Details for Section IV.A: Allowing Direct Sale in The First Stage

This online appendix contains the details for deriving the revenue-maximizing mechanisms allowing sale in the first stage. As in the main text, we will proceed with two cases, the first with single-round shortlisting and the second with sequential shortlisting.

A Single-round Shortlisting

A.1 Mechanisms

The first-stage mechanism is characterized by the selling rule $p_i(\boldsymbol{\alpha})$, the shortlisting rule $A^g(\boldsymbol{\alpha})$, and payment rule $x_i(\boldsymbol{\alpha}), i = 1, 2, ..., N$. Given the reported profile $\boldsymbol{\alpha}$, the selling rule $p_i(\boldsymbol{\alpha}) : [\underline{\alpha}, \overline{\alpha}]^N \to [0, 1]$, assigns a selling probability to each buyer i, where $\sum_{i \in \mathbf{N}} p_i(\boldsymbol{\alpha}) \leq 1$; if the object is unsold in the first stage, the shortlisting rule, $A^g : [\underline{\alpha}, \overline{\alpha}]^N \to [0, 1]$, assigns a probability to each subgroup $g \in 2^{\mathbf{N}}$ for information acquisition, where $\sum_{g \in 2^{\mathbf{N}}} A^g(\boldsymbol{\alpha}) = 1$. The payment rule $x_i : [\underline{\alpha}, \overline{\alpha}]^N \to \mathbb{R}$, specifies bidder i's first-stage payment given the reported profile $\boldsymbol{\alpha}$.

Given the first-stage reported profile $\boldsymbol{\alpha}$, and that group g is shortlisted for information acquisition, the second-stage mechanism is characterized by $p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$, the probability with which the asset is allocated to buyer $i \in g$, and $t_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$, the payment to the seller made by buyer $i \in g, \forall g \in 2^{\mathbf{N}}$.

We will identify the revenue-maximizing mechanism in two steps. First, we establish a revenue bound by considering a relaxed problem in which the second-stage signal s_i is known to the shortlisted buyer i. In this relaxed problem, we ignore the second-stage incentive compatibility condition (IC) and individual rationality condition (IR). Second, we will identify a feasible mechanism (satisfying IC and IR in both stages) in the original setting, which achieves the above revenue bound.

A.2 A Revenue Upper Bound with Public $\mathbf{s}$

We will first identify an upper bound for the expected revenue in a relaxed setting with public $\mathbf{s}$ for the shortlisted buyers. In this relaxed setting, the mechanisms are specified exactly the same as in Section II.A. We drop the IC and IR constraints for the shortlisted bidders in the second stage so that all shortlisted bidders must incur entry costs to learn their second-stage signals as in our original setup, and regardless of their second-stage signals, they must participate in the second-stage selling mechanism and report their second-stage signals truthfully.

As a result, the highest possible expected revenue achievable in this relaxed setting imposes an upper bound for the expected revenue that can be obtained in our original setup, where the bidders' second-stage IC and IR constraints must both be satisfied. We next proceed to identify this bound.

Given the announced $\boldsymbol{\alpha}$ and s_i, let the interim winning probability and expected payment be, respectively, $P_i^g(\boldsymbol{\alpha}, s_i) = E_{\mathbf{s}_{-i}^g} p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$ and $T_i^g(\boldsymbol{\alpha}, s_i) = E_{\mathbf{s}_{-i}^g} t_i^g(\boldsymbol{\alpha}, \mathbf{s}^g)$, where $\mathbf{s}_{-i}^g = \mathbf{s}^g \backslash \{s_i\}$, $\forall i \in g$ and $\forall g \in 2^{\mathbf{N}}$. Let g_i denote a shortlisted subgroup that contains bidder i. For shortlisted bidder $i \in g_i$ with type α_i, her interim expected payoff when she reports $\hat{\alpha}_i$ and others report truthfully is given by

$$\begin{aligned}\pi_i(\alpha_i, \hat{\alpha}_i) &= -E_{\boldsymbol{\alpha}_{-i}} x_i(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) + \\ & E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) E_{s_i} u(\alpha_i, s_i) + \\ \left[\begin{array}{c} \left[1 - \sum_j p_j(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) \left[E_{s_i}\left(u(\alpha_i, s_i) P_i^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, s_i) - T_i^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, s_i)\right) - c\right] \end{array} \right] \end{array} \right\}.\end{aligned} \tag{40}$$

The IC condition requires $\pi_i(\alpha_i, \alpha_i) \geq \pi_i(\alpha_i, \hat{\alpha}_i)$. Standard arguments such as envelope theorem (cf. Theorem 2 in Milgrom and Segal (2002)) lead to the following result:

$$\begin{aligned}\frac{d\pi_i(\alpha_i, \alpha_i)}{d\alpha_i} &= \frac{\partial \pi_i(\alpha_i, \hat{\alpha}_i)}{\partial \alpha_i}\Big|_{\hat{\alpha}_i = \alpha_i} \\ &= E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(\alpha_i; \boldsymbol{\alpha}_{-i}) E_{s_i} u_1(\alpha_i, s_i) + \\ \left[\begin{array}{c} \left[1 - \sum_j p_j(\alpha_i; \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{g_i} A^{g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}) \left[E_{s_i} u_1(\alpha_i, s_i) P_i^{g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}, s_i)\right] \end{array} \right] \end{array} \right\}.\end{aligned} \tag{41}$$

Therefore, we have

$$\begin{aligned}&\pi_i(\alpha_i, \alpha_i) \\ =\ & \pi_i(\underline{\alpha}, \underline{\alpha}) + E_{\boldsymbol{\alpha}_{-i}} \int_{\underline{\alpha}}^{\alpha_i} \left\{ \begin{array}{c} p_i(y; \boldsymbol{\alpha}_{-i}) E_{s_i} u_1(y, s_i) + \\ \left[\begin{array}{c} \left[1 - \sum_j p_j(y; \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{g_i} A^{g_i}(y, \boldsymbol{\alpha}_{-i}) \left[E_{s_i} u_1(y, s_i) P_i^{g_i}(y, \boldsymbol{\alpha}_{-i}, s_i)\right] \end{array} \right] \end{array} \right\} dy \\ =\ & \pi_i(\underline{\alpha}, \underline{\alpha}) + E_{\alpha_{-i}} \int_{\underline{\alpha}}^{\alpha_i} \int u_1(y, s_i) \cdot \left\{ \begin{array}{c} p_i(y; \boldsymbol{\alpha}_{-i}) + \\ \left[\begin{array}{c} \left[1 - \sum_j p_j(y; \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{g_i} A^{g_i}(y, \boldsymbol{\alpha}_{-i}) \left[P_i^{g_i}(y; \boldsymbol{\alpha}_{-i}, s_i)\right] \end{array} \right] \end{array} \right\} dG_i(s_i) dy.\end{aligned} \tag{42}$$

Taking expectation, we have

$$\begin{aligned}&E\pi_i(\alpha_i, \alpha_i) \\ =\ & \pi_i(\underline{\alpha}, \underline{\alpha}) + \int_{\underline{\alpha}}^{\overline{\alpha}} \int_{\underline{\alpha}}^{\alpha_i} E_{s_i} \left[E_{\alpha_{-i}} \left\{ \begin{array}{c} p_i(y; \boldsymbol{\alpha}_{-i}) u_1(y, s_i) + \\ \left[\begin{array}{c} \left[1 - \sum_j p_j(y; \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{g_i} \left[A^{g_i}(y, \boldsymbol{\alpha}_{-i}) P_i^{g_i}(y, \boldsymbol{\alpha}_{-i}, s_i) u_1(y, s_i)\right] \end{array} \right] \end{array} \right\} \right] dy dF(\alpha_i)\end{aligned}$$

$$= \pi_i(\underline{\alpha},\underline{\alpha}) + E_{\alpha_i}\left[\frac{1-F(\alpha_i)}{f(\alpha_i)}E_{s_i}\left[E_{\alpha_{-i}}\left[\begin{array}{c} p_i(\alpha_i;\boldsymbol{\alpha}_{-i})u_1(\alpha_i,s_i)+ \\ \left[\begin{array}{c}\left[1-\sum_j p_j(\alpha_i;\boldsymbol{\alpha}_{-i})\right] \\ \cdot\sum_{g_i}\left[A^{g_i}(\alpha_i,\boldsymbol{\alpha}_{-i})P_i^{g_i}(\alpha_i,\boldsymbol{\alpha}_{-i},s_i)u_1(\alpha_i,s_i)\right]\end{array}\right]\end{array}\right]\right]\right]$$

$$= \pi_i(\underline{\alpha},\underline{\alpha}) + E_{\boldsymbol{\alpha}}\left\{E_{s_i}\left[\frac{1-F(\alpha_i)}{f(\alpha_i)}\left\{\begin{array}{c} p_i(\boldsymbol{\alpha})u_1(\alpha_i,s_i)+ \\ \left[\begin{array}{c}\left[1-\sum_j p_j(\boldsymbol{\alpha})\right] \\ \cdot\sum_{g_i}\left[A^{g_i}(\boldsymbol{\alpha})P_i^{g_i}(\boldsymbol{\alpha},s_i)u_1(\alpha_i,s_i)\right]\end{array}\right]\end{array}\right\}\right]\right\}. \tag{43}$$

