1,X}^{s*}\right)^{-1} = \lambda_1^{s*}$$

$$c_{1,Y}^{s*} : \left(\frac{1}{\alpha^\alpha \left(1-\alpha\right)^{1-\alpha}} \left(c_{1,X}^{s*}\right)^\alpha \left(c_{1,Y}^{s*}\right)^{1-\alpha}\right)^{1-\sigma} \left(1-\alpha\right) \left(c_{1,Y}^{s*}\right)^{-1} = p^*\lambda_1^{s*}$$

$$R : B_1^b = B_1^s$$

$$R^* : B_1^{b*} = B_1^{s*}$$

$$B_1^b : \frac{\lambda_1^b - \mu}{R} = \beta \lambda_2^b$$

$$B_1^s : \frac{\lambda_1^s}{R} = \beta \lambda_2^s$$

$$B_1^{b*} : \frac{\lambda_1^{b*} - \mu^*}{R^*} = \beta \lambda_2^{b*}$$

$$B_1^{s*} : \frac{\lambda_1^{s*}}{R^*} = \beta \lambda_2^{s*}$$

$$F_1 : \frac{\lambda_1^s (1 + \tau)}{R^W} = \beta \lambda_2^s$$

$$F_1^* : \frac{\lambda_1^* (1 + \tau^*)}{R^W} = \beta \lambda_2^*$$

$$\lambda_1^b : c_{1,X}^b + p c_{1,Y}^b + \frac{B_1^b}{R} = x_1^b + p y_1^b + B_0^b$$

$$\lambda_2^b : c_{2,X}^b = x_2^b + B_1^b + T$$

$$\lambda_1^s : c_{1,X}^s + p c_{1,Y}^s + \frac{B_1^s}{R} + (1 + \tau) \frac{F_1}{R^W} + \Gamma = x_1^s + p y_1^s + B_0^s + F_0$$

$$\lambda_2^s : c_{2,X}^s = x_2^s + B_1^s + F_1 - T$$

$$\lambda_1^{b*} : c_{1,X}^{b*} + p^* c_{1,Y}^{b*} + \frac{B_1^{b*}}{R^*} = x_1^{b*} + p^* y_1^{b*} + B_0^{b*}$$

$$\lambda_2^{b*} : c_{2,X}^{b*} = x_2^{b*} + B_1^{b*} + T^*$$

$$\lambda_1^{s*} : c_{1,X}^{s*} + p^* c_{1,Y}^{s*} + \frac{B_1^{s*}}{R^*} + (1 + \tau^*) \frac{F_1^*}{R^W} + \Gamma^* = x_1^{s*} + p^* y_1^{s*} + B_0^{s*} + F_0^*$$

$$\lambda_2^{s*} : c_{2,X}^{s*} = x_2^{s*} + B_1^{s*} + F_1^*$$

$$\mu : -\frac{B_1^b}{R} = \kappa \left(x_1^b + p y_1^b \right) \ \text{ or } \mu = 0$$

$$\mu^* : -\frac{B_1^{b*}}{R^*} = \kappa \left(x_1^{b*} + p^* y_1^{b*} \right) \ \text{ or } \mu^* = 0$$

$$p : c_{1,Y}^b + c_{1,Y}^s = y_1^s + y_1^b$$

$$p^* : c_{1,Y}^{b*} + c_{1,Y}^{s*} = y_1^{b*} + y_1^{s*}$$

$$T : \lambda_2^s = \lambda_2^b$$

$$T^* : \lambda_2^{s*} = \lambda_2^{b*}$$

$$R^W : F_1 = F_1^*$$

A.2 Planner's problem

The home country policymaker will choose τ to maximize the sum of home saver and borrower welfare subject to the saver and borrower budget constraints in each period and the borrower borrowing constraint. The Lagrangian is given by (where for brevity we have gone ahead and made the substitution based on the domestic bond market clearing $B_t = B_t^b = -B_t^s$ for $t = 0, 1$):

