Online Appendix

Uncovering the Effects of the Zero Lower Bound with an Endogenous Financial Wedge

Dan Cao, Georgetown University
Wenlan Luo, Tsinghua University
Guangyu Nie, Shanghai University of Finance and Economics

Contents

A Multiple Equilibria with \((k_0, b_{-1})\) as State Variables A-2

B Two-Period Economy with Natural Borrowing Limit A-3
  B.1 Last-Period Equilibrium .................................................. A-4
  B.2 Derivations of the AS-AD curves ........................................ A-4
  B.3 Graphical Representations for Proposition 1 (Part 1) .............. A-7

C Proof of Proposition 1 A-9

D Proof of Proposition 2 A-13
  D.1 Thresholds for Collateral Constraint and Equilibrium Existence .......... A-14
  D.2 Equilibrium Properties with Binding Collateral Constraint ............ A-19
  D.3 Equilibrium Properties with Non-binding Collateral Constraint ......... A-34
  D.4 Equilibrium Non-Existence ................................................ A-35

E Investment Friction and Endogenous Asset Price A-36
  E.1 Natural Borrowing Limit .................................................. A-36
  E.2 Tighter Borrowing Limit .................................................. A-38

F Proof of Proposition 3 A-40
  F.1 Equilibrium Properties .................................................. A-40
  F.2 Threshold for Binding Irreversibility Constraint ..................... A-43
  F.3 Region with binding irreversibility Constraint ........................ A-44
  F.4 Region with Non-binding Irreversibility Constraint .................... A-45
  F.5 AS-AD Representation .................................................. A-48

G Proof of Proposition 4 A-48
  G.1 Equilibrium Properties with Binding Collateral Constraint ............ A-49
  G.2 Thresholds for Binding Collateral Constraint and Irreversibility .... A-50
  G.3 Region with Non-binding Collateral Constraint ........................ A-59
  G.4 Region with Binding Collateral Constraint and Binding Irreversibility Constraint A-60
A.5 Regions with Binding Collateral Constraint and Non-binding Irreversibility Con-straint . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . A-63
A.6 AS-AD Representation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . A-73

H More Details from the Quantitative Model A-74
H.1 Complete Setup . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . A-74
H.2 Global Solution Method . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . A-77
H.3 Policy Functions and ZLB duration from the Quantitative Model . . . . . . . . . . A-81
H.4 Asset Prices in the Data and in the Model . . . . . . . . . . . . . . . . . . . . . . . . A-84
H.5 Numerical Error Analyses . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . A-85
H.6 Comparisons with Piecewise-linear Solutions . . . . . . . . . . . . . . . . . . . . . . A-86

I Representative Agent Model with Exogenous Wedges A-89

A Multiple Equilibria with \((k_0, b_{-1})\) as State Variables

One key reason why we solve the wealth-recursive equilibrium using \((k_0, \omega_0)\) as state variables, instead of a probably more straightforward choice \((k_0, b_{-1})\) is that, there can be multiple equilibria if we use the latter as state variables. Figure A.1 provides two such examples. The left panel in the figure plots the policy functions in \(\omega_0\) fixed a value of \(k_0\) (1.85). In the top-left figure, \(b_{-1}\) is generated using equation (4). We see that \(b_{-1}\) is non-monotone in \(\omega_0\). Therefore, if we use \((k_0, b_{-1})\) as the state variables, then when \(b_{-1}\) lies between \(-1.806\) and \(-1.805\), there are two equilibria with binding ZLB and binding borrowing constraint, but with different values of output, labor supply, and markup. Similarly, the right panel in the figure shows that when \(b_{-1}\) lies between \(-1.90\) and \(-1.89\), there are two equilibria, one with binding ZLB and the other one with non-binding ZLB. For the same parameter values, Proposition 2 shows that there exists an unique equilib-rium with \((k_0, \omega_0)\) as state variables.

Why are there multiple equilibria with \((k_0, b_{-1})\) as state variables? To understand this issue, we use the definition of wealth share (4) at \(t = 0\):

\[
\omega_0 = \frac{R_0^K k_0 + b_{-1}}{R_0^K k_0}.
\]

When agents in the economy expect higher \(R_0^K\), the entrepreneurs’ wealth share is higher (\(b_{-1}\) is negative), which leads them to invest more. This implies higher aggregate demand and hence higher relative price of entrepreneurs’ output, higher wage and higher level of
labor supply from the households. All these factors support a higher value $R^K_0$ because

$$R^K_0 = 1 - \delta + P^e_0 A_0^{1-a} \left( \frac{L_0}{k_0} \right)^{1-a},$$

making the expectation self-fulfilling. On the other hand, if agents expect lower $R^K_0$ then the same factors imply a lower value of $R^K_0$. For this mechanism to generate multiple equilibria, we need investment to be significantly responsive to entrepreneurs’ wealth share. This is more likely to be the case when the borrowing constraint binds as shown by strictly positive multipliers in Figure A.1. The figure also shows that, when two equilibria exists with the same value of $(k_0, b_{-1})$, one equilibrium features higher $R^K_0$, output, labor supply, and entrepreneurs’ wholesale price than the other.

![Figure A.1: Multiple Equilibria when using $(k_0, b_{-1})$ as States](image)

*Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$, $m = 0.9$ and $c = 21$. $k_0 = 1.85$."

**B Two-Period Economy with Natural Borrowing Limit**

In this appendix, we first provide the explicit solution for equilibrium in the last period $t = 1$ of the two-period economy. Then, we derive the expression for the AS-AD curves at $t = 0$ and their properties.
B.1 Last-Period Equilibrium

In the last period, there are no borrowing or lending, and thus we have $b_1 = 0$. The entrepreneurs makes no further investment either, i.e., $k_2 = 0$. We assume the markup takes its steady state value, $X_1 = X^*$. In the last period, the entrepreneurs and households consume all their wealth. From equations (3a),(3b), and (5a), (6a) at $t = 1$, we obtain

$$c_1 = R^K_1 k_1 \omega_1, \quad (A.1a)$$

and

$$c'_1 = R^K_1 k_1 (1 - \omega_1) + \left(1 - \frac{\alpha}{X^*}\right) Y_1. \quad (A.1b)$$

Given $\omega_1$ and $k_1$, for a labor supply $L_1$, we can solve $w_1$ and $R^K_1$ from (6d) and (6e), and $c_1$ and $c'_1$ from (A.1a) and (A.1b). Lastly, from (5b) at $t = 1$, we solve for $L_1$ from the following equation:

$$\frac{1 - \alpha}{X^*} L^{-\alpha}_1 - \left(1 - \frac{\alpha}{X^*} \omega_1\right) L^{1-\alpha}_1 = (1 - \omega_1) (1 - \delta) \left(\frac{k_1}{A_1}\right)^{1-\alpha}. \quad (A.1c)$$

It follows that $L_1$ is decreasing in $k_1$ and increasing in $\omega_1$.

B.2 Derivations of the AS-AD curves

By the result that $R^K_1 = R_0$ and the expression for $R^K_1$ in (6e), we obtain

$$R^K_1 = 1 - \delta + \frac{\alpha}{X^*} A^{1-\alpha}_1 \left(\frac{k_1}{L_1}\right)^{a-1} = R_0.$$ 

So

$$\frac{k_1}{L_1} = \left[(R_0 - 1 + \delta) \frac{X^*}{A^{a-1}_1} \right]^{\frac{1}{a-1}}. \quad (A.2)$$

From the households’ Euler equation and intra-temporal condition at $t = 1$, we obtain

$$c'_0 = \frac{c'_1}{\beta R_0} = \frac{w_1}{\beta R_0}.$$ 

In addition

$$w_1 = \frac{1 - \alpha}{X^*} A^{1-\alpha}_1 \left(\frac{k_1}{L_1}\right)^a.$$
Plugging the expression for $k_1/L_1$, (A.2), into this equation for $w_1$, we obtain the expression for $c'_0$:

$$c'_0 = \frac{1}{\beta R_0} \frac{1 - \frac{\lambda}{X^*}}{A_1} \left( R_0 - 1 + \frac{\delta}{\alpha} \right)^{\frac{\alpha}{1-\alpha}}. \quad (A.3)$$

From the households’ intra-temporal condition at $t = 0$,

$$w_0 = c'_0 = \frac{w_1}{\beta R_0}. \quad (A.3)$$

Therefore,

$$\frac{1 - \frac{\lambda}{X_0}}{A_0} A_0^{1-\alpha} \left( \frac{k_0}{L_0} \right)^{\frac{\alpha}{L_1}} = \frac{1}{\beta R_0} \frac{1 - \frac{\lambda}{X^*}}{A_1} \left( \frac{k_1}{L_1} \right)^{\frac{\alpha}{L_1}}$$

or equivalently,

$$\frac{L_0}{k_0} = \left( \beta R_0 \frac{X^*}{X_0} \frac{A_0^{1-\alpha}}{A_1} \right)^{\frac{1}{\alpha}} \frac{L_1}{k_1} = \frac{1}{A_0} \left( \beta R_0 \frac{A_0}{A_1} \frac{X^*}{X_0} \right)^{\frac{1}{\alpha}} \left( (R_0 - 1 + \frac{\delta}{\alpha}) \frac{\lambda}{X^*} \right)^{\frac{1-\alpha}{\alpha}}, \quad (A.4)$$

where the second inequality is obtained from (A.2). Plugging this expression for $L_0/k_0$ into the expression for $R^K_0$, (6e) at $t = 0$, we arrive at

$$R^K_0 = 1 - \delta + \frac{A_0}{X_0} \alpha \left( \frac{L_0}{k_0} \right)^{1-\alpha} = 1 - \delta + \left( \frac{A_0}{A_1} \beta R_0 \right)^{1-\alpha} \left( \frac{X^*}{X_0} \right)^{\frac{1}{\alpha}} \left( (R_0 - 1 + \delta) \frac{\lambda}{X^*} \right)^{\frac{1-\alpha}{\alpha}}. \quad (A.5)$$

From the entrepreneurs’ budget constraint

$$c_0 + \frac{1}{R_0} c_1 = R^K_0 k_0 \omega_0,$$

and the Euler equation

$$c_1 = \gamma R_0 c_0,$$

we obtain

$$c_0 = \frac{1}{1 + \gamma} R^K_0 k_0 \omega_0.$$

Combining this with the last expression for $R^K_0$, we obtain

$$c_0 = \frac{1}{1 + \gamma} R^K_0 \omega_0 k_0, \quad (A.6)$$

where $R^K_0$ given in (A.5) is the value of each unit of capital at time 0.
Lastly, the investment of the entrepreneurs is given by

$$I_0 = k_1 - (1 - \delta) k_0.$$  \hfill (A.7)

From the feasibility constraint at \( t = 1 \),

$$c_1 + c'_1 = A_1^{1-\alpha} k_1 \left( \frac{L_1}{k_1} \right)^{1-\alpha} + (1 - \delta) k_1$$

$$= k_1 \left( 1 - \delta + (R_0 - 1 + \delta) \frac{X^*}{\alpha} \right).$$

Therefore,

$$k_1 = \frac{c_1 + c'_1}{1 - \delta + \frac{X^*}{\alpha} (R_0 - 1 + \delta)},$$  \hfill (A.8)

with

$$c_1 = \gamma R_0 c_0,$$$$
c'_1 = \frac{1 - \alpha}{X^*} A_1 \left( (R_0 - 1 + \delta) \frac{X^*}{\alpha} \right)^{\frac{\alpha}{\alpha - 1}},$$

and the expressions for \( c_0 \) is given by (A.6).

For the aggregate supply equation, we first obtain the expression for \( w_0 X_0 \) from (6d):

$$w_0 X_0 = A_0^{1-\alpha} (1 - \alpha) \left( \frac{k_0}{L_0} \right)^{\alpha}.$$

Then from the expression for \( L_0/k_0 \) in (A.4) and the expression for \( Y_0^{AS} \) as a function of \( w_0 X_0 \) from (9), we obtain:

$$Y_0^{AS} = \frac{X^*}{\alpha} \left( \beta R_0 \frac{A_0 X^*}{A_1 X_0} \right)^{\frac{1-\alpha}{\alpha}} (R_0 - 1 + \delta) k_0.$$  \hfill (A.9)

**Lemma 1.** When the ZLB does not bind, the aggregate supply curve is upward sloping in \( R_0 \). There exists \( \bar{R}_0 \) such that aggregate demand curve is downward sloping when \( R_0 \leq \bar{R}_0 \).

**Proof.** Given the ZLB is not binding, we can set \( X_0 = X^* \) in the AS curve (A.9) and AD curve (8). Then it is easy to see that the AS curve is increasing in \( R_0 \). The AD curve
Since \( L^P_0 \) becomes
\[
Y^P_0 = \left[ \frac{1}{\beta R_0} + \frac{1}{1 - \delta + \frac{X^*}{\alpha} (R_0 - 1 + \delta)} \right] \left[ \frac{1 - \alpha}{X^*} A_1 \left( \frac{R_0 - 1 + \delta}{X^*} \right) \right] \frac{\delta}{\alpha} \gamma \omega_0 \left[ 1 - \delta + \left( \frac{A_0}{A_1} \frac{R_0}{\alpha} \right) \frac{1 - \alpha}{\alpha} \right] \frac{1}{\beta} \frac{\delta}{\alpha} (R_0 - 1 + \delta) k_0 - (1 - \delta) k_0.
\]

On the right-hand side, the first term is decreasing in \( R_0 \), and the second term is increasing in \( R_0 \). Taking derivative of \( Y^P_0 \) with respect to \( R_0 \), we have
\[
\frac{\partial Y^P_0}{\partial R_0} = - \left[ \frac{1}{\beta R_0^2} + \frac{X^*}{\alpha} (R_0 - 1 + \delta) \right] \left[ \frac{1 - \alpha}{X^*} A_1 \left( \frac{R_0 - 1 + \delta}{X^*} \right) \right] \frac{\delta}{\alpha} \gamma \omega_0 \left[ 1 - \delta + \left( \frac{A_0}{A_1} \frac{R_0}{\alpha} \right) \frac{1 - \alpha}{\alpha} \right] \frac{1}{\beta} \frac{\delta}{\alpha} (R_0 - 1 + \delta) k_0
\]
\[
- \left[ \frac{1}{\beta R_0} + \frac{1}{1 - \delta + \frac{X^*}{\alpha} (R_0 - 1 + \delta)} \right] \left[ \frac{1 - \alpha}{X^*} A_1 \left( \frac{R_0 - 1 + \delta}{X^*} \right) \right] \frac{\delta}{\alpha} \gamma \omega_0 \left[ 1 - \delta + \left( \frac{A_0}{A_1} \frac{R_0}{\alpha} \right) \frac{1 - \alpha}{\alpha} \right] \frac{1}{\beta} \frac{\delta}{\alpha} (R_0 - 1 + \delta) k_0
\]
\[
+ \left( \frac{\gamma R_0}{1 - \delta + \frac{X^*}{\alpha} (R_0 - 1 + \delta)} \right) \frac{1 - \alpha}{\alpha} \omega_0 k_0 \left[ \frac{1 - \alpha}{\alpha} \frac{A_0}{A_1} \left( \frac{R_0 A_0}{A_1} \frac{1 - \alpha}{\alpha} \right) (R_0 - 1 + \delta) + \left( \frac{R_0 A_0}{A_1} \frac{1 - \alpha}{\alpha} \right) \right].
\]

Since \( \lim_{R_0 \to 1 - \delta} \frac{\partial Y^P_0}{\partial R_0} = -\infty \), and \( \lim_{R_0 \to \infty} \frac{\partial Y^P_0}{\partial R_0} = \infty \), and the expression of \( \frac{\partial Y^P_0}{\partial R_0} \) is continuous, there exists an \( R_0 \) such that the AD curve is downward sloping when \( R_0 \leq R_0 \).

Lemma 2. When the ZLB binds, both AS and AD curves, in \( Y_0 \) and \( P^e_0 \), are upward sloping.

Proof. With binding ZLB, the AS curve can be expressed as
\[
Y^S_0 = \left( P^e_0 \right)^{1-\alpha} \frac{1}{\beta X^* A_0} \frac{A_1}{\alpha} \frac{\delta}{\alpha} X^* k_0,
\]
which is increasing in \( P^e_0 \). For the AD curve, inserting \( R_0 = 1 \) into (A.3) and using \( c_1' = \beta R_0 c_0' \), we find that both \( c_0' \) and \( c_1' \) are independent of \( P^e_0 \). On the other hand, by (A.6) and (A.7), we find that \( c_0 \) and \( I_0 \) are increasing in \( P^e_0 \). Thus the AD curve is also increasing in \( P^e_0 \). 

B.3 Graphical Representations for Proposition 1 (Part 1)

When the ZLB does not bind, i.e., \( R_0 > 1 \), our specification of monetary policy, (2) implies \( X_0 = X^* \). The AD and AS expressions, (8) and (A.9), can be represented by output \( Y_0 \) as functions of interest rate \( R_0 \). Lemma 1 in Appendix B.2 shows that the aggregate supply
curve is upward slopping and the aggregate demand curve is downward slopping when \( R_0 \) is not too high.

Why is the AS curve upward sloping? In period 0, since both the TFP \( A_0 \) and capital stock \( k_0 \) are given, \( Y_0^S \) responds to \( R_0 \) through the labor supply. A higher \( R_0 \) reduces \( c_0' \) in two ways. First, it encourages saving and discourages consumption through the inter-temporal substitution effect. Second, with \( R_0 = R^K_1 \), a higher \( R_0 \) implies a lower capital-labor ratio in the last period \( t = 1 \) by (6e), and thus lower \( c_1' \) and \( \omega_1 \) by (5b) and (6d), which reduces \( c_0' \) through the income effect. Combining both effects, labor cost \( w_0 \) in the first period becomes cheaper by the labor supply equation (5b), which boosts the aggregate supply.

The intuition for why the AD curve is downward sloping is more involved since it is the summation of three variables: the households’ consumption \( (c_0') \), entrepreneurs’ consumption \( (c_0) \), and investment \( (I_0) \). Investment demand \( I_0 \) is determined such that the capital stock at \( t = 1 \), \( k_1 = (1 - \delta) k_0 + I_0 \), suffices to serve the consumption demand \( c_1 \) and \( c_1' \), with the marginal product of capital pinned down by \( R_0 \) in equilibrium. Combining this equation with the inter-temporal optimal choices \( c_1 = \gamma R_0 c_0 \) and \( c_1' = \beta R_0 c_0' \), in Appendix B.2, we show that the AD curve can be written as

\[
Y_0^{AD} = \left[ 1 + \frac{\beta R_0}{1 - \delta + \frac{X^*}{\alpha} (R_0 - 1 + \delta)} \right] c_0' + \left( 1 + \frac{\gamma R_0}{1 - \delta + \frac{X^*}{\alpha} (R_0 - 1 + \delta)} \right) c_0 - (1 - \delta) k_0. \tag{A.10}
\]

On the right-hand side, the first component is associated with the households’ consumption and is decreasing in \( R_0 \). As discussed earlier, a higher \( R_0 \) depresses \( c_0' \) and \( c_1' \), and thus depresses \( I_0 \) by reducing the demand in period 1. The second component is associated with the entrepreneurs’ consumption and is increasing in \( R_0 \). This is because as labor cost \( w_0 \) becomes cheaper, the returns to capital and thus to the entrepreneurs’ wealth become higher. However, we show that as long as the entrepreneur wealth share \( \omega_0 \) and interest rate \( R_0 \) are not too high (as guaranteed by the conditions given in Proposition 1), the change in the second component is dominated by that of the first one, leaving the AD curve downward sloping. This is the case we focus on here.

Part 1 of Proposition 1 shows that \( R_0 \) is decreasing in \( k_0 \) and increasing in \( \omega_0 \), which also implies that the ZLB tends to bind when \( k_0 \) is high or when \( \omega_0 \) is low. The intuition for these results can be analyzed by plotting the AS and AD curves in Figure A.2 (output on the x-axis and interest rate on the y-axis).\(^{31}\)

---

\(^{31}\)The standard representations of the AS-AD curves plot price-level against output. We cannot strictly follow these representations in this case because the wholesale price level is constant at \( 1/X^* \). We can do
Consider an exogenous increase in the initial capital, $k_0$. As implied by (A.9), given $R_0$, output increases in $k_0$, and the AS curve shifts to the right. This result is unsurprising since larger $k_0$ implies greater production capacity. The AD curve, on the contrary, shifts to the left. We show in Appendix B.2 that given $R_0$, $c'_0$ does not depend on $k_0$. Although $c_0$ and $k_1$ are increasing in $k_0$, they increase by a smaller amount compared to the increase in $(1-\delta)k_0$. Thus the aggregate demand is decreasing in $k_0$. As a result, $R_0$ is lower in equilibrium. This is illustrated in the left panel of Figure A.2.

For an exogenous increase in the entrepreneurs’ wealth share, $\omega_0$, the AS curve is not affected. For the AD curve, since the entrepreneurs now have more wealth, $c_0$ increases. In addition, the entrepreneurs also increase their consumption in period 1, $c_1$, which, in order to clear the good market in period 1, requires higher capital holding $k_1$ and thus higher investment. Given $R_0$, $c'_0$ is does not depend on $\omega_0$. Therefore, in sum, the AD curve shifts to the right. In equilibrium, both $R_0$ and output $Y_0$ are higher. This is illustrated in the right panel of Figure A.2.