Thus

$$\sum_{i=1}^{N} E\pi_i(\alpha_i,\alpha_i) = \sum_{i=1}^{N}\pi_i(\underline{\alpha},\underline{\alpha}) + E_{\boldsymbol{\alpha}}\left\{\begin{array}{c}\sum_i\left[p_i(\boldsymbol{\alpha})\frac{1-F(\alpha_i)}{f(\alpha_i)}E_{s_i}u_1(\alpha_i,s_i)\right]+ \\ \left[\begin{array}{c}\left[1-\sum_j p_j(\boldsymbol{\alpha})\right] \\ \cdot\sum_g A^g(\boldsymbol{\alpha})E_{\mathbf{s}}\sum_{i\in g}\left[p_i^g(\boldsymbol{\alpha},\mathbf{s}^g)\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)\right]\end{array}\right]\end{array}\right\}. \tag{44}$$

The total expected surplus from the two-stage mechanism is

$$TS = E_{\boldsymbol{\alpha}}\left\{\begin{array}{c}\sum_i p_i(\boldsymbol{\alpha})E_{s_i}u(\alpha_i,s_i) \\ +\left[1-\sum_i p_i(\boldsymbol{\alpha})\right]\sum_g\left\{A^g(\boldsymbol{\alpha})E_{\mathbf{s}}\left[\sum_{i\in g}p_i^g(\boldsymbol{\alpha},\mathbf{s}^g)u(\alpha_i,s_i)-|g|c\right]\right\}\end{array}\right\}. \tag{45}$$

The seller's expected revenue is thus given by

$$\begin{aligned} & ER \\ = & TS - \sum_{i=1}^{N} E\pi_i(\alpha_i,\alpha_i) \\ = & E_{\boldsymbol{\alpha}}\left\{\begin{array}{c}\sum_i\left[p_i(\boldsymbol{\alpha})E_{s_i}\left(u(\alpha_i,s_i)-\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)\right)\right]+ \\ \left[1-\sum_i p_i(\boldsymbol{\alpha})\right]\sum_g A^g(\boldsymbol{\alpha})E_{\mathbf{s}}\left[\sum_{i\in g}p_i^g(\boldsymbol{\alpha},\mathbf{s}^g)\left(u(\alpha_i,s_i)-\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)\right)-|g|c\right]\end{array}\right\} \\ & -\sum_{i=1}^{N}\pi_i(\underline{\alpha},\underline{\alpha}). \end{aligned} \tag{46}$$

Clearly, to maximize ER, the seller should set $\pi_i(\underline{\alpha},\underline{\alpha}) = 0$ for all $i = 1, 2, ..., N$.

Recall that the *virtual value adjusted by the second-stage signal is defined* in (3):

$$w(\alpha_i,s_i) = u(\alpha_i,s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i).$$

From the expression of the expected revenue, we can derive the optimal allocation rules in both stages as follows. At the second stage, given the revealed $\boldsymbol{\alpha}$ and the shortlisted group g, $\forall\mathbf{s}^g$, $p_i^{*g}(\boldsymbol{\alpha},\mathbf{s}^g)$

takes the same form as in (4)[36]

$$p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g) = \begin{cases} 1 & \text{if } i = \arg\max_{j \in g}\{w(\alpha_j, s_j)\} \text{ and } w(\alpha_i, s_i) \geq 0 \\ 0 & \text{otherwise} \end{cases} \forall g, \forall i \in g.$$

Recall that the expected *virtual surplus* (the virtual value less the entry cost) is defined in (5):

$$w^{*g}(\boldsymbol{\alpha}) = E_{\mathbf{s}}\left[\sum_{i \in g} p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g) w(\alpha_i, s_i) - |g|c\right].$$

At the first stage, contingent on the revealed $\boldsymbol{\alpha}$, the optimal shortlisting rule is given in (6):[37]

$$A^{*g}(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } g = \arg\max_{\tilde{g}}\{w^{*\tilde{g}}(\boldsymbol{\alpha})\} \text{ and } w^{*g}(\boldsymbol{\alpha}) \geq 0 \\ 0 & \text{otherwise} \end{cases} \forall g.$$

Recall that $g^*(\boldsymbol{\alpha})$ denotes the set of bidders admitted under the optimal shortlisting rule. The highest revenue generated from the second-stage sale is

$$R_2^{g^*(\boldsymbol{\alpha})}(\boldsymbol{\alpha}) = E_{\mathbf{s}}\left[\sum_{i \in g^*(\boldsymbol{\alpha})} p_i^{g^*(\boldsymbol{\alpha})}(\boldsymbol{\alpha}, \mathbf{s}^{g^*(\boldsymbol{\alpha})})\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) - |g^*(\boldsymbol{\alpha})|c\right], \tag{47}$$

and the highest revenue generated from the first-stage sale is

$$\begin{aligned} R_1^*(\boldsymbol{\alpha}) &= \max_{\{i=1,2,..,N\}} E_{s_i}\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) \\ &= E_s\left(u(\alpha_{(1)}, s) - \frac{1 - F(\alpha_{(1)})}{f(\alpha_{(1)})} u_1(\alpha_{(1)}, s)\right), \end{aligned} \tag{48}$$

where $\alpha_{(1)}$ denotes the highest first-stage type among all buyers, and s is distributed uniformly over $[0, 1]$.[38]

Clearly, the optimal first-stage selling probabilities are:

$$p_i^*(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } \alpha_i \geq \alpha_j, \forall j \text{ and } R_1^*(\boldsymbol{\alpha}) \geq R_2^{g^*(\boldsymbol{\alpha})}(\boldsymbol{\alpha}), \\ 0 & \text{otherwise,} \end{cases} \forall i. \tag{49}$$

In other words, given first-stage type profile $\boldsymbol{\alpha}$, the object is sold in the first stage if and only if

[36] Ties occur with probability zero and are hence ignored.

[37] Again ties occur with probability zero and are hence ignored.

[38] Assumptions 1 and 2 imply $u_{11} \leq 0$, and we have $u_1 > 0$ and $(\frac{1-F(\cdot)}{f(\cdot)})' \leq 0$. These imply that the buyer with $\alpha_{(1)}$ possesses the highest expected virtual value.

by doing so it generates higher expected revenue than that from first-stage optimal shortlisting and second-stage optimal selling mechanism.

Allocation rule $(p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g),\ A^{*g}(\boldsymbol{\alpha}),\ p_i^*(\boldsymbol{\alpha}))$ gives rise to the following bound for the seller's expected revenue:

$$ER^{**} = E_{\boldsymbol{\alpha}} \left\{ \begin{array}{c} \sum_i \left[p_i^*(\boldsymbol{\alpha}) E_{s_i} w(\alpha_i, s_i)\right] + \\ \left[1 - \sum_i p_i^*(\boldsymbol{\alpha})\right] \sum_g A^{*g}(\boldsymbol{\alpha}) E_{\mathbf{s}} \left[\sum_{i \in g} p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g) w(\alpha_i, s_i) - |g| c\right] \end{array} \right\}. \tag{50}$$

We have the following property for the first stage selling probabilities.

Lemma 6. Given $\boldsymbol{\alpha}_{-i}$, if $p_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}) = 1$, then $p_i^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) = 1$ for $\tilde{\alpha}_i > \alpha_i$.

Proof of Lemma 6: To establish this result, it suffices to show that for any g_i, if

$$E_{s_i}\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) \geq R_2^{g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}),$$

then

$$E_{s_i}\left(u(\tilde{\alpha}_i, s_i) - \frac{1 - F(\tilde{\alpha}_i)}{f(\tilde{\alpha}_i)} u_1(\alpha_i, s_i)\right) \geq R_2^{g_i}(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}), \text{ for any } \tilde{\alpha}_i > \alpha_i.$$

We first consider the case $g_i = \{i\}$. In this case,

$$\begin{aligned}
& E_{s_i}\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) \geq R_2^{g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}) \\
\iff\ & E_{s_i}\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right) \geq E_{s_i} \max\left\{\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right), 0\right\} - c \\
\iff\ & c \geq -E_{s_i} \min\left\{\left(u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)\right), 0\right\}.
\end{aligned}$$

Since $u(\alpha_i, s_i) - \frac{1 - F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i)$ increases in α_i, we have

$$c \geq -E_{s_i} \min\left\{\left(u(\tilde{\alpha}_i, s_i) - \frac{1 - F(\tilde{\alpha}_i)}{f(\tilde{\alpha}_i)} u_1(\tilde{\alpha}_i, s_i)\right), 0\right\},$$

which further leads to

$$E_{s_i}\left(u(\tilde{\alpha}_i, s_i) - \frac{1 - F(\tilde{\alpha}_i)}{f(\tilde{\alpha}_i)} u_1(\tilde{\alpha}_i, s_i)\right) \geq R_2^{g_i}(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}).$$

Hence we have $p_i^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) = 1$ for $\tilde{\alpha}_i > \alpha_i$.

We now turn to the case $g_i \supset \{i\}$. In this case, define

$$\xi = \max_{j \in g_i \setminus \{i\}} \left\{u(\alpha_j, s_j) - \frac{1 - F(\alpha_j)}{f(\alpha_j)} u_1(\alpha_j, s_j)\right\} \vee 0.$$

We have

$$\begin{aligned}
& E_{s_i}\left(u(\alpha_i,s_i)-\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)\right)\geq R_2^{g_i}(\alpha_i,\boldsymbol{\alpha}_{-i}) \\
\iff\ & E_{s_i}\left(u(\alpha_i,s_i)-\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)\right)\geq E_{s_i,\xi}\max\left\{u(\alpha_i,s_i)-\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i),\xi\right\}-|g_i|c \\
\iff\ & |g_i|c+E_{s_i,\xi}\min\left\{\left(u(\alpha_i,s_i)-\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)\right),\xi\right\}\geq 0.
\end{aligned}$$

Since $u(\alpha_i,s_i)-\frac{1-F(\alpha_i)}{f(\alpha_i)}u_1(\alpha_i,s_i)$ increases in α_i, we have

$$|g_i|c+E_{s_i,\xi}\min\left\{\left(u(\tilde{\alpha},s_i)-\frac{1-F(\tilde{\alpha})}{f(\tilde{\alpha})}u_1(\tilde{\alpha},s_i)\right),\xi\right\}\geq 0,$$

which further leads to

$$E_{s_i}\left(u(\tilde{\alpha}_i,s_i)-\frac{1-F(\tilde{\alpha}_i)}{f(\tilde{\alpha}_i)}u_1(\tilde{\alpha}_i,s_i)\right)\geq R_2^{g_i}(\tilde{\alpha}_i,\boldsymbol{\alpha}_{-i}),$$

hence $p_i^*(\tilde{\alpha}_i,\boldsymbol{\alpha}_{-i})=1$ for $\tilde{\alpha}_i>\alpha_i$. □

It is clear that ER^{**} provides an upper bound for the seller expected revenue in the original setting with private $\mathbf{s}$.