$$
\begin{aligned}
W = & \; u\left(c_{1,X}^s, c_{1,Y}^s\right) + \beta u\left(c_{2,X}^s\right) + u\left(c_{1,X}^b, c_{1,Y}^b\right) + \beta u\left(c_{2,X}^b\right) \\
& - \lambda_1^b \left(c_{1,X}^b + pc_{1,Y}^b + \frac{B_1}{R} - x_1^b - py_1^b - B_0\right) \\
& - \beta \lambda_2^b \left(c_{2,X}^b - x_2^b - B_1 - T\right) \\
& - \lambda_1^s \left(c_{1,X}^s + pc_{1,Y}^s - \frac{B_1}{R} + \frac{(1+\tau)F_1}{R^W} + \Gamma - x_1^s - py_1^s + B_0\right) \\
& - \beta \lambda_2^s \left(c_{2,X}^s - x_2^s + B_1 - F_1 + T\right) \\
& - \mu \left(-\frac{B_1}{R} - \kappa\left(x_1^b + py_1^b\right)\right)
\end{aligned}
$$

We can go ahead and make the substitution $\Gamma = -\frac{\tau F_1}{R^W}$, since the planner internalizes the fact that tax revenues are rebated back to savers lump sum.

When taking the derivative $\frac{dW}{d\tau}$, terms involving the derivatives of $c_{1,X}^s$, $c_{1,Y}^s$, $c_{2,X}^s$, $c_{1,X}^b$, $c_{1,Y}^b$, $c_{2,X}^b$, B_1 will cancel since households are already optimizing with respect to these variables. This leaves the derivatives of p, F_1, R, R^W since those variables are not internalized by households, or they are not fully internalized by households (in the case of F_1, households do not internalize the rebating of the tax revenue)

$$
\begin{aligned}
\frac{dW}{d\tau} &= \left(-\lambda_1^b\left(c_{1,Y}^b - y_1^b\right) - \lambda_1^s\left(c_{1,Y}^s - y_1^s\right)\right)\frac{dp}{d\tau} \\
&\quad + \left(\lambda_1^b - \lambda_1^s\right)\frac{B_1}{(R)^2}\frac{dR}{d\tau} + \left(-\lambda_1^s\frac{1}{RW} + \beta\lambda_2^s\right)\frac{dF_1}{d\tau} \\
&\quad + \lambda_1^s\frac{F_1}{(R^w)^2}\frac{dR^w}{d\tau} - \mu\frac{B_1}{(R)^2}\frac{dR}{d\tau} \\
&\quad + \kappa\mu y_1^b\frac{dp}{d\tau}
\end{aligned}
$$

And after some rearranging this becomes:

$$
\begin{aligned}
\frac{dW}{d\tau} &= \left(-\lambda_1^s\frac{1}{R^w} + \beta\lambda_2^s\right)\frac{dF_1}{d\tau} \\
&\quad + \beta R\left(\lambda_2^b - \lambda_2^s\right)\left(c_{1,Y}^s - y_1^s\right)\frac{dp}{d\tau} + \beta\left(\lambda_2^b - \lambda_2^s\right)\frac{B_1}{R}\frac{dR}{d\tau} \\
&\quad + \lambda_1^s\frac{F_1}{(R^w)^2}\frac{dR^w}{d\tau} \\
&\quad + \left(\kappa y_1^b - \left(c_{1,Y}^b - y_1^b\right)\right)\mu\frac{dp}{d\tau}
\end{aligned}
$$

The first line in this derivative represents the cost to τ in terms of distorting interemporal substitution. Since $\frac{dF_1}{d\tau} < 0$, raising τ makes savers hold fewer foreign bonds, the distortion to increasing τ is negative when $-\lambda_1^s\frac{1}{R^w} + \beta\lambda_2^s > 0$, and the marginal utility of consumption is high in the second period and savers want to hold more bonds F_1. The second line in this derivative represents the planner's use of τ for domestic redistribution, either by changing p, and thus affecting saver and borrower welfare due to differences in consumption and endowments of the non-traded good, $c_{1,Y}^b - y_1^b$ and $c_{1,Y}^s - y_1^s$. Or by changing R and thus affecting saver and borrower welfare given that borrowers hold a stock of bonds $B < 0$. The use of the transfer T is meant to eliminate the planner's motive to use τ for domestic redistribution. The second period transfer from savers to borrowers will maximize home country welfare by setting $\lambda_2^b = \lambda_2^s$. Thus the transfer will eliminate the first two lines of

this derivative.