C Proof of Proposition 1

Combining the expressions for $c'_0$, $c_0$, and $I_0$ in (A.3), (A.6) and (A.7), the optimal inter-temporal choices, $c_1 = \gamma R_0 c_0$ and $c'_1 = \gamma R_0 c'_0$, we can reduce the whole system into one equation with one unknown: $R_0$ or $X_0$ depending on whether the ZLB is binding or not as follows:

so when the ZLB binds and the wholesale price level varies.
\[
\frac{1}{k_0} = \frac{(1 - \delta) \left[ (1 - \delta) \left( 1 - \frac{X^*}{a} \right) \left( 1 - \frac{1 + \gamma}{a} \omega_0 \right) + \left( \frac{X^*}{a} - \frac{X^* + \gamma}{1 + \gamma} \omega_0 \right) R_0 \right]}{1 - \frac{\alpha}{X^+} A_1 \left[ \frac{X^*}{a} \left( R_0 - 1 + \delta \right) \right]^{\frac{\alpha}{\alpha + \gamma}} \left[ 1 + \frac{X^*}{a} + \frac{1}{\beta \alpha} \left( 1 - \delta \right) \left( 1 - \frac{X^*}{a} \right) \right]} \\
\frac{1 - \frac{\alpha}{X^+} A_1 \left[ \frac{X^*}{a} \left( R_0 - 1 + \delta \right) \right]^{\frac{\alpha}{\alpha + \gamma}} \left[ 1 + \frac{X^*}{a} + \frac{1}{\beta \alpha} \left( 1 - \delta \right) \left( 1 - \frac{X^*}{a} \right) \right]}{1 - \frac{\alpha}{X^+} A_1 \left[ \frac{X^*}{a} \left( R_0 - 1 + \delta \right) \right]^{\frac{\alpha}{\alpha + \gamma}} \left[ 1 + \frac{X^*}{a} + \frac{1}{\beta \alpha} \left( 1 - \delta \right) \left( 1 - \frac{X^*}{a} \right) \right]}. \\
\]

(A.11)

**Lemma 3.** When the ZLB does not bind, \( R_0 \) is decreasing in \( k_0 \) and increasing in \( \omega_0 \) if \( \omega_0 \leq \frac{X^* + X^* \gamma}{X^* + \alpha \gamma} \).

**Proof.** The monetary policy rule in equation (2) implies that, when \( R_0 > 1, X_0 = X^* \). Inserting \( X_0 = X^* \) into (A.11), we obtain

\[
\frac{1}{k_0} = \frac{(1 - \delta) \left[ (1 - \delta) \left( 1 - \frac{X^*}{a} \right) \left( 1 - \frac{1 + \gamma}{a} \omega_0 \right) + \left( \frac{X^*}{a} - \frac{X^* + \gamma}{1 + \gamma} \omega_0 \right) R_0 \right]}{1 - \frac{\alpha}{X^+} A_1 \left[ \frac{X^*}{a} \left( R_0 - 1 + \delta \right) \right]^{\frac{\alpha}{\alpha + \gamma}} \left[ 1 + \frac{X^*}{a} + \frac{1}{\beta \alpha} \left( 1 - \delta \right) \left( 1 - \frac{X^*}{a} \right) \right]} \\
\frac{1 - \frac{\alpha}{X^+} A_1 \left[ \frac{X^*}{a} \left( R_0 - 1 + \delta \right) \right]^{\frac{\alpha}{\alpha + \gamma}} \left[ 1 + \frac{X^*}{a} + \frac{1}{\beta \alpha} \left( 1 - \delta \right) \left( 1 - \frac{X^*}{a} \right) \right]}{1 - \frac{\alpha}{X^+} A_1 \left[ \frac{X^*}{a} \left( R_0 - 1 + \delta \right) \right]^{\frac{\alpha}{\alpha + \gamma}} \left[ 1 + \frac{X^*}{a} + \frac{1}{\beta \alpha} \left( 1 - \delta \right) \left( 1 - \frac{X^*}{a} \right) \right]}. \\
\]

In the equation above, for the right-hand side, the denominator is decreasing in \( R_0 \), and the numerators are increasing in \( R_0 \) if \( \omega_0 \leq \frac{X^* + X^* \gamma}{X^* + \alpha \gamma} \). As a result, its right-hand side is increasing in \( R_0 \). As \( R_0 \to \infty \), the right-hand side goes to infinity. As \( R_0 \to 1 - \delta \), the right-hand side goes to zero. Thus there is a unique solution of \( R_0 \) between \([1 - \delta, +\infty)\). As \( k_0 \) increases, \( R_0 \) decreases. When \( k_0 \) is large enough and hits the threshold \( \tilde{k}_0 (\omega_0) \) as in Lemma 5, the ZLB is binding, and we switch to the other system with \( R_0 = 1 \) and \( X_0 \) being the unknowns.

To see how \( R_0 \) responds to \( \omega_0 \), we can rewrite the equation above as follows:

\[
(1 - \delta) \left[ (1 - \delta) \left( 1 - \frac{X^*}{a} \right) \left( 1 - \frac{1 + \gamma}{a} \omega_0 \right) + \left( \frac{X^*}{a} - \frac{X^* + \gamma}{1 + \gamma} \omega_0 \right) R_0 \right] \\
\frac{1 - \frac{\alpha}{X^+} A_1 \left[ \frac{X^*}{a} \left( R_0 - 1 + \delta \right) \right]^{\frac{\alpha}{\alpha + \gamma}} \left[ 1 + \frac{X^*}{a} + \frac{1}{\beta \alpha} \left( 1 - \delta \right) \left( 1 - \frac{X^*}{a} \right) \right]}{1 - \frac{\alpha}{X^+} A_1 \left[ \frac{X^*}{a} \left( R_0 - 1 + \delta \right) \right]^{\frac{\alpha}{\alpha + \gamma}} \left[ 1 + \frac{X^*}{a} + \frac{1}{\beta \alpha} \left( 1 - \delta \right) \left( 1 - \frac{X^*}{a} \right) \right]}. \\
\]

\[
- \frac{1 - \frac{\alpha}{X^+} A_1 \left[ \frac{X^*}{a} \left( R_0 - 1 + \delta \right) \right]^{\frac{\alpha}{\alpha + \gamma}} \left[ 1 + \frac{X^*}{a} + \frac{1}{\beta \alpha} \left( 1 - \delta \right) \left( 1 - \frac{X^*}{a} \right) \right]}{1 - \frac{\alpha}{X^+} A_1 \left[ \frac{X^*}{a} \left( R_0 - 1 + \delta \right) \right]^{\frac{\alpha}{\alpha + \gamma}} \left[ 1 + \frac{X^*}{a} + \frac{1}{\beta \alpha} \left( 1 - \delta \right) \left( 1 - \frac{X^*}{a} \right) \right]}. \\
\]

Denote its left-hand side as \( F (R_0, \omega_0) \). A careful examination shows that \( F \) is increasing in \( R_0 \) if \( \omega_0 \leq \frac{X^* + X^* \gamma}{X^* + \alpha \gamma} \), and decreasing in \( \omega_0 \). By the Implicit Function Theorem, \( \partial R_0 / \partial \omega_0 > 0 \).
Lemma 4. When the ZLB binds and $\alpha < \frac{X^*}{1 + X^*}$, $X_0$ is increasing in $k_0$, and is decreasing in $\omega_0$. Output $Y_0$ is decreasing in $k_0$, and increasing in $\omega_0$.

Proof. By the monetary policy rule (2), when $R_0 = 1$, $X_0 > X^*$. By setting $R_0 = 1$ in (A.11) and after some calculation, we have the following equation to pin down $X_0$:

$$
\frac{1 - \alpha}{X^*} A_1 \left( \frac{\delta X^*}{\alpha} \right)^{\frac{\alpha}{\gamma}} \left( 1 + \frac{\delta X^*}{a \beta} + \frac{1}{\beta} (1 - \delta) \right) \frac{1}{k_0} = \left( 1 - \delta + \frac{\delta X^*}{\alpha} \right) \delta \left( \frac{1}{\beta A_0} \right)^{\frac{\alpha - 1}{\alpha}} \left( \frac{X_0}{X^*} \right)^{-\frac{1}{\alpha}} \left( \frac{X_0}{\alpha} - 1 \right) + \left[ \left( 1 - \delta + \frac{\delta X^*}{\alpha} \right) \left( 1 - \frac{1}{1 + \gamma} \omega_0 \right) - \frac{\gamma}{1 + \gamma} \omega_0 \right] \left[ (1 - \delta) + \delta \left( \frac{1}{\beta A_0} \right)^{\frac{\alpha - 1}{\alpha}} \left( \frac{X_0}{X^*} \right)^{-\frac{1}{\alpha}} \right].
$$

(A.13)

The left-hand side is independent of $X_0$. If $\alpha < \frac{X^*}{1 + X^*}$, the right-hand side is decreasing in $X_0$. As $k_0$ increases, the left hand side decreases, so $X_0$ must increase. Since the right-hand side is decreasing in $\omega_0$, when $\omega_0$ increases, $X_0$ decreases.

For output, inserting the expression for $Y_0$ from the AS curve (A.9) into (A.13), we arrive at

$$
\frac{1 - \alpha}{X^*} A_1 \left( \frac{\delta X^*}{\alpha} \right)^{\frac{\alpha}{\gamma}} \left( 1 + \frac{\delta X^*}{a \beta} + \frac{1}{\beta} (1 - \delta) \right) = \left[ \left( 1 - \delta + \frac{\delta X^*}{\alpha} \right) \left( 1 - \frac{1}{1 + \gamma} \omega_0 \right) - \frac{\gamma}{1 + \gamma} \omega_0 \right] \left[ (1 - \delta) k_0 + Y_0 \right] + \left( 1 - \delta + \frac{\delta X^*}{\alpha} + \gamma \right) \frac{1}{1 + \gamma} \omega_0 \left( 1 - \frac{\alpha}{X_0} \right) Y_0.
$$

(A.14)

Since the left-hand side of (A.14) is constant, and $X_0$ is increasing in $k_0$, with larger $k_0$, $Y_0$ must decrease to equate (A.14). Thus, $Y_0$ is decreasing in $k_0$ when the ZLB binds. To see how $Y_0$ responds to $\omega_0$, we can insert the expressions of $c'_0$, $c_0$ and $I_0$ from (A.3), (A.3) and (A.7) into the AD curve (8), and get

$$
Y_0 = \left( \frac{1}{\gamma} + \frac{1}{(1 - \delta) (1 - \frac{X^*}{\alpha}) + \frac{X^*}{\alpha}} \right) \frac{\gamma}{1 + \gamma} \omega_0 \left[ 1 - \delta + \delta \left( \frac{\beta A_0}{A_1} \right)^{\frac{1}{\alpha}} \left( \frac{X_0}{X^*} \right)^{-\frac{1}{\alpha}} \right] \left[ (1 - \delta) k_0 \right] 0. \quad \Box
$$

Lemma 5. Given an initial wealth distribution $\omega_0$, there exists a cutoff value $\hat{k}_0(\omega_0)$ such that when $k_0 < \hat{k}_0(\omega_0)$, the ZLB is not binding; and when $k_0 > \hat{k}_0(\omega_0)$, the ZLB is binding. $\hat{k}_0(\omega_0)$

A-11
is increasing in \( \omega_0 \).

**Proof.** Inserting \( R_0 = 1 \) and \( X_0 = X^* \) in equation (A.11), we obtain the expression for \( \hat{k}_0 (\omega_0) \) as

\[
k_0 (\omega_0) = \frac{\frac{1-\alpha}{X^*} A_1 \left( \frac{\delta X^*}{\alpha} \right)^{\frac{1}{\alpha-1}} \left( 1 + \frac{\delta X^*}{\alpha} + \frac{1}{\beta} (1-\delta) \right)}{(1-\delta) \left( 1 - \delta + \frac{\delta X^*}{\alpha} \right) - \left( \frac{1-\delta + \frac{\delta X^*}{\alpha} + \gamma}{1+\gamma} \right) \frac{1-\delta}{\omega_0}.}
\]

(A.15)

which is increasing in \( \omega_0 \) and \( A_1 \).

We first show that when \( k_0 < \hat{k}_0 (\omega_0) \), the ZLB is not binding in any equilibrium. Otherwise, given a binding ZLB, \( X_0 \) is increasing in \( k_0 \) by Lemma 4, which implies \( X_0 < X^* \) when \( k_0 < \hat{k}_0 (\omega_0) \). This contradicts the restriction \( X_0 \geq X^* \). Thus, any equilibrium with \( k_0 < \hat{k}_0 (\omega_0) \) features non-binding ZLB. From equation (A.12), we can see that a unique equilibrium exists in this region without a binding ZLB.

Similarly, we can show that when \( k_0 \geq \hat{k}_0 (\omega_0) \), the ZLB is binding in any equilibrium. Otherwise, given \( R_0 > 1, R_0 \) is decreasing in \( k_0 \) by Lemma 3, which implies \( R_0 < 1 \) when \( k_0 > \hat{k}_0 (\omega_0) \). This contradicts the ZLB restriction. Thus the ZLB is binding when \( k_0 \geq \hat{k}_0 (\omega_0) \).

\( \square \)

**Lemma 6.** When \( m = 1 \), given \( \omega_0 \), there is an upper bound of initial capital \( \bar{k}_0 (\omega_0) \), such that an equilibrium does not exist when \( k_0 > \bar{k}_0 (\omega_0) \). \( \bar{k}_0 (\omega_0) \) is increasing in \( \omega_0 \) and \( A_1 \).

**Proof.** By setting \( X_0 \to \infty \) in (A.13), we obtain the expression for \( \bar{k}_0 (\omega_0) \):

\[
\bar{k}_0 (\omega_0) = \frac{\frac{1-\alpha}{X^*} A_1 \left( \frac{\delta X^*}{\alpha} \right)^{\frac{1}{\alpha-1}} \left( 1 + \frac{\delta X^*}{\alpha} + \frac{1}{\beta} (1-\delta) \right)}{(1-\delta) \left( 1 - \delta + \frac{\delta X^*}{\alpha} \right) - \left( \frac{1-\delta + \frac{\delta X^*}{\alpha} + \gamma}{1+\gamma} \right) \frac{1-\delta}{\omega_0}}.
\]

(A.16)

By Lemma 4, \( X_0 \) increases with \( k_0 \). Since at \( k_0 = \bar{k}_0 (\omega_0) \), \( X_0 \) cannot be increased further, an equilibrium does not exist with binding ZLB when \( k_0 > \bar{k}_0 (\omega_0) \). Since by Lemma 5, \( X_0 = X^* \), Lemma 4 implies that \( \hat{k}_0 (\omega_0) < \bar{k}_0 (\omega_0) \). In Lemma 5, we also show that there is no equilibrium with \( R_0 > 1 \) when \( k_0 > \hat{k}_0 (\omega_0) \). Thus there is no equilibrium when \( k_0 > \bar{k}_0 (\omega_0) \). We can easily see from (A.16) that \( \bar{k}_0 (\omega_0) \) is increasing in \( \omega_0 \) and \( A_1 \).

The intuition for this non-existence result is as follows – to simplify the discussion we assume exogenous labor supply. At the ZLB, the return to capital \( R_1^K \) is equal to 1. Therefore, by (6e) for \( t = 1, k_1 \) is bounded from above. Thus output and the consumption of households and entrepreneurs are all bounded from above by the market clearing condition (3a) in period 1. Then, from the Euler equations of the entrepreneurs and households,
and \( c_0 \) are also bounded from above. Therefore, if \( k_0 \) is sufficiently high,

\[
(1 - \delta)k_0 > c'_0 + c_0 + k_1
\]

violating \( (3a) \) in period 0 even if output falls to zero.

Combining the results from Lemmas 3 to 6, we obtain a complete proof of Proposition 1.

**D Proof of Proposition 2**

The equilibrium in the last period, \( t = 1 \), is the same as for the natural borrowing limit and is provided in Appendix B.1. Here, we focus on equilibrium at \( t = 0 \).

**Proposition 2.** Assume that \( m < 1 \), \( \alpha < \frac{X^*}{1+X^*} \) and \( \omega_0 \) is smaller than a threshold \( \varpi \), which depends on model parameters and is given in Appendix D (equation (A.21a)). There is a cutoff value \( \tilde{k}_0(\omega_0) > 0 \) such that when \( 0 < k_0 \leq \tilde{k}_0(\omega_0) \), there exists a unique equilibrium and when \( k_0 > \tilde{k}_0(\omega_0) \) there does not exist an equilibrium. In addition,

1. if \( \omega_0 < \Lambda_0(m, \gamma, \beta, \alpha, X^*) \),

\( \tilde{k}_0(\omega_0) < \hat{k}_0(\omega_0) \), where \( \hat{k}_0(\omega_0) \) is defined in Proposition 1.

2. the collateral constraint is binding if and only if \( \omega_0 \leq \omega_0^{CC}(k_0) \), for some cutoff function \( \omega_0^{CC}(k_0) \).

**Proof.** We note that the proof for equilibrium uniqueness turns out to be rather challenging. For example, in the region of state space featuring an equilibrium with binding ZLB and binding collateral constraint, not only we need to establish equilibrium uniqueness with this binding pattern, we also need to rule out equilibria with all possible combinations of binding or non-binding ZLB and collateral constraint. What adds to the complexity is that the systems of equations determining equilibria with different binding patterns are fundamentally different. The non-existence result for \( k_0 > \tilde{k}_0(\omega_0) \) is a novel result for this class of New Keynesian models. It is similar to the natural borrowing limit case analyzed in Subsection 2.3 and we provide more details in Appendix D.4. It points to the difficulties in solving the infinite-horizon version of these models which we and other researchers have encountered. In particular, if one solves the model using an iterative algorithm such as policy iterations, at some point the algorithm might not be able to find...
an equilibrium in certain region of the state space. One way to get around this issue of equilibrium non-existence is to add investment frictions to the model as in Subsection 2.5 and Section 3.

The detailed proof is provided in the remainder of this section and we proceed as follows.

1. Describe the cutoff function $\omega_0^{CC}(k_0)$ of $\omega_0$ for a binding collateral constraint, and the cutoff function $\bar{k}_0(\omega_0)$ of $k_0$ for equilibrium existence.

2. We show that when $\omega_0 \leq \omega_0^{CC}(k_0)$ and $k_0 < \bar{k}_0(\omega_0)$, there exists a unique equilibrium with binding collateral constraint, while such an equilibrium does not exist when $\omega_0 > \omega_0^{CC}(k_0)$ or $k_0 \geq \bar{k}_0(\omega_0)$

3. We show that when $\omega_0 > \omega_0^{CC}(k_0)$ and $k_0 < \bar{k}_0(\omega_0)$, there exists a unique equilibrium with non-binding collateral constraint, while such an equilibrium does not exist when $\omega_0 \leq \omega_0^{CC}(k_0)$ or $k_0 \geq \bar{k}_0(\omega_0)$.

### D.1 Thresholds for Collateral Constraint and Equilibrium Existence

In this first step of the proof, we construct the threshold functions $\omega_0^{CC}(.)$ and $\bar{k}_0(.)$ for binding collateral constraint and equilibrium existence. We use these thresholds later in Step 2 and Step 3 of the proof.

#### D.1.1 Cutoff Value for a Binding Collateral Constraint

The following two lemmas help us identify the cutoff value of a binding collateral constraint in the $\{k_0, \omega_0\}$ space.

**Lemma 7.** With natural borrowing limit $(m = 1)$, given $\alpha < \frac{X^*}{1+X^*}$ and $\omega_0 < \frac{X^*+X^*\gamma}{X^*+\alpha\gamma}$, in the unique equilibrium constructed in Proposition 1, the leverage ratio $-\frac{b_0}{R_k^1k_1}$ is decreasing in $\omega_0$.

**Proof.** From the entrepreneurs’ budget and their optimal choice $c_0 = \frac{1}{1+\gamma}R_0^k\omega_0k_0$, their leverage ratio is

$$-\frac{b_0}{R_k^1k_1} = 1 - \frac{\gamma}{1+\gamma} \frac{R_0^k\omega_0k_0}{k_1}.$$  

Showing that the leverage ratio is decreasing in $\omega_0$ is equivalent to showing that $\frac{R_0^k\omega_0k_0}{k_1}$ is increasing in $\omega_0$.