A.3 Revenue-Maximizing Selling Mechanism in The Original Setting

We will establish that ER^{**} can be achieved by a feasible mechanism (satisfying IC and IR in both stages) in the original setting. To this end, we will first establish necessary conditions implied by IC conditions in both stages.

We start with the second stage. Suppose group g is shortlisted, and the profile $\tilde{\boldsymbol{\alpha}}$ reported in the first stage is revealed as public information to the shortlisted bidders. First, suppose $\boldsymbol{\alpha}$ is truthfully reported at the first stage and group g is shortlisted. Assume that they follow the recommendation and incur the information acquisition cost c to discover $\mathbf{s}^g$.

We are now ready to consider the implication of the first-stage IC. The lie correction strategy of (9) and (10) still hold. Let $\pi_i(\alpha_i,\hat{\alpha}_i)$ be the expected payoff (net of the entry cost) for a type-α_i bidder who reports $\hat{\alpha}_i$ in the first stage. In particular, we have

$$\begin{aligned}
& \pi_i(\alpha_i,\hat{\alpha}_i) \\
= \ & E_{\boldsymbol{\alpha}_{-i}}\left\{\begin{array}{c} p_i(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})E_{s_i}u(\alpha_i,s_i)+\left[\begin{array}{c}[1-\sum_j p_j(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})] \\ \cdot\sum_{g_i}A^{g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})[\tilde{\pi}_i^{g_i}(\alpha_i,\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})-c]\end{array}\right] \\ -x_i(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})\end{array}\right\}
\end{aligned}$$

$$
= E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) E_{s_i} u(\alpha_i, s_i) + \\ \left[\begin{array}{c} [1 - \sum_j p_j(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i})] \\ \cdot \sum_{g_i} A^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) \left[E_{s_i} \left(u(\alpha_i, s_i) P_i^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \hat{s}_i) - T_i^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \hat{s}_i) \right) - c \right] \end{array} \right] \end{array} \right\}
$$

$$
-x_i(\hat{\alpha}_i), \tag{51}
$$

where $\hat{s}_i = \sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$ and $x_i(\hat{\alpha}_i) = E_{\boldsymbol{\alpha}_{-i}} x_i(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$.

By similar arguments leading to (25), we have

$$
\frac{\partial \pi_i(\alpha_i, \hat{\alpha}_i)}{\partial \alpha_i} \tag{52}
$$

$$
= \int E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) u_1(\alpha_i, s_i) + \\ [1 - \sum_j p_j(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i})] \sum_{g_i} [A^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) u_1(\alpha_i, s_i) P_i^{g_i}(\boldsymbol{\alpha}_{-i}, \hat{\alpha}_i, \sigma_i(y, \hat{\alpha}_i, s_i))] \end{array} \right\} dG_i(s_i),
$$

which gives the next lemma immediately.

Lemma 7. Suppose $\boldsymbol{\alpha}_{-i}$ is truthfully revealed from the first stage and the second-stage mechanism is incentive-compatible given a truthfully revealed $\boldsymbol{\alpha}$. If buyer i with type α_i reports $\hat{\alpha}_i$ in the first stage, then i's first-stage expected payoff can be expressed as

$$
\pi_i(\alpha_i, \hat{\alpha}_i) - \pi_i(\hat{\alpha}_i, \hat{\alpha}_i) \tag{53}
$$

$$
= \int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i} E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) u_1(y, s_i) + \\ [1 - \sum_j p_j(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i})] \sum_{g_i} [A^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) u_1(y, s_i) P_i^{g_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \sigma_i(y, \hat{\alpha}_i, s_i))] \end{array} \right\} dy.
$$

Applying the envelop theorem and using (52), we have

$$
\begin{aligned}
\frac{d\pi_i(\alpha_i, \alpha_i)}{d\alpha_i} &= \frac{\partial \pi_i(\alpha_i, \hat{\alpha}_i)}{\partial \alpha_i}|_{\hat{\alpha}_i = \alpha_i} \\
&= \int E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(\boldsymbol{\alpha}) u_1(\alpha_i, s_i) + \\ [1 - \sum_j p_j(\boldsymbol{\alpha})] \sum_{g_i} [A^{g_i}(\boldsymbol{\alpha}) u_1(\alpha_i, s_i) P_i^{g_i}(\boldsymbol{\alpha}, s_i)] \end{array} \right\} dG_i(s_i),
\end{aligned} \tag{54}
$$

which leads to the next lemma.

Lemma 8. If the two-stage mechanism is incentive compatible, then buyer i's expected payoff (as a function of her pre-entry type) can be expressed as

$$
\pi_i(\alpha_i, \alpha_i) - \pi_i(\underline{\alpha}, \underline{\alpha}) \tag{55}
$$

$$
= \int_{\underline{\alpha}}^{\alpha_i} \int E_{\boldsymbol{\alpha}_{-i}} \left\{ \begin{array}{c} p_i(y; \boldsymbol{\alpha}_{-i}) u_1(y, s_i) + \\ [1 - \sum_j p_j(y; \boldsymbol{\alpha}_{-i})] \sum_{g_i} [A^{g_i}(y, \boldsymbol{\alpha}_{-i}) u_1(y, s_i) P_i^{g_i}(y, \boldsymbol{\alpha}_{-i}, s_i)] \end{array} \right\} dG_i(s_i) dy.
$$

As shown by (42) and (55), bidders' first-stage expected payoffs do not depend on whether information $\mathbf{s}$ is public or private. Moreover, with truthful revelation, the total expected surplus TS from the two-stage mechanism is given by (45). The seller's expected revenue is thus given by

$$\begin{aligned}
& ER \\
= \; & TS - \sum_{i=1}^{N} E\pi_i(\alpha_i, \alpha_i) \\
= \; & E_{\boldsymbol{\alpha}} \left\{ \begin{array}{c} \sum_i \left[p_i(\boldsymbol{\alpha}) E_{s_i} \left(u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i) \right) \right] + \\ [1 - \sum_i p_i(\boldsymbol{\alpha})] \sum_g A^g(\boldsymbol{\alpha}) E_{\mathbf{s}} \left[\sum_{i \in g} p_i^g(\boldsymbol{\alpha}, \mathbf{s}^g) \left(u(\alpha_i, s_i) - \frac{1-F(\alpha_i)}{f(\alpha_i)} u_1(\alpha_i, s_i) \right) - |g| c \right] \end{array} \right\} \\
& - \sum_{i=1}^{N} \pi_i(\underline{\alpha}, \underline{\alpha}). \qquad (56)
\end{aligned}$$

which coincides with the seller expected revenue with public $\mathbf{s}$, i.e. the expression in (46).

It is clear that if allocation rule $(p_i^*(\boldsymbol{\alpha}), A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ defined in (49), (6), and (4) can be supported by some appropriately defined payment rule $(\tilde{x}_i^{*g}(\boldsymbol{\alpha}), \tilde{t}_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ which also ensures $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$, then the revenue bound ER^{**} in (50) can be achieved. As a result, these allocation and payment rules constitute a revenue-maximizing two-stage selling mechanism in the original setting.

We next proceed to show such payment rule $(\tilde{x}_i^{*g}(\boldsymbol{\alpha}), \tilde{t}_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ exists. To this end, we need to utilize the properties of the allocation rule $(p_i^*(\boldsymbol{\alpha}), A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$, which are revealed by Corollaries 1, 2 and Lemma 6.

Note that $u(\alpha_i, s_i)$ increases in s_i and by Assumption 1, $u_1(\alpha_i, s_i)$ (weakly) decreases with s_i. This implies that $w(\alpha_i, s_i)$ increases with s_i. By the final good allocation rule (4), the winning probability $P_i^{*g}(\boldsymbol{\alpha}, s_i)$ is weakly increasing in s_i. By Lemma 2 in Myerson (1981), the second-stage mechanism is incentive compatible (given $\boldsymbol{\alpha}$ and g). Thus, given the truthfully revealed $\boldsymbol{\alpha}$ and shortlisted group g, a second-stage payment rule, say, $\tilde{t}_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g), \forall i \in g, \forall g$, can be constructed to truthfully implement the second-stage allocation rule $p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g), \forall i \in g, \forall g$ while maintaining the second-stage IR constraints (to participate in the second-stage mechanism), i.e. $\tilde{\pi}_i^g(\boldsymbol{\alpha}, \alpha_i; s_i, s_i) \geq 0$ on equilibrium path. This resembles the Myerson (1981) setting with asymmetric bidders. Note $\tilde{t}_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g)$ coincides with $t_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g)$ for the case without first-stage sale.

Recall that we use $\tilde{\pi}_i^{*g_i}(\alpha_i, \hat{\alpha}_i; \boldsymbol{\alpha}_{-i})$ to denote the second-stage expected payoff to buyer i of type α_i if she announces $\hat{\alpha}_i$ and is shortlisted in group g_i, given that everyone else announces $\boldsymbol{\alpha}_{-i}$ truthfully at the first stage. Lemma 7 must hold given the second stage mechanism is IC upon truthful revelation in the first stage.