The third line of this derivative represents the planner's use of τ for terms-of-trade manipulation.

And finally the fourth line in the derivative represents the crisis management motive for the use of τ. Note that this motive is only relevant when the constraint is binding and thus $\mu > 0$. By increasing τ the planner will reduce exports of the traded good in the first period and thus increase the relative price of the non-traded good, $\frac{dp}{d\tau} > 0$. This increase in the relative price of the non-traded good will lead to a loosening of the home country borrowing constraint by increasing the value of the borrower's collateral, κy_1^b. But at the same time the increase in p will either tighten or loosen the borrowing constraint depending on whether the borrower consumes more or less of the non-traded good than their endowment, $c_{1,Y}^b - y_1^b$.

The maximization condition reduces to (after factoring in the equilibrium conditions $R = \frac{R^w}{1+\tau}$ and the saver's first order condition with respect to F_1):

$$\frac{dW}{d\tau} = \frac{\lambda_1^s}{R^w} \left(\tau \frac{dF_1}{d\tau} + \frac{F_1}{R^w} \frac{dR^w}{d\tau} \right) + \left(\kappa y_1^b - \left(c_{1,Y}^b - y_1^b \right) \right) \mu \frac{dp}{d\tau} = 0$$

A.3 A domestic tax instead of a foreign tax

In the model presented in the main text, the policy instrument was a tax on the purchase of foreign bonds. Thus a $\tau > 0$ was a tax on net capital outflows. Instead let's imagine that the policy instrument was a tax on savers's purchases of the domestic bond B, and thus the policy instrument is a domestic tax and not a capital control. In this case, most of the model will remain the same, the only change is to the borrower and saver budget constraints:

For savers the budget constraint is:

$$c_{1,X}^s + p c_{1,Y}^s + (1+\tau) \frac{B_1^s}{R} + \frac{F_1}{R^W} + \Gamma = x_1^s + p y_1^s + B_0^s$$

$$c_{2,X}^s = x_2^s + B_1^s + F_1 - T$$

And the lump-sum rebate $\Gamma = -\tau \frac{B_1^s}{R}$ is returned to the saver. For borrowers the budget constraint is:

$$c_{1,X}^b + pc_{1,Y}^b + \frac{B_1^b}{R} = x_1^b + py_1^b + B_0^b$$

$$c_{2,X}^b = x_2^b + B_1^b + T$$

Obviously this change in the model will affect four first order conditions, the first order condition of home and foreign savers with respect to F_1 and the first order condition of home and foreign borrowers with respect to B_1^b. These new first order conditions are:

$$B_1^b : \frac{\lambda_1^b - \mu}{R} = \beta \lambda_2^b$$

$$B_1^{b*} : \frac{\lambda_1^{b*} - \mu^*}{R^*} = \beta \lambda_2^{b*}$$

$$B_1^s : \frac{(1+\tau)\lambda_1^s}{R} = \beta \lambda_2^s$$

$$B_1^{s*} : \frac{(1+\tau^*)\lambda_1^{s*}}{R^*} = \beta \lambda_2^{s*}$$

$$F_1 : \frac{\lambda_1^s}{R^W} = \beta \lambda_2^s$$

$$F_1^* : \frac{\lambda_1^*}{R^W} = \beta \lambda_2^*$$

Notice that iwhen the tax τ was applied to the purchase of F_1, the τ put a wedge between home and foreign saver's interest rates, but the interest rate for home country savers and borrowers was the same. When instead the tax τ is applied to the savers purchase of B the

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τ puts a wege between home saver and borrower interest rates, but home and foreign savers face the same interst rates.

