Online Appendix

Bargaining under the Illusion of Transparency

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1 Online Appendix A

Proof of Proposition 1. In the proof below I denote the maximal equilibrium revenue

that the seller can achieve, given friction Δ, in the seller-offer protocol by $V_{So}(\Delta)$ and in an alternating-offer protocol by $V_A(\Delta)$. Finally, to simplify notation I still let α denote α_1. The seller's expected revenue given a TIOLI offer in $t = 1$ is $V_M = \alpha V_F + (1-\alpha)\max\{qh, l\}$. Consider the four possible sequential bargaining protocols. If the buyer makes all offers, the seller's equilibrium payoff is zero. If they alternate with the seller making the first offer, the informed seller type's payoff is bounded by $(1-e^{-\Delta})V_F$, the uninformed type's by $(1-e^{-\Delta})\max\{qh, l\}$, because any higher offer will be rejected by the respective buyer type.

Alternating Offers. Consider the protocol where the buyer makes the first offer. Three types of equilibria can arise: fully-revealing, semi-revealing and pooling. Consider full revelation. Type θ buyer at $t = 1$ names p_θ. Note $p_h = e^{-\Delta}h$ must hold because the seller must accept any price higher and, given revelation, reject any lower price. Furthermore, $p_l \in [e^{-\Delta}l, \min\{l, p_h\}]$, since any lower price will be rejected, and any higher price will violate individual rationality or separation. A tighter upper bound on p_l may hold, and will be considered in Proposition 2, but ignoring it here just strengthens the argument. For buyer separation to be incentive compatible:

$$(1-e^{-\Delta})h \geq (1-\alpha)(h-p_l) \tag{1.1}$$

must hold. Re-writing this one obtains that $\Delta > \Delta_{\min} = \ln h - \ln(\alpha h + (1-\alpha)p_l)$. Consider now the seller's ex-ante expected revenue. Suppose $l \geq qh$. Simple algebra shows that given a binding Eq.(1.1) for $V_A(\Delta) > V_M$, it must be that:

$$\Delta < \Delta_{\max} = \ln(qh) - \ln(l - p_l(1-q) + q\alpha(h-l)).$$

Since $p_l \leq l$ it follows, however, that $\Delta_{\max} \leq \Delta_{\min}$. Suppose now that $qh > l$. Simple algebra shows that given a binding Eq.(1.1) for $V_A(\Delta) \geq V_M$, it must be that:

$$\Delta \leq \Delta_{\max} = \ln(qh) - \ln\left(l\alpha(1-q) + qh - p_l(1-q)\right).$$

Since $(qh-l)(1-\alpha) \geq (1-q\alpha)(p_l - l)$, it again follows that $\Delta_{\max} \leq \Delta_{\min}$.[1]

Consider semi-revelation with serious offers. The relevant case is where the high type mixes at $t=1$ between revelation, p_h , and pooling, p_l, doing the latter with probability y. Again $p_h = e^{-\Delta}h$ must hold. If the uninformed seller type does not mix, the result follows directly from the above discussion. Suppose he accepts p_l with probability z. If $p_2 = l$, then $p_l = e^{-\Delta}l$ must hold. Now for the high type buyer to mix the following indifference must hold:

$$(1-e^{-\Delta})h = (1-\alpha)(z(h - e^{-\Delta}l) + (1-z)e^{-\Delta}(h-l)).$$

The seller's maximal revenue here is affected by z only through its impact on Δ ensuring that the above equality holds. Specifically, the seller's expected revenue is:

$$\alpha e^{-\Delta}(qh + (1-q)l) + (1-\alpha)((q-qy)e^{-\Delta}h + (1-q+qy)e^{-\Delta}l).$$

The minimal necessary separating friction $\underline{\Delta}(z)$ is given when $z=0$. Straightforward algebra shows that $e^{-\underline{\Delta}(0)}V_F \leq V_M$, hence the same is true for all $z>0$. If $p_2 = h$, it needs to be that initially $qh > l$. For the high type to mix, the indifference condition here is:

$$(1-e^{-\Delta})h = (1-\alpha)z(h - p_l),$$

where $p_l = \frac{qy}{qy+(1-q)}e^{-\Delta}h$ must hold for the uninformed seller type to mix. It is easy to see that $V_A^0(\Delta) \leq hqe^{-\Delta} + \alpha(1-q)e^{-\Delta}l \leq V_M$. Finally, in a pooling equilibrium, $V_A(\Delta) \leq \max\{l, \alpha(1-q)l + e^{-\Delta}qh\} \leq V_M$. Note also if there is an equilibrium where the buyer makes a non-serious offer, the revenue result holds a fortiori.

Seller-Offers. For $V_{So}(\Delta) > V_M$, two facts must hold.[2] First, the uninformed seller needs to sell to both types at different prices with positive probability. Second, there has to be some pooling between the uninformed and the informed seller conditional on $\theta = h$ since otherwise the high type buyer always learns the seller's type. Absent such pooling, the seller's revenue becomes separable in α, and hence the result follows immediately from Lemma 1. Let such a pooling price at $t=1$ be $p_{1,h}$.

Consider the case where the seller follows a pure pricing strategy. Let the price named by the informed seller conditional on $\theta = l$ in $t=1$ be $p_{1,l}$. In $t=2$, the uninformed seller's

[1] Note that if in a fully-revealing equilibrium the seller did not accept the buyer's initial offer, then, by revelation, the upper-bound on the seller's revenue would remain unchanged.