Construct the first-stage payment rule as follows:

$$\tilde{x}_i^*(\boldsymbol{\alpha}) = p_i^*(\boldsymbol{\alpha}) E_{s_i} u(\alpha_i, s_i) + [1 - \sum_j p_j^*(\boldsymbol{\alpha})] \sum_{g_i} A^{*g_i}(\alpha_i, \boldsymbol{\alpha}_{-i}) [\tilde{\pi}_i^{*g_i}(\alpha_i, \alpha_i; \boldsymbol{\alpha}_{-i}) - c] \qquad (57)$$

$$-\int_{\underline{\alpha}}^{\alpha_i}\int u_1(y,s_i)\left\{\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i})+ \\ [1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})]\sum_{g_i}\left[A^{*g_i}(y,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i)\right]\end{array}\right\}dG_i(s_i)dy.$$

Substituting (57) into (51), we can verify that

$$\begin{aligned}&\pi_i^*(\alpha_i,\alpha_i) \qquad (58)\\ =\;&\int_{\underline{\alpha}}^{\alpha_i}\int u_1(y,s_i)\cdot E_{\boldsymbol{\alpha}_{-i}}\left\{\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i})+ \\ [1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})]\sum_{g_i}\left[A^{*g_i}(y,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i)\right]\end{array}\right\}dG_i(s_i)dy,\end{aligned}$$

which is precisely equation (55) with $\pi_i^*(\underline{\alpha},\underline{\alpha})=0$. Note that $\pi_i^*(\alpha_i,\alpha_i)\geq 0$, so IR is satisfied in the first stage.

Proposition 6. Under Assumptions 1 and 2, the optimal first-stage selling probabilities (49), the first-stage optimal shortlisting rules (6) and the second-stage optimal final good allocation (4) are IR and IC implementable by payments $(\tilde{x}_i^*(\boldsymbol{\alpha}),\, \tilde{t}_i^{*g}(\boldsymbol{\alpha},\mathbf{s}^g))$. Moreover, $\pi_i^*(\underline{\alpha},\underline{\alpha})=0$.

Proof of Proposition 6: Following the above discussions, it remains to show the first-stage IC. Suppose that all buyers except i report their types $\boldsymbol{\alpha}_{-i}$ truthfully. Consider buyer i with α_i contemplating to misreport $\hat{\alpha}_i<\alpha_i$. The deviation payoff is

$$\Delta=\pi_i^*(\alpha_i,\hat{\alpha}_i)-\pi_i^*(\alpha_i,\alpha_i)=[\pi_i^*(\alpha_i,\hat{\alpha}_i)-\pi_i^*(\hat{\alpha}_i,\hat{\alpha}_i)]+[\pi_i^*(\hat{\alpha}_i,\hat{\alpha}_i)-\pi_i^*(\alpha_i,\alpha_i)]. \qquad (59)$$

Since (55) is satisfied by the construction of $x_i^*(\boldsymbol{\alpha})$, we have

$$\begin{aligned}&\pi_i^*(\hat{\alpha}_i,\hat{\alpha}_i)-\pi_i^*(\alpha_i,\alpha_i)\\ =\;&-\int_{\hat{\alpha}_i}^{\alpha_i}\int u_1(y,s_i)\cdot E_{\boldsymbol{\alpha}_{-i}}\left\{\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i})+[1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})] \\ \cdot\sum_{g_i}\left[A^{*g_i}(y,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i)\right]\end{array}\right\}dG_i(s_i)dy.\end{aligned}$$

By Lemma 7,

$$\begin{aligned}&\pi_i^*(\alpha_i,\hat{\alpha}_i)-\pi_i^*(\hat{\alpha}_i,\hat{\alpha}_i)\\ =\;&\int_{\hat{\alpha}_i}^{\alpha_i}E_{s_i}E_{\boldsymbol{\alpha}_{-i}}\left\{\begin{array}{c} p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})u_1(y,s_i)+[1-\sum_j p_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})] \\ \cdot\sum_{g_i}\left[A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})u_1(y,s_i)P_i^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))\right]\end{array}\right\}dy.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Delta\;=\;&\int_{\hat{\alpha}_i}^{\alpha_i}\int E_{s_i}\left[u_1(y,s_i)E_{\boldsymbol{\alpha}_{-i}}\left[\begin{array}{c} p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})+(1-\sum_j p_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})) \\ \cdot\sum_{g_i}\left[A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\sigma_i(y,\hat{\alpha}_i,s_i))\right]\end{array}\right]\right]dy\\ &-\int_{\hat{\alpha}_i}^{\alpha_i}\int E_{s_i}\left[u_1(y,s_i)E_{\boldsymbol{\alpha}_{-i}}\left[\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i})+[1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})] \\ \cdot\sum_{g_i}\left[A^{*g_i}(y,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i)\right]\end{array}\right]\right]dy\end{aligned}$$

$$
\begin{aligned}
= & \underbrace{\left\{ \begin{array}{l} \int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i}\left[u_1(y,s_i)E_{\boldsymbol{\alpha}_{-i}}\left[\begin{array}{c} p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})+(1-\sum_j p_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})) \\ \cdot\sum_{g_i} A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\sigma_i(y,\hat{\alpha}_i,s_i)) \end{array}\right]\right] dy \\ -\quad \int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i}\left[u_1(y,s_i)\cdot E_{\boldsymbol{\alpha}_{-i}}\left[\begin{array}{c} p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})+(1-\sum_j p_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})) \\ \cdot\sum_{g_i} A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \end{array}\right]\right] dy \end{array}\right\}}_{\text{First Term}} \\
& +\underbrace{\left\{ \begin{array}{l} \int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i}\left[u_1(y,s_i)\cdot E_{\boldsymbol{\alpha}_{-i}}\left[\begin{array}{c} p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})+(1-\sum_j p_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})) \\ \cdot\sum_{g_i} A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \end{array}\right]\right] dy \\ -\int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i}\left[u_1(y,s_i)\cdot E_{\boldsymbol{\alpha}_{-i}}\left[\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i})+(1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})) \\ \cdot\sum_{g_i} A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \end{array}\right]\right] dy \end{array}\right\}}_{\text{Second Term}} \\
& +\underbrace{\left\{ \begin{array}{l} \int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i}\left[u_1(y,s_i)\cdot E_{\boldsymbol{\alpha}_{-i}}\left[\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i})+(1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})) \\ \cdot\sum_{g_i} A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \end{array}\right]\right] dy \\ -\int_{\hat{\alpha}_i}^{\alpha_i} E_{s_i}\left[u_1(y,s_i)\cdot E_{\boldsymbol{\alpha}_{-i}}\left[\begin{array}{c} p_i^*(y;\boldsymbol{\alpha}_{-i})+(1-\sum_j p_j^*(y;\boldsymbol{\alpha}_{-i})) \\ \cdot\sum_{g_i} A^{*g_i}(y,\boldsymbol{\alpha}_{-i})P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \end{array}\right]\right] dy \end{array}\right\}}_{\text{Third Term}}.
\end{aligned}
$$

For any s_i and any $y \in [\hat{\alpha}_i, \alpha_i]$, $\sigma_i(y,\hat{\alpha}_i,s_i) \geq s_i$, where $\sigma_i(y,\hat{\alpha}_i,s_i)$ is the lie correction strategy. From Corollary 1 (ii), we have $P_i^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\sigma_i(y,\hat{\alpha}_i,s_i)) - P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \leq 0$, which implies that the first term in Δ is nonpositive.

We now consider the second term in Δ when $y > \hat{\alpha}_i$. If $p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) = 1$, we must have $p_i^*(y;\boldsymbol{\alpha}_{-i}) = 1$ by Lemma 6. In this case, the two terms in the square brackets are identical. If $p_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) = 0$, we must have $p_i^*(y;\boldsymbol{\alpha}_{-i}) = 0$ or 1. If $p_i^*(y;\boldsymbol{\alpha}_{-i}) = 0$, the two terms in the two pairs of square brackets are identical. If $p_i^*(y;\boldsymbol{\alpha}_{-i}) = 1$, then the term in the first pair of square brackets must be smaller than the term in the second pair of square brackets. Thus, the second term in Δ must be nonpositive.

We now consider the third term in Δ when $y > \hat{\alpha}_i$. By Corollary 2, the optimal shortlisting rule implies that given $\boldsymbol{\alpha}_{-i}$, when buyer i is admitted with a higher α_i, she must be admitted to a group with a weakly smaller size. If y and $\hat{\alpha}_i$ are admitted in the same group, then $A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) = A^{*g_i}(y,\boldsymbol{\alpha}_{-i})$ and this term in Δ is zero.

We now turn to the case where $g^*(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) \supset g^*(y,\boldsymbol{\alpha}_{-i}) \supset \{i\}$. Note that $A^{*g_i}(\cdot,\boldsymbol{\alpha}_{-i})$ is 1 for the shortlisted group, and 0 for all other groups. Therefore,

$$
\begin{aligned}
& \sum_{g_i}[A^{*g_i}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}) - A^{*g_i}(y,\boldsymbol{\alpha}_{-i})]P_i^{*g_i}(y,\boldsymbol{\alpha}_{-i},s_i) \\
= \ & P_i^{*g^*(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}(y,\boldsymbol{\alpha}_{-i},s_i) - P_i^{*g^*(y,\boldsymbol{\alpha}_{-i})}(y,\boldsymbol{\alpha}_{-i},s_i) \\
\leq \ & 0,
\end{aligned}
$$

which implies that the third term in Δ is nonpositive. Since $g^*(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}) \supset g^*(y, \boldsymbol{\alpha}_{-i}) \supset \{i\}$, we must have $P_i^{*g^*(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}(y, \boldsymbol{\alpha}_{-i}, s_i) \leq P_i^{*g^*(y,\boldsymbol{\alpha}_{-i})}(y, \boldsymbol{\alpha}_{-i}, s_i)$, i.e. entrant i wins with a smaller probability if a strictly bigger group is shortlisted.