In this new model, the planner's problem can be written as:

$$
\begin{aligned}
W \;=\;\; & u\left(c_{1,X}^s, c_{1,Y}^s\right) + \beta u\left(c_{2,X}^s\right) + u\left(c_{1,X}^b, c_{1,Y}^b\right) + \beta u\left(c_{2,X}^b\right) \\
& -\lambda_1^b\left(c_{1,X}^b + p c_{1,Y}^b + \frac{B_1}{R} - x_1^b - p y_1^b - B_0\right) \\
& -\beta\lambda_2^b\left(c_{2,X}^b - x_2^b - B_1 - T\right) \\
& -\lambda_1^s\left(c_{1,X}^s + p c_{1,Y}^s - \frac{(1+\tau)B_1}{R} + \frac{F_1}{R^W} + \Gamma - x_1^s - p y_1^s + B_0\right) \\
& -\beta\lambda_2^s\left(c_{2,X}^s - x_2^s + B_1 - F_1 + T\right) \\
& -\mu\left(-\frac{B_1}{R} - \kappa\left(x_1^b + p y_1^b\right)\right)
\end{aligned}
$$

Where we can go ahead and make the substitution $\Gamma = -\tau\frac{B_1^s}{R}$. Furthermore we can go ahead and make the substitution $B_t = B_t^b = -B_t^s$ for $t = 0,1$)

When taking the derivative $\frac{dW}{d\tau}$, terms involving the derivatives of $c_{1,X}^s$, $c_{1,Y}^s$, $c_{2,X}^s$, $c_{1,X}^b$, $c_{1,Y}^b$, $c_{2,X}^b$, F_1 will cancel since households are already optimizing with respect to these variables. This leaves the derivatives of p, B_1, R, R^W since those variables are not internalized by households, or they are not fully internalized by households (in the case of B_1, savers do not internalize the rebating of the tax revenue)

$$
\begin{aligned}
\frac{dW}{d\tau} \;=\;\; & \left(-\lambda_1^b\left(c_{1,Y}^b - y_1^b\right) - \lambda_1^s\left(c_{1,Y}^s - y_1^s\right)\right)\frac{dp}{d\tau} \\
& + \left(\lambda_1^b - \lambda_1^s\right)\frac{B_1}{(R)^2}\frac{dR}{d\tau} + \left(-\lambda_1^b\frac{1}{R} + \beta\lambda_2^b + \mu\frac{1}{R} + \lambda_1^s\frac{1}{R} - \beta\lambda_2^s\right)\frac{dB_1}{d\tau} \\
& + \lambda_1^s\frac{F_1}{(R^w)^2}\frac{dR^w}{d\tau} - \mu\frac{B_1}{(R)^2}\frac{dR}{d\tau} \\
& + \kappa\mu y_1^b\frac{dp}{d\tau}
\end{aligned}
$$

and after some rearranging this becomes:

$$
\begin{aligned}
\frac{dW}{d\tau} &= \left(\lambda_1^s \frac{1}{R} - \beta\lambda_2^s\right) \frac{dB_1}{d\tau} \\
&+ \left(\left((\lambda_1^b - \mu) - \lambda_1^s\right)\left(c_{1,Y}^s - y_1^s\right)\frac{dp}{d\tau} + \left((\lambda_1^b - \mu) - \lambda_1^s\right)\frac{B_1}{(R)^2}\frac{dR}{d\tau}\right. \\
&+ \lambda_1^s \frac{F_1}{(R^w)^2}\frac{dR^w}{d\tau} \\
&+ \mu\left(\kappa y_1^b - \left(c_{1,Y}^b - y_1^b\right)\right)\frac{dp}{d\tau}
\end{aligned}
$$

The second period transfer from borrowers to savers ensures that $\lambda_2^s = \lambda_2^b$, which after applying borrower and saver first order conditions with respect to B_1, when τ is small in absolute value, the transfer ensures that $\lambda_1^b - \mu \approx \lambda_1^s$. This this derivative can be further simplified to:

$$
\frac{dW}{d\tau} = \left(\lambda_1^s \frac{1}{R} - \beta\lambda_2^s\right)\frac{dB_1}{d\tau} + \lambda_1^s \frac{F_1}{(R^w)^2}\frac{dR^w}{d\tau} + \mu\left(\kappa y_1^b - \left(c_{1,Y}^b - y_1^b\right)\right)\frac{dp}{d\tau}
$$