[2] In the case where $\alpha, \rho = 0$, there is a unique PBE.

price, given price discrimination, must be $p_2 = l$, and the informed seller has a dominant strategy. For the informed seller to pool on $p_{1,h}$ in $t = 1$, $p_{1,h} \geq \max\{e^{-\Delta}h, p_{1,l}\}$ must hold. For the high type buyer to accept $p_{1,h}$ the following IC constraint must hold:

$$h - p_{1,h} \geq e^{-\Delta}(1-\alpha)(h-l). \tag{1.2}$$

Given a binding Eq.(1.2), and setting $p_{1,l} = l$, we obtain an upper-bound on revenue given pure strategies. Let this be $\widehat{V}_{So}(\Delta)$.Simple algebra shows that:

$$\text{if } l \geq qh, \text{ then } \widehat{V}_{So}(\Delta) - V_M = \left(e^{-\Delta} - 1\right)(1-\alpha)(l - qh) \leq 0,$$
$$\text{if } qh > l, \text{ then } \widehat{V}_{So}(\Delta) - V_M = e^{-\Delta}(1-\alpha)(l-qh) \leq 0.$$

Consider the case where the seller's pricing strategy involves mixing. If the uninformed seller mixes between h and l in $t = 2$, the IC constraint on $p_{1,h}$, Eq.(1.2), can be relaxed. For this to be possible, $qh > l$ must hold. The high type buyer is indifferent between accepting or rejecting $p_{1,h}$ if:

$$h - p_{1,h} = (1-\alpha)ke^{-\Delta}(h-l),$$

where k is the probability that the uninformed seller names l in $t = 2$. For the uninformed seller to mix in $t = 2$, it has to be true that $\frac{q(1-j)}{q(1-j)+(1-q)} = l/h$ where j is the probability that the high type buyer accepts $p_{1,h}$. This is independent of α. If $k = 0$, then $p_{1,h} = h$. Here it is straightforward that the seller's revenue is below V_M. Since $p_{1,h}$ is maximal for $k = 0$, and all other prices and j are independent of k, the same holds for all k provided $p_{1,h} \geq e^{-\Delta}h$ so that the informed seller is still willing to pool. Hence again revenue is bounded from above by V_M for all α. Considering mixing in $t = 1$ alone will not relax the IC constraint on the maximal price at which the high type is willing to buy in $t = 1$. Hence it cannot boost the revenue above $V_{So}(\Delta)$.

Proof Proposition 2. Consider a fully revealing equilibrium with serious offers. Let $\alpha^\rho = (1-\widehat{\rho})\alpha + \widehat{\rho}$ replace α in Eq.(1.1). Hence, the separating friction here must satisfy: $\Delta \geq \ln h - \ln(\alpha^\rho h + (1-\alpha^\rho)p_l)$. This constraint implies a positive lower bound on Δ for all $\widehat{\rho} < 1$. In this equilibrium, $\widehat{\rho}$ imposes an upper bound on p_l of the following form:

$$p_l \leq \alpha^\rho e^{-\Delta} l + (1-\alpha^\rho) l$$

because an informed seller type cannot reject any initial offer greater than $e^{-\Delta}l$ from the low type in a perfect equilibrium. As before, $p_l = e^{-\Delta}l$ is always feasible for any $\widehat{\rho}$. Simple substitution shows, however, that, given binding incentive constraints, the lower bound on Δ in this equilibrium class, Δ^ρ_{min}, is decreasing in $\widehat{\rho}$ with $\Delta^1_{\text{min}} = 0$. The revenue condition is the same as before, except the constraint on p_l, now implies that Δ^ρ_{max} is decreasing in $\widehat{\rho}$

. Nevertheless, $\Delta^{\rho}_{\max} \geq \ln V_F - \ln V_M$ is true for all ρ. This implies that there exists ρ_A such that if $\widehat{\rho} > \rho_A$, then $\Delta^{\rho}_{\min} < \Delta^{\rho}_{\max}$. The corollary then follows form that fact that under $\Delta = 0$ or $\Delta = \infty$ no equilibrium can generate a revenue higher than V_M for any $\widehat{\rho} < 1$.

Proof of Proposition 3. Let again $\alpha^{\rho} = (1-\widehat{\rho})\alpha + \widehat{\rho}$. Consider equilibria without mixing and with price discrimination and pooling in $t = 1$ conditional on $\theta = h$. Pooling again requires that:

$$p_{1,h} \geq e^{-\Delta} h. \tag{1.3}$$

For the high type buyer to accept $p_{1,h}$ the IC constraint is now:

$$h - p_{1,h} \geq e^{-\Delta}(1-\alpha^{\rho})(h-l). \tag{1.4}$$

Combining Eq.(1.3) and Eq.(1.4) one gets the following constraint:

$$\Delta \geq \Delta^{\rho}_{\min} = \ln(h + (1-\alpha^{\rho})(h-l)) - \ln h,$$

where $\Delta^{\rho}_{\min}$ is decreasing in $\widehat{\rho}$ and becomes 0 as $\widehat{\rho}$ goes to 1.[3] Let's denote the equilibrium revenue, given a binding Eq.(1.4) and setting $p_{1,l} = l$, by $\widehat{V}^{\rho}_{So}(\Delta)$. Suppose that $qh \geq l$. Simple algebra shows that $\widehat{V}^{\rho}_{So}(\Delta) > V_M$ for all $\rho > (qh-l)/(qh-ql)$ as long as $\Delta \in (\Delta^{\rho}_{\min}, \infty)$. Suppose that $l > qh$. Here, $\widehat{V}^{\rho}_{So}(\Delta) > V_M$ requires that $\Delta < \Delta^{\rho}_{\max} = \ln(l - qh + q\rho(h-l)) - \ln(l-qh)$. The condition that $\Delta^{\rho}_{\max} > \Delta^{\rho}_{\min}$ is equivalent to $\rho > (l - qh)/(l + qh\alpha(1-\alpha)^{-1})$. Hence, there exists $\rho_{S_0} < 1$ such that if $\widehat{\rho} > \rho_{S_0}$, then there exists $\widehat{\Delta}^{\rho}_{\min} < \widehat{\Delta}^{\rho}_{\max}$ such that $V^{\rho}_{So}(\Delta) > V_M$ for all $\Delta \in (\ \widehat{\Delta}^{\rho}_{\min},\ \widehat{\Delta}^{\rho}_{\max})$.[4] Finally, note that if $\Delta = \infty$, the uninformed seller's maximal revenue is bounded by $\max\{qh, l\}$. If $\Delta = 0$, pooling requires $p_{1,h} = h$ which violates Eq.(1.3) for any $\widehat{\rho} < 1$.

Proof of Proposition 6. Let $\rho = 1 - (1-l)^{\ln\frac{1}{\delta}}$. Given $\alpha = 0$, with a change of variables, one can write:

$$\gamma(l,\delta) = \left(1 - \delta(1-l)^{\ln\frac{1}{\delta}}\right) \frac{1 - \delta(1-l)^{\ln\frac{1}{\delta}} - \sqrt{1-\delta}}{\delta^2((1-l)^{\ln\frac{1}{\delta}})^2 + \delta(1 - 2(1-l)^{\ln\frac{1}{\delta}})}.$$

[3] Note that $p_{1,h} \geq l + (h-l)(\alpha + \rho - \alpha\rho) \geq p_{1,l}$ hence separation aways holds.

[4] Note that $\widehat{V}^{\rho}_{So}(\Delta)$ is maximal at $\Delta^{\rho}_{\min}$. This is true because $\partial\widehat{V}^{\rho}_{So}(\Delta)/\partial\Delta < 0$ whenever $l \geq hq$ and provided that $\rho > \rho_{So}$ also when $qh > l$.

Consider $\gamma_\delta(l,\delta)$ given by:

$$\gamma_\delta(l,\delta) = \frac{\overbrace{\delta\left(1-l\right)^{\ln\frac{1}{\delta}}\left(2-\delta\left(1-l\right)^{\ln\frac{1}{\delta}}-2\sqrt{1-\delta}\right)-\left(2-\delta-2\sqrt{1-\delta}\right)}^{VI}}{-2\delta^2\sqrt{1-\delta}\left(\delta\left(1-l\right)^{2\ln\frac{1}{\delta}}-2\left(1-l\right)^{\ln\frac{1}{\delta}}+1\right)^2} *$$
$$* \overbrace{\left(\left(\delta-2\right)\left(1-l\right)^{\ln\frac{1}{\delta}}+2(1-\delta)\left(\ln\left(1-l\right)\right)\left(1-l\right)^{\ln\frac{1}{\delta}}+1\right)}^{VII}.$$

I first show that Term VI is positive. Simple re-arrangements show that this is equivalent to:

$$\frac{1}{2-2\sqrt{1-\delta}} < \frac{1-\delta\left(1-l\right)^{\ln\frac{1}{\delta}}}{\delta-\delta^2\left(1-l\right)^{2\ln\frac{1}{\delta}}}. \tag{1.5}$$

The LHS of Eq.(1.5) is independent of l. For any fixed δ, consider now the value of l which minimizes the RHS of Eq.(1.5). The derivative of the RHS of Eq.(1.5) with respect to l is:

$$\delta\frac{\ln\frac{1}{\delta}}{\left(\delta\left(1-l\right)^{2\ln\frac{1}{\delta}}-1\right)^2\left(1-l\right)}\left(1-l\right)^{\ln\frac{1}{\delta}}\left(\delta\left(1-l\right)^{2\ln\frac{1}{\delta}}-2\left(1-l\right)^{\ln\frac{1}{\delta}}+1\right).$$

The first-order condition is solved implicitly by l^*:

$$\left(1-l^*\right)^{\ln\frac{1}{\delta}} = \frac{1}{\delta}\left(1-\sqrt{1-\delta}\right).$$

To see that l^* actually minimizes the RHS of Eq.(1.5), note that $\delta\left(1-l\right)^{2\ln\frac{1}{\delta}}-2\left(1-l\right)^{\ln\frac{1}{\delta}}+1$ is increasing in l, it is negative if $l\to 0$, and positive if $l\to 1$. Substituting then $\left(1-l^*\right)^{\ln\frac{1}{\delta}} = \frac{1}{\delta}\left(1-\sqrt{1-\delta}\right)$ into the RHS of Eq.(1.5), one obtains that it is bounded from below by:

$$\frac{1-\left(1-\sqrt{1-\delta}\right)}{\delta-\left(1-\sqrt{1-\delta}\right)^2} = \frac{1}{2-2\sqrt{1-\delta}}.$$