A similar argument can be used to rule out deviating to $\hat{\alpha}_i > \alpha_i$. □

Proposition 6 reveals that allocation rule $(p_i^*(\boldsymbol{\alpha}), A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ and payment rule $(\tilde{x}_i^{*g}(\boldsymbol{\alpha}), \tilde{t}_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ constitute a feasible (both IC and IR) two-stage mechanism and entail $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$. Clearly, by (56) this mechanism achieves the revenue bound in ER^{**} (50). Therefore, these rules constitute the revenue-maximizing two-stage selling mechanism in the original setting.

Proposition 7. Under Assumptions 1 and 2, allocation rule $(p_i^*(\boldsymbol{\alpha}), A^{*g}(\boldsymbol{\alpha}), p_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ and payment rule $(\tilde{x}_i^{*g}(\boldsymbol{\alpha}), \tilde{t}_i^{*g}(\boldsymbol{\alpha}, \mathbf{s}^g))$ constitute a revenue-maximizing two-stage selling mechanism in the original setting, which achieves revenue bound ER^{**} in (50).

B Sequential Shortlisting

Now we move to the setting where the seller may conduct sequential shortlisting. The mechanism is specified in the same way as in Section III except that in the first stage the selling probability to buyer i is $p_i(\mathbf{m}_1)$, where $\sum_{i \in \mathbf{N}} p_i(\mathbf{m}_1) \leq 1$; and only if the object is unsold in the first stage, each subgroup $g_1 \in 2^{\mathbf{N}}$ would be shortlisted with probability $A^{g_1}(\mathbf{m}_1|g_0)$ for information acquisition, where $\sum_{g \in 2^{\mathbf{N}}} A^g(\mathbf{m}_1|g_0) = 1$. Here, we follow the same notation as in Section III.

Our analysis proceeds as follows. We first consider a relaxed environment where the agents are only endowed with private information $\boldsymbol{\alpha}$, where s_i's become known to bidders once they are discovered. The optimal solution for this relaxed environment provides an upper bound for the seller's expected revenue in the original environment where the discovered s_i's are private information to the shortlisted bidders. We will establish that this upper bound is actually achievable in the original environment.

B.1 The Relaxed Environment

For a given mechanism and message sequence $(\mathbf{m}_k, k = 1, 2, ..., M)$, the probability of a shortlisting outcome $\mathbf{g} = (g_1, g_2, ..., g_M)$ is given by

$$\Pr(\mathbf{g}|(\mathbf{m}_i)_{i=1}^M) = \Pi_{k=1}^M A^{g_k}(\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_k|g_0, g_1, g_2, ..., g_{k-1}).$$

As s_i becomes known to bidder i once discovered in the relaxed environment, we have that for $k \geq 2$, $m_{k,i} = s_i$, $i \in g_{k-1}$,and $m_{k,i} = \phi$, $i \notin g_{k-1}$. We use $\mathbf{m}_k^s$, $k \geq 2$ to denote these true types from stages 2 to $M+1$. Agent i's expected payoff when i is endowed with α_i but announces $\hat{\alpha}_i$ is given by:

$\pi_i(\alpha_i, \hat{\alpha}_i)$

$$= E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \begin{array}{c} p_i(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) E_{s_i} u(\alpha_i, s_i) + [1-\sum_j p_j(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})] \\ \cdot \left[\begin{array}{c} \sum_{\forall \mathbf{g} \text{ s.t. } i\in G_{\mathbf{g}}} \left[\begin{array}{c} \Pr(\mathbf{g}|(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left[u(\alpha_i, s_i) p_i^{G_{\mathbf{g}}}((\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) - c \right] \end{array} \right] \\ -\sum_{\forall \mathbf{g}} \left[\Pr(\mathbf{g}|(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \sum_{k=1}^{M} t_{k+1,i}((\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{k+1}^{\mathbf{s}}) \right] \end{array} \right] \end{array} \right\}$$
$$-E_{\boldsymbol{\alpha}_{-i}}[t_{1,i}((\hat{\alpha}_i,\boldsymbol{\alpha}_{-\mathbf{i}}))].$$

Incentive compatibility together with the envelop theorem gives:

$$\frac{d\pi_i(\alpha_i,\alpha_i)}{d\alpha_i} \tag{60}$$
$$= E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \left[\begin{array}{c} p_i(\alpha_i;\boldsymbol{\alpha}_{-i}) E_{s_i} u_1(\alpha_i, s_i) + [1-\sum_j p_j(\alpha_i;\boldsymbol{\alpha}_{-i})] \cdot \\ \sum_{\forall \mathbf{g}, i\in G_{\mathbf{g}}} \left[\Pr(\mathbf{g}|(\alpha_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) p_i^{G_{\mathbf{g}}}((\alpha_i,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) u_1(\alpha_i, s_i) \right] \end{array} \right] \right\}.$$

Thus, we have

$$\pi_i(\alpha_i,\alpha_i) = \pi_i(\underline{\alpha},\underline{\alpha}) + \tag{61}$$
$$E_{\boldsymbol{\alpha}_{-i}} \int_{\underline{\alpha}}^{\alpha_i} E_{\mathbf{s}} \left\{ \left[\begin{array}{c} p_i(y;\boldsymbol{\alpha}_{-i}) E_{s_i} u_1(y, s_i) + [1-\sum_j p_j(y;\boldsymbol{\alpha}_{-i})] \\ \cdot \sum_{\forall \mathbf{g} \text{ s.t. } i\in G_{\mathbf{g}}} \left[\Pr(\mathbf{g}|(y,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) p_i^{G_{\mathbf{g}}}((y,\boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) u_1(y, s_i) \right] \end{array} \right] \right\} dy.$$

The expected social surplus given $\boldsymbol{\alpha}$ is as follows:

$$TS = E_{\boldsymbol{\alpha}} \left\{ \begin{array}{c} \sum_i [p_i(\boldsymbol{\alpha}) E_{s_i} u(\alpha_i, s_i)] + [1-\sum_i p_i(\boldsymbol{\alpha})] \\ \cdot \sum_{\forall \mathbf{g}} E_{\mathbf{s}} \left[\Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \sum_{i\in G_{\mathbf{g}}} \left[p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) u(\alpha_i, s_i) - c \right] \right] \end{array} \right\}. \tag{62}$$

The seller seeks to maximize the expected revenue:

$$ER = TS - \sum_i E_{\boldsymbol{\alpha}_i} [\pi_i(\alpha_i,\alpha_i)].$$

By the standard procedure, we can rewrite the seller's objective as follows.

Lemma 9. The seller's objective is to maximize:

$$ER = E_{\boldsymbol{\alpha}} \left\{ \begin{array}{c} \sum_i [p_i(\boldsymbol{\alpha}) E_{s_i} w(\alpha_i, s_i)] + [1-\sum_i p_i(\boldsymbol{\alpha})] \\ \cdot \sum_{\forall \mathbf{g}} E_{\mathbf{s}} \left[\Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \sum_{i\in G_{\mathbf{g}}} \left[p_i^{G_{\mathbf{g}}}(\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) w(\alpha_i, s_i) - c \right] \right] \end{array} \right\}$$
$$-\sum_i \pi_i(\underline{\alpha},\underline{\alpha}), \tag{63}$$

where $w(\alpha_i, s_i) = u(\alpha_i, s_i) - u_1(\alpha_i, s_i)\frac{1-F(\alpha_i)}{f(\alpha_i)}$.

Define $\Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{s}) = \Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}})$, and for any $G \in 2^N$, define

$$\Pr(G|\boldsymbol{\alpha}, \mathbf{s}) = \sum_{\forall \mathbf{g} \text{ s.t. } G_{\mathbf{g}} = G} \Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{s}),$$

where, as before, $G_{\mathbf{g}}$ denotes the set of all agents shortlisted in sequence $\mathbf{g}$.

Note we have

$$\sum_{\forall \mathbf{g} \text{ s.t. } i \in G_{\mathbf{g}}} \Pr(\mathbf{g}|\boldsymbol{\alpha}, \mathbf{s}) = \sum_{\forall G \text{ s.t. } i \in G} \Pr(G|\boldsymbol{\alpha}, \mathbf{s}).$$

To maximize the expected revenue ER, at the final allocation stage, given the revealed $\boldsymbol{\alpha}$ and the shortlisted group G, $\forall \mathbf{s}^G$, the optimal allocation rule is given by[39]

$$p_i^{*G}(\boldsymbol{\alpha}, \mathbf{s}^G) = \begin{cases} 1 & \text{if } i = \arg\max_{j \in G}\{w(\alpha_j, s_j)\} \text{ and } w(\alpha_i, s_i) \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \forall G, \forall i \in G, \tag{64}$$

which maximizes the virtual value among the bidders within the shortlisted group G.