First, consider the case in which ZLB is not binding, i.e. $R_0 > 1$. Using the expression
for $k_1$ from (A.8) we obtain

$$\frac{R^k_0 \omega_0 k_0}{k_1} = \frac{\omega_0 R^k_0 k_0 \left[ 1 - \delta + \frac{X^*}{\alpha} (R_0 - 1 + \delta) \right]}{1 + \gamma \omega R^k_0 k_0 + \frac{1 - a}{X^*} A_1 \left( (R_0 - 1 + \delta) \frac{X^*}{\alpha} \right)^{\frac{\alpha}{\alpha - 1}}}$$

$$= \frac{1 - \delta + \frac{X^*}{\alpha} (R_0 - 1 + \delta)}{\omega_0 R^k_0 k_0} \frac{\gamma R_0}{1 + \gamma} + \frac{\frac{1 - a}{X^*} A_1 \left( (R_0 - 1 + \delta) \frac{X^*}{\alpha} \right)^{\frac{\alpha}{\alpha - 1}}}{\omega_0 R^k_0 k_0}.$$ 

Differentiating the last line in $\omega_0$, the derivative $\frac{d}{d\omega_0} \left[ \frac{R^k_0 \omega_0 k_0}{k_1} \right]$ has the same sign as

$$\frac{X^*}{\alpha} dR_0 \left[ \frac{\gamma R_0}{1 + \gamma} + \frac{\frac{1 - a}{X^*} A_1 \left( (R_0 - 1 + \delta) \frac{X^*}{\alpha} \right)^{\frac{\alpha}{\alpha - 1}}}{\omega_0 R^k_0 k_0} \right] - \left[ \gamma \frac{dR_0}{d\omega_0} + \frac{\frac{1 - a}{X^*} A_1 \left( (R_0 - 1 + \delta) \frac{X^*}{\alpha} \right)^{\frac{\alpha}{\alpha - 1}}}{\omega_0 R^k_0 k_0} \right] \left[ 1 - \frac{X^*}{\alpha} (R_0 - 1 + \delta) \right]. \quad (A.17)$$

By Proposition 1, given $\omega_0 < \frac{X^* + X^* \gamma}{X^* + \alpha \gamma}$, $R_0$ increases with $\omega_0$: $\frac{dR_0}{d\omega_0} > 0$. In addition, by setting $X_0 = X^*$ in (A.5), $R^K_0$ becomes

$$R^K_0 = 1 - \delta + \left( \frac{A_0}{A_1} \beta R_0 \right)^{\frac{1 - a}{\alpha}} (R_0 - 1 + \delta),$$

which is strictly increasing in $R_0$ and hence in $\omega_0$. Therefore,

$$\frac{d}{d\omega_0} \frac{\frac{1 - a}{X^*} A_1 \left( (R_0 - 1 + \delta) \frac{X^*}{\alpha} \right)^{\frac{\alpha}{\alpha - 1}}}{\omega_0 R^k_0 k_0} < 0.$$ 

To show that (A.17) is positive, we only need to show that

$$\frac{X^*}{\alpha} dR_0 \gamma R_0 \frac{dR_0}{1 + \gamma d\omega_0} > \gamma \frac{dR_0}{1 + \gamma d\omega_0} \left[ 1 - \frac{X^*}{\alpha} (R_0 - 1 + \delta) \right].$$

This inequality holds because $\frac{X^*}{\alpha} > 1$.

Now, consider the case in which ZLB is binding, i.e. $R_0 = 1$ and $X_0 > X^*$. Using the
expression for \( k_1 \) from (A.8), we obtain

\[
\frac{R_0^k \omega_0 k_0}{k_1} = \frac{\omega_0 R_0^k k_0 \left[ 1 - \delta + \frac{X^*}{\alpha} \delta \right]}{1 + \frac{\gamma}{1 + \gamma} \omega_0 R_0^k k_0 + \frac{1 - \alpha}{X^*} A_1 \left( \frac{\delta X^*}{\alpha} \right)^{\frac{\gamma}{1 + \gamma}}}
\]

So leverage is decreasing in \( \omega_0 \) if \( \omega_0 R_0^k \) is increasing in \( \omega_0 \). By setting \( R_0 = 1 \) in (A.5), \( R_0^k \) becomes

\[
R_0^k = 1 - \delta + \left( \frac{A_0}{A_1} \right)^{\frac{1 - \alpha}{\alpha}} \left( \frac{X^*}{X_0} \right)^{\frac{\gamma}{1 + \gamma}} \delta \left( \frac{X^*}{X_0} \right)^{\frac{1}{\alpha}}.
\]

Proposition 1 shows that when \( \alpha < \frac{X^*}{1 + X^*} \), \( X_0 \) is decreasing with \( \omega_0 \). Therefore, \( R_0^k \) is increasing in \( \omega_0 \). Hence, \( \omega_0 R_0^k \) is increasing in \( \omega_0 \) as desired. \( \square \)

**Lemma 8.** Given \( m < 1, \alpha < \frac{X^*}{1 + X^*} \) and \( \omega_0 < \frac{X^* + X^* \gamma}{X^* + \alpha \gamma} \), there is a cutoff value of wealth, \( \omega_0^{CC} (k_0) \) such that in the unique equilibrium of the model with natural borrowing limit constructed in Proposition 1, the leverage ratio \( \frac{b_0}{R_1 k_1} \) is at \( \omega_0 = \omega_0^{CC} (k_0) \).

**Proof.** By Lemma 7, with natural borrowing limit, the leverage ratio \( \frac{b_0}{R_1 k_1} \) is decreasing in \( \omega_0 \). Thus we can derive the expression of \( \omega_0^{CC} (k_0) \) by setting \( \frac{b_0}{R_1 k_1} = m \).

At \( \omega_0^{CC} (k_0) \), we have \( \omega_1 = 1 - m \) and \( R_1^k = R_0 \). After some calculations, given \( k_0 \), we obtain the following system of two equations with two unknowns, \( \{ \omega_0^{CC}, R_0 \} \) or \( \{ \omega_0^{CC}, X_0 \} \) depending on whether the ZLB is binding,

\[
k_1 = \frac{\gamma}{1 + \gamma} \left[ (1 - \delta) \left( 1 - \frac{\omega_0}{1 + \gamma} \right) + \left( \frac{X_0}{\alpha} - \frac{\omega_0}{1 + \gamma} \right) \left( \frac{X^*}{X_0} \right)^{\frac{\gamma}{1 + \gamma}} \left( \frac{A_0}{A_1} \beta R_0 \right)^{\frac{1 - \alpha}{\alpha}} \right] k_0 \quad (A.18a)
\]

\[
- \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{\gamma}{1 + \gamma}} \left[ \frac{1}{1 - \delta} \left( \frac{\omega_0}{1 + \gamma} \right) \right] \left( \frac{X^*}{X_0} \right)^{\frac{1}{\alpha}} \left( \frac{A_0}{A_1} \beta R_0 \right)^{\frac{1 - \alpha}{\alpha}} (R_0 - 1 + \delta) \]

in which \( k_1 \) is a decreasing function of \( R_0 \):

\[
k_1 = \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1}{1 + \gamma}} \left( \frac{A_0}{A_1} \beta R_0 \right)^{\frac{1 - \alpha}{\alpha}} (R_0 - 1 + \delta)^{\frac{1}{1 + \gamma}}
\]
We consider the two cases, binding or non-binding ZLB separately.

Case 1: Non-binding ZLB

We first consider the case that the ZLB is not binding at \( \omega^0_{CC} (k_0) \). So \( X_0 = X^* \) in the two equations above. Given \( k_0, \{ R_0, \omega^0_{CC} \} \) are the two unknowns. After some calculations, we can express \( \omega^0_{CC} \) and \( k_0 \) as functions of \( R_0 \):

\[
\omega^0_{CC} = \frac{1 - \delta + \frac{X^*}{\alpha} \left( \beta R_0 \frac{A_0}{A_1} \right)^{\frac{1 - \delta}{1 - \alpha}} \left( R_0 - 1 + \delta \right)}{1 + \frac{X^*}{\alpha} \left( R_0 - \frac{A_0}{A_1} \right)^{\frac{1 - \delta}{1 - \alpha}} \left( R_0 - 1 + \delta \right)}
\]

\[
k_0 = \frac{1}{\beta R_0} \frac{X^*}{\alpha} \left[ \frac{X^*}{\alpha} - 1 + m + \beta + \frac{\beta}{\gamma} \right] R_0 - \frac{\left( X^* - 1 \right) \left( 1 - \delta \right)}{1 - \delta + \frac{X^*}{\alpha} \delta \left( \beta \frac{A_0}{A_1} \right)^{\frac{1 - \delta}{1 - \alpha}}}
\]

Notice that \( R_0 \) is strictly decreasing in \( k_0 \). So we can write \( \omega^0_{CC} \) as a function of \( k_0 \). By varying the value of \( R_0 \), we can trace out \( \omega^0_{CC} (k_0) \). In particular, when \( k_0 \) equals to

\[
\hat{k}^0_{CC} = \frac{\left( X^* - 1 \right) \delta + m}{\beta} + 1 + \frac{1}{\gamma} \left( 1 - m \right) \frac{1 - \frac{\delta}{\alpha} A_1 \left( X^* \right)^{\frac{1 - \delta}{1 - \alpha}}}{1 - \delta + \frac{X^*}{\alpha} \delta \left( \beta \frac{A_0}{A_1} \right)^{\frac{1 - \delta}{1 - \alpha}}}
\]

\( R_0 = 1 \) at \( \omega_0 = \omega^0_{CC} \left( \hat{k}^0_{CC} \right) \). Thus out result here with non-binding ZLB only applies for \( k_0 < \hat{k}^0_{CC} \).

Case 2: Binding ZLB

Now consider the case that the ZLB is binding at \( \omega^0_{CC} (k_0) \). We set \( R_0 = 1 \) in equations (A.18a) and (A.18b). Given \( k_0, \{ X_0, \omega^0_{CC} \} \) are the two unknowns. After some calculations, we can express \( \omega^0_{CC} \) and \( k_0 \) as functions of \( X_0 \):

\[
\omega^0_{CC} = A_0 \frac{1 - \delta + \delta \left( \beta \frac{A_0}{A_1} \right)^{\frac{1 - \delta}{1 - \alpha}} \left( X_0 \right)^{\frac{1}{\alpha}}}{1 - \delta + \delta \left( \beta \frac{A_0}{A_1} \right)^{\frac{1 - \delta}{1 - \alpha}} \left( X_0 \right)^{\frac{1}{\alpha}}}
\]

\[
k_0 = A_1 \frac{1 - \frac{\delta}{\alpha} A_1 \left( \delta \frac{X^*}{\alpha} \right)^{\frac{1 - \delta}{1 - \alpha}}}{1 - \delta + \delta \left( \beta \frac{A_0}{A_1} \right)^{\frac{1 - \delta}{1 - \alpha}} \left( X_0 \right)^{\frac{1}{\alpha}}}
\]
in which $\Lambda_0$ and $\Lambda_1$ are constants:

$$\Lambda_0 = \frac{1 + \gamma}{\gamma} \frac{(1 - m)}{m - \delta + \frac{\delta X^*}{\alpha}} + \frac{1 + \gamma - m}{\gamma}, \quad \text{(A.18f)}$$

$$\Lambda_1 = \frac{m - \delta + \frac{\delta X^*}{\alpha} + 1 + \frac{1 - m}{\gamma}}{m (1 - \delta) + \left(\frac{X^*}{\alpha} - 1 + m\right) \delta}. \quad \text{(A.18g)}$$

Notice that $X_0$ is increasing in $k_0$. So we can write $\omega^\text{CC}_0$ as function of $k_0$. By varying the value of $X_0$, we can trace out $\omega^\text{CC}_0(k_0)$. In particular, when $k_0$ equals to $\hat{k}_0^\text{CC}$ in (A.18d), $X_0 = X^*$. Thus our result here with binding ZLB only applies for $k_0 > \hat{k}_0^\text{CC}$. Furthermore, when $k_0 = \Lambda_1 \frac{1 - \Delta A_1 (\frac{\delta X^*}{\alpha})^{\frac{1 + \gamma}{\gamma}}}{1 - \delta}$, $\omega^\text{CC}_0(k_0) = \Lambda_0$, and $X_0 \to +\infty$. Thus there is no solution for $\omega^\text{CC}_0(k_0)$ when $k_0 > \Lambda_1 \frac{1 - \Delta A_1 (\frac{\delta X^*}{\alpha})^{\frac{1 + \gamma}{\gamma}}}{1 - \delta}$. This happens because the $\{k_0, \omega_0\}$ lies in the region where no equilibrium exists as shown in Proposition 1.

To sum up, we have completed the calculation for $\omega^\text{CC}_0(k_0)$. When $k_0 \leq \hat{k}_0^\text{CC}$ in (A.18d), $\omega^\text{CC}_0(k_0)$ is given by (A.18c), and when $\hat{k}_0^\text{CC} < k_0 \leq \Lambda_1 \frac{1 - \Delta A_1 (\frac{\delta X^*}{\alpha})^{\frac{1 + \gamma}{\gamma}}}{1 - \delta}$, $\omega^\text{CC}_0(k_0)$ is given by (A.18e). When $k_0 > \Lambda_1 \frac{1 - \Delta A_1 (\frac{\delta X^*}{\alpha})^{\frac{1 + \gamma}{\gamma}}}{1 - \delta}$, $\omega^\text{CC}_0(k_0)$ is not defined. See the dashed line in Figure 3 for an example. \hfill \Box

### D.1.2 Threshold for Equilibrium Existence

We define $\bar{k}_0(\omega_0)$ as follows. When $\omega_0 \geq \Lambda_0$ given in equation (A.18f), we set $\bar{k}_0(\omega_0) = \bar{k}_0(\omega_0)$ in equation (A.16), the cutoff value for equilibrium existence in the natural borrowing limit case. When $\omega_0 < \Lambda_0$, $\bar{k}_0(\omega_0)$ is solved by the following two equations in which the marginal product of capital, $r^K_1 = R^K_1 - (1 - \delta) \in (\delta, +\infty)$, is used as an auxiliary variable:

$$k_1 + \frac{1}{\beta} \frac{1 - \Delta A_1}{X^*} \left(\frac{X^*}{\alpha}\right)^{\frac{1 + \gamma}{\gamma}} \left(r^K_1\right)^{\frac{\alpha}{\gamma - 1}} = (1 - \delta) \left(1 - \frac{\omega_0}{1 + \gamma}\right) \bar{k}_0,$$  \hfill (A.18h)

$$\left(1 - m (1 - \delta) - m r^K_1 \right) k_1 = \frac{\gamma}{1 + \gamma} (1 - \delta) \omega_0 \bar{k}_0.$$  \hfill (A.18i)

In both equations, $k_1$ is a decreasing function of $r^K_1$ as below:

$$k_1 = \frac{1 - \Delta A_1}{m (1 - \delta) + \left(\frac{X^*}{\alpha} - 1 + m\right) r^K_1}.$$  \hfill (A.18j)

A-18
Equations (A.18h) and (A.18i) imply that $r^K_1$ is a decreasing function of $\omega_0$:

$$r^K_1 = \frac{\left(\frac{1+\gamma}{\omega_0} - 1\right)(1 - m(1 - \delta)) - \left(\gamma + \frac{\gamma}{\beta}m(1 - \delta)\right)}{(\frac{1+\gamma}{\omega_0} - 1)m + \gamma \left(\frac{X^*}{\alpha} - 1 + m\right)}.$$ (A.18k)

Inserting this expression into (A.18h), we see that $\bar{k}_0(\omega_0)$ is increasing, and thus $r^K_1$ is decreasing in $k_0$ at $k_0 = \bar{k}_0(\omega_0)$. In particular, at $\omega_0 = \Lambda_0$, the equation above implies $r^K_1 = \delta$, and by (A.18i) we have $\bar{k}_0(\omega_0) = \bar{k}_0(\omega_0)$ at $\omega_0 = \Lambda_0$. This shows that the function $\bar{k}_0(\omega_0)$ is continuous at $\omega_0 = \Lambda_0$.

When $\omega_0 < \Lambda_0$, by the expression of $\bar{k}_0(\omega_0)$ in (A.16), we have

$$\frac{(1 - \delta)\gamma\omega_0}{\frac{\gamma}{\beta}1-A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}} \bar{k}_0(\omega_0) = \frac{1 + \beta + \delta \left(\frac{X^*}{\alpha} - 1\right)}{(1 + \gamma) - \left(1 - \delta + \delta \frac{X^*}{\alpha}\right)} - \gamma,$$ (A.18l)

Similarly, by (A.18i), (A.18j) and (A.18k), we have

$$\frac{\gamma}{\frac{\gamma}{\beta}1-A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{1}{\alpha-1}}} \bar{k}_0(\omega_0) = \frac{1 - m(1 - \delta) + \frac{\beta m + m^2(1 - \delta)}{\frac{X^*}{\alpha} - (1 - m)}}{(1 + \gamma) - \left(1 - m(1 - \delta) + \frac{m^2(1 - \delta)}{\frac{X^*}{\alpha} - (1 - m)}\right)} - \gamma,$$ (A.18m)

whose left-hand side is the same as that of (A.18i).

Given $\omega_0 < \Lambda_0$, we can show that the first term on the right-hand-side of (A.18l) is larger than that of (A.18m). Besides, from (A.18k), we have $r^K_1 > \delta$ at $k_0 = \bar{k}_0(\omega_0)$, and then $\delta^{\frac{\alpha}{\alpha-1}} > [r^K_1]^{\frac{\alpha}{\alpha-1}}$. Thus by comparing (A.18l) and (A.18m), we see that $\bar{k}_0(\omega_0) < \bar{k}_0(\omega_0)$. Applying the Implicit Function Theorem to equations (A.18k) and (A.18m), we can show that $\bar{k}_0(\omega_0)$ is decreasing in $m$ when $\omega_0 < \Lambda_0$. By (A.18h) and (A.18k), it follows that $\bar{k}_0(\omega_0)$ is increasing in $A_1$.

### D.2 Equilibrium Properties with Binding Collateral Constraint

Having defined the thresholds $\omega_0^{CC}(\cdot)$ and $\bar{k}_0(\cdot)$, now we proceed with Step 2 for the proof of Proposition 2. When the collateral constraint is binding, $b_0 = -mR^k_1k_1$, and by (4), $\omega_1 = 1 - m$. Using the equations in Appendix B.1, we can explicitly solve for the
equilibrium at $t = 1$ given $R^K_1$. For example, by (6e) and (A.1c),

$$
\frac{k_1}{L_1} = A_1 \left[ \frac{X^*}{\alpha} \left( R^K_1 - 1 + \delta \right) \right]^{\frac{1}{\alpha - \delta}},
$$

$$
L_1 = \frac{\frac{1}{\alpha} - \frac{m\alpha(1-\delta)}{X^*} + 1 - \frac{\alpha}{X^*} (1 - m)}{X^* (R^K_1 - 1 + \delta)}.
$$

Now, back to time $t = 0$. Denote the marginal product of capital as $r^K_0 = R^K_0 - (1 - \delta)$. We can express the other variables as functions of $\{r^K_1, R_0\}$ or $\{r^K_1, X_0\}$ depending on whether the ZLB is binding. In particular,

$$
k_1 = \frac{1-\alpha}{X^*} A_1 \left[ \frac{X^*}{\alpha} \left( r^K_1 \right)^{\frac{\alpha}{\alpha - \delta}} \right] \frac{1}{\beta R_0} \left( \frac{A_0}{A_1} \beta R_0 \right)^{\frac{1-\alpha}{\alpha}} r^K_1
$$

(A.19a)

which is decreasing in $r^K_1$.

By (5b) for period 0 and 1, (5c) and (6e), we have

$$
r^K_0 = \left( \beta R_0 \frac{A_0}{A_1} \right)^{\frac{1-\alpha}{\alpha}} \left( \frac{X^*}{X_0} \right)^{\frac{1}{\alpha}} r^K_1.
$$

(A.19b)

Equilibria with binding collateral constraint can be written as systems of two equations and two unknowns, $\{r^K_1, R_0\}$ or $\{r^K_1, X_0\}$:

$$
k_1 = \left[ (1 - \delta) \left( 1 - \frac{\omega_0}{1 + \gamma} \right) + \left( \frac{X_0}{\alpha} - \frac{\omega_0}{1 + \gamma} \right) \left( \frac{X^*}{X_0} \right)^{\frac{1}{\alpha}} \left( \frac{A_0}{A_1} \beta R_0 \right)^{\frac{1-\alpha}{\alpha}} r^K_1 \right] k_0
$$

$$
- \frac{1-\alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{\alpha}{\alpha - \delta}} \left( \frac{1}{\beta R_0} \left( r^K_1 \right)^{\frac{\alpha}{\alpha - \delta}} \right),
$$

(A.19c)

and

$$
\left( 1 - \frac{m (1 - \delta + r^K_1)}{R_0} \right) k_1 = \frac{\gamma}{1 + \gamma} \left( 1 - \delta + r^K_0 \right) \omega_0 k_0,
$$

(A.19d)

in which $k_1$ and $r^K_0$ are given in equations (A.19a) and (A.19b). Equation (A.19c) is derived by the feasibility condition (3a) in period 0, while equation (A.19d) is derived by applying $c_0 = \frac{1}{1+\gamma} R^K_0 k_0 \omega_0$ and $b_0 = -mR^K_1 k_1$ (binding collateral constraint) to the entrepreneurs’ budget constraint in period 0.

A solution to the system of equations (A.19c) and (A.19d) corresponds to an equilibrium with binding collateral constraint if the multiplier $\mu_0$ implied by (7) is positive, i.e.,
if

\[ R_0 \leq R_1^K. \tag{A.19e} \]

In the next subsection, we characterize the properties of the solution to (A.19c) and (A.19d), depending on whether the ZLB is binding. We temporarily ignore the requirement (A.19e) and get back to it later in Subsection D.2.3.

### D.2.1 Equilibrium with Non-binding ZLB and Binding Collateral Constraint

**Lemma 9.** Assume that the collateral constraint is binding, and \( \omega_0 < \frac{X^*}{\alpha} \). Given \( k_0 \), there is a cutoff value of \( \omega_0 \), \( \omega_0 (k_0) \) such that if \( \omega_0 \geq \omega_0 (k_0) \), there exists a unique solution to (A.19c) and (A.19d) with non-binding ZLB, and \( R_0 \) is increasing in \( \omega_0 \). If \( \omega_0 < \omega_0 (k_0) \), there does not exist such a solution.

**Proof.** **Step 1: Equilibrium Representation**

Setting \( X_0 = X^* \), \( r_1^K \) can be expressed as functions of \( R_0 \) in both (A.19c) and (A.19d). Equation (A.19c) becomes

\[
\frac{1 - \frac{\alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1 - \delta}{\delta}}}{\frac{\max(1 - \delta)}{X^*} + \left[ 1 - \frac{\alpha}{X^*} (1 - m) \right] r_1^K} + \frac{1 - \frac{\alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{\alpha}{\delta}}}{\beta R_0} = \left[ (1 - \delta) \left( 1 - \frac{\omega_0}{1 + \gamma} \right) + \left( \frac{X^*}{\alpha} - \frac{\omega_0}{1 + \gamma} \right) \left( \frac{A_0}{A_1} \right) \frac{1}{r_1^K} \right] k_0 \left( r_1^K \right)^{\frac{\alpha}{\delta}} \tag{A.20a} \]

in which \( r_1^K \) is a decreasing function of \( R_0 \). Denote this implicit function as \( r_1^K = f_1 (R_0) \).