This expression for $\frac{dW}{d\tau}$ is nearly identical to the one when the policy variable τ was a capital control applied to the purchase of foreign bonds, the only difference is that the cost of the tax τ, in terms of lost intertemporal smoothing is $\left(\lambda_1^s \frac{1}{R} - \beta\lambda_2^s\right)\frac{dB_1}{d\tau}$, whereas before it was $\left(-\lambda_1^s \frac{1}{R^w} + \beta\lambda_2^s\right)\frac{dF_1}{d\tau}$. It is also important to note the sign change from $\left(-\lambda_1^s \frac{1}{R^w} + \beta\lambda_2^s\right)\frac{dF_1}{d\tau}$ to $\left(\lambda_1^s \frac{1}{R} - \beta\lambda_2^s\right)\frac{dB_1}{d\tau}$; recall that $B_1 = B_1^h = -B_1^s$. So in the capital control model, $\frac{dF_1}{d\tau} < 0$, the saver would hold less F_1 after an increase in τ. In this version with the τ on B_1^s, $\frac{dB_1^s}{d\tau} < 0$, and the saver holds less B_1^s after an increase in τ, but that means that $\frac{dB_1}{d\tau} > 0$. After one further substitution of the saver's first-order condition with respect to B_1, total home country welfare is maximized when:

$$
\tau = \frac{RF_1}{(R^W)^2}\frac{dR^W}{dB_1} + \frac{\mu}{\lambda_1^s}R\left(\kappa y_1^b - \left(c_{1,Y}^b - y_1^b\right)\right)\frac{dp}{dB_1}
$$

Notice that this looks nearly identical to the expression for τ in the text when τ is applied to the purchase of foreign bonds. The only difference is the derivatives dR^w and dp are with respect to dB_1, not dF_1. Earlier we found that $\frac{dR^W}{dF_1} < 0$ and $\frac{dp}{dF_1} < 0$. The same logic can be used to see that $\frac{dR^W}{dB_1} < 0$ and $\frac{dp}{dB_1} < 0$ (recall that $B_1 = -B_1^s$, as B_1 gets larger, borrowers borrow less and thus savers shift from buying domestic bonds to buying foreign bonds, similarly as B_1 gets larger, borrowers borrow less, domestic absorbtion decreases, and the price p decreases). Thus when the τ is applied to domestic bonds and not foreign bonds, the sign of the optimal τ is opposite. In the expression for the optimal τ on foreign bonds, all term entered the expression with a negative sign, for the optimal τ on domestic bonds, the sign is positive.

Thus when $\mu = 0$ and the policy maker is concerned with terms of trade manipulation alone, $\tau = \frac{RF_1}{(R^W)^2} \frac{dR^W}{dB_1} > 0$ when $F_1 < 0$. When the country is a net debtor it will want to increase foreign savings to drive down the world interest rate. When the tax τ was applied to foreign bonds this was done by setting $\tau < 0$ to encourage the purchase of foreign bonds. When the tax τ is instead applied to domestic bonds, foreign savings is encouraged by setting $\tau > 0$, raising the price of domestic bonds and thus encouraging the purchase of foreign bonds.

Similarly the optimal crisis management τ has the opposite sign when taxes are applied to domestic bonds. Here $\frac{\mu}{\lambda_1^s} R \left(\kappa y_1^b - \left(c_{1,Y}^b - y_1^b \right) \right) \frac{dp}{dB_1} < 0$ when $\mu > 0$. When $\mu > 0$ the policy maker wants to discourage foreign asset purchases to increase domestic absorption. When the tax is applied to foreign bonds, this is accomplished by setting $\tau > 0$ to raise the price of foreign bonds, when the tax is applied to domestic bonds this is accomplished by setting $\tau < 0$ to lower the price of domestic bonds and thus discourage the purchase of foreign bonds.

Thus whether the tax τ is applied to domestic bonds or foreign bonds, when $F_1 < 0$ the policy maker is still torn between two motives for setting τ, the terms of trade manipulation motive, dR^W, and the crisis management motive dp.