Analogously to Corollary 1, under Assumptions 1 and 2, we can establish the following properties of the optimal final-stage allocation rule:

Corollary 4. (i) $p_i^{*G_i}(\boldsymbol{\alpha}, \mathbf{s}^{G_i})$ increases in both α_i and s_i, $\forall i \in G_i$, $\forall g_i$, $\boldsymbol{\alpha}_{-i}$, and $\mathbf{s}_{-i}^{G_i}$, which implies that $P_i^{*G_i}(\alpha_i, \boldsymbol{\alpha}_{-i}, s_i) := E_{\boldsymbol{\alpha}_{-i}}[p_i^{*G_i}(\boldsymbol{\alpha}, \mathbf{s}^{G_i})]$ increases in both α_i and s_i, $\forall g_i$, $\boldsymbol{\alpha}_{-i}$; (ii) If $\alpha_i > \hat{\alpha}_i$, $s_i < \hat{s}_i$ and $u(\alpha_i, s_i) \geq u(\hat{\alpha}_i, \hat{s}_i)$, then $p_i^{*G_i}(\alpha_i, \boldsymbol{\alpha}_{-i}, s_i, \mathbf{s}_{-i}^{G_i}) \geq p_i^{*G_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \hat{s}_i, \mathbf{s}_{-i}^{G_i})$, which implies $P_i^{*G_i}(\alpha_i, \boldsymbol{\alpha}_{-i}, s_i) \geq P_i^{*G_i}(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}, \hat{s}_i), \forall g_i$, $\boldsymbol{\alpha}_{-i}$.

By substituting (64) into ER in (63), we have the following result.

Lemma 10. For any $\{\Pr(G), \forall G \in 2^{\Omega}\}$ derived from any shortlisting rule, to maximize the expected revenue ER, the seller sets $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$ and allocates the object to the shortlisted bidder whose virtual value is the highest, provided that it is positive. Ties are randomly broken. In this case,

$$ER = E_{\boldsymbol{\alpha}} \left\{ \begin{array}{c} \sum_i p_i(\boldsymbol{\alpha}) E_{s_i} w(\alpha_i, s_i) \\ + \left[1 - \sum_i p_i(\boldsymbol{\alpha})\right] E_{\mathbf{s}} \left[\sum\limits_{G \in 2^{\mathbf{N}}} \Pr(G|\boldsymbol{\alpha}, \mathbf{s}) \left[\max\{w_i^+(\alpha_i, s_i)\}_{i \in G} - \sum_{i \in G} c \right] \right] \end{array} \right\}. \tag{65}$$

B.2 Optimal Shortlisting

Lemma 4 and Proposition 3 still hold. Therefore, the same optimal sequential shortlisting rule of Proposition 3 in Section III.A remains valid.

[39] Ties occur with probability zero and are hence ignored.

For any shortlisting procedure, we define the expected *virtual surplus* (the virtual value less the entry cost) as follows

$$w^*(\boldsymbol{\alpha}) = E_{\mathbf{s}}\left[\sum_{G\in 2^{\mathbf{N}}} Pr(G|\boldsymbol{\alpha},\mathbf{s})\left(\max\{w_i^+(\alpha_i,s_i)\}_{i\in G} - \sum_{i\in G} c\right)\right].$$

For the optimal shortlisting rule described in Proposition 3, we define

$$R_2^*(\boldsymbol{\alpha}) = E_{\mathbf{s}}\left[\sum_{G\in 2^{\mathbf{N}}} \Pr{}^*(G|\boldsymbol{\alpha},\mathbf{s})\left(\max\{w_i^+(\alpha_i,s_i)\}_{i\in G} - \sum_{i\in G} c\right)\right], \tag{66}$$

where $\Pr^*(G|\boldsymbol{\alpha},\mathbf{s})$ denote the probability that the set of bidders G is admitted under the optimal shortlisting rule.

Recall that we let

$$\begin{aligned} R_1^*(\boldsymbol{\alpha}) &= \max_{\{i=1,2,..,N\}} E_{s_i} w(\alpha_i, s_i) \\ &= E_s w(\alpha_{(1)}, s), \end{aligned} \tag{67}$$

where $\alpha_{(1)}$ denotes the highest first-stage type among all agents, and s is uniformly distributed over [0,1].

It is clear that the optimal first-stage selling probabilities are:

$$\tilde{p}_i^*(\boldsymbol{\alpha}) = \begin{cases} 1 & \text{if } \alpha_i \geq \alpha_j, \forall j \text{ and } R_1^*(\boldsymbol{\alpha}) \geq R_2^*(\boldsymbol{\alpha}), \\ 0 & \text{otherwise,} \end{cases} \quad \forall i. \tag{68}$$

In other words, given the first-stage type profile $\boldsymbol{\alpha}$, the object would be sold in the first stage if and only if expected revenue generated from the first-stage sale to the buyer with the highest first-stage type is higher than that from the optimal sequential shortlisting rule and final stage optimal selling mechanism.

For the first-stage selling probabilities, we have the following property.

Lemma 11. For given $\boldsymbol{\alpha}_{-i}$, if $\tilde{p}_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}) = 1$, then $\tilde{p}_i^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) = 1$ for $\tilde{\alpha}_i > \alpha_i$.

Proof of Lemma 11: Note $\tilde{p}_i^*(\alpha_i, \boldsymbol{\alpha}_{-i})$ is either zero or one, and it can be one only if α_i is the highest first stage signal among all buyers. To establish the wanted result, it suffices to show that if α_i is the highest first stage signal among all buyers, and $R_1^*(\alpha_i, \boldsymbol{\alpha}_{-i}) \geq R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i})$, then we would have $R_1^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) \geq R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i})$, for any $\tilde{\alpha}_i > \alpha_i$.

Recall that $R_1^*(\boldsymbol{\alpha}) = \max_{\{i=1,2,..,N\}} E_{s_i} w(\alpha_i, s_i)$ and

$$R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}) = E_{\mathbf{s}} \left[\sum_{G \in 2^{\mathbf{N}}} \Pr^*(G|\boldsymbol{\alpha}, \mathbf{s}) \left(\max\{w_i^+(\alpha_i, s_i)\}_{i \in G} - \sum_{i \in G} c \right) \right].$$

We use G_i to denote a non-empty shortlisted group. Note that G_i must contain buyer i. Moreover, G_i must consist of a group of buyers with the highest first stage types.

For any $\boldsymbol{\alpha}_{-i}$,

$$\begin{aligned} & R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}) \\ = \; & E_{\mathbf{s}} \left[\Pr^*(G_i = \{i\}|\alpha_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[w_i^+(\alpha_i, s_i) - c \right] \right] \\ & + E_{\mathbf{s}} \left[\sum_{k=2}^{N} \Pr^*(|G_i| = k|\alpha_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[\begin{array}{c} \max\{w_i^+(\alpha_i, s_i), w_j^+(\alpha_j, s_j)\}_{j \in G_i \setminus \{i\}} \\ -|G|c \end{array} \right] \right], \end{aligned}$$

and

$$\begin{aligned} & R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) \\ = \; & E_{\mathbf{s}} \left[\Pr^*(G_i = \{i\}|\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[w_i^+(\tilde{\alpha}_i, s_i) - c \right] \right] \\ & + E_{\mathbf{s}} \left[\sum_{k=2}^{N} \Pr^*(|G_i| = k|\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[\begin{array}{c} \max\{w_i^+(\tilde{\alpha}_i, s_i), w_j^+(\alpha_j, s_j)\}_{j \in G_i \setminus \{i\}} \\ -|G|c \end{array} \right] \right]. \end{aligned}$$

Denote $\xi_{G_i \setminus \{i\}} = \max_{j \in G_i \setminus \{i\}} \{w_j^+(\alpha_j, s_j)\}$. We have

$$\begin{aligned} & R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}) \\ = \; & E_{\mathbf{s}} \left[\Pr^*(G_i = \{i\}|\alpha_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[w_i^+(\alpha_i, s_i) - c \right] \right] \\ & + E_{\mathbf{s}} \left[\sum_{k=2}^{N} \Pr^*(|G_i| = k|\alpha_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[\max\{w_i^+(\alpha_i, s_i), \xi_{G_i \setminus \{i\}}\} - |G|c \right] \right], \end{aligned}$$

and

$$\begin{aligned} & R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) \\ = \; & E_{\mathbf{s}} \left[\Pr^*(G_i = \{i\}|\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[w_i^+(\tilde{\alpha}_i, s_i) - c \right] \right] \\ & + E_{\mathbf{s}} \left[\sum_{k=2}^{N} \Pr^*(|G_i| = k|\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[\max\{w_i^+(\tilde{\alpha}_i, s_i), \xi_{G_i \setminus \{i\}}\} - |G|c \right] \right]. \end{aligned}$$

Define

$$\begin{aligned}
& R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \\
= \ & \Pr{}^*(G_i = \{i\} | \tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[w_i^+(\tilde{\alpha}_i, s_i) - c \right] \\
& + \sum_{k=2}^{N} \Pr{}^*(|G_i| = k | \tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[\max\{w_i^+(\tilde{\alpha}_i, s_i), \xi_{G_i \setminus \{i\}}\} - |G|c \right],
\end{aligned}$$

and

$$\begin{aligned}
& R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \\
= \ & \Pr{}^*(G_i = \{i\} | \alpha_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[w_i^+(\alpha_i, s_i) - c \right] \\
& + \sum_{k=2}^{N} \Pr{}^*(|G_i| = k | \alpha_i, \boldsymbol{\alpha}_{-i}, \mathbf{s}) \left[\max\{w_i^+(\alpha_i, s_i), \xi_{G_i \setminus \{i\}}\} - |G|c \right].
\end{aligned}$$

We next show that we have

$$R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) - R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}) \leq E_{\mathbf{s}} \left\{ w_i^+(\tilde{\alpha}_i, s_i) - w_i^+(\alpha_i, s_i) \right\}. \tag{69}$$