Consider Term VII. If $\delta=0$, it equals 1. If $\delta\to 1$, it converges to 0. Consider now the derivative of Term VII with respect to δ. This is given by:

$$-\frac{1}{\delta}\left(\ln\left(1-l\right)-1\right)\left(1-l\right)^{\ln\frac{1}{\delta}}\left[\delta+2\ln\left(1-l\right)-2\delta\ln\left(1-l\right)\right]. \tag{1.6}$$

The sign of Eq.(1.6) is the same as that of $\delta+2\ln\left(1-l\right)-2\delta\ln\left(1-l\right)$ which is negative iff $\delta<\frac{2\ln(1-l)}{2\ln(1-l)-1}\in(0,1)$. Hence, Term VII is positive at $\delta=0$, remains positive for a while as δ increases, then becomes negative and converges from below to 0 as $\delta\to 1$. Given the continuity of $\gamma_\delta(l,\delta)$ in l, it then follows that there exists $\delta_l^*>0$ such that for all $\delta<\delta_l^*$,

$\gamma_\delta(l,\delta) > 0$, and for all $\delta > \delta_l^*$, $\gamma_\delta(l,\delta)$.

Proof Proposition 7. Fix T. Given verifiable leakage, the skimming property must still hold on the path till leakage. Let p_t be the price in round t in the absence of leakage.

Part I. Let $\alpha \geq \rho > 0$. Note that $p_T = \theta_T/2$. I proceed by induction. Let $t < T$ and suppose for all $s, \in \{t+1, ...T-1\}$, $\theta_{s+1} = \theta_s$. At t, conditional on no leakage until (including) t, p_t must maximize:

$$\max_{p_t}(\theta_t - \theta_{t+1})p_t + \theta_{t+1}^2(0.5\textstyle\sum_{s=1}^{T-t} e^{-\Delta s}\alpha(1-\alpha)^{s-1} + 0.25e^{-\Delta(T-t)}(1-\alpha)^{T-t}),$$

subject to: $\theta_{t+1} - p_t = e^{-\Delta(T-t)}(1-\rho)^{T-t}(\theta_{t+1}/2)$. Let $g = e^{-\Delta(T-t)}$, $f = (1-\rho)^{T-t}$, $n = (1-\alpha)^{T-t}$, and $b = \frac{\alpha e^{-\Delta}}{1-e^{-\Delta}(1-\alpha)} \in [0,1]$. Expressing this in θ_{t+1}, the FOC is $(\theta_t - 2\theta_{t+1})$ $(1-0.5gf) + \theta_{t+1}(b(1-gn) + 0.5gn) = 0$ with the unique solution of

$$\theta_{t+1} = \theta_t \min\{\frac{1-0.5gf}{2-gf-b(1-gn)-0.5gn}, 1\}.$$

This is internal iff:

$$A(\alpha,\rho,T-t,\Delta) \equiv 1 - 0.5gf - b + gn(b-0.5) > 0. \tag{1.7}$$

Note that $\lim_{\Delta\to 0} A(\alpha,\rho,T-t,\Delta) = 0.5(n-f)$ independent of T and t.[5] Hence, $A(\alpha,\rho,T-t,0) \leq 0$ iff $\alpha \geq \rho$. Let me describe three further properties of the function $A(\alpha,\rho,T-t,\Delta)$. **1**. $A(\alpha,\rho,T-t,\Delta)$ is strictly increasing in Δ since $\partial A(\alpha,\rho,T-t,\Delta)/\partial\Delta$ is given by:

$$-\frac{\partial b}{\partial \Delta}(1-gn) - \frac{\partial g}{\partial \Delta}n(0.5(1+\frac{f}{n}) - b) > 0,$$

where the inequality follows because (i), $1 - gn > 0$; (ii), $0.5(1+\frac{f}{n}) > b$, since $f \geq n$ $\Leftrightarrow \alpha \geq \rho$; and (iii), $\frac{\partial b}{\partial\Delta}, \frac{\partial g}{\partial\Delta} < 0$. **2**. $A(\alpha,\rho,T-t,\Delta)$ is strictly increasing in ρ since f is strictly decreasing in ρ. **3**. $A(\alpha,\rho,T-t,\Delta)$ is strictly decreasing in α since $b(1-gn)+gn$ is strictly increasing in α and n is strictly decreasing in α. There then exists $L^*(\alpha,\rho,T) \geq 0$ such that the seller never bargains iff $\Delta \leq L^*(\alpha,\rho,T)$. Furthermore, $L^*(\alpha,\rho,T)$ is increasing in α, decreasing in ρ. Since the seller never bargains in equilibrium iff $A(\alpha,\rho,T-t,\Delta) \leq 0$ for all $t < T$, it follows that if the seller never bargains given a horizon T then the same holds given any horizon $T' < T$.