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A.4   Borrowers also hold foreign bonds

Now let's consider the same problem as before, except where borrowing agents also can hold foreign bonds. In this case the stock of foreign bonds held by borrowers is F_1^b and the stock held by savers is F_1^s, where $F_1 = F_1 = F_1^b + F_1^s$. The borrower budget constraints and borrowing constraint will change in this new model, and this will add one first order condition to the problem, the borrower's first order condition with respect to F_1^b. The new and modified equations are:

$$\lambda_1^b : c_{1,X}^b + p c_{1,Y}^b + \frac{B_1^b}{R} + \frac{(1+\tau) F_1^b}{R^W} + \Gamma^b = x_1^b + p y_1^b + B_0^b$$

$$\lambda_2^b : c_{2,X}^b = x_2^b + B_1^b + F_1^b + T$$

$$\mu : -\frac{B_1^b}{R} - \frac{F_1^b}{R^W} = \kappa \left(x_1^b + p y_1^b \right) \text{ or } \mu = 0$$

$$F_1^b : \frac{\lambda_1^b (1+\tau) - \mu}{R^W} = \beta \lambda_2^b$$

The home country policymaker will choose τ to maximize the sum of home saver and borrower welfare subject to the saver and borrower budget constraints in each period and the borrower borrowing constraint. The Lagrangian is given by (where for brevity we have gone ahead and made the substitution based on the domestic bond market clearing $B_t = B_t^b = -B_t^s$ for $t = 0, 1$):

$$
\begin{aligned}
W \;=\;& u\left(c^s_{1,X}, c^s_{1,Y}\right) + \beta u\left(c^s_{2,X}\right) + u\left(c^b_{1,X}, c^b_{1,Y}\right) + \beta u\left(c^b_{2,X}\right) \\
& -\lambda^b_1\left(c^b_{1,X} + pc^b_{1,Y} + \frac{B_1}{R} + \frac{(1+\tau)\,F^b_1}{R^W} + \Gamma^b - x^b_1 - py^b_1 - B_0\right) \\
& -\beta\lambda^b_2\left(c^b_{2,X} - x^b_2 - B_1 - F^b_1 - T\right) \\
& -\lambda^s_1\left(c^s_{1,X} + pc^s_{1,Y} - \frac{B_1}{R} + \frac{(1+\tau)\,F^s_1}{R^W} + \Gamma^s - x^s_1 - py^s_1 + B_0\right) \\
& -\beta\lambda^s_2\left(c^s_{2,X} - x^s_2 + B_1 - F^s_1 + T\right) \\
& -\mu\left(-\frac{B_1}{R} - \frac{F^b_1}{R^W} - \kappa\left(x^b_1 + py^b_1\right)\right)
\end{aligned}
$$

We can go ahead and make the substitution $\Gamma^b = -\frac{\tau F^b_1}{R^W}$ and $\Gamma^s = -\frac{\tau F^s_1}{R^W}$, since the planner internalizes the fact that tax revenues are rebated back to savers lump sum.

When taking the derivative $\frac{dW}{d\tau}$, terms involving the derivatives of $c^s_{1,X}$, $c^s_{1,Y}$, $c^s_{2,X}$, $c^b_{1,X}$, $c^b_{1,Y}$, $c^b_{2,X}$, B_1 will cancel since households are already optimizing with respect to these variables. This leaves the derivatives of p, F^b_1, F^s_1, R, R^W since those variables are not internalized by households, or they are not fully internalized by households (in the case of F^b_1 and F^s_1, households do not internalize the rebating of the tax revenue)

$$
\begin{aligned}
\frac{dW}{d\tau} \;=\;& \left(-\lambda^b_1\left(c^b_{1,Y} - y^b_1\right) - \lambda^s_1\left(c^s_{1,Y} - y^s_1\right)\right)\frac{dp}{d\tau} \\
& + \left(\lambda^b_1 - \lambda^s_1\right)\frac{B_1}{(R)^2}\frac{dR}{d\tau} + \left(-\lambda^s_1\frac{1}{R^W} + \beta\lambda^s_2\right)\frac{dF^s_1}{d\tau} + \left(-\left(\lambda^b_1 - \mu\right)\frac{1}{R^W} + \beta\lambda^b_2\right)\frac{dF^b_1}{d\tau} \\
& + \lambda^s_1\frac{F^s_1}{(R^w)^2}\frac{dR^w}{d\tau} + \lambda^b_1\frac{F^b_1}{(R^w)^2}\frac{dR^w}{d\tau} - \mu\frac{F^b_1}{(R^w)^2}\frac{dR^w}{d\tau} - \mu\frac{B_1}{(R)^2}\frac{dR}{d\tau} \\
& + \kappa\mu y^b_1\frac{dp}{d\tau}
\end{aligned}
$$