Note that $\forall \mathbf{s}$, there exists one and only one group that can be shortlisted with probability 1. Note $\tilde{p}_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}) = 1$ means that at least buyer i should be shortlisted when buyer i's first stage type is α_i. Give this, we have that at least buyer i should be shortlisted when buyer i's first stage type is $\tilde{\alpha}_i > \alpha_i$. We use $G_i(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})$ and $G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})$ to denote the shortlisted groups, respectively. We must have $G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \subseteq G_i(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})$. Note that the optimal shortlisting rule and selling rule mean that

$$\begin{aligned}
& E_{\mathbf{s}} \max \left[\{w_i^+(\alpha_i, s_i), \xi_{G_i(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \setminus \{i\}}\} - |G_i(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})|c \right] \\
\geq \ & E_{\mathbf{s}} \max \left[\{w_i^+(\alpha_i, s_i), \xi_{G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \setminus \{i\}}\} - |G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})|c \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
& R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) - R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}) \\
= \ & E_{\mathbf{s}} R_2^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) - E_{\mathbf{s}} R_2^*(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \\
= \ & E_{\mathbf{s}} \left[\max\{w_i^+(\tilde{\alpha}_i, s_i), \xi_{G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \setminus \{i\}}\} - |G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})|c \right] \\
& - E_{\mathbf{s}} \left[\max\{w_i^+(\alpha_i, s_i), \xi_{G_i(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \setminus \{i\}}\} - |G_i(\alpha_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})|c \right] \\
\leq \ & E_{\mathbf{s}} \left[\max\{w_i^+(\tilde{\alpha}_i, s_i), \xi_{G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \setminus \{i\}}\} - |G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})|c \right] \\
& - E_{\mathbf{s}} \left[\max\{w_i^+(\alpha_i, s_i), \xi_{G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s}) \setminus \{i\}}\} - |G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}; \mathbf{s})|c \right]
\end{aligned}$$

$$
\begin{aligned}
&= E_{\mathbf{s}} \left\{ \begin{array}{c} \left[\max\{w_i^+(\tilde{\alpha}_i, s_i), \xi_{G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i};\mathbf{s})\backslash\{i\}}\} - |G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i};\mathbf{s})|c\right] \\ -\left[\max\{w_i^+(\alpha_i, s_i), \xi_{G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i};\mathbf{s})\backslash\{i\}}\} - |G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i};\mathbf{s})|c\right] \end{array} \right\} \\
&= E_{\mathbf{s}} \left(\max\{w_i^+(\tilde{\alpha}_i, s_i), \xi_{G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i};\mathbf{s})\backslash\{i\}}\} - \max\{w_i^+(\alpha_i, s_i), \xi_{G_i(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i};\mathbf{s})\backslash\{i\}}\} \right) \\
&\leq E_{\mathbf{s}} \left(w_i^+(\tilde{\alpha}_i, s_i) - w_i^+(\alpha_i, s_i) \right),
\end{aligned}
$$

which gives (69).

We thus have that if $\tilde{p}_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}) = 1$, then $\tilde{p}_i^*(\tilde{\alpha}_i, \boldsymbol{\alpha}_{-i}) = 1$ for $\tilde{\alpha}_i > \alpha_i$. □

B.4 Incentive Compatibility in the Original Setting

We are now ready to show that the optimal first-stage selling rule (68), the optimal sequential shortlisting procedure described in Proposition 3, and the final-stage optimal allocation rule (64) are truthfully implementable by some well constructed payment rules.

We use $(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{M+1})$ to denote the announcements of agents at different stages. We denote the shortlisting rule of Proposition 3 by $\mathbf{A}^* = \{A^{*g_k}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{k-1}; g_1, g_2, ..., g_{k-1})$, $k = 1, 2, ..., M$, $\forall \mathbf{g} = (g_1, g_2, ..., g_M)\}$, and denote the allocation rule of (64) by $\mathbf{p}^* = \{p_i^{*G_{\mathbf{g}}}(\hat{\boldsymbol{\alpha}}, \mathbf{m}_2, ..., \mathbf{m}_{M+1}), i \in \mathbf{N}$, $\forall \mathbf{g} = (g_1, g_2, ..., g_M)\}$. In addition,

$$
\Pr{}^*(\mathbf{g}|(\mathbf{m}_i)_{i=1}^M) = \Pi_{k=1}^M A^{*g_k}(\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_k | g_0, g_1, g_2, ..., g_{k-1}),
$$

which is the probability that sequence $\mathbf{g}$ is shortlisted given messages reported $(\mathbf{m}_i)_{i=1}^M$.

The analysis on IC and IR for stage $k \in \{2, ..., N+1\}$ are identical to those of Section III.B. We now focus on IC and IR in stage 1.

By the same logic as in Lemma 4 of Esö and Szentes (2007) and Lemma 2 of Liu, Liu, and Lu (2020), when agent i reports $\hat{\alpha}_i$ at stage 1, she will report lie correction $\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$. In the proof of Corollary 1 in Esö and Szentes (2007) and Corollary 2 in Liu, Liu, and Lu (2020), they show that $w_i(\alpha_i, s_i) \lesseqgtr w_i(\hat{\alpha}_i, \sigma_i(\alpha_i, \hat{\alpha}_i, s_i))$ if and only if $\alpha_i \lesseqgtr \hat{\alpha}_i$.

Let $r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$ denote the rank of $\hat{\alpha}_i$ in $(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$, and $\mathbf{m}^{\boldsymbol{\sigma}_i}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}$ denote the stage $r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})$ reports in which agent i's report is $\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)$. Further assume that all shortlisted agents receive a subsidy of c from the seller besides the stage-1 transfer $t_{1,i}(\cdot)$ to make sure that they have the incentive to conduct the due diligence. At stage 1, agent i's expected payoff when i is of type α_i but announces $\hat{\alpha}_i$ is:

$$
\pi_i(\alpha_i, \hat{\alpha}_i) \tag{70}
$$

$$
= \quad E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left[\begin{array}{c} \tilde{p}_i^*(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) E_{s_i} u(\alpha_i, s_i) + \left[1 - \sum_j \tilde{p}_j^*(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i})\right] \\ \cdot \left[\begin{array}{c} \sum_{h=r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}^{\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left[u(\alpha_i, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}^{\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) - c \right] \end{array} \right] \\ - \sum_{h=r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}^{\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot\, t_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1,i}^*((\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}^{\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)}) \end{array} \right] \end{array} \right] \\ -t_{1,i}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})) \end{array} \right],
$$

where $\mathbf{g}_{k,h}$ is defined in the proof of Proposition 4.

By similar arguments in establishing (52), we have

$$
\frac{d\pi_i(\alpha_i, \hat{\alpha}_i)}{d\alpha_i} =
$$

$$
E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \begin{array}{c} \tilde{p}_i^*(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i}) E_{s_i} u_1(\alpha_i, s_i) + \left[1 - \sum_j \tilde{p}_j^*(\hat{\alpha}_i; \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{h=r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}^{\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left(u_1(\alpha_i, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\hat{\alpha}_i, \boldsymbol{\alpha}_{-i})+1}^{\sigma_i(\alpha_i, \hat{\alpha}_i, s_i)}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \right) \end{array} \right] \end{array} \right\}. \quad (71)
$$

We are now ready to pin down the transfer $t_{1,i}^*(\cdot)$ (net of the entry subsidy c) that induces truthful revelation in stage 1. By the envelop theorem, optimality of truthful revelation requires

$$
\begin{aligned}
& \frac{d\pi_i(\alpha_i, \alpha_i)}{d\alpha_i} && (72) \\
= \quad & E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \begin{array}{c} \tilde{p}_i^*(\alpha_i, \boldsymbol{\alpha}_{-i}) E_{s_i} u_1(\alpha_i, s_i) + \left[1 - \sum_j \tilde{p}_j^*(\alpha_i, \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{h=r(\alpha_i, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (\alpha_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\alpha_i, \boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left(u_1(\alpha_i, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((\alpha_i, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(\alpha_i, \boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \right) \end{array} \right] \end{array} \right\}.
\end{aligned}
$$

Recall that we set $\pi_i(\underline{\alpha}, \underline{\alpha}) = 0$. We thus have

$$
\begin{aligned}
& \pi_i(\alpha_i, \alpha_i) && (73) \\
= \quad & \int_{\underline{\alpha}}^{\alpha_i} E_{\boldsymbol{\alpha}_{-i}} E_{\mathbf{s}} \left\{ \begin{array}{c} \tilde{p}_i^*(y, \boldsymbol{\alpha}_{-i}) E_{s_i} u_1(y, s_i) + \left[1 - \sum_j \tilde{p}_j^*(y, \boldsymbol{\alpha}_{-i})\right] \\ \cdot \sum_{h=r(y, \boldsymbol{\alpha}_{-i})}^{N} \left[\begin{array}{c} \Pr{}^*(\mathbf{g}_{1,h} | (y, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(y, \boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \\ \cdot \left(u_1(y, s_i) p_i^{*G_{\mathbf{g}_{1,h}}}((y, \boldsymbol{\alpha}_{-i}), \mathbf{m}_2^{\mathbf{s}}, ..., \mathbf{m}_{r(y, \boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}}, ..., \mathbf{m}_{M+1}^{\mathbf{s}}) \right) \end{array} \right] \end{array} \right\} dy.
\end{aligned}
$$

By (70) and (73), we set

$$
t_{1,i}^*(\hat{\boldsymbol{\alpha}}) \quad (74)
$$

$$= E_{\mathbf{s}}\left\{\begin{array}{c}\tilde{p}_i^*(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})E_{s_i}u(\alpha_i,s_i)+\left[1-\sum_j \tilde{p}_j^*(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})\right]\\ \cdot\left[\begin{array}{c}\sum_{h=r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{1,h}|(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}^{\mathbf{s}}_{r(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1})\\ \cdot\left(u(\hat{\alpha}_i,s_i)p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}^{\mathbf{s}}_{r(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1})-c\right)\end{array}\right]\\ -\sum_{h=r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{1,h}|(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}^{\mathbf{s}}_{r(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1})\\ \cdot\, t^*_{r(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})+1,i}((\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}^{\mathbf{s}}_{r(\hat{\alpha}_i,\hat{\boldsymbol{\alpha}}_{-i})+1})\end{array}\right]\end{array}\right]\end{array}\right\}$$

$$-\int_{\underline{\alpha}}^{\hat{\alpha}_i} E_{\mathbf{s}}\left\{\begin{array}{c}\tilde{p}_i^*(y,\hat{\boldsymbol{\alpha}}_{-i})E_{s_i}u_1(y,s_i)+\left[1-\sum_j \tilde{p}_j^*(y,\hat{\boldsymbol{\alpha}}_{-i})\right]\\ \cdot\sum_{h=r(y,\hat{\boldsymbol{\alpha}}_{-i})}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{1,h}|(y,\hat{\boldsymbol{\alpha}}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}^{\mathbf{s}}_{r(y,\hat{\boldsymbol{\alpha}}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1})\\ \cdot\left(u_1(y,s_i)p_i^{*G_{\mathbf{g}_{1,h}}}((y,\hat{\boldsymbol{\alpha}}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}^{\mathbf{s}}_{r(y,\hat{\boldsymbol{\alpha}}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1})\right)\end{array}\right]\end{array}\right\}dy.$$

The following proposition establishes IC in stage 1.