We can easily verify that \( \lim_{R_0 \to 0} f_1 (R_0) \to +\infty \), and \( \lim_{R_0 \to +\infty} f_1 (R_0) \to 0 \).

We can write this equation in the form of

\[
\frac{1 - \frac{\alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1 - \delta}{\delta}}}{X^* - \psi_0 (R_0) - \psi_1 (R_0) r_1^K + \psi_2 (r_1^K)^{\frac{1 - \delta}{\delta}} + \psi_3 (R_0) \left( r_1^K \right)^{\frac{1 - \delta}{\delta}} + \psi_4 (R_0) \left( r_1^K \right)^{1 + \frac{1 - \delta}{\delta}},} \tag{A.20b} \]

where \( \psi_0^1, \psi_1^1, \psi_2^1, \psi_3^1, \psi_4^1 > 0 \). Denote its right-hand side as \( F_1 (r_1^K, R_0) \).

Equation (A.19d) becomes

\[
1 = \frac{m \left( 1 - \delta + r_1^K \right)}{R_0} + \frac{\gamma}{1 + \gamma} \omega_0 k_0 \left[ 1 - \delta + \left( \beta R_0 \frac{A_0}{A_1} \right)^{\frac{1 - \delta}{\delta}} \right] \left[ \frac{\max(1 - \delta)}{X^*} + \left( 1 - \frac{\alpha}{X^*} (1 - m) \right) r_1^K \right] \frac{1 - \frac{\alpha}{X^*} A_1 \left( X^*/\alpha \right)^{\frac{1 - \delta}{\delta}}}{\beta R_0} \left( r_1^K \right)^{\frac{\beta}{\delta}}, \tag{A.20c} \]

\[ \text{A-21} \]
which can be similarly written as

\[
\frac{1 - \alpha}{X^\ast} A_1 \left( \frac{X^\ast}{\alpha} \right)^{\frac{1}{\alpha - 1}} = \psi_0^2 (R_0) + \psi_2^2 (R_0) r_1^K + \psi_3^2 (r_1^K) \frac{1}{\alpha} + \psi_4^2 (R_0) \left( r_1^K \right) \frac{1}{\alpha - 1} + \psi_5^2 (R_0) \left( r_1^K \right)^{1 + \frac{1}{\alpha - 1}},
\]

(A.20d)

where \( \psi_0^2, \psi_2^2, \psi_3^2, \psi_4^2 > 0 \). Thus there exists a unique solution for \( r_1^K \) as a function of \( R_0 \). Denote this implicit function as \( r_1^K = f_2 (R_0) \). We can also easily verify that that \( \lim_{R_0 \to m(1-\delta)} f_2 (R_0) \to 0 \), and as \( \lim_{R_0 \to +\infty} f_2 (R_0) \to 0 \). Thus \( f_2 (R_0) \) is not monotone.

**Step 2: Equilibrium Existence**

First, we show that, given \( \omega_0 < \frac{X^\ast}{\alpha} \), as \( R_0 \to +\infty \), \( f_2 (R_0) \) is asymptotically higher than \( f_1 (R_0) \).

As \( R_0 \to +\infty \), \( f_1 (R_0) \) and \( f_2 (R_0) \) both converge to zero. We can derive the following asymptotic behaviors as \( R_0 \to +\infty \):

\[
[f_1 (R_0)]^{\frac{1}{1 - \alpha}} \propto \frac{1}{\left( \frac{X^\ast}{\alpha} - \omega_0 \right) \left( \frac{A_0}{A_1} \right)^{\frac{1}{\alpha - 1}} k_0 \alpha (1 - \delta)} R_0^{\frac{\alpha - 1}{\alpha}}.
\]

\[
[f_2 (R_0)]^{\frac{1}{1 - \alpha}} \propto \frac{1 + \gamma}{\omega_0} \frac{\left( \beta \frac{A_0}{A_1} \right)^{\frac{1 - 1}{\alpha}} k_0 \alpha (1 - \delta)} R_0^{\frac{\alpha - 1}{\alpha}}.
\]

If \( \omega_0 < \frac{X^\ast}{\alpha} \), \( f_2 (R_0) \) is asymptotically higher than \( f_1 (R_0) \).

From the last two steps, we obtain \( f_1 (R_0) > f_2 (R_0) \) at \( R_0 = m(1 - \delta) \) and \( f_1 (R_0) < f_2 (R_0) \) when \( R_0 \) is sufficiently high. By the Intermediate Value Theorem, the two functions will cross at least once. This guarantees the existence of a solution \( (R_0, r_1^K) \) for the two equations (A.20a) and (A.20c) without considering the ZLB.

**Step 3: Equilibrium Uniqueness**

We show at any intersection of \( f_1, f_2 \), i.e. \( f_1 (R_0) = f_2 (R_0) \), the slope of \( f_2 \) must be steeper than the one for \( f_1 \), i.e. \( f_1' (R_0) < f_2' (R_0) \).

By the Implicit Function Theorem,

\[
f_1' (R_0) = -\frac{\partial F_1 / \partial R_0}{\partial F_1 / \partial r_1^K},
\]

\[A-22\]
in which
\[
\frac{\partial F_1}{\partial r_1^k} = -\psi_1^1 (R_0) + \frac{\alpha}{1 - \alpha} \psi_2^1 \left( r_1^k \right)^{\frac{1}{1-\alpha} - 1} + \frac{1}{1 - \alpha} \psi_3^1 (R_0) \left( r_1^k \right)^{\frac{1}{1-\alpha}} + \left( 1 + \frac{1}{1 - \alpha} \right) \psi_4^1 (R_0) \left( r_1^k \right)^{\frac{1}{1-\alpha}},
\]
\[
\frac{\partial F_1}{\partial R_0} = -\frac{d}{dR_0} \psi_0^1 (R_0) - \frac{d}{dR_0} \psi_1^1 (R_0) r_1^k + \frac{d}{dR_0} \psi_3^1 (R_0) \left( r_1^k \right)^{\frac{1}{1-\alpha}} + \frac{d}{dR_0} \psi_4^1 (R_0) \left( r_1^k \right)^{1 + \frac{1}{1-\alpha}}.
\]

We can easily check that \( \frac{\partial F_1}{\partial R_0} > 0 \). Since \( r_1^k \) is a decreasing function of \( R_0 \), by the implicit function theorem, \( \frac{\partial F_1}{\partial r_1^k} > 0 \).

Similarly,
\[
f_2'(R_0) = -\frac{\partial F_2 / \partial R_0}{\partial F_2 / \partial r_1^k},
\]
in which
\[
\frac{\partial F_2}{\partial r_1^k} = \psi_2^1 (R_0) + \frac{\alpha}{1 - \alpha} \psi_2^2 \left( r_1^k \right)^{\frac{1}{1-\alpha} - 1} + \frac{1}{1 - \alpha} \psi_3^2 (R_0) \left( r_1^k \right)^{\frac{1}{1-\alpha}} + \left( 1 + \frac{1}{1 - \alpha} \right) \psi_4^2 (R_0) \left( r_1^k \right)^{\frac{1}{1-\alpha}},
\]
\[
\frac{\partial F_2}{\partial R_0} = \frac{d}{dR_0} \psi_0^2 (R_0) + \frac{d}{dR_0} \psi_1^2 (R_0) r_1^k + \frac{d}{dR_0} \psi_3^2 (R_0) \left( r_1^k \right)^{\frac{1}{1-\alpha}} + \frac{d}{dR_0} \psi_4^2 (R_0) \left( r_1^k \right)^{1 + \frac{1}{1-\alpha}}.
\]

After lengthy algebras using these expressions, we find that \( f_1'(R_0) < f_2'(R_0) \) is implied by
\[
\left[ \frac{d}{dR_0} \psi_3^1 (R_0) \left( r_1^k \right)^{\frac{1}{1-\alpha}} + \frac{d}{dR_0} \psi_4^1 (R_0) \left( r_1^k \right)^{1 + \frac{1}{1-\alpha}} \right] 
\times \left[ \frac{\alpha}{1 - \alpha} \psi_2^2 \left( r_1^k \right)^{\frac{1}{1-\alpha} - 1} + \frac{1}{1 - \alpha} \psi_3^2 (R_0) \left( r_1^k \right)^{\frac{1}{1-\alpha}} + \left( 1 + \frac{1}{1 - \alpha} \right) \psi_4^2 (R_0) \left( r_1^k \right)^{\frac{1}{1-\alpha}} \right] 
> \left[ \frac{d}{dR_0} \psi_3^2 (R_0) \left( r_1^k \right)^{\frac{1}{1-\alpha}} + \frac{d}{dR_0} \psi_4^2 (R_0) \left( r_1^k \right)^{1 + \frac{1}{1-\alpha}} \right] 
\times \left[ \frac{\alpha}{1 - \alpha} \psi_2^1 \left( r_1^k \right)^{\frac{1}{1-\alpha} - 1} + \frac{1}{1 - \alpha} \psi_3^1 (R_0) \left( r_1^k \right)^{\frac{1}{1-\alpha}} + \left( 1 + \frac{1}{1 - \alpha} \right) \psi_4^1 (R_0) \left( r_1^k \right)^{\frac{1}{1-\alpha}} \right].
\]

Inserting the expressions of \( \psi_1^1, \psi_2^1, \psi_3^1, \psi_4^1 \) and \( \psi_2^2, \psi_3^2, \psi_4^2 \) above and after some calculations, we see that the inequality above holds.\(^\text{32}\)

Combining the previous three steps together, we can see that with binding collateral constraint and non-binding ZLB, a solution to \((A.20a)\) and \((A.20c)\) exists and is unique (without checking whether the implied \( R_0 \) satisfies the ZLB).

**Step 4: Cutoff of \( \omega_0 \) for ZLB**

An example of equations \((A.20a)\) and \((A.20c)\) are given in Figure A.3. In particular,

\(^\text{32}\)The calculations are relatively straightforward since \( F_1 (r_1^k, R_0) \) and \( F_2 (r_1^k, R_0) \) take the same form, and many common terms cancel out.
by checking equations (A.20a) and (A.20c), we see that, as $\omega_0$ increases both curves shift to the right, and the equilibrium $R_0$ increases. In other words, $R_0$ increases with $\omega_0$. Thus we can identify the cutoff for binding ZLB, $\hat{\omega}_0(k_0)$, such that given $k_0$, $R_0 = 1$ at $\omega_0 = \hat{\omega}_0(k_0)$. The expression of $\hat{\omega}_0(k_0)$ can be solved implicitly by imposing $R_0 = 1$ in (A.20a) and (A.20c).

![Figure A.3: Equilibria with Binding Collateral Constraint and no ZLB](image)

Note: This figure is generated by setting $\beta = 0.99, \gamma = 0.98, \alpha = 0.35, \delta = 0.025, A_0 = 1, A_1 = 1.005, m = 0.8$ and $\epsilon = 21$. $k_0 = 0.1$.

We can show that at $\omega_0 = \hat{\omega}_0(k_0)$, $r^K_1$ is decreasing in $k_0$. In particular, there is a constant

$$
\hat{k}_0 = \frac{1}{m(1-\delta) + \left(\frac{X^*}{\alpha} - (1-m)\right)\left(\frac{1}{m} - (1-\delta)\right)} + \frac{1}{\beta} \left[\frac{1-\alpha}{X^* A_1} \left(\frac{1}{m} - (1-\delta)\right)\right]^{\frac{\bar{n}_T}{\bar{n}}},
$$

such that when $k_0 = \hat{k}_0$, $\omega_0(\hat{k}_0) = 0$, and $r^K_1 = \frac{1}{m} - (1-\delta)$. As $k_0 \to +\infty$, $\lim_{k_0 \to +\infty} \hat{\omega}_0(k_0) = \frac{(1+\gamma)(1-m(1-\delta))}{1+\gamma\left(\frac{2}{\beta} - 1\right)m(1-\delta)}$, and $\lim_{k_0 \to +\infty} r^K_1 [k_0, \hat{\omega}_0(k_0)] = 0$. See the blue solid line in Figure A.4 for an example of $\hat{\omega}_0(k_0)$.
D.2.2 Equilibrium with Binding ZLB and Binding Collateral Constraint

Now we explore the situation when both the collateral constraint and the ZLB are binding.

**Lemma 10.** Assume the collateral constraint is binding, and $\omega_0$ is smaller than

$$\omega = \min \left\{ (1 + \gamma) \frac{1 - \alpha}{\alpha} X^*, H (\gamma, \beta, \delta, m, \alpha, X^*) \right\},$$

where $H$ is a function defined in (A.21g). Given $k_0$, there is a cutoff value of $\omega_0$, $\underline{\omega}_0 (k_0) < \hat{\omega}_0 (k_0)$ defined in Lemma 9 such that if $\underline{\omega}_0 (k_0) < \omega_0 < \hat{\omega}_0 (k_0)$, there exists a unique solution to (A.19c) and (A.19d) with binding ZLB, and in this solution, $X_0$ is decreasing in $\omega_0$. If $\omega_0 \geq \hat{\omega}_0 (k_0)$ or $\omega_0 \leq \underline{\omega}_0 (k_0)$, there does not exist such an solution.

**Proof.** **Step 1: Equilibrium Representation**

In this case, we represent the system as functions of $\{r_1^K, r_0^K\}$. Setting $R_0 = 1$, and from

---

**Figure A.4: Cut-offs with Binding Collateral Constraint**

*Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$, $m = 0.5$ and $\epsilon = 21$.**
(A.19b) $X_0$ can be expressed as a function of $\{r^K_1, r^K_0\}$ as below:

$$X_0 = X^* \left( \frac{A_0}{A_1} \right)^{1-\alpha} \left( \frac{r^K_1}{r^K_0} \right) \alpha. \tag{A.21b}$$

The counterpart for the restriction $X_0 \geq X^*$ is

$$r^K_0 \leq \left( \frac{A_0}{A_1} \right)^{1-\alpha} r^K_1. \tag{A.21c}$$

Equation (A.19c) becomes

$$k_1 + \frac{11 - \alpha}{\beta X^* A_1} \left( \frac{X^*}{\alpha} \right)^{-\frac{\alpha-1}{\alpha}} \left( \frac{r^K_1}{r^K_0} \right)^{-\frac{\alpha-1}{\alpha}}
= (1 - \delta) \left( 1 - \frac{\omega_0}{1 + \gamma} \right) k_0 + \left( \frac{X^*}{\alpha} \beta \frac{A_0}{A_1} \right)^{1-\alpha} \left( \frac{r^K_1}{r^K_0} \right) - \frac{\omega_0}{1 + \gamma} r^K_0 k_0. \tag{A.21d}$$

If $\omega_0 < (1 + \gamma) \frac{1-\alpha}{\alpha} X^*$, we can show that $r^K_0$ is a decreasing function of $r^K_1$. Denote $r^K_0 = g_1 (r^K_1)$. There is an upper bound for $r^K_1$, denoted as $\hat{r}^K_1$, such that $g_1 (\hat{r}^K_1) = 0$, and $X_0 \to \infty$ at $\hat{r}^K_1$.

Equation (A.19d) becomes

$$\left( 1 - m (1 - \delta) - mr^K_1 \right) k_1 = \frac{\gamma}{1 + \gamma} \omega_0 k_0 \left( 1 - \delta + r^K_0 \right) \tag{A.21e}$$

in which $r^K_0$ is decreasing in $r^K_1$ as well. Denote this function as $r^K_0 = g_2 (r^K_1)$. Similarly, in (A.21e), there is also an upper bound for $r^K_1$, denoted as $\hat{r}^K_1$, such that $g_2 (\hat{r}^K_1) = 0$, and $X_0 \to \infty$ at the upper bound $\hat{r}^K_1$.

**Step 2: Equilibrium Existence**

We show that given $\omega_0 < \hat{\omega}_0 (k_0)$, defined in Lemma 9, and $\hat{r}^K_1 > \hat{r}^K_1$, there exists an equilibrium with both binding collateral constraint and ZLB.

The intuition of this result can be seen in Figure A.5. The black dashed line corresponds to $X_0 = X^*$ below which we have $X_0 \geq X^*$. When $\omega_0 < \hat{\omega}_0 (k_0)$, by equations (A.20a) and (A.20c) in Subsection D.2.1 (also see Figure A.3), with $R_0 = 1$ and $X_0 = X^*$, $r^K_1$ in (A.21d) is smaller than $r^K_1$ in (A.21e). Correspondingly, in Figure A.5, Point A, the intersection of $g_1 (r^K_1)$ and $X_0 = X^*$ lies to the lower left of Point B, the intersection of $g_2 (r^K_1)$ and $X_0 = X^*$. In other words, at $r^K_1 = r^K_{1,B}$, the value at point B, $g_1 (r^K_{1,B}) < g_2 (r^K_{1,B})$. Now with $\hat{r}^K_1 > \hat{r}^K_1$, we see that $g_1 (\hat{r}^K_1) > g_2 (\hat{r}^K_1)$. Since both $g_1 (r^K_1)$ and $g_2 (r^K_1)$ are continuous,
by the Intermediate Value Theorem, they intersect at least once at some $r^K_1 \in [r^K_{1,\beta}, r^K_1]$ with $X_0 > X^\ast$. Thus there exists an equilibrium with both binding ZLB and collateral constraint in this range.

![Figure A.5: Equilibria with Binding Collateral Constraint and ZLB](image)

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$, $m = 0.8$ and $\epsilon = 21$. $\omega_0 = 0.08$ and $k_0 = 1.27$.

**Step 3: Equilibrium Uniqueness**

When $\omega_0$ is smaller than the value in equation (A.21g), the slope of $g_1 (r^K_1)$ is higher than the slope of $g_2 (r^K_1)$ when they intersect.

Using implicit function theorem, the derivatives of $g_1 (r^K_1)$ and $g_2 (r^K_1)$ are

$$
\frac{\partial g_1}{\partial r^K_1} (r^K_1) = \frac{\frac{\partial k_1}{\partial r^K_1} - \frac{1}{\beta} A_1 \left( \frac{X^\ast}{\alpha} \right)^{\frac{1}{1-\gamma}} (r^K_1)^{\frac{1}{1-\gamma}} - X_0 k_0 r^K_0}{(1-\alpha) X_0 - \omega_0} k_0,
$$

$$
\frac{\partial g_2}{\partial r^K_1} (r^K_1) = \frac{-m k_1 + (1 - m (1 - \delta) - m r^K_1) \frac{\partial k_1}{\partial r^K_1}}{\frac{1}{1-\gamma} \omega_0 k_0}.
$$

We will show that given (A.21c) and $\delta \leq r^K_1 \leq \frac{1-m(1-\delta)}{m}$,

$$
\frac{\partial g_1}{\partial r^K_1} (r^K_1) > \frac{\partial g_2}{\partial r^K_1} (r^K_1).
$$
holds at their intersection for $\omega_0$ sufficiently small. The lower bound of $r^K_1$, $\delta$ is derived by the relation $R^K_1 \geq R = 1$, while its upper bound, $\frac{1 - m (1 - \delta)}{m}$ is from equation (A.21e). Indeed, the inequality can be rewritten as

$$
\left( \left( \left[ \frac{-\partial k_1}{\partial r^K_1} \right] + \frac{1}{\beta} A_1 \left( \frac{X^*}{\alpha} \right) \frac{1}{1 + \gamma} \left( r^K_1 \right) \gamma \frac{1 + \gamma}{1 + \gamma} \left( m k_1 + (1 - m (1 - \delta) - m r^K_1) \left[ \frac{-\partial k_1}{\partial r^K_1} \right] \right) \right) \right) \omega_0 
\leq \left( \frac{1 - \alpha}{\alpha} \left( m k_1 + (1 - m (1 - \delta) - m r^K_1) \left[ \frac{-\partial k_1}{\partial r^K_1} \right] \right) - \frac{\gamma}{1 + \gamma} \omega_0 k_0 r^K_1 \right) X_0, 
$$

(A.21f)
in which the expression of $k_1$ is from (A.19a).