This simplifies to:

$$\frac{dW}{d\tau} = \left(-\lambda_1^s \frac{1}{R^w} + \beta\lambda_2^s\right)\frac{dF_1^s}{d\tau} + \left(-\left(\lambda_1^b - \mu\right)\frac{1}{R^W} + \beta\lambda_2^b\right)\frac{dF_1^b}{d\tau}$$
$$+\beta R\left(\lambda_2^b - \lambda_2^s\right)\left(c_{1,Y}^s - y_1^s\right)\frac{dp}{d\tau} + \beta\left(\lambda_2^b - \lambda_2^s\right)\frac{B_1}{R}\frac{dR}{d\tau}$$
$$+\lambda_1^s \frac{F_1}{(R^w)^2}\frac{dR^w}{d\tau}$$
$$+\left(\kappa y_1^b - \left(c_{1,Y}^b - y_1^b\right)\right)\mu\frac{dp}{d\tau}$$

This simplifies to:

$$\frac{dW}{d\tau} = \left(-\lambda_1^s \frac{1}{R^w} + \beta\lambda_2^s\right)\frac{dF_1^s}{d\tau} + \left(-\left(\lambda_1^b - \mu\right)\frac{1}{R^W} + \beta\lambda_2^b\right)\frac{dF_1^b}{d\tau}$$
$$+\beta R\left(\lambda_2^b - \lambda_2^s\right)\left(c_{1,Y}^s - y_1^s\right)\frac{dp}{d\tau} + \beta\left(\lambda_2^b - \lambda_2^s\right)\frac{B_1}{R}\frac{dR}{d\tau}$$
$$+\lambda_1^s \frac{F_1^s}{(R^w)^2}\frac{dR^w}{d\tau} + \lambda_1^b \frac{F_1^b}{(R^w)^2}\frac{dR^w}{d\tau} - \mu\frac{F_1^b}{(R^w)^2}\frac{dR^w}{d\tau}$$
$$+\left(\kappa y_1^b - \left(c_{1,Y}^b - y_1^b\right)\right)\mu\frac{dp}{d\tau}$$

And since $\lambda_2^b = \lambda_2^s$, then through the first order conditions with respect to B, $\lambda_1^s = \lambda_1^b - \mu$

$$\frac{dW}{d\tau} = \tau\left(\lambda_1^s \frac{dF_1^s}{d\tau} + \lambda_1^b \frac{dF_1^b}{d\tau}\right) + \lambda_1^s \frac{F_1}{R^w}\frac{dR^w}{d\tau} + \left(\kappa y_1^b - \left(c_{1,Y}^b - y_1^b\right)\right)R^w\mu\frac{dp}{d\tau} = 0$$

Where this becomes:

$$\frac{dW}{d\tau} = \tau\left(\lambda_1^s \frac{dF_1}{d\tau} + \mu\frac{dF_1^b}{d\tau}\right) + \lambda_1^s \frac{F_1}{R^w}\frac{dR^w}{d\tau} + \left(\kappa y_1^b - \left(c_{1,Y}^b - y_1^b\right)\right)R^w\mu\frac{dp}{d\tau} = 0$$

Notice this is nearly identical to the expression for the optimal τ in the earlier model. The only difference is that in the previous model where only savers held foreign bonds, the coefficient of τ in this expression was $\lambda_1^s \frac{dF_1}{d\tau}$, now it is $\lambda_1^s \frac{dF_1}{d\tau} + \mu\frac{dF_1^b}{d\tau}$. When the constraint

doesn't bind the two are identical.