Proposition 8. We have $\pi_i(\alpha_i,\alpha_i)\geq\pi_i(\alpha_i,\hat{\alpha}_i),\forall\alpha_i,\hat{\alpha}_i$.

Proof of Proposition 8: Without loss of generality, we consider $\hat{\alpha}_i<\alpha_i$. By (71), we have

$$\begin{aligned}&\pi_i(\alpha_i,\hat{\alpha}_i)-\pi_i(\hat{\alpha}_i,\hat{\alpha}_i)\\ =&\int_{\hat{\alpha}_i}^{\alpha_i}E_{\boldsymbol{\alpha}_{-i}}E_{\mathbf{s}}\left\{\begin{array}{c}\tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})E_{s_i}u_1(y,s_i)+\left[1-\sum_j \tilde{p}_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})\right]\times\\ \sum_{h=r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{1,h}|(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}^{\sigma_i(y,\hat{\alpha}_i,s_i)}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1})\\ \cdot\left(u_1(y,s_i)p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}^{\sigma_i(y,\hat{\alpha}_i,s_i)}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1})-c\right)\end{array}\right]\end{array}\right\}dy.\end{aligned}$$

By (74) and (70), we have (73). Therefore,

$$\begin{aligned}&\pi_i(\alpha_i,\alpha_i)-\pi_i(\hat{\alpha}_i,\hat{\alpha}_i)\\ =&\int_{\hat{\alpha}_i}^{\alpha_i}E_{\boldsymbol{\alpha}_{-i}}E_{\mathbf{s}}\left\{\begin{array}{c}\tilde{p}_i^*(y,\boldsymbol{\alpha}_{-i})E_{s_i}u_1(y,s_i)+\left[1-\sum_j \tilde{p}_j^*(y,\boldsymbol{\alpha}_{-i})\right]\\ \cdot\sum_{h=r(y,\boldsymbol{\alpha}_{-i})}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{1,h}|(y,\boldsymbol{\alpha}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}^{\mathbf{s}}_{r(y,\boldsymbol{\alpha}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1})\\ \cdot\left(u_1(y,s_i)p_i^{*G_{\mathbf{g}_{1,h}}}((y,\boldsymbol{\alpha}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}^{\mathbf{s}}_{r(y,\boldsymbol{\alpha}_{-i})+1},...,\mathbf{m}^{\mathbf{s}}_{M+1})\right)\end{array}\right]\end{array}\right\}dy.\end{aligned}$$

Consider

$$\begin{aligned}\Delta &= \pi_i(\alpha_i,\alpha_i)-\pi_i(\alpha_i,\hat{\alpha}_i)\\ &= [\pi_i(\alpha_i,\alpha_i)-\pi_i(\hat{\alpha}_i,\hat{\alpha}_i)]+[\pi_i(\hat{\alpha}_i,\hat{\alpha}_i)-\pi_i(\alpha_i,\hat{\alpha}_i)]\\ &= \int_{\hat{\alpha}_i}^{\alpha_i}E_{\boldsymbol{\alpha}_{-i}}E_{\mathbf{s}}\left\{P(y,\boldsymbol{\alpha}_{-i},\mathbf{s})\right\}dy-\int_{\hat{\alpha}_i}^{\alpha_i}E_{\boldsymbol{\alpha}_{-i}}E_{\mathbf{s}}\left\{P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s})\right\}dy,\end{aligned}$$

where

$$P(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) = \tilde{p}_i^*(y,\boldsymbol{\alpha}_{-i}) + \left[\underbrace{\begin{array}{c}\left(1-\sum_j \tilde{p}_j^*(y,\boldsymbol{\alpha}_{-i})\right)\\ \cdot \sum\limits_{\forall h=r(y,\boldsymbol{\alpha}_{-i})}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{1,h}|(y,\boldsymbol{\alpha}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{r(y,\boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}})\\ \cdot p_i^{*G_{\mathbf{g}_{1,h}}}((y,\boldsymbol{\alpha}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{r(y,\boldsymbol{\alpha}_{-i})+1}^{\mathbf{s}},...,\mathbf{m}_{M+1}^{\mathbf{s}})\end{array}\right]\end{array}}_{\tilde{P}(y,\boldsymbol{\alpha}_{-i},\mathbf{s})}\right]$$

is agent i's winning probability given her type y and that she reports truthfully; and

$$P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s}) = \tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) + \left[\underbrace{\begin{array}{c}\left(1-\sum_j \tilde{p}_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})\right)\\ \cdot \sum\limits_{\forall h=r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})}^{N}\left[\begin{array}{c}\Pr{}^*(\mathbf{g}_{1,h}|(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}^{\sigma_i(y,\hat{\alpha}_i,s_i)},...,\mathbf{m}_{M+1}^{\mathbf{s}})\\ \cdot p_i^{*G_{\mathbf{g}_{1,h}}}((\hat{\alpha}_i,\boldsymbol{\alpha}_{-i}),\mathbf{m}_2^{\mathbf{s}},...,\mathbf{m}_{r(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i})+1}^{\sigma_i(y,\hat{\alpha}_i,s_i)},...,\mathbf{m}_{M+1}^{\mathbf{s}})\end{array}\right]\end{array}}_{\tilde{P}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s})}\right]$$

is agent i's winning probability given her type y and that she reports $\hat{\alpha}_i$ in stage 1 and corrects her lie when shortlisted.

By Lemma 11, we always have $\tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) \leq \tilde{p}_i^*(y;\boldsymbol{\alpha}_{-i})$. Moreover, $\tilde{p}_i^*(\cdot;\cdot) \in \{0,1\}$ by the optimal first-stage selling rule. Fix $\boldsymbol{\alpha}_{-i}$ and $\mathbf{s}$. We consider the following cases.

Case I: If $\tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) = 1$, then $\tilde{p}_i^*(y;\boldsymbol{\alpha}_{-i}) = 1$ by Lemma 11. This implies $P(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) = P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s})$, $\forall y$.

Case II: $\tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) = 0$ and $\tilde{p}_i^*(y;\boldsymbol{\alpha}_{-i}) = 1$. This implies $P(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) = 1 \geq P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s})$, $\forall y$.

Case III: $\tilde{p}_i^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i}) = 0$ and $\tilde{p}_i^*(y;\boldsymbol{\alpha}_{-i}) = 0$. In this case, we have

$$\begin{aligned} P(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) &= \left(1-\sum_j \tilde{p}_j^*(y;\boldsymbol{\alpha}_{-i})\right)\tilde{P}(y,\boldsymbol{\alpha}_{-i},\mathbf{s}), \\ P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s}) &= \left(1-\sum_j \tilde{p}_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})\right)\tilde{P}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s}). \end{aligned} \tag{75}$$

By the same arguments as in the proof of Proposition 4, we have

$$\tilde{P}(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) \geq \tilde{P}(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s}). \tag{76}$$

By our first-stage selling rule, if buyer i does not obtain the object, the other buyers' first-stage winning chances become smaller when buyer i's first-stage type becomes higher. The reason is that the change improves the expected revenue generated from optimal shortlisting, but does not change the expected revenue from the first-stage sale. We thus have $\sum_j \tilde{p}_j^*(y,\boldsymbol{\alpha}_{-i}) \leq \sum_j \tilde{p}_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})$, which means

$$\left(1-\sum_j \tilde{p}_j^*(y,\boldsymbol{\alpha}_{-i})\right) \geq \left(1-\sum_j \tilde{p}_j^*(\hat{\alpha}_i;\boldsymbol{\alpha}_{-i})\right). \tag{77}$$

Therefore, by (75), (76), and (77), we also have $P(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) \geq P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s})$, $\forall y$, for case III.

Aggregating all cases, we always have $P(y,\boldsymbol{\alpha}_{-i},\mathbf{s}) \geq P(\hat{\alpha}_i,\boldsymbol{\alpha}_{-i},\mathbf{s}), \forall y > \hat{\alpha}_i$, which immediately means $\Delta \geq 0$ when $\hat{\alpha}_i < \alpha_i$. The case of $\hat{\alpha}_i > \alpha_i$ can be similarly demonstrated. We have thus established IC for the first stage. The first-stage IR holds by construction since we set $\pi_i(\underline{\alpha},\underline{\alpha}) = 0$ and π_i is increasing in the first-stage type by (72). $\square$