Part II. Let $\rho \geq \alpha$ and $\Delta > 0$. Consider any $t < T$. Let $V_t(\theta_t)$ be the seller's value function in round t conditional on no leakage until then. Given the buyer's strategy, this is

[5] The second-order condition is always satisfied since $-2 + gf + b(1-gn) + 0.5gn < 0$.

given by:

$$V_t(\theta_t) = (\theta_t - \theta_{t+1})(\theta_{t+1} - e^{-\Delta}(1-\rho)(\theta_{t+1} - p_{t+1}(\theta_{t+1})))+ \\ + e^{-\Delta}(\alpha\theta_{t+1}^2/2 + (1-\alpha)V_{t+1}(\theta_{t+1})),$$

where $p_{t+1}(\theta_{t+1})$ is the equilibrium price, in the absence of leakage, in $t+1$. Wlog let $p_t \leq \theta_t$ for any t. Consider now a perturbation of θ_{t+1}. Its impact on $V_t(\theta_t)$ is given by:

$$(\theta_t - 2\theta_{t+1})(1 - e^{-\Delta}(1-\rho)) - e^{-\Delta}(1-\rho)p_{t+1}(\theta_{t+1})+ \\ + (\theta_t - \theta_{t+1})e^{-\Delta}(1-\rho)p'_{t+1}(\theta_{t+1}) + \alpha e^{-\Delta}\theta_{t+1} + e^{-\Delta}(1-\alpha)V'_{t+1}(\theta_{t+1}).$$

I proceed by contradiction. Suppose that $\theta_{t+1} = \theta_t$. An upper-bound on $V'_{t+1}(\theta_{t+1})$ is given by p_{t+1}. Hence, the above impact is bounded from above by:

$$-\theta_{t+1}(1 - e^{-\Delta}) + e^{-\Delta}(\rho - \alpha)(p_{t+1}(\theta_{t+1}) - \theta_{t+1}) < 0,$$

where the inequality follows from the fact that $\Delta > 0$ and $\rho \geq \alpha$. Hence lowering θ_{t+1} would raise the seller's payoff, a contradiction .

Proof of Corollary 3. Note that $R(\rho, \alpha, \Delta) = \alpha \sum_{t=0}^{\infty}(1-\alpha)^t(\lambda\gamma)^t = \frac{\alpha}{1-(1-\alpha)\lambda\gamma}$. Since $\lambda\gamma$ is increasing in α and decreasing in ρ the first part follows and $\lim_{\Delta\to 0} \frac{\alpha}{1-(1-\alpha)\lambda\gamma} = 1$ for any $\rho, \alpha > 0$.

Proof of Propostion 10. Lemma 1 in Online Appendix B characterizes $\lim_{\Delta\to 0}\gamma(\Delta)$. The proof of Proposition 5 shows that $\lim_{\Delta\to 0} V_S^{\alpha,\rho}(\Delta) = \lim_{\Delta\to 0}[\alpha(\Delta)V_F + (1-\alpha(\Delta))\gamma(\Delta)/2]$.

Proof of Corollary 4. Under the single-offer scheme, since at any point the seller is either informed or not, by dynamic consistency, the seller either immediately quotes the static monopoly price or simply waits. The seller's optimal expected revenue is then the maximum of the static monopoly profit and the present value of leakage. The total probability that leakage occurs over ω amount of real time is $1 - \lim_{\Delta\to 0}(1 - \xi\Delta^{\phi})^{\omega/\Delta}$. Hence, if $\phi > 1$, the revenue under the single-offer scheme is V_M; if $\phi = 1$, it is $\max\{V_M, \int_0^{\infty} V_F \xi e^{-t(\xi+1)}dt\}$; if $\phi < 1$, it is V_F .

1.1 Online Appendix B

Lemma 1 (Lemma for Proposition 10) *Consider any $\beta, \xi \in (0, \infty)$. Let $\kappa < 0.5$, then $\lim_{\Delta\to 0} \gamma(\Delta) = 1$. Let $\kappa > 0.5$, then $\lim_{\Delta\to 0}\gamma(\Delta) = 1$ when $\phi < 1$, $\lim_{\Delta\to 0}\gamma(\Delta) = \frac{\xi}{\xi+1}$ when $\phi = 1$, and $\lim_{\Delta\to 0}\gamma(\Delta) = 0$ when $\phi > 1$. Let $\kappa = 0.5$, then $\lim_{\Delta\to 0} \gamma(\Delta) = \frac{\beta}{\beta+1}$ if $\phi > 1$ and $\lim_{\Delta\to 0} \gamma(\Delta) = 1$ when $\phi < 1$.*

Proof. In the proof below, I use the facts that $\lim_{\Delta\to 0}(1-e^{-\Delta})\Delta^z = 0$ if $z > -1$ and $\lim_{\Delta\to 0}(1-e^{-\Delta})\Delta^z = 1$ if $z = -1$. After substituting in $\alpha = \xi\Delta^\phi$ and $\rho = \beta\Delta^\kappa$, the seller's initial price can be separated into two terms with the appropriate signs:

$$\overbrace{\frac{(1-(1-\beta\Delta^\kappa)e^{-\Delta})^2 - 0.5\xi\Delta^\phi e^{-\Delta}}{e^{-2\Delta}(1-\beta\Delta^\kappa)^2 - e^{-\Delta}(1-2\beta\Delta^\kappa) - \xi\Delta^\phi e^{-\Delta}}}^{\text{Term } I}$$

$$\overbrace{(\frac{(1-e^{-\Delta})\left(1-e^{-\Delta}(1-\beta\Delta^\kappa)\right)^2 + 0.25\xi^2\Delta^{2\phi}e^{-2\Delta}}{(e^{-2\Delta}(1-\beta\Delta^\kappa)^2 - e^{-\Delta}(1-2\beta\Delta^\kappa) - \xi\Delta^\phi e^{-\Delta})^2}}^{\text{Term } II})^{1/2}.$$