After some calculations, we can show a stronger result as below:

$$
\left( \left( \left[ \frac{-\partial k_1}{\partial r^K_1} \right] + \frac{1}{\beta} A_1 \left( \frac{X^*}{\alpha} \right) \frac{1}{1 + \gamma} \left( r^K_1 \right) \gamma \frac{1 + \gamma}{1 + \gamma} \left( m k_1 + (1 - m (1 - \delta) - m r^K_1) \left[ \frac{-\partial k_1}{\partial r^K_1} \right] \right) \right) \right) \omega_0 
\leq \frac{1 - \alpha}{\alpha} \left( m + \frac{1 - \frac{m}{X^*} (1 - m) (1 - m (1 - \delta) - m r^K_1)}{X^*} + \frac{1 - \frac{m}{X^*} (1 - m) r^K_1}{X^*} \right) k_1 X^*, 
$$

and this inequality holds if

$$
\omega_0 < H(\gamma, \beta, \delta, m, \alpha, X^*) \equiv \min \left\{ G(\delta), G \left( \frac{1 - m (1 - \delta)}{m} \right) \right\}, 
$$

(A.21g)
in which

$$
G \left( r^K_1 \right) = \frac{1 - \alpha}{\alpha} \left( \frac{m^2 \alpha^2 (1 - \delta)}{X^*} + \left( 1 - \frac{m}{X^*} (1 - m) \right) \frac{1}{1 + \gamma} \frac{1}{1 - \alpha} m a (1 - \delta) \frac{1}{r^K_1} \right) X^* / \left\{ \left( 1 + \gamma \frac{\gamma}{\beta} - 1 \right) m (1 - \delta) \right\} \frac{1}{1 + \gamma} \frac{1}{1 - \alpha} \frac{1}{X^*} \frac{m a (1 - \delta)}{r^K_1} 
+ \left( \frac{\gamma}{\beta} (X^* - \alpha (1 - m)) - \alpha m \right) \frac{1}{1 - \alpha} \frac{1}{1 + \gamma} \frac{1}{X^*} \frac{m (1 - \delta)}{r^K_1} 
+ \frac{1}{1 + \gamma} \frac{1}{1 - \alpha} \left( 1 - \frac{m}{X^*} (1 - m) \right) \frac{1 + \gamma \frac{2 \alpha \gamma}{\beta} - 1}{1 + \gamma} \frac{m (1 - \delta)}{r^K_1} + \frac{1}{1 + \gamma} \frac{1}{1 - \alpha} \frac{1 - 2 \alpha m^2 \alpha (1 - \delta)}{X^*}. 
$$

As a result, given $k_0, \omega_0 < \hat{\omega}_0 (k_0)$ from Subsection D.2.1, and $r^K_1 > \hat{r}_1^K$, an equilibrium with binding collateral constraint and ZLB exists and is unique. Otherwise, if there are multiple equilibria in this region, $g_1 \left( r^K_1 \right)$ and $g_2 \left( r^K_1 \right)$ cross for multiple times, and then one of these equilibria features $\frac{dg_1}{dr^K_1} \leq \frac{dg_2}{dr^K_1}$ which contradicts the slope comparison above.

**Step 4: Equilibrium Non-Existence with Binding ZLB and Collateral Constraint**

In Figure A.6, we show the comparative statics results for decreasing $\omega_0$. We see that as $\omega_0$ decreases, $g_1 \left( r^K_1 \right)$ shifts to the right, while $g_2 \left( r^K_1 \right)$ shifts to the left, making the equilibrium $r^K_1$ higher and $r^K_0$ lower. From (A.21b), $X_0$ also increases. Thus $X_0 decreases in \omega_0$. 

A-28
We also see that \( \frac{\partial \hat{r}_1^K}{\partial \omega_0} > 0 \) and \( \frac{\partial \tilde{r}_1^K}{\partial \omega_0} < 0 \). When \( \omega_0 \) drops to the level \( \omega_0 (k_0) \) such that \( \hat{r}_1^K = \tilde{r}_1^K \), at the intersection of \( g_1 (r_1^K) \) and \( g_2 (r_1^K) \), \( r_0^K = 0 \), and \( X_0 \) goes to infinity.

![Figure A.6: Comparative Statics with Binding Collateral Constraint and ZLB](image)

Note: This figure is generated by setting \( \beta = 0.99 \), \( \gamma = 0.98 \), \( \alpha = 0.35 \), \( \delta = 0.025 \), \( A_0 = 1 \), \( A_1 = 1.005 \), \( m = 0.8 \) and \( \epsilon = 21 \), and \( k_0 = 1.27 \).

Now we solve the cut-off value \( \omega_0 (k_0) \) by setting \( \hat{r}_1^K = \tilde{r}_1^K \). Accordingly, by setting \( r_0^K = 0 \) in equations (A.21d) and (A.21e), we have

\[
\frac{1}{m} \left( 1 - \alpha X^* \right) \left( \frac{\omega_0}{1 + \gamma} \right) = \frac{1}{m} \left( 1 - \alpha X^* \right) \left( \frac{\omega_0}{1 + \gamma} \right) \left( 1 - \delta \right) \left( 1 - \frac{\omega_0}{1 + \gamma} \right) k_0,
\]

\[
\left( 1 - m \left( 1 - \delta \right) - mr_1^K \right) k_1 = \frac{\gamma}{1 + \gamma} \left( 1 - \delta \right) \omega_0 k_0.
\]

Notice that these two equations are exactly the same as equations (A.18h) and (A.18i) when we define \( \bar{k}_0 (\omega_0) \). Thus \( \omega_0 (k_0) \) is the inverse function of \( \bar{k}_0 (\omega_0) \) and is increasing in \( k_0 \).

We can show that \( \forall k_0 > 0, \omega_0 (k_0) < \omega_0 (k_0) \), the cutoff for binding ZLB in Lemma 9. First, when

\[
k_0 = \bar{k}_0 = \frac{1 - \alpha X^* A_1 \left( \frac{\omega_0}{1 + \gamma} \right) \left( 1 - \delta \right) \left( 1 - \frac{\omega_0}{1 + \gamma} \right)^{\frac{\delta}{\alpha}}}{m \left( 1 - \delta \right) + \left( \frac{\omega_0}{1 + \gamma} \right) \left( 1 - \delta \right) \left( 1 - \delta \right) \left( 1 - \frac{\omega_0}{1 + \gamma} \right)^{\frac{\delta}{\alpha}}},
\]

\( \omega_0 (k_0) = 0 \), and the implied \( r_1^K = \frac{1}{m} \left( 1 - \delta \right) \). Notice \( k_0 > \bar{k}_0 \) in equation (A.20e).

As \( k_0 \to +\infty \), \( \lim_{k_0 \to +\infty} \omega_0 (k_0) = \frac{(1 + \gamma)(1 - m(1 - \delta))}{1 + \gamma + (\beta - 1)m(1 - \delta)} \), and \( \lim_{k_0 \to +\infty} r_1^K \left[ k_0, \omega_0 (k_0) \right] = 0. \)
Lastly, $\omega_0(k_0)$ and $\hat{\omega}_0(k_0)$ never cross. Otherwise, $r^K_0 = 0$ at their intersection. To sum up, we have $\omega_0(k_0) < \hat{\omega}_0(k_0)$. See the red dashed line in Figure A.4 for an example of $\omega_0(k_0)$.

Now we show that, when $\omega_0 < \omega_0(k_0)$, there is no solution to (A.19c) and (A.19d) with binding collateral constraint and binding ZLB. We keep our definition of Point B as the intersection of $g_2(r^K_1)$ and $r^K_0 = (\beta A_0 A_1) \frac{1}{x} r^K_1$ in equation (A.21c). Since $\omega_0 < \omega_0(k_0) < \hat{\omega}_0(k_0)$, we have $g_1(r^K_{1,B}) < g_2(r^K_{1,B})$ at $r^K_1 = r^K_{1,B}$. We also have $g_1(\hat{r}^K_1) < g_2(\hat{r}^K_1)$. Then in the range $[r^K_{1,B}, \hat{r}^K_1]$, if $g_1(r^K_1)$ and $g_2(r^K_1)$ intersect, they would intersect at least twice, and at one of the intersections, the condition $\frac{dg_1}{dr^K_1} > \frac{dg_2}{dr^K_1}$ would be violated. Consequently, there is no solution with binding collateral constraint and ZLB in this region.

We can use similar argument to rule out any solution to (A.19c) and (A.19d) with binding collateral constraint and binding ZLB when $\omega_0 \geq \hat{\omega}_0(k_0)$. In this case, Point B lies to the lower-left of Point A in Figure A.5, and $\hat{r}^K_1 < \hat{r}^K_1$. Therefore, if $g_1(r^K_1)$ and $g_2(r^K_1)$ intersect, they would intersect at least twice. Then at one of the intersections, the condition $\frac{dg_1}{dr^K_1} > \frac{dg_2}{dr^K_1}$ would be violated.  

\section*{D.2.3 Equilibrium Existence and Uniqueness with Binding Collateral Constraint}

In the previous subsections, we impose a binding collateral constraint and characterize the equilibrium properties without checking that whether $\mu_0 \geq 0$, or equivalently (A.19e). It is possible that $R^K_1 < R_0$ and borrowing to the limit may not be the entrepreneurs’ optimal choice. In this subsection, we check whether this is the case.

The following lemma is useful in proving the equilibrium existence and uniqueness with binding collateral constraint.

\begin{lemma}

The derivative of the excess return $R^K_1 - R_0$ is negative at $\omega_0 = \omega_0^{CC}(k_0)$ given in Lemma 8.

\end{lemma}

\begin{proof}

It is equivalent to show that $r^K_1 - R_0$ is decreasing in $\omega_0$ at $\omega_0 = \omega_0^{CC}(k_0)$. Notice that since the collateral constraint is binding at $\omega_0 = \omega_0^{CC}(k_0)$, the equilibrium properties with binding collateral constraint from Lemmas 9 and 10 still apply at $\omega_0 = \omega_0^{CC}(k_0)$.

If the ZLB is binding at $\omega_0 = \omega_0^{CC}(k_0)$, $R_0 = 1$, and $R^K_1 = 1 - \delta + r^K_1$ is decreasing in $\omega_0$ as shown in Figure A.6, which implies a negative derivative of $R^K_1 - R_0$ at $\omega_0 = \omega_0^{CC}(k_0)$.

\end{proof}
If the ZLB is not binding, we write the system that determines $r^K_1$ and $R_0$ as

$$F_1 \left( r^K_1, R_0; \omega_0 \right) = 0,$$
$$F_2 \left( r^K_1, R_0; \omega_0 \right) = 0,$$

where $F_1$ and $F_2$ are defined in equations (A.20b) and (A.20d). By the Implicit Function Theorem,

$$\begin{bmatrix}
\frac{\partial F_1}{\partial r^K_1} & \frac{\partial F_1}{\partial R_0} \\
\frac{\partial F_2}{\partial r^K_1} & \frac{\partial F_2}{\partial R_0}
\end{bmatrix} \begin{bmatrix}
\frac{dr^K_1}{d\omega_0} \\
\frac{dR_0}{d\omega_0}
\end{bmatrix} = -\begin{bmatrix}
\frac{\partial F_1}{\partial \omega_0} \\
\frac{\partial F_2}{\partial \omega_0}
\end{bmatrix},$$

we have

$$\begin{bmatrix}
\frac{dr^K_1}{d\omega_0} \\
\frac{dR_0}{d\omega_0}
\end{bmatrix} = -\frac{1}{\frac{\partial F_1}{\partial r^K_1} \frac{\partial F_2}{\partial R_0} - \frac{\partial F_1}{\partial R_0} \frac{\partial F_2}{\partial r^K_1}} \begin{bmatrix}
\frac{\partial F_2}{\partial \omega_0} & -\frac{\partial F_1}{\partial \omega_0} \\
-\frac{\partial F_2}{\partial \omega_0} & \frac{\partial F_1}{\partial \omega_0}
\end{bmatrix} \begin{bmatrix}
\frac{\partial F_1}{\partial \omega_0} \\
\frac{\partial F_2}{\partial \omega_0}
\end{bmatrix}.$$

In Subsection D.2.1, we have shown that

$$\frac{\partial F_1}{\partial R_0} \frac{\partial F_2}{\partial r^K_1} - \frac{\partial F_1}{\partial r^K_1} \frac{\partial F_2}{\partial R_0} < 0.$$

Thus to show $\frac{dr^K_1}{d\omega_0} - \frac{dR_0}{d\omega_0} < 0$, we need to show that

$$\frac{\partial F_2}{\partial R_0} \frac{\partial F_1}{\partial \omega_0} - \frac{\partial F_1}{\partial R_0} \frac{\partial F_2}{\partial \omega_0} + \frac{\partial F_2}{\partial r^K_1} \frac{\partial F_1}{\partial \omega_0} - \frac{\partial F_1}{\partial r^K_1} \frac{\partial F_2}{\partial \omega_0} < 0.$$

We can see that $\frac{\partial F_1}{\partial R_0} > 0$, $\frac{\partial F_1}{\partial r^K_1} > 0$, $\frac{\partial F_2}{\partial \omega_0} < 0$, $\frac{\partial F_2}{\partial r^K_1} > 0$, and $\frac{\partial F_2}{\partial R_0} > 0$. But the sign of $\frac{\partial F_2}{\partial R_0}$ is not clear. In particular, we have $\frac{\partial F_2}{\partial \omega_0} = -\gamma \frac{\partial F_1}{\partial \omega_0}$. Thus to show the inequality above, it is sufficient to show that

$$\frac{\partial F_2}{\partial R_0} \frac{\partial F_1}{\partial \omega_0} + \gamma \frac{\partial F_1}{\partial R_0} \frac{\partial F_1}{\partial \omega_0} + \frac{\partial F_2}{\partial r^K_1} \frac{\partial F_1}{\partial \omega_0} + \gamma \frac{\partial F_1}{\partial r^K_1} \frac{\partial F_1}{\partial \omega_0} < 0.$$

Since $\frac{\partial F_1}{\partial \omega_0} < 0$, it is then sufficient to show that

$$\frac{\partial F_2}{\partial R_0} + \gamma \frac{\partial F_1}{\partial R_0} + \frac{\partial F_2}{\partial r^K_1} + \gamma \frac{\partial F_1}{\partial r^K_1} > 0.$$
Since \( \frac{\partial F_1}{\partial r_1} > 0 \), we can show a stronger result

\[
\frac{\partial F_2}{\partial R_0} + \gamma \frac{\partial F_1}{\partial R_0} + \frac{\partial F_2}{\partial r_1^K} > 0.
\]

This expression can be written as

\[
\frac{\partial F_2}{\partial R_0} + \gamma \frac{\partial F_1}{\partial R_0} + \frac{\partial F_2}{\partial r_1^K} = \frac{d}{dR_0} \psi_0^2 (R_0) - \gamma \frac{d}{dR_0} \psi_0^1 (R_0) + \psi_1^2 (R_0)
\]

\[
+ \left[ \frac{d}{dR_0} \psi_1^2 (R_0) - \gamma \frac{d}{dR_0} \psi_1^1 (R_0) \right] r_1^K
\]

\[
+ \left[ \frac{d}{dR_0} \psi_2^2 (R_0) + \gamma \frac{d}{dR_0} \psi_2^1 (R_0) \right] \left( r_1^K \right)^{1 + \frac{1}{\gamma}}
\]

\[
+ \left[ \frac{d}{dR_0} \psi_2^3 (R_0) + \gamma \frac{d}{dR_0} \psi_2^4 (R_0) \right] \left( r_1^K \right)^{1 + \frac{1}{\gamma}}
\]

\[
+ \left( 1 + \frac{1}{1 - \alpha} \right) \psi_2^0 (R_0) \left( r_1^K \right)^{1 - \frac{1}{\gamma}},
\]

in which the expressions \( \psi_0^1, \psi_1^1, \psi_1^3, \psi_2^3, \psi_2^4, \psi_2^2 > 0 \) can be found in equations (A.20b) and (A.20d).

Since \( \frac{d}{dR_0} \psi_0^1 (R_0), \frac{d}{dR_0} \psi_1^1 (R_0), \frac{d}{dR_0} \psi_1^3 (R_0), \frac{d}{dR_0} \psi_2^3 (R_0), \frac{d}{dR_0} \psi_2^4 (R_0) > 0 \), it is then sufficient to show that

\[
\frac{d}{dR_0} \psi_0^2 (R_0) - \gamma \frac{d}{dR_0} \psi_0^1 (R_0) + \psi_1^2 (R_0) + \left[ \frac{d}{dR_0} \psi_1^2 (R_0) - \gamma \frac{d}{dR_0} \psi_1^1 (R_0) \right] r_1^K > 0.
\]

Inserting the expressions of \( \psi_0^1 (R_0), \psi_1^1 (R_0), \psi_0^2 (R_0), \psi_1^2 (R_0) \) into the expression above, and use the fact that \( 1 - \delta + r_1^K = R_0 \) at \( \omega_0^{CC} (k_0) \), it remains to show that

\[
\left[ \frac{\gamma}{\beta} (1 - \delta) m + \frac{\gamma}{\beta} \left( \frac{X^*}{\alpha} - (1 - m) \right) \right] r_1^K \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\gamma}} > 0,
\]

which holds naturally. Thus \( r_1^K - R_0 \) is decreasing in \( \omega_0 \) at \( \omega_0^{CC} (k_0) \) when the ZLB is not binding. \( \square \)

**Lemma 12.** When \( \omega_0 \leq \omega_0^{CC} (k_0) \) and \( k_0 < \bar{k}_0 (\omega_0) \), there is a unique equilibrium with binding collateral constraint. When \( \omega_0 > \omega_0^{CC} (k_0) \) or \( k_0 \geq \bar{k}_0 (\omega_0) \), there is no such an equilibrium.

**Proof.** In Lemma 10, we show \( \bar{k}_0 (\omega_0) \) is the inverse function of \( \omega_0 (k_0) \). And the region
with \( k_0 < \tilde{k}_0 (\omega_0) \) is equivalent to the region with \( \omega_0 > \min \left\{ 0, \omega^a_0 (k_0) \right\} \). When \( \omega_0 \leq \min \left\{ 0, \omega^a_0 (k_0) \right\} \), Lemma 10 shows that there is no equilibrium with binding collateral constraint and binding ZLB. Lemma 10 also shows that \( \omega^a_0 (k_0) < \hat{\omega}_0 (k_0) \) in Lemma 9. By Lemma 9, there is also no equilibrium with binding collateral constraint and non-binding ZLB when \( \omega_0 \leq \omega^a_0 (k_0) \).

When \( \omega_0 \in \left[ \min \left\{ 0, \omega^a_0 (k_0) \right\}, \omega^{CC}_0 (k_0) \right] \), by Lemmas 9 and 10, we impose a binding collateral constraint and establish the existence of an equilibrium in this region. It remains to verify that \( \mu_0 \geq 0 \). In other words, borrowing up to the limit is indeed the entrepreneurs’ optimal choice. Equivalently we need to show \( R^K_1 \geq R_0 \). We prove this by contradiction.

Assume that in an equilibrium at some \( k_0 = k^a_0 \) and \( \omega_0 = \omega^a_0 \in \left[ \min \left\{ 0, \omega^a_0 (k_0) \right\}, \omega^{CC}_0 (k_0) \right] \), the implied \( \mu_0 < 0 \). Since the equilibrium value of \( \mu_0 \) is continuous in \( \omega_0 \), as \( \omega_0 \) decreases, the value of \( \mu_0 \) also moves continuously. Now if \( k_0 \geq k_0 \) in (A.21h), we can decrease \( \omega_0 \) to \( \omega_0 (k_0) \). Since the ZLB is binding at \( \omega_0 (k_0) \), and \( R^K_1 > \delta \) at \( \omega_0 (k_0) \), \( \mu_0 > 0 \) at \( \omega_0 (k_0) \). Then as \( \omega_0 \) drops, the value of \( \mu_0 \) switches from negative to positive, and it must be zero at a certain value of \( \omega_0 \in \left[ \omega^a_0 (k_0), \omega^b_0 \right] \). Call this value \( \omega^b_0 \). Then at both \( (k^a_0, \omega^a_0) \) and \( (k^a_0, \omega^{CC}_0 (k^a_0)) \), the leverage ratio is exactly \( m \), and \( R^K_1 = R_0 \). This violates the fact that \( \omega^{CC}_0 (k_0) \) is uniquely determined as in Lemma 13.

If \( k^a_0 < k_0 \) in (A.21h), as \( \omega_0 \) drops from \( \omega^a_0 \), it will hit \( \omega_0 = 0 \). If the ZLB is binding at \( \{k^a_0, \omega_0 = 0\} \), then by equation (A.21e), \( R^K_1 = \frac{1}{m} > 1 = R_0 \). If the ZLB is not binding at \( \{k^a_0, \omega_0 = 0\} \), then by equation (A.20c), \( R^K_1 = \frac{R_0}{m} > R_0 \). In either case, we have \( \mu_0 > 0 \) at \( \{k^a_0, \omega_0 = 0\} \), and then we can find a value \( \omega^b_0 > 0 \) at which \( \mu_0 = 0 \) and obtain a contradiction.

Next we show that when \( \omega_0 > \omega^{CC}_0 (k_0) \), the implied excess return \( R^K_1 - R_0 < 0 \), and thus borrowing up to the limit is not the optimal choice for the entrepreneurs. We also show this by contradiction. Suppose that at some \( k_0 = k^a_0 \) and \( \omega_0 = \omega^c_0 > \omega^{CC}_0 (k_0) \), the implied excess return \( R^K_1 - R_0 > 0 \). Since the excess return moves continuously with \( \omega_0 \), \( R^K_1 - R_0 = 0 \) at \( \omega^{CC}_0 (k_0) \), and its derivative is negative at \( \omega^{CC}_0 (k_0) \) by Lemma 11, then there must exist a \( \omega^d_0 \in (\omega^{CC}_0 (k_0), \omega^c_0) \) such that \( R^K_1 - R_0 = 0 \) at \( \omega_0 = \omega^d_0 \), and the leverage ratio is \( m \). But again this violates the fact \( \omega^{CC}_0 (k_0) \) is uniquely determined as in Lemma 13. Therefore, there is no equilibrium with binding collateral constraint given \( \omega_0 > \omega^{CC}_0 (k_0) \).

Lastly, when \( k_0 \geq \tilde{k}_0 (\omega_0) \), it must be that \( \omega_0 \leq \omega^a_0 (k_0) \). By Lemma 10, an equilibrium with binding collateral constraint does not exist. □
D.3 Equilibrium Properties with Non-binding Collateral Constraint

Now we proceed to Step 3 in the proof of Proposition 2.