Step I. Re-arranging terms, the limit of Term I becomes:

$$\lim_{\Delta\to 0}(1 + \frac{-1+0.5\xi\Delta^\phi + e^\Delta}{e^{-\Delta}(1-\beta\Delta^\kappa)^2 - (1-2\beta\Delta^\kappa) - \xi\Delta^\phi}). \tag{1.8}$$

Applying L'Hôpital's rule, the limit of the second term in Eq.(1.8) equals:

$$\lim_{\Delta\to 0}\frac{-(0.5\xi\phi\Delta^{\phi-1} + e^\Delta)}{1-2\beta\Delta^\kappa + \beta^2\Delta^{2\kappa} - 2\kappa\beta(1-e^{-\Delta})\Delta^{\kappa-1} + \xi\phi\Delta^{\phi-1} - 2\kappa\beta^2\Delta^{2\kappa-1}}. \tag{1.9}$$

A. Consider $\phi > 1$. If $\kappa > 0.5$, Eq.(1.9) converges to -1; if $\kappa < 0.5$, to 0. Hence, Term I converges to 0 and 1 respectively. **B.** Consider $\phi = 1$. If $\kappa > 0.5$, Term I converges to $1 - \frac{0.5\xi+1}{1+\xi} = \frac{0.5\xi}{1+\xi}$; if $\kappa < 0.5$, to 1. **C.** Consider $\phi < 1$. Multiply both the numerator and the denominator of Eq.(1.9) by $\Delta^{1-\phi}$ to obtain:

$$\lim_{\Delta\to 0}\frac{-0.5\xi\phi}{-2\kappa\beta(1-e^\Delta)\Delta^{\kappa-\phi} + \xi\phi - 2\kappa\beta^2\Delta^{2\kappa-\phi}}.$$

If $2\kappa > \phi$, the above goes to -0.5, so Term I to 0.5. If $2\kappa < \phi$, the above goes to 0, so Term I to 1. Finally, if $2\kappa = \phi$, the above goes to $\frac{-0.5\xi}{\xi-\beta^2}$, and Eq.(1.8) to $\frac{0.5\xi-\beta^2}{\xi-\beta^2}$. **D**. Finally, consider the case of $\kappa = 0.5$. It follows that if $\phi > 1$, Term I converges to $\frac{\beta^2}{-1+\beta^2}$, and if $\phi < 1$, it converges to 0.5.

Step II. Let's separate Term II into the two additive parts. Consider the first additive part. Applying L'Hôpital's rule to the first part of Term II inside the bracket one gets that limit is the limit of the fraction where the numerator is:

$$\left(\Delta^\kappa\beta e^{-\Delta} - e^{-\Delta} + 1\right)\left(3(1-e^{-\Delta}) - 2\Delta^\kappa\beta + 2\Delta^{\kappa-1}\kappa\beta(1-e^{-\Delta}) + 3\Delta^\kappa\beta e^{-\Delta}\right),$$

and the denominator is:

$$2((2\Delta^{\kappa}\beta-1)(1-e^{-\Delta})-\Delta^{\phi}\xi+\Delta^{2\kappa}\beta^2e^{-\Delta})*$$
$$(-1+2\beta(\Delta^{\kappa-1}\kappa-\Delta^{\kappa})(1-e^{-\Delta})+\Delta^{\phi}\xi-\Delta^{\phi-1}\phi\xi-2\Delta^{2\kappa}\beta^2e^{-\Delta}+2\Delta^{2\kappa-1}\kappa\beta^2e^{-\Delta}+2\Delta^{\kappa}\beta e^{-\Delta}).$$

A. Consider $\kappa \geq 1$. Multiplying both the numerator and the denominator by Δ^{-1}, one obtains that the former converges to zero and the latter to a non-zero amount. Hence, the ratio goes to 0. **B.** Consider $\kappa \in (0.5, 1)$. Multiply both the numerator and the denominator by $\Delta^{-2\kappa}$ and simplifying terms. The numerator converges to $(\beta)(2\kappa\beta+\beta)$. Let $\phi > 2\kappa$. It follows that $-\Delta^{\phi-1}\phi\xi+\Delta^{2\kappa-1}2\kappa\beta^2 = \Delta^{2\kappa-1}(-\Delta^{z}\phi\xi+2\kappa\beta^2)$ for some $z > 0$. Hence, since the denominator goes to infinity, the ratio goes to zero. Let $\phi < 2\kappa$. It follows that $-\Delta^{\phi-1}\phi\xi+\Delta^{2\kappa-1}2\kappa\beta^2 = \Delta^{\phi-1}(-\phi\xi+\Delta^{z}2\kappa\beta^2)$ for some $z > 0$. Hence, the ratio again goes to 0. Consider $\phi = 2\kappa$, the ratio again converges to 0. **C.** If $\kappa = 0.5$, then again by multiplying both sides by $\Delta^{-2\kappa}$, the ratio goes to $\beta^2/\left(-1+\beta^2\right)^2$ if $\phi > 1$ and to 0 if $\phi < 1$. **D.** Finally, if $\kappa < 0.5$, multiplying both sides by $\Delta^{-2\kappa}$, we get again that the numerator goes to a positive finite amount for any $\beta > 0$ and, given the above, that the ratio goes to zero.