Lemma 13. Given \( m < 1, \alpha < \frac{X^*}{1+X^*} \) and \( \omega_0 \) is smaller than

\[
\min \left\{ \frac{X^* + X^* \gamma}{X^* + \alpha \gamma}, (1 + \gamma) \frac{1 - \alpha}{\alpha} X^*, H(\gamma, \beta, \delta, m, \alpha, X^*) \right\},
\]

when \( \omega_0 \geq \omega_0^{CC}(k_0) \) and \( k_0 < \bar{k}_0(\omega_0) \) there exists a unique equilibrium with non-binding collateral constraint; when \( \omega_0 < \omega_0^{CC}(k_0) \) or \( k_0 \geq \bar{k}_0(\omega_0) \), such an equilibrium does not exist.

Proof. Notice that an equilibrium with non-binding collateral constraint is equivalent to an equilibrium with natural borrowing limit defined in Proposition 1. Therefore, by Lemma 7, the leverage ratio \( -\frac{b_0}{R_1^1 k_1} \) is decreasing in \( \omega_0 \). Recall that, the value of \( \omega_0^{CC}(k_0) \) is determined by setting \( -\frac{b_0}{R_1^1 k_1} = m \). When \( \omega_0 \geq \omega_0^{CC}(k_0) \), the leverage ratio is smaller than \( m \), and the collateral constraint is not binding. Then an equilibrium exists and is unique given \( k_0 < \bar{k}_0(\omega_0) \) by Proposition 1. In this region, \( \bar{k}_0(\omega_0) = \bar{k}_0(\omega_0) \) in equation (A.16) by construction.

When \( \omega_0 < \omega_0^{CC}(k_0) \) and \( k_0 \leq \bar{k}_0(\omega_0) \), assuming a non-binding collateral constraint, the implied leverage ratio would be larger than \( m \) by Lemma 7, which violates the collateral constraint. Thus there is no equilibrium with non-binding collateral constraint in that region.

When \( k_0 \geq \bar{k}_0(\omega_0) \), there are two cases, \( \bar{k}_0(\omega_0) \leq k_0 < \bar{k}_0(\omega_0) \) and \( k_0 \geq \bar{k}_0(\omega_0) \). By Proposition 1, an equilibrium with non-binding collateral constraint does not exist in the latter case.

In the former case, we know both \( \bar{k}_0(\omega_0) \) and \( \bar{k}_0(\omega_0) \) are increasing and \( \bar{k}_0(\omega_0) < \bar{k}_0(\omega_0) \) when \( \omega_0 < \Lambda_0 \) by Lemma 6 and by the construction of \( \bar{k}_0(\omega_0) \) in Subsection D.1.2. By Lemma 10, \( \bar{k}_0(\omega_0) \) is the inverse of \( \omega_0(\omega_0) \). We now show that \( \omega_0(\omega_0) < \omega_0^{CC}(\omega_0) \) for all \( \omega_0 > 0 \) where both functions are well-defined.

Indeed, if \( k_0 \geq \bar{k}_0^{CC} \) defined in (A.18d), the ZLB is binding at \( \omega_0^{CC}(k_0) \) by Lemma 8. Lemma 10 shows that \( X_0 \) is decreasing in \( \omega_0 \) when both the collateral constraint and ZLB are binding. Since \( X_0 \) is finite and determined by (A.18e) at \( \omega_0 = \omega_0^{CC}(k_0) \) and \( X_0 = +\infty \) at \( \omega_0 = \omega_0(\omega_0) \) defined in Lemma 10, it must be the case that \( \omega_0(\omega_0) < \omega_0^{CC}(\omega_0) \).

If \( k_0 < \bar{k}_0^{CC} \), the ZLB is not binding at \( \omega_0^{CC}(k_0) \) by Lemma 8. Since Lemma 9 shows that \( R_0 \) is increasing in \( \omega_0 \) with binding collateral constraint and non-binding ZLB, \( \omega_0^{CC}(k_0) > \hat{\omega}_0(\omega_0) \), the cutoff value for a binding ZLB given by Lemma 9. Lemma 10 further shows \( \omega_0(\omega_0) < \hat{\omega}_0(\omega_0) \). Hence, \( \omega_0(\omega_0) < \omega_0^{CC}(\omega_0) \).
We have established that $\omega_0(k_0) < \omega_0^{CC}(k_0)$. Since $\omega_0(k_0)$ is the inverse function of $\bar{k}_0(\omega_0)$, and when $\bar{k}_0(\omega_0) \leq k_0 < \bar{k}_0(\omega_0)$, it must be the case that $\omega_0 \leq \omega_0(k_0)$, which implies $\omega_0 < \omega_0^{CC}(k_0)$. So for these $k_0$, an equilibrium with natural borrowing limit exists, but by Lemma 7, the implied leverage ratio would be larger than $m$ which violates the collateral constraint.

Combining the results from Lemmas 12, 13 and our definition for $\bar{k}_0(\omega_0)$ in Subsection D.1.2, we complete the proof of Proposition 2.

D.4 Equilibrium Non-Existence

With tighter borrowing limit, Proposition 2 shows that there does not exist an equilibrium when capital $k_0$ is sufficiently high, similar to the natural borrowing limit case analyzed in Subsection 2.3. In Figure A.7, we plot the threshold for equilibrium existence, $\bar{k}_0$, as a function of $\omega_0$ under different borrowing limits $m$. The collateral constraint binds when $\omega_0$ is low, and we see $\bar{k}_0$ is lower when $m$ is lower. The reason for equilibrium non-existence when capital goes beyond $\bar{k}_0$ is insufficient demand. When the collateral constraint binds with smaller $m$, the entrepreneurs’ consumption and investment are more constrained. Thus from the market clear condition (3a), the threshold for equilibrium nonexistence, $\bar{k}_0$, is smaller. For large $\omega_0$, the collateral constraint does not bind, so $\bar{k}_0$ equals $\bar{k}_0$ independent of the value of $m$. In Appendix D.1.2, we also show that $\bar{k}_0$ is increasing in $A_1$ and hence, equilibrium is less likely to exist when $A_1$ is low.

![Figure A.7: Upper Bound of Capital](image)

Note: This figure is generated by setting $\beta = 0.99, \gamma = 0.98, \alpha = 0.35, \delta = 0.025, A_0 = 1, A_1 = 1.005$ and $\epsilon = 21$. 
E Investment Friction and Endogenous Asset Price

In this appendix, we provide the formal statements of the equilibrium characterizations for 2-period economy with irreversible investment under the natural borrowing limit and tighter borrowing limits similar to Propositions 1 and 2. The next two appendices provide the proofs for these results.

E.1 Natural Borrowing Limit

Similar to Subsection 2.3, we focus on the case with natural borrowing limit first. The collateral constraint (1) does not bind in equilibrium, and \( \mu_0 = 0 \).

**Equilibrium Characterizations** The Proposition 3 below shows that an equilibrium always exists, is unique, and has intuitive properties. The state space \( \{k_0, \omega_0\} \) can be partitioned into different regions with either binding or non binding ZLB and binding or non-binding investment irreversibility constraint. Figure A.8 shows what the different regions look like. The investment irreversibility constraint binds when \( k_0 \) is sufficiently high and the ZLB constraint binds when \( \omega_0 \) is sufficiently low. Interestingly, unlike in the previous model without investment friction, the ZLB does not necessarily bind when \( k_0 \) is high. Indeed, the proposition below shows that for \( \omega_0 \) sufficiently high, the ZLB does not bind for any \( k_0 > 0 \).

**Proposition 3.** With the investment irreversibility constraint, \( m = 1 \), and \( \omega_0 \) smaller than

\[
\min \left\{ \frac{1}{X^*} + \frac{1}{\gamma} \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \right\} \frac{1 + \gamma \left( 1 - \frac{1}{\alpha} \right) X^*}{\gamma},
\]

an equilibrium always exists and is unique.\(^{33}\) Besides, there is a threshold of \( k_0, k_0^* (\omega_0) \) such that when \( k_0 < k_0^* (\omega_0) \), the irreversibility constraint does not bind and \( q_0 = 1 \); and when \( k_0 \geq k_0^* (\omega_0) \), the irreversibility constraint binds and \( q_0 \) is decreasing in \( k_0 \).

In addition, denote \( \omega_0^* \) as

\[
\omega_0^* = \frac{X^*}{\alpha} \left[ \left( \frac{A_0}{A_1} \beta \right)^{\frac{1-\alpha}{\alpha}} \frac{1}{\beta} (1 - \delta) \right] \frac{1}{\gamma} \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \left[ (1 - \delta) + \left( \frac{A_0}{A_1} \beta \right)^{\frac{1-\alpha}{\alpha}} \right]. \tag{A.22}
\]

\(^{33}\)With our calibrated parameters, the value of the upper bound of \( \omega_0 \) is 2.4, which is much larger than the typical values of \( \omega_0 \).
Figure A.8: Regions for Irreversibility and ZLB When \( m = 1 \)

Note: This figure is generated by setting \( \beta = 0.99 \), \( \gamma = 0.98 \), \( \alpha = 0.35 \), \( \delta = 0.025 \), \( A_0 = 1 \), \( A_1 = 0.95 \), and \( \epsilon = 21 \).

1. If \( \omega_0 > \omega_0^* \): the ZLB does not bind \( \forall k_0 > 0 \). \( R_0 \) is decreasing in \( k_0 \) when \( k_0 < k_0^* (\omega_0) \), and is independent of \( k_0 \) when \( k_0 \geq k_0^* (\omega_0) \).

2. If \( \omega_0 \leq \omega_0^* \): there exists another threshold \( \hat{k}_0 (\omega_0) \) which is smaller than \( k_0^* (\omega_0) \), such that:
   
   (i) when \( k_0 < \hat{k}_0 (\omega_0) \), the ZLB does not bind and \( R_0 \) is decreasing in \( k_0 \);
   (ii) when \( k_0 \in [\hat{k}_0 (\omega_0), k_0^* (\omega_0)] \), the ZLB binds, and \( X_0 \) is increasing in \( k_0 \);
   (iii) and when \( k_0 \geq k_0^* (\omega_0) \), ZLB binds and \( X_0 \) is independent of \( k_0 \).

Proof. The proof is given in Appendix F. We also present the AS-AD representation of the equilibrium in Appendix F.5.

One distinguishing feature of this model is that an equilibrium exists for any \( k_0 > 0 \). As discussed in Proposition 1, Part 3, in equilibrium, the rate of return on each unit of capital invested at time 0 is bounded from below by the ZLB and by (6e), which puts an upper bound on aggregate capital supply at time 1 if there is no capital adjustment cost. However, with investment irreversibility, \( k_1 \) can be large and \( R^K_1 \) in (A.23b) can be very low. The entrepreneurs are still happy to hold units of capital at time 0 because the price
$q_0$ endogenous drops yielding high return to holding capital. More formally, we show that as $k_0$ goes to infinity, either $R_0$ converges to 1 or a constant strictly greater than 1, and

$$q_0 = \phi(\omega_0) k_0^{\alpha - 1}$$

for some explicit function $\phi$. Therefore $\lim_{k_0 \to \infty} q_0 = 0$, warranting that the return to each unit of capital is exactly $R_0$, as implied by (11).

E.2 Tighter Borrowing Limit

As in Subsection 2.4, the state-space $\{k_0, \omega_0\}$ can be partitioned into regions depending on whether each of the three constraints - the collateral constraint, ZLB, and investment irreversibility constraint - binds. Figure A.9 shows the different regions for a particular set of parameters. The ZLB and the investment irreversibility constraint tend to bind when $k_0$ is high and when $\omega_0$ is low, and the collateral constraint tends to bind when $k_0$ is low and when $\omega_0$ is low.

![Figure A.9: Regions for Irreversibility, ZLB and Collateral Constraint](image)

**Note:** This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 0.99$, $m = 0.7$ and $\epsilon = 21$.

The following proposition provides a complete characterization of equilibrium at time 0.
Proposition 4. With \( m < 1 \) and the investment irreversibility constraint, given \( \{k_0, \omega_0\} \), an equilibrium always exists and is unique. In addition, there is a threshold value for \( k_0, k^{**}_0(\omega_0) \), such that

1. If \( k_0 \geq k^{**}_0(\omega_0) \), the investment irreversibility constraint binds. As \( k_0 \) increases, \( q_0 \) decreases, but \( R_0, X_0 \) and the multiplier for the collateral constraint, \( \mu_0 \) remain constant. In this region, the collateral constraint is binding if and only if \( \omega_0 \leq \omega^{CC}_{0, Irr} \), a constant whose value depends on the parameters.

2. If \( k_0 < k^{**}_0(\omega_0) \), the investment irreversibility constraint does not bind. There is a cutoff value of wealth, \( \omega^{CC}_0(k_0) \) such that the collateral constraint is binding if and only if \( \omega_0 \leq \omega^{CC}_0(k_0) \).

Proof. The proof is given in Appendix G. We also present the AS-AD representation of the equilibrium in Appendix G.6.

Similar to the case with natural borrowing limit, we can also show that as \( k_0 \) goes to infinity, either \( R_0 \) converges to 1 or a constant strictly greater than 1, and

\[ q_0 = \phi^{CC}(\omega_0)k_0^{\alpha-1} \]

for some explicit function \( \phi^{CC} \). So \( \lim_{k_0 \to \infty} q_0 = 0 \), inducing the entrepreneurs to hold on to their old units of capital because the return to each unit of capital is higher than \( R_0 \), as implied by (11).

Policy Functions In the six left panels of Figure A.10, we plot several variables as functions of \( k_0 \) fixing \( \omega_0 \) for \( m = 1 \) and \( m = 0.7 \). The shapes of the policy functions look similar under these two values of \( m \). The interest rate \( R_0 \) is decreasing in \( k_0 \), and when the ZLB binds, markup \( X_0 \) increases from its steady state value \( X^* \). As \( k_0 \) increases above some threshold the irreversibility constraint binds and \( X_0, \mu_0 \) become constant. Capital price is decreasing in \( k_0 \). The bottom right panel shows that when \( m \) decreases from 1 to 0.7, output decreases (weakly), but the magnitude of the decrease is only significant when both collateral constraint and ZLB bind (for \( k_0 \) greater than 0.09). Similarly, the six right panels of Figure A.10 show several variables as functions of \( \omega_0 \) fixing \( k_0 \) for \( m = 1 \) and \( m = 0.7 \). All variables, except for the multiplier on the collateral constraint, are increasing in \( \omega_0 \). In addition, output decreases significantly when \( m \) decreases from 1 to 0.7 if both collateral constraint and ZLB bind (for \( \omega_0 \) less than 0.2).
Figure A.10: Policy Functions with Irreversibility Constraint, \( m = 1 \)

Note: This figure is generated by setting \( \beta = 0.99, \gamma = 0.98, \alpha = 0.35, \delta = 0.025, A_0 = 1, A_1 = 0.99, \) and \( \epsilon = 21. \) In the left panel, \( \omega_0 = 0.1. \) In the right panel, \( k_0 = 0.1. \)

F Proof of Proposition 3

To prove Proposition 3, we proceed as follows:

1. Describe the threshold for a binding irreversibility constraint, \( k^*_0 (\omega_0). \)

2. Show that an equilibrium with binding irreversibility constraint exists, and is unique, if and only if \( k_0 \geq k^*_0 (\omega_0). \)

3. Show that an equilibrium with non-binding irreversibility constraint exists, and is unique, if and only if \( k_0 < k^*_0 (\omega_0). \)

Combining the results from the previous three steps, we can prove Proposition 3.

F.1 Equilibrium Properties

We first describe some equilibrium results which are useful for the following proof of Proposition 3.

**Last Period** In the last period, period 1, there is no return of investment, the irreversibility constraint is binding, i.e., \( k_2 = (1-\delta)k_1, \) and the capital price \( q_1 = 0. \) We still set
the markup $X_1 = X^*$. The market clearing condition (3a) implies $c_1 + c'_1 = Y_1$. Given 
\{k_1, \omega_1\}, we can solve for the equilibrium in closed-form. In particular,

\[
L_1 = \frac{X^*}{1 - \frac{X^*}{X^*} \omega_1}, \quad \text{(A.23a)}
\]

\[
R^K_1 = \frac{\alpha}{X^*} \left( \frac{X^*}{1 - \frac{X^*}{X^*} \omega_1} \right)^{1-\alpha} k_1^{\alpha-1}. \quad \text{(A.23b)}
\]

**First Period** In period 0, the gross return for holding capital $k_1$ is $R^K_0$. With natural bor-
rowing limit, by the no-arbitrage condition,

\[
R_0 = \frac{R^K_1}{q_0}. \quad \text{(A.24a)}
\]

Given $k_1$, $R_0$ and $q_0$, in the last period, we can derive the follow expressions:

\[
Y_1 = \frac{X^*}{\alpha} q_0 R_0 k_1, \quad \text{(A.24b)}
\]

\[
c'_1 = \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} q_0 R_0 \right)^{\frac{\alpha}{\alpha-1}},
\]

\[
c_1 = \frac{X^*}{\alpha} q_0 R_0 k_1 - \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} q_0 R_0 \right)^{\frac{\alpha}{\alpha-1}}.
\]

Using the optimal consumption choices $c_0 = \frac{c_1}{\gamma R_0}$ and $c'_0 = \frac{c'_1}{\gamma R_0}$, we obtain the expressions for consumption at $t = 0$:

\[
c_0 = \frac{1}{\gamma} \frac{X^*}{\alpha} q_0 k_1 - \frac{1 - \alpha}{\gamma R_0} \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} q_0 R_0 \right)^{\frac{\alpha}{\alpha-1}}, \quad \text{(A.24c)}
\]

\[
c'_0 = \frac{1}{\beta R_0} \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} q_0 R_0 \right)^{\frac{\alpha}{\alpha-1}}. \quad \text{(A.24d)}
\]

The return to capital at $t = 0$ is

\[
R^K_0 = (1 - \delta) q_0 + \left( \beta R_0 A_0 \right)^{\frac{1-\alpha}{\alpha}} \left( \frac{X_0}{X^*} \right)^{-\frac{1}{\alpha}} q_0 R_0, \quad \text{(A.24e)}
\]
and the aggregate output is

\[ Y_0 = \frac{1}{\gamma} X^* \frac{1}{\alpha} q_0 k_1 - \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \frac{1}{R_0} \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} q_0 R_0 \right)^{\frac{\alpha}{\alpha - 1}}. \] (A.24f)

By the market clearing condition of used capital, (10d), and the budget constraint of the entrepreneurs (10c) in period 0 and 1, we have

\[ c_0 + \frac{c_1}{R_0} = \omega_0 R_0^k k_0. \]

Then the entrepreneurs’ optimal consumption at \( t = 0 \) is

\[ c_0 = \frac{1}{1 + \gamma} \omega_0 R_0^k k_0. \]

We can use the following two equations to represent the equilibrium conditions with two unknowns: \( \{ k_1, R_0 \}, \{ q_0, R_0 \}, \{ k_1, X_0 \} \) or \( \{ q_0, X_0 \} \) depending on whether the ZLB or the irreversibility constraint is binding.\(^{34}\)

The first equation is derived from (A.24c) for \( c_0 \):

\[ \frac{1}{\gamma} \omega_0 R_0^k k_0 = \frac{X^*}{\alpha} q_0 k_1 - \frac{1}{R_0} \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} q_0 R_0 \right)^{\frac{\alpha}{\alpha - 1}}, \] (A.24g)

in which \( R_0^k \) is from equation (A.24e).

The second equation is derived by the feasibility condition at \( t = 0 \) and equations (A.24c), (A.24d) and (A.24f):

\[ \frac{1}{\gamma} \frac{X^*}{\alpha} q_0 k_1 - \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \frac{1}{R_0} \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} q_0 R_0 \right)^{\frac{\alpha}{\alpha - 1}} + k_1 = (1 - \delta) k_0 + \frac{X^*}{\alpha} \left( \frac{1}{\beta R_0} A_0 \frac{X_0}{X^*} \right)^{\frac{\alpha - 1}{\alpha}} q_0 R_0 k_0. \] (A.24h)

\(^{34}\)Or we can represent the system in a more rigorous way with 4 unknowns: \( \{ k_1, q_0, R_0, X_0 \} \) with another two complementary conditions:

\[ (1 - q_0) [k_1 - (1 - \delta) k_0] = 0, \]

\[ (R_0 - 1) (X_0 - X^*) = 0. \]
F.2 Threshold for Binding Irreversibility Constraint

Assume $\omega_0 < \frac{1-\alpha X^*}{\left(\frac{1}{\gamma} - \frac{1}{\beta}\right)\frac{\gamma}{1+\gamma}}$. Here we construct the threshold of $k_0$ for a binding irreversibility constraint, $k_0^* (\omega_0)$. Intuitively, since $k_0^* (\omega_0)$ is the cutoff of binding irreversibility, we should have both $q_0 = 1$ and $k_1 = (1 - \delta) k_0$ when $k_0 = k_0^* (\omega_0)$. Its expression also depends on whether the ZLB is binding at $k_0 = k_0^* (\omega_0)$. We give the expression of $k_0^* (\omega_0)$ here and verify it later.

In particular, we will show that, when $\omega_0 > \omega_0^*$ in equation (A.22), the ZLB is not binding at $k_0 = k_0^* (\omega_0)$, and its expression is

$$k_0^* (\omega_0) = \frac{1-\alpha A_1 \left( X^* \right)^{\frac{1}{\alpha}}}{\alpha} \frac{X^*}{\alpha - 1 + \gamma (1 - \frac{1}{\beta}) \omega_0} \left( A_0 A_1^{\alpha} \right)^{\frac{1-\alpha}{\alpha}} \frac{1}{\alpha},$$

(A.25)

which is increasing in $\omega_0$.

When $\omega_0 \leq \omega_0^*$, the ZLB is binding at $k_0 = k_0^* (\omega_0)$, and its expression is

$$k_0^* (\omega_0) = \frac{1-\alpha X^* A_1 \left( X^* \right)^{\frac{1}{\alpha}}}{\alpha^2} \frac{X^*}{(1 - \delta) \frac{\gamma}{1+\gamma} \omega_0} \left[ (1 - \delta) + \left( \beta A_0 A_1^{\alpha} \right)^{\frac{1-\alpha}{\alpha}} \left( \frac{X^*}{X^* (\omega_0)} \right) \right],$$

(A.26)

in which $X^* (\omega_0)$ is given implicitly by the following equation:

$$\left( 1 - \delta \right) \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \frac{\gamma}{1+\gamma} \omega_0 + \frac{1}{\beta} \left( 1 - \delta \right) \frac{X^*}{\alpha} = \left( \frac{A_0}{A_1} \beta \right)^{\frac{1-\alpha}{\alpha}} \left( \frac{X^* (\omega_0)}{X^*} \right)^{\frac{1}{\alpha}}.$$

(A.27)

When $\omega_0 < \frac{1-\alpha X^*}{\left(\frac{1}{\gamma} - \frac{1}{\beta}\right)\frac{\gamma}{1+\gamma}}$, we can easily check that $X^* (\omega_0)$ is decreasing in $\omega_0$, and then $k_0^* (\omega_0)$ is increasing in $\omega_0$ when $\omega_0 \leq \omega_0^*$. We can also show that $k_0^* (\omega_0)$ is continuous at $\omega_0 = \omega_0^*$. See the red dashed line in Figure A.8 as one example of $k_0^* (\omega_0)$.
F.3 Region with binding irreversibility Constraint

**Lemma 14.** With natural borrowing limit and \( \omega_0 < \frac{1 - \alpha}{(1 - \frac{1}{\gamma})^{1/\gamma}} \), an equilibrium with binding irreversibility constraint exists and is unique if and only if \( k_0 \geq k_0^* (\omega_0) \) given in Subsection F.2. When \( k_0 \geq k_0^* (\omega_0), q_0 \) is decreasing in \( k_0 \), and

1. if \( \omega_0 \leq \omega_0^* \) in (A.22), ZLB is binding, and \( X_0 = X_0^* (\omega_0) \) in equation (A.27) which is independent of \( k_0 \);
2. if \( \omega_0 > \omega_0^* \), ZLB is not binding, and \( R_0 = R_0^* (\omega_0) \) in equation (A.28b) which is also independent of \( k_0 \).

**Proof.** When the irreversibility constraint is binding, \( k_1 = (1 - \delta) k_0 \). Inserting it into equations (A.24g) and (A.24h) and after some calculation, we have the following equation with \( R_0 \) or \( X_0 \) being the only unknown:

\[
\frac{1}{\beta} (1 - \delta) \frac{X^*}{\alpha} + \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) (1 - \delta) \frac{\gamma}{1 + \gamma} \omega_0 = X_0 - \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \frac{\gamma}{1 + \gamma} \omega_0 \left( \beta R_0 \frac{A_0}{A_1} \right)^{\frac{1 - \alpha}{\alpha}} \left( \frac{X_0}{X^*} \right)^{-\frac{1}{\alpha}} R_0. \tag{A.28a}
\]

Notice that the solution here does not depend on \( k_0 \).

**Non-binding ZLB**

If the ZLB is not binding, we set \( X_0 = X^* \) in (A.28a), in which \( R_0 \) is the only unknown. The solution is \( R_0 = R_0^* (\omega_0) \) as below, which is independent of \( k_0 \).

\[
R_0^* (\omega_0) = \left( \frac{A_0}{A_1} \beta \right)^{\alpha - 1} \left[ \frac{(1 - \delta) \left( \frac{1}{\beta} \frac{X^*}{\alpha} + \frac{\gamma}{1 + \gamma} \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \omega_0 \right)}{\frac{X^*}{\alpha} - \frac{\gamma}{1 + \gamma} \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \omega_0} \right]^\alpha. \tag{A.28b}
\]

When \( \omega_0 < \frac{\frac{X^*}{\alpha}}{\left( \frac{1}{1 - \frac{1}{\beta}} \frac{1}{1 + \gamma} \right)} \), \( R_0^* (\omega_0) \) is increasing in \( \omega_0 \), and \( R_0^* (\omega_0) \geq 1 \) if and only if \( \omega_0 \geq \omega_0^* \) in (A.22).

The capital price \( q_0 \) is

\[
q_0 = \frac{A_1^{1 - \alpha}}{X^* R_0^* (\omega_0)} \left[ \frac{\frac{1}{\gamma} (1 - \delta) - \left( \frac{A_1}{A_0} \right)^{\frac{\alpha - 1}{\alpha}} \left[ R_0^* (\omega_0) \right]^{\frac{1}{\alpha}}}{\left( \frac{1}{\gamma} - \frac{1}{\beta} \right)^{\frac{1 - \alpha}{\alpha}} X^*} \right]^{\alpha - 1} k_0^{k_0^* - 1},
\]

which is decreasing in \( k_0 \). We find in the equation above, \( q_0 \leq 1 \) if and only if \( k_0 \geq k_0^* (\omega_0) \) in equation (A.25). As a result, there exists a unique equilibrium with binding
irreversibility constraint and non-binding ZLB if and only if \( \omega_0 > \omega_0^* \) and \( k_0 \geq k_0^*(\omega_0) \).

**Binding ZLB**

Now consider the case when both the ZLB and the irreversibility constraint are binding. Then we set \( R_0 = 1 \) in (A.28a), and \( X_0 \) becomes the only unknown. After some calculation, equation (A.28a) becomes (A.27), in which the solution \( X_0^* = X_0^*(\omega_0) \) is independent of \( k_0 \). When \( \omega_0 < \frac{1 - \alpha}{1 - \beta} X^* \), \( X_0^* (\omega_0) \) is decreasing in \( \omega_0 \), and \( X_0^*(\omega_0) \geq X^* \) if and only if \( \omega_0 \leq \omega_0^* \).

The capital price \( q_0 \) is

\[
q_0 = A_1^{1-\frac{\gamma}{\alpha}} \frac{1}{X^*} \left[ (1 - \delta) - \frac{\gamma}{1+\gamma} \omega_0 \frac{\hat{k}_0}{X^*} \left( \left. \frac{1}{\hat{k}_0} \right( X_0^*(\omega_0) \right) \right]^{\frac{1-\alpha}{\alpha}} k_0^{\frac{1-\alpha}{\alpha}} \frac{1}{X^*} \left[ (1 - \delta) + \left( \frac{\beta A_0}{A_1} \right) \frac{1-\alpha}{\alpha} \left( X_0^*(\omega_0) \right) \right]^{-\frac{1}{\alpha}} \left[ \frac{1-\alpha}{\alpha} \frac{1}{X^*} \right]^{\frac{1-\alpha}{\alpha}},
\]

which is decreasing in \( k_0 \). We find in the equation above, \( q_0 \leq 1 \) if and only if \( k_0 \geq k_0^*(\omega_0) \) in equation (A.26). As a result, there exists a unique equilibrium with binding irreversibility constraint and binding ZLB if and only if \( \omega_0 \leq \omega_0^* \) and \( k_0 \geq k_0^*(\omega_0) \).

To sum up, when \( k_0 \geq k_0^*(\omega_0) \), there exists a unique equilibrium with binding irreversibility constraint. Otherwise, there does not exist such an equilibrium.

**F.4 Region with Non-binding Irreversibility Constraint**

**F.4.1 Region with Non-binding Irreversibility and Binding ZLB**

**Lemma 15.** With natural borrowing limit and \( \omega_0 \) smaller than \( \frac{1+\gamma}{\gamma} \frac{1-\alpha}{\alpha} X^* \) \( \min \left\{ \frac{X^*}{1+\frac{\gamma}{\alpha} X^*}, \frac{1}{\hat{k}_0} \right\} \), an equilibrium with non-binding irreversibility and binding ZLB exists and is unique if and only if \( \omega_0 \leq \omega_0^* \), and \( \hat{k}_0 (\omega_0) \leq k_0 < k_0^*(\omega_0) \).\(^a\) In addition, in this region, \( X_0 \) is increasing in \( k_0 \) and is decreasing in \( \omega_0 \). \( k_0 \) is decreasing in \( k_0 \).

**Proof.** When the ZLB is binding and the irreversibility constraint is not binding, using equations (A.24g) and (A.24h) and after some calculation, we have the following equation

\(^a\)With our calibrated parameters, the value of this upper bound of \( \omega_0 \) is 2.91. \( \omega_0^* \) is given in (A.22), \( k_0^*(\omega_0) \) is in (A.26) and \( \hat{k}_0 (\omega_0) \) is in (A.29b).
with $X_0$ being the only unknown:

\[
(1 - \delta) \left(1 - \frac{\frac{\alpha}{X^*} \gamma}{1 + \gamma} \omega_0\right) + \left(\frac{X_0}{\alpha} - \frac{1 + \frac{\alpha}{X^*} \gamma}{1 + \gamma} \omega_0\right) \left(\frac{A_0}{A_1} \beta\right)^{1-\frac{\delta}{\alpha}} \left(\frac{X^*}{X_0}\right)^{\frac{1}{\gamma}}.
\] (A.29a)

\[
= \left(\frac{\alpha}{X^*} + \frac{1}{\beta} \right) 1 - \frac{\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{1-\delta}{\gamma}} \frac{1}{k_0}.
\]

If $\omega_0 < \frac{1 - \alpha}{\alpha} X^* + \frac{1 + \gamma}{1 + \gamma} \omega_0$, by the implicit function theorem, $X_0$ is decreasing in $\omega_0$ and increasing in $k_0$. Given $\omega_0$, $X_0 = X^*$ at $k_0 = \hat{k}_0(\omega_0)$ given below

\[
\hat{k}_0(\omega_0) = \frac{\left(\frac{\alpha}{X^*} + \frac{1}{\beta} \right) 1 - \frac{\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{1-\delta}{\gamma}} \frac{1}{k_0}}{1 - \delta + \frac{X^*}{\alpha} \left(\frac{A_0}{A_1} \beta\right)^{1-\frac{\delta}{\alpha}} \gamma \omega_0 \left[\frac{1}{1 - \delta} \left(\frac{A_0}{A_1} \beta\right)^{1-\frac{\delta}{\alpha}} \left(\frac{X^*}{X_0}\right)^{\frac{1}{\gamma}}\right]}. \quad \text{(A.29b)}
\]

Thus there is no equilibrium with binding ZLB and non-binding irreversibility constraint when $k_0 < \hat{k}_0(\omega_0)$.

The ratio $\frac{k_1}{k_0}$ can be expressed as

\[
\frac{k_1}{k_0} = \frac{1 - \frac{\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{1-\delta}{\gamma}} \frac{1}{k_0}}{1 - \frac{\alpha}{X^*} A_1 \left(\frac{X^*}{\alpha}\right)^{\frac{1-\delta}{\gamma}} \frac{1}{k_0} + \frac{\alpha}{X^*} \frac{1 + \gamma}{1 + \gamma} \omega_0 \left[1 - \delta + \left(\frac{A_0}{A_1} \beta\right)^{1-\frac{\delta}{\alpha}} \left(\frac{X^*}{X_0}\right)^{\frac{1}{\gamma}}\right]}, \quad \text{(A.29c)}
\]

which is decreasing in $k_0$. In particular, $\frac{k_1}{k_0} = 1 - \delta$ when $k_0 = k_0^* (\omega_0)$ in (A.29b). Thus there is no equilibrium with binding ZLB and non-binding irreversibility constraint when $k_0 \geq k_0^* (\omega_0)$.

After some calculation, we find $\hat{k}_0(\omega_0) \leq k_0^* (\omega_0)$ if and only if $\omega_0 \leq \omega_0^*$. When $\omega_0 > \omega_0^*$, the set $\hat{k}_0(\omega_0) \leq k_0 < k_0^* (\omega_0)$ is empty. When $\omega_0 \leq \omega_0^*$ and $\hat{k}_0(\omega_0) \leq k_0 < k_0^* (\omega_0)$, since $X_0$ is increasing in $k_0$, $X_0 \in [X^*, \hat{X}_0^*(\omega_0)]$, in which $\hat{X}_0^*(\omega_0)$ is given in equation (A.27). Thus we know an equilibrium with non-binding irreversibility and binding ZLB exists and is unique if and only if $\omega_0 \leq \omega_0^*$ and $\hat{k}_0(\omega_0) \leq k_0 < k_0^* (\omega_0)$.

**F.4.2 Region with Non-binding Irreversibility and Non-binding ZLB**

**Lemma 16.** With natural borrowing limit and $\omega_0 < \frac{1 + \gamma}{1 + \frac{1 + X^*}{\alpha}} \frac{X^*}{\alpha}$, an equilibrium with non-binding irreversibility and non-binding ZLB exists if and only if $\{k_0, \omega_0\}$ lies in one of the following two regions: $\{\omega_0 > \omega_0^*, k_0 < k_0^* (\omega_0)\}$ or $\{\omega_0 \leq \omega_0^*, k_0 < \hat{k}_0(\omega_0)\}$.\(^{36}\) In addition, in both regions, $R_0$ is decreasing in $k_0$ and increasing in $\omega_0$. The ratio $\frac{k_1}{k_0}$ is decreasing in $k_0$.

\(^{36}\) $\omega_0^*$ is given in (A.22), $k_0^* (\omega_0)$ is in (A.25) and $\hat{k}_0(\omega_0)$ is in (A.29b).
Proof. When the ZLB is not binding, setting $X_0 = X^*$ in (A.24g) and (A.24h) and after some calculation, we get the following equation with $R_0$ as the only unknown:

$$
\left[ \frac{1}{\gamma} - \Pi_0 \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \right] \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} \right) \frac{1}{R_0^{\alpha - 1}} \\
= \left[ (1 - \delta) \left( \Pi_0 - \frac{1}{1 + \gamma} \omega_0 \right) + \left( \frac{X^*}{\alpha} \right) \Pi_0 - \frac{1}{1 + \gamma} \omega_0 \right] \left( \frac{A_0}{A_1} \right) \frac{1}{R_0^{\alpha - 1}} \right] k_0,
$$

(A.30a)

in which $\Pi_0$ is a constant:

$$
\Pi_0 = \frac{1}{\gamma} \frac{X^*}{1 + \gamma X^*}.
$$

(A.30b)

Given $\omega_0 < \frac{1 + \gamma X^*}{1 + \gamma X^*} X^*$, by the implicit function theorem, $R_0$ is decreasing in $k_0$ and increasing in $\omega_0$. In particular, $R_0 = 1$ when $k_0 = \hat{k}_0 (\omega_0)$ in (A.29b). This suggests that there is no equilibrium with non-binding irreversibility and non-binding ZLB when $k_0 \geq \hat{k}_0 (\omega_0)$.

In addition, the ratio $k_1/k_0$ is

$$
k_1/k_0 = \Pi_1 \left( 1 - \delta \right) \left( 1 - \frac{\gamma}{1 + \gamma} \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \omega_0 \right) + \Pi_1 \left( \frac{X^*}{\alpha} - \frac{\gamma}{1 + \gamma} \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \omega_0 \right) \left( \frac{A_0}{A_1} \right) \frac{1}{R_0^{\alpha - 1}} \right] k_0,
$$

in which $R_0$ is derived from the previous equation, and $\Pi_1$ is a constant:

$$
\Pi_1 = \frac{1}{\left( 1 + \frac{1}{\gamma} \frac{X^*}{\alpha} \right) \left( 1 - \frac{1}{\gamma} \frac{X^*}{\alpha} \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \right)}.
$$

(A.30c)

We see that $k_1/k_0$ is decreasing in $k_0$. In particular, $k_1/k_0 = 1 - \delta$ when $k_0 = k_0^* (\omega_0)$ in (A.25). This suggests that there is no equilibrium with non-binding irreversibility and non-binding ZLB when $k_0 \geq k_0^* (\omega_0)$.

To sum up, for an equilibrium with non-binding irreversibility and non-binding ZLB to exist, we should have $k_0 < k_0^* (\omega_0)$ and $k_0 < \hat{k}_0 (\omega_0)$. After some calculation, we find $\hat{k}_0 (\omega_0) \leq k_0^* (\omega_0)$ if and only if $\omega_0 \leq \omega_0^*$. Thus the region for the existence of such an equilibrium is $\{ \omega_0 > \omega_0^*, k_0 < k_0^* (\omega_0) \}$ and $\{ \omega_0 \leq \omega_0^*, k_0 < \hat{k}_0 (\omega_0) \}$. Given such an equilibrium exists, it is unique since we have a unique solution of $R_0$ from equation (A.30a).

Combining the results of Lemmas 14, 15 and 16, we complete the proof for Proposition 3.
F.5 AS-AD Representation

If irreversibility constraint (10a) is not binding, \( q_0 = 1 \) from (10e) and locally, the equilibrium and the AS-AD curves are the same as the ones without irreversible investment (except for the last period equilibrium).\(^{37}\) So in this section we focus on the case in which irreversibility constraint (10a) binds and, thus, \( q_0 < 1 \).

When the irreversibility constraint binds, \( I_0 = 0 \), and the aggregate demand is given by summing up \( c_0 \) in (A.24c) and \( c'_0 \) in (A.24d):

\[
Y_{0}^{AD} = \frac{1}{\gamma} \frac{X^*}{\alpha} q_0 (1 - \delta) k_0 - \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) \frac{1}{R_0} \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} q_0 R_0 \right)^{\frac{\alpha}{\alpha-1}}. \tag{A.31}
\]

Using the production function, we derive the AS curve as follows:

\[
Y_0^{AS} = \frac{X^*}{\alpha} \left( \frac{1}{\beta R_0} \frac{A_1 X_0}{A_0 X^*} \right)^{\frac{\alpha-1}{\alpha}} q_0 R_0 k_0. \tag{A.32}
\]

By Proposition 3, when the investment irreversibility constraint binds, both \( R_0 \) and \( X_0 \) are independent of \( k_0 \). As a result, we can plot the AS-AD curves as functions of \( q_0 \). Notice that both curves are increasing in \( q_0 \). For the AS curve in (A.32), a higher \( q_0 \) increases \( R_1^K \) by the non-arbitrage condition in (A.24a) and depresses wage at \( t = 1 \), \( w_1 \), which reduces the household’s lifetime wealth. As a results, the household chooses to supply more labor in period 0, which increases output. For the AD curve, on the one hand, a higher \( q_0 \) increases \( R_1^K \) and depresses the households’ consumption \( c'_0 \); on the other hand, the entrepreneurs enjoy higher \( c_0 \) since the households’ higher labor supply increases the entrepreneurs’ wealth. By (A.31), we see the net effect of \( q_0 \) on the aggregate demand is positive. We plot the AS-AD curves with binding ZLB and irreversibility constraint in Figure A.11. We see that as \( k_0 \) increases, \( Y_0 \) increases and \( q_0 \) decreases. As \( \omega_0 \) increases, both \( Y_0 \) and \( q_0 \) increase.

G Proof of Proposition 4

To prove Proposition 4, we proceed in the following steps.

1. Describe the threshold of \( \omega_0 \) for a binding collateral constraint, \( \omega_0^{CC} (k_0) \), and the

\(^{37}\)There is some difference though. In the benchmark model without investment irreversibility, \( R_1^K = 1 - \delta + \frac{\alpha}{\alpha_1} \left( \frac{k_1}{\alpha_1 L_1} \right)^{\alpha-1} \); while with irreversibility, \( R_1^K = \frac{\alpha}{\alpha_1} \left( \frac{k_1}{\alpha_1 L_1} \right)^{\alpha-1} \) since capital price in the last period is 0. But this difference does not change the results qualitatively. We choose to omit this part here.
threshold of $k_0$ for a binding irreversibility constraint, $k_{0}^{**} (\omega_0)$. 

2. Show an equilibrium with binding irreversibility constraint exists and is unique if and only if $k_0 \geq k_{0}^{**} (\omega_0)$.

3. Show an equilibrium with non-binding irreversibility constraint and non-binding collateral constraint exists and is unique if and only if $k_0 < k_{0}^{**} (\omega_0)$ and $\omega_0 > \omega_0^{CC} (k_0)$.

4. Show an equilibrium with non-binding irreversibility constraint and binding collateral constraint exists and is unique if and only if $k_0 < k_{0}^{**} (\omega_0)$ and $\omega_0 \leq \omega_0^{CC} (k_0)$.

G.1 Equilibrium Properties with Binding Collateral Constraint

The equilibrium in the last period is determined in Appendix F.1. In addition, when the collateral constraint binds at $t = 0$, by the definition of wealth share in equation (4), $\omega_1 = 1 - m$, and by equation (A.23a) the labor supply at $t = 1$ is constant and we denote it as $L^c_1$: 
\[
L_1^{cc} = \frac{1 - \frac{1}{X^*}}{1 - (1 - m) \frac{1}{X^*}}.
\]  
(A.33a)

The consumptions at \( t = 1 \) are:

\[
\begin{align*}
    c_1 &= (1 - m) \frac{\alpha}{X^*} (A_1 L_1^{cc})^{1 - \alpha} k_1^a, \\
    c_1' &= \frac{1 - \alpha}{X^*} A_1^{1 - \alpha} (L_1^{cc})^{-\alpha} k_1^a.
\end{align*}
\]

In the first period,

\[
\begin{align*}
    c_0 &= \frac{1}{1 + \gamma} \omega_0 R_0^K k_0, \\
    c_0' &= \frac{c_1'}{\beta R_0}, \\
    Y_0 &= \left( \beta \frac{A_0}{A_1} R_0 \frac{X^*}{X_0} \right)^{1 - \alpha} \left( \frac{k_1}{A_1 L_1^{cc}} \right)^{1 - \alpha} k_0, \\
    R_0^K &= (1 - \delta) q_0 + \frac{\alpha}{X_0} \left( \beta R_0 \frac{A_0}{A_1} \frac{X^*}{X_0} \right)^{1 - \alpha} \left( \frac{k_1}{A_1 L_1^{cc}} \right)^{1 - \alpha}.
\end{align*}
\]  
(A.33b)

### G.2 Thresholds for Binding Collateral Constraint and Irreversibility

#### G.2.1 Threshold for Binding Collateral Constraint

We first show that in the model with natural borrowing limit in Subsection E.1, the leverage ratio \( -\frac{b_1}{R_1^K k_1} \) is decreasing in \( \omega_0 \). Now with \( m < 1 \), if the collateral constraint is not binding, the equilibrium is the same as the natural borrowing limit model. Then there is a cutoff value \( \omega_0^{CC} (k_0) \) such that \( -\frac{b_1}{R_1^K k_1} = m \) at \( \omega_0 = \omega_0^{CC} (k_0) \).