Step III. Consider now the second additive part from Term II. Applying L'Hôpital's rule, and simplifying terms, the limit equals:

$$\lim_{\Delta\to 0}\frac{-\Delta^{2\phi-1}\xi^2\left(\Delta-\phi\right)}{4\left(e^{-\Delta}-\Delta^{\phi}\xi+\Delta^{2\kappa}\beta^2e^{-\Delta}-1\right)}* \\ *\frac{1}{\left(-1+\Delta^{\phi}\xi-\Delta^{\phi-1}\phi\xi-2\Delta^{2\kappa}\beta^2+2\Delta^{2\kappa-1}\kappa\beta^2+2\Delta^{\kappa}\beta\right)}. \tag{1.10}$$

By multiplying both the numerator and the denominator by $\Delta^{-\phi}$ and eliminating vanishing terms, Eq. (1.10) becomes:

$$\lim_{\Delta\to 0}\frac{-\Delta^{\phi}\xi^2+\Delta^{\phi-1}\xi^2\phi}{4\left(-(1-e^{-\Delta})\Delta^{-\phi}-\xi+\Delta^{2\kappa-\phi}\beta^2\right)\left(-1+\Delta^{\phi}\xi-\Delta^{\phi-1}\phi\xi+2\Delta^{2\kappa-1}\kappa\beta^2\right)}. \tag{1.11}$$

A. Consider $\phi > 1$. The above goes to zero both if $2\kappa \geq \phi$ and if $2\kappa < \phi$. **B.** Consider $\phi = 1$. If $\kappa < 0.5$, the above converges to 0. If $\kappa > 0.5$, the above converges to $\frac{\xi^2}{4(-1-\xi)(-1-\xi)} = (\frac{0.5\xi}{1+\xi})^2$ where I again used the fact that $\lim_{\Delta\to 0}(1-e^{-\Delta})\Delta^{-1} = 1$. **C.** Consider $\phi < 1$. If $2\kappa > \phi$, the above converges to $(\frac{1}{2})^2$; if $2\kappa < \phi$, to 0; if $2\kappa = \phi$, to $(\frac{0.5\xi}{-\xi+\beta^2})^2$. **Step IV.** Collecting the terms and accounting for the sign of the denominator of Term II inside the bracket, depending on κ and ϕ, the result follows .

Claim 1 *Fix any $\Delta > 0$ and $\rho \geq \alpha \geq 0$. The perfect equilibrium characterized by Propositions 5 and 8 is the unique limit in terms of prices and payoffs of the sequence of perfect* *equilibria of the finite T-horizon games as $T \to \infty$.*

Proof. Consider a finite $T-$horizon game. The informed seller's stationary strategy is just as before. The proof of Proposition 7 shows that the skimming property holds and if the first-order condition for the seller's best response requirement in round t is satisfied, then in that round the corresponding second-order condition is also satisfied. Furthermore, the seller's equilibrium price and revenue at the beginning of the final round T is unique and linear. Suppose then that $p_t = \gamma_t \theta_t$ and $V_t(\theta_t) = \phi_t \theta_t^2$, where $V_t(\theta_t)$ is the uninformed seller type's ex ante expected equilibrium value function at round t when the highest remaining buyer type is θ_t. Let's proceed by induction. By the skimming property, given the marginal buying-type's indifference condition , $\theta_{t+1} - p_t = e^{-\Delta}(1-\rho)(\theta_{t+1} - p_{t+1})$ and the above induction hypothesis, the seller's round t dynamic optimization condition, given state variable θ_t is

$$\max_{\theta_{t+1} \le \theta_t} (\theta_t - \theta_{t+1})\theta_{t+1}(1 - e^{-\Delta}(1-\rho) + e^{-\Delta}(1-\rho)\gamma_{t+1}) + \\ e^{-\Delta}(\alpha 0.5\theta_{t+1}^2 + (1-\alpha)V_{t+1}(\theta_{t+1})). \tag{1.12}$$

Solving this, after some re-arrangements, one obtains that the unique solution is given by:

$$\theta_{t+1} = \theta_t \max\{\frac{1 - e^{-\Delta}(1 - \gamma_{t+1} - \rho + \gamma_{t+1}\rho)}{2(1 - e^{-\Delta}(1 + 0.5\alpha - \gamma_{t+1} - \rho + \phi_{t+1}(1-\alpha) + \gamma_{t+1}\rho))}, 1\}. \tag{1.13}$$

Given the proof of Proposition 7, Part II. if $\Delta > 0$ and $\rho \ge \alpha$, then this solution must be internal.

Substituting the above iteration back into the seller's round t best response condition, given the induction hypothesis, the uninformed seller's type round $t+1$ value function can be written as $V_{t+1}(\theta_{t+1}) =$

$$\frac{\theta_{t+1}^2}{4} \frac{\gamma_{t+2}^2 e^{-2\Delta}(1-\rho)^2 + 2\gamma_{t+2}e^{-\Delta}(1-\rho)(1 - e^{-\Delta}(1-\rho)) + (1 - (1-\rho)e^{-\Delta})^2}{1 - e^{-\Delta}(1 + 0.5\alpha - \gamma_{t+2} - \rho + \phi_{t+2} + \gamma_{t+2}\rho - \alpha\phi_{t+2})}. \tag{1.14}$$

Hence, both aspects of the induction hypothesis are verified and there is a unique solution in prices and payoffs given any finite T.

Note that, given the marginal buying type's indifference condition and Eq. (1.13), after some algebra, one obtains the following expression determining γ_t:

$$\gamma_{t+1} = \frac{1}{2} \frac{(1 - e^{-\Delta}(1 - \gamma_{t+1} - \rho + \gamma_{t+1}\rho))(1 - e^{-\Delta}(1-\rho)(1 - \gamma_{t+2}))}{1 - e^{-\Delta}(1 + 0.5\alpha - \gamma_{t+2} - \rho + \phi_{t+2} + \gamma_{t+2}\rho - \alpha\phi_{t+2})}. \tag{1.15}$$

Diving Eq.(1.14) by Eq.(1.15), it follows that $\frac{V_{t+1}(\theta_{t+1})}{\gamma_{t+1}} = \frac{\theta_{t+1}^2}{2}$. In turn, the above condition then implies the following difference equation:

$$\gamma_t = \Phi(\gamma_{t+1}) \equiv \frac{0.5(1-e^{-\Delta}(1-\rho)(1-\gamma_{t+1}))^2}{1-e^{-\Delta}(1+0.5\alpha-\gamma_{t+1}-\rho+0.5\gamma_{t+1}+\gamma_{t+1}\rho-0.5\alpha\gamma_{t+1})}. \tag{1.16}$$

Let me first show that $\Phi(\gamma)$ is increasing. Consider its first derivative. This is given by:

$$\begin{aligned}
&\overbrace{-\frac{1}{2}e^{-\Delta}\frac{1-e^{-\Delta}(1-\rho)(1-\gamma)}{(1-e^{-\Delta}(1-\rho)(1-\gamma)-0.5e^{-\Delta}(\alpha+\gamma-\alpha\gamma))^2}}^{Term\ 0}* \\
&*\overbrace{(0.5\alpha(1+e^{-\Delta}(1-\gamma)(1-\rho))+e^{-\Delta}(1-\rho)^2+}^{Term\ 1} \\
&+0.5e^{-\Delta}(1-\gamma)(1-\rho)+\rho+\gamma(1-\rho)\rho e^{-\Delta}-1.5
\end{aligned} \tag{1.17}$$

Note that Term 0 is always negative. I now show that Term I is also negative. Note that Term I is increasing in α. To then obtain an upper bound, given any ρ, set $\alpha=\rho$. After some rearrangements, one obtains that Term 1 can be bounded from above by:

$$-1.5(1-\rho)(1-e^{-\Delta})-0.5e^{-\Delta}(\gamma(1-\rho)+\rho(1-\gamma)-\rho^2(1-\gamma))<0.$$

Let me show that $\Phi(\gamma)$ is also convex. Consider now the second derivative of Eq.(1.16). After some algebra, one obtains that this is given by:

$$\frac{e^{-2\Delta}}{2}\frac{0.5\alpha^2-\alpha(1-e^{-\Delta}(1-\rho))+0.5e^{-2\Delta}(1-\rho)^2-e^{-\Delta}(1-\rho)+0.5}{(0.5\gamma e^{-\Delta}(1+\alpha)-0.5\alpha e^{-\Delta}-e^{-\Delta}(1-\rho)-\gamma e^{-\Delta}\rho+1)^3}. \tag{1.18}$$

The above denominator is always positive. This is true because $0.5\gamma e^{-\Delta}+e^{-\Delta}(1-\gamma)(\rho-0.5\alpha)+1-e^{-\Delta}>0$ as long as $\rho-0.5\alpha\geq 0$. Consider now the numerator. Note that $\alpha(0.5\alpha-e^{-\Delta}\rho+e^{-\Delta}-1)$ is decreasing in α given the assumption that $\rho\geq\alpha$. It follows that $\alpha-e^{-\Delta}\rho+e^{-\Delta}-1<(1-e^{-\Delta})(\rho-1)<0$. To obtain a lower bound on the numerator, set $\alpha=\rho$. After some re-arrangements, one obtains that the numerator is bounded from below by $-e^{-\Delta}(1-\rho)^2+0.5e^{-2\Delta}(1-\rho)^2>0$, where the inequality follows since $0.5-e^{-\Delta}(1-0.5e^{-\Delta})>0$. Finally, note that $0<\Phi(0)$ and $\Phi(1)=(2-e^{-\Delta})^{-1}\leq 1$.

Since Φ is increasing and convex on $[0,1]$, it then has a unique fixed point on $[0,1]$.

Consider now a fixed t, as $T\to\infty$, the value of $\gamma_{t,T}$ converges to the unique fixed point of Φ. Solving for this unique fixed point, one obtains that this point $\overline{\gamma}(\Delta,\rho,\alpha)$ is given by:

$$\frac{(1-(1-\rho)e^{-\Delta})^2-0.5\alpha e^{-\Delta}-\sqrt{(1-e^{-\Delta})(1-e^{-\Delta}(1-\rho))^2+0.25\alpha^2e^{-2\Delta}}}{e^{-2\Delta}(1-\rho)^2-e^{-\Delta}(1-2\rho)-\alpha e^{-\Delta}}. \tag{1.19}$$

Note that the above equals the expression for $\gamma(\Delta,\rho,\alpha)$ in the proof of Proposition 5.

In turn, since $\frac{V_{t+1}(\theta_{t+1})}{\gamma_{t+1}} = \frac{\theta_{t+1}^2}{2}$, the uninformed seller's equilibrium continuation value also converges to that under Proposition 5. Hence, for any given Δ, the seller's price sequence and the payoffs also converge to those identified by Proposition 5 .