**Lemma 17.** With \( m = 1 \), the irreversibility constraint and

\[
\omega_0 \leq \frac{1 + \gamma}{\gamma} \frac{1 - \alpha}{\alpha} X^* \min \left\{ \frac{\frac{X^*}{\alpha}}{1 + \frac{1}{\gamma} \frac{X^*}{\alpha} \left( \frac{1}{\gamma} - \frac{1}{\beta} \right)}, \frac{1}{\frac{1}{\gamma} - \frac{1}{\beta}} \right\},
\]

the leverage ratio \( -\frac{b_0}{R_1^K k_1} \) is decreasing in \( \omega_0 \).

**Proof.** When \( m = 1 \), we have shown in Proposition 3 that the irreversibility constraint is binding if and only if \( k_0 \geq k_0^* (\omega_0) \) given in Subsection F.2.
Case I: Non-binding Irreversibility Constraint

When $k_0 < k_0^* (\omega_0)$, the irreversibility constraint is not binding, and $R_0 = R_1^k$. From the entrepreneurs’ budget and their optimal choice $c_0 = \frac{1}{1+\gamma} R_0^k \omega_0 k_0$, their leverage ratio is

$$- \frac{b_0}{R_0^k k_1} = 1 - \frac{\gamma}{1+\gamma} \frac{R_0^k \omega_0 k_0}{k_1}.$$ 

Showing that the leverage ratio is decreasing in $\omega_0$ is equivalent to showing that $\frac{R_0^k \omega_0 k_0}{k_1}$ is increasing in $\omega_0$.

First, consider the case in which ZLB is not binding, i.e. $R_0 > 1$. Substituting in the expression for $k_1$ using (A.24b), (A.24c) and (A.24d), we obtain

$$\frac{R_0^k \omega_0 k_0}{k_1} = \frac{X^* R_0}{\gamma R_0 + \frac{1-a}{X^* A_1 (\frac{X^*}{\alpha} R_0)^{\frac{a}{a-1}}}}.$$ 

Differentiating both sides of the equation above to $\omega_0$, the derivative $\frac{d}{d\omega_0} \left[ \frac{R_0^k \omega_0 k_0}{k_1} \right]$ has the same sign as the derivative of the right-hand side, which, after some calculation, is

$$\frac{X^*}{\alpha} \left[ \frac{dR_0}{d\omega_0} \frac{1-a}{X^* A_1 (\frac{X^*}{\alpha} R_0)^{\frac{a}{a-1}}} - \frac{d}{d\omega_0} \left( \frac{1-a}{X^* A_1 (\frac{X^*}{\alpha} R_0)^{\frac{a}{a-1}}} R_0 \right) \right]. \quad (A.34)$$

By Lemma 16, $R_0$ increases with $\omega_0$: $\frac{dR_0}{d\omega_0} > 0$. In addition, by setting $X_0 = X^*$ and $q_0 = 1$ in (A.24e), $R_0^k$ becomes

$$R_0^k = 1 - \delta + \left( \beta R_0 \frac{A_0}{A_1} \right)^{\frac{1-a}{a}} R_0,$$

which is strictly increasing in $R_0$ and hence in $\omega_0$. Therefore,

$$\frac{d}{d\omega_0} \left( \frac{1-a}{X^* A_1 (\frac{X^*}{\alpha} R_0)^{\frac{a}{a-1}}} \frac{X^* R_0}{\omega_0 R_0^k k_0} \right) < 0,$$

then expression (A.34) is positive.

Now, consider the case when $k_0 < k_0^* (\omega_0)$ and ZLB is binding, i.e. $R_0 = 1$ and $X_0 >
$X^\ast$. Then we obtain

$$R_k^k = \frac{R_0^k \omega_0 k_0}{k_1} = \frac{X^\ast}{\alpha} \frac{\gamma}{1 + \gamma} + \frac{1 - \frac{\alpha}{X^\ast} A_1 \frac{X^\ast}{\omega_0 R_0^k k_0}}{\alpha}.$$ 

By setting $R_0 = 1$ and $q_0 = 1$ in (A.24e), $R_0^k$ becomes

$$R_0^k = 1 - \delta + \left( \beta \frac{A_0}{A_1} \right) \frac{1 - \frac{\alpha}{X^\ast}}{1 + \gamma} \left( \frac{X_0}{X^\ast} \right)^{-\frac{1}{\alpha}}.$$ 

Lemma 15 shows that when $\omega_0 < \frac{1}{\alpha} X^\ast \frac{1 + \gamma}{1 + \gamma}$, $X_0$ is decreasing in $\omega_0$. Therefore, $R_0^k$ is increasing in $\omega_0$. Hence, $\frac{R_0^k \omega_0 k_0}{k_1}$ is increasing in $\omega_0$ as desired.

**Case II: Binding Irreversibility Constraint**

When $k_0 > k^\ast_0 (\omega_0)$, the irreversibility constraint is binding, $q_0 < 1$, $k_1 = (1 - \delta) k_0$, and $R_0 = \frac{R_1^k}{q_0}$. From the entrepreneurs’ budget and their optimal choice $c_0 = \frac{1}{1 + \gamma} R_0^k \omega_0 k_0$, their leverage ratio is

$$- \frac{b_0}{R_1^k k_1} = 1 - \frac{\gamma}{1 + \gamma} \frac{R_0^k \omega_0 k_0}{q_0 k_0} = 1 - \frac{\gamma}{1 + \gamma q_0 (1 - \delta)}.$$

Replacing $R_0^k$ by its expression in (A.24e), we have

$$- \frac{b_0}{R_1^k k_1} = 1 - \frac{\gamma}{1 + \gamma} \omega_0 \left[ 1 + \left( \beta R_0 \frac{A_0}{A_1} \right) \frac{1 - \frac{\alpha}{X^\ast}}{1 + \gamma} \left( \frac{X_0}{X^\ast} \right)^{\frac{1}{\alpha}} \frac{1}{1 - \delta} R_0 \right].$$

We show in Lemma 14 that given $\omega_0 < \frac{1}{\alpha} X^\ast \frac{1 + \gamma}{1 + \gamma}$, when $k_0 > k^\ast_0 (\omega_0)$ and $\omega_0 \geq \omega^\ast_0$ in equation (A.22), $X_0 = X^\ast$ and $R_0 = R_0^\ast (\omega_0)$ in (A.28b) which is increasing in $\omega_0$. Thus in the equation above, $- \frac{b_0}{R_1^k k_1}$ is decreasing in $\omega_0$. When $k_0 > k^\ast_0 (\omega_0)$ and $\omega_0 < \omega^\ast_0$, by Lemma 14, $R_0 = 1$ and $X_0 = X^\ast (\omega_0)$ in equation (A.27) which is decreasing in $\omega_0$. Thus in the equation above, $- \frac{b_0}{R_1^k k_1}$ is also decreasing in $\omega_0$.
Figure A.12: Regions for Irreversibility, ZLB and Collateral Constraint

Note: Both figures are generated by setting $\beta = 0.99$, $\gamma = 0.6$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $m = 0.7$ and $\epsilon = 21$. In Case I, $A_1 = 1$; in Case II, $A_1 = 0.93$. 
Lemma 18. With $m = 1$, the irreversibility constraint and

$$\omega_0 \leq \frac{1 + \gamma}{\gamma} \frac{1 - \alpha}{\alpha} X^* \min \left\{ \frac{X^*}{1 \gamma \frac{1 - \alpha}{\alpha}}, \frac{1}{1 - \gamma} \right\},$$

there is a threshold value of $\omega_0$, $\omega_0^{CC} (k_0)$ such that the leverage ratio $-\frac{b_1}{R_k k_1} = m$ at $\omega_0 = \omega_0^{CC} (k_0)$. In particular, define a constant

$$\mathcal{R}^{CC} = \left( \frac{1}{\rho} \frac{A_1}{A_0} \right)^{\frac{1}{\alpha}} (1 - \delta) \left( \frac{1}{\gamma} (1 - m) \frac{\alpha}{X^*} + \frac{1}{\beta} (1 - \frac{\alpha}{X^*} (1 - m)) \right). \quad (A.35a)$$

1. If $\mathcal{R}^{CC} \geq 1$, $\omega_0^{CC} (k_0) = \left\{ \begin{array}{ll} \omega_{0,\text{noZLB}}^{CC} (k_0), & k_0 \in [0, k_{0,\text{Irr}}^{CC}] ; \\ \omega_{0,\text{Irr}}^{CC}, & k_0 \in (k_{0,\text{Irr}}, +\infty) . \end{array} \right.$

2. If $\mathcal{R}^{CC} < 1$, $\omega_0^{CC} (k_0) = \left\{ \begin{array}{ll} \omega_{0,\text{noZLB}}^{CC} (k_0), & k_0 \in [0, \tilde{k}_0] ; \\ \omega_{0,\text{ZLB}}^{CC} (k_0), & k_0 \in (\tilde{k}_0, k_{0,\text{Irr}}^{CC}] ; \\ \omega_{0,\text{Irr}}^{CC}, & k_0 \in (k_{0,\text{Irr}}, +\infty) . \end{array} \right.$

in which $\omega_{0,\text{noZLB}}^{CC} (k_0)$ is given in equation (A.35d). $\omega_{0,\text{ZLB}}^{CC} (k_0)$ is from (A.35g), $\tilde{k}_0$ is a constant in (A.35e),

$$\omega_{0,\text{Irr}}^{CC} = \left\{ \begin{array}{ll} \frac{1 - \delta + \frac{1}{\gamma} \left[ \frac{\alpha}{X^*} + \frac{1}{\gamma} \right] (1 - m) (1 - \delta)}{1 - \delta + \frac{1}{\gamma} \left[ \frac{\alpha}{X^*} + \frac{1}{\gamma} \right] (1 - m) (1 - \delta)}, & \text{if } \mathcal{R}^{CC} \geq 1 ; \\ \frac{1 - \delta + \frac{1}{\gamma} \left[ \frac{\alpha}{X^*} + \frac{1}{\gamma} \right] (1 - m) (1 - \delta)}{1 - \delta + \frac{1}{\gamma} \left[ \frac{\alpha}{X^*} + \frac{1}{\gamma} \right] (1 - m) (1 - \delta)}, & \text{if } \mathcal{R}^{CC} < 1 . \end{array} \right. \quad (A.35b)$$

and

$$k_{0,\text{Irr}}^{CC} = \left\{ \begin{array}{ll} \left[ (1 - \delta) \left( \frac{1 - m}{\gamma} + \frac{1}{\gamma} \left[ \frac{\alpha}{X^*} + \frac{1}{\gamma} \right] \right) \right]^{\frac{\delta}{\gamma}} \frac{\alpha^\prime A_0 L_1^{CC}}{X^* (1 - \delta)}, & \text{if } \mathcal{R}^{CC} \geq 1 ; \\ \frac{1 - \delta + \frac{1}{\gamma} \left[ \frac{\alpha}{X^*} + \frac{1}{\gamma} \right] (1 - \delta)}{1 - \delta + \frac{1}{\gamma} \left[ \frac{\alpha}{X^*} + \frac{1}{\gamma} \right] (1 - \delta)}^{\frac{\delta}{\gamma}} \frac{\alpha^\prime A_0 L_1^{CC}}{X^* (1 - \delta)}, & \text{if } \mathcal{R}^{CC} < 1 . \end{array} \right. \quad (A.35c)$$

and $L_1^{CC}$ is a constant from (A.33a). See the black dotted line in Figure A.12 for an example.

Proof. Proposition 3 shows that the irreversibility constraint is binding if and only if $k_0 \geq k_0^* (\omega_0)$. We first consider the case with non-binding irreversibility constraint.

Case I: Non-binding irreversibility constraint at $\omega_0^{CC} (k_0)$

Case 1.1: Non-binding irreversibility constraint and non-binding ZLB at $\omega_0^{CC} (k_0)$
Using the condition $\frac{b_0}{R_1 k_1} = m$ and (A.30a), we find the interest rate at $\omega_0^{CC}$ is

$$R_0^{CC} = (1 - \delta)^{\alpha} \left( \frac{A_0}{A_1 \beta} \right)^{\alpha - 1} \left( \frac{\omega_0^{CC} - \Pi_2}{\frac{X^*}{\alpha} \Pi_2 - \omega_0^{CC}} \right)^{\alpha},$$

in which $\Pi_2$ is constant:

$$\Pi_2 = \frac{\Pi_0}{1 + \frac{X^*}{\alpha - 1 + m} \left[ \frac{1}{\gamma} - \Pi_0 \left( \frac{1 - \delta}{\beta} \right) \right]}, \quad (A.35c)$$

and $\Pi_0$ is a constant defined in (A.30b). We see that $R_0^{CC}$ is increasing in $\omega_0$ when $\omega_0 \in \left( \Pi_2, \frac{X^*}{\alpha} \Pi_2 \right)$. Inserting the expression of $R_0^{CC}$ into (A.30a), we can derive the expression of $k_0$ as below:

$$k_0 = \left[ \frac{1}{\gamma} - \Pi_0 \left( \frac{1 - \delta}{\beta} \right) \right] \frac{1 - \delta}{\frac{X^*}{\alpha} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{\alpha}{1 - \alpha}}} \left( \frac{R_0^{CC} \Pi_2 - \omega_0^{CC}}{\left( \frac{X^*}{\alpha} - 1 \right) \Pi_0 - \Pi_2 \frac{X^* - 1}{1 + \gamma}} \right) \omega_0^{CC}. \quad (A.35d)$$

Notice that $k_0$ is decreasing in $\omega_0^{CC}$, and thus we denote its inverse function as $\omega_0^{CC, noZLB} (k_0)$, which is decreasing. At $k_0 = 0$, $\omega_0^{CC} = \frac{X^*}{\alpha} \Pi_2$, and $R_0^{CC} \to +\infty$. As $k_0$ increases, $\omega_0^{CC}$ and $R_0^{CC}$ decrease. In particular, when $k_0$ reaches the cutoff value $\tilde{k}_0$ as below:

$$\tilde{k}_0 = \frac{\left( \frac{1}{\beta} + \frac{1 + \frac{1 - m}{\gamma}}{\frac{X^*}{\alpha} - 1 + m} \right) \frac{1 - \delta}{\frac{X^*}{\alpha} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{\alpha}{1 - \alpha}}} \left( \frac{X^*}{\alpha} - 1 \right) \Pi_0 - \Pi_2 \frac{X^* - 1}{1 + \gamma}}{1 - \delta + \frac{X^*}{\alpha} \left( \beta \frac{A_0}{A_1} \right)^{\frac{1 - \delta}{\alpha}}}, \quad (A.35e)$$

we have $R_0^{CC} (\tilde{k}_0) = 1$, and

$$\omega_0^{CC} (\tilde{k}_0) = \Pi_2 \left( 1 + \frac{\left( \frac{X^*}{\alpha} - 1 \right)}{1 + \left( 1 - \delta \right) \left( \beta \frac{A_0}{A_1} \right)^{\frac{a - 1}{a}}} \right). \quad (A.35f)$$

Thus when $k_0 > \tilde{k}_0$, our assumption that the irreversibility constraint and ZLB are both non-binding at $\omega_0^{CC} (k_0)$ does not hold.

Case 1.2: Non-binding irreversibility constraint and binding ZLB at $\omega_0^{CC} (k_0)$

Given $k_0$, we have two equations, $-\frac{b_0}{R_1 k_1} = m$ and (A.29a) with two unknowns: $\{X_0, \omega_0^{CC}\}$. 

A-55
After some calculations, we can express both $\omega_0^{CC}$ and $k_0$ as functions of $X_0$ as below:

$$
k_0 = \Pi_3 \left( \frac{\alpha}{\alpha} + \frac{1}{1} \right) \frac{1}{\alpha} A_1 \left( \frac{X_0}{X_0} \right)^{\frac{\alpha}{\alpha-1}} \times 1 - \delta + \frac{\alpha}{\alpha} \left( \frac{X_0}{X_0} \right)^{\frac{1}{1-\delta}} \times \left( \frac{\alpha}{\alpha} \right)^{\frac{\alpha}{\alpha-1}},
$$

$$
\omega_0^{CC} = \Pi_2 \left[ 1 + \left( \frac{\alpha}{\alpha} - 1 \right) \left( \frac{X_0}{X_0} \right)^{\frac{1}{1-\delta}} \right],
$$

(A.35g)

in which $\Pi_2$ is defined in (A.29a), and $\Pi_3$ is a constant:

$$
\Pi_3 = \frac{1 + \left( \frac{\alpha}{\alpha} - 1 + m \right)}{1 - m} \frac{1 - \Pi_0 \left( 1 - \gamma \right)}{1 - \Pi_0 \left( 1 - \gamma \right)}.
$$

Notice that in (A.35d), $k_0$ is increasing in $X_0$. Thus by varying the value of $X_0$ in the range $[X^*, +\infty)$, we can trace out $\omega_0^{CC} (k_0)$. In particular, when $X_0 = X^*$, the values of $k_0$ and $\omega_0^{CC}$ are the same as in equations (A.35e) and (A.35f), suggesting that $\omega_0^{CC} (k_0)$ is continuous at $k_0 = \tilde{k}_0$.

**Case II: Binding irreversibility constraint at $\omega_0^{CC} (k_0)$**

We have proved in Lemma 17 that with binding irreversibility constraint, the leverage ratio is independent of $k_0$ and is decreasing in $\omega_0$. Thus we only need to find the cutoff value of $\omega_0$, $\omega_0^{CC}_{Irr}$, such that the leverage ratio $\frac{b_0}{R^k_{Irr}} = m$ at $\omega_0 = \omega_0^{CC}_{Irr}$. One question is, whether the ZLB is binding at $\omega_0 = \omega_0^{CC}_{Irr}$.

**Case 2.1 Binding irreversibility constraint and non-binding ZLB at $\omega_0^{CC} (k_0)$**

We find that when $\beta^{CC} \geq 1$, the ZLB is not binding at $\omega_0^{CC}_{Irr}$. Using $k_1 = (1 - \delta) k_0$, $R_0 = \frac{R^k_0}{q_0}$, the entrepreneurs’ optimal choice $c_0 = \frac{1}{1 + \gamma} R^k_0 \omega_0 k_0$, and $R^k_0$ in (A.24e), we have

$$
\frac{\gamma}{1 + \gamma} \omega_0^{CC}_{Irr} \left[ 1 + \left( \frac{\beta A_0}{A_1} \right)^{\frac{1}{1-\gamma}} \left( \frac{R^k_0}{1 - \delta} \right) \left( \frac{\omega_0^{CC}_{Irr}}{\omega_0^{CC}_{Irr}} \right) \right] = 1 - m,
$$

in which $R^k_0 (\cdot)$ is given in (A.28b). Replacing $R^k_0 \left( \omega_0^{CC}_{Irr} \right)$ by its expression, we have

$$
\omega_0^{CC}_{Irr} = \frac{(1 - m) \frac{X^*}{\alpha} \left( \frac{1 + \gamma}{\gamma} \right)}{\frac{X^*}{\alpha} \left( 1 + \frac{1}{\beta} \right) + (1 - m) \left( \frac{1}{\gamma} - \frac{1}{\beta} \right)}.
$$

(A.35h)

We find that $R^k_0 \left( \omega_0^{CC}_{Irr} \right) \geq 1$ if and only if $\beta^{CC} \geq 1$.
The value of $k_{0,\text{irr}}^{CC}$ can be derived by $k_0^* \left( \omega_{0,\text{irr}}^{CC} \right)$ from equation (A.25):

$$k_{0,\text{irr}}^{CC} = \left[ (1 - \delta) \left( \frac{1 - m}{\gamma} + \frac{1 - \alpha}{\alpha \beta L_{1}^{CC}} \right) \right]^{\frac{1}{\alpha - 1}} \frac{\alpha \beta A_0 L_{1}^{CC}}{X^* \left( 1 - \delta \right)}. \quad (A.35i)$$

When $k_0 \geq k_{0,\text{irr}}^{CC}$ and $\omega_0 = \omega_{0,\text{irr}}^{CC}$, the leverage ratio equals to $m$.

**Case 2.2 Binding irreversibility constraint and binding ZLB at $\omega_0^{CC} (k_0)$**

If $\mathcal{R}^{CC} < 1$, then the ZLB is binding at $\omega_{0,\text{irr}}^{CC}$. Using $k_1 = (1 - \delta) k_0$, $R_0 = \frac{R_k}{q_0}$, the entrepreneurs’ optimal choice $c_0 = \frac{1}{1 + \gamma} R_k \omega_0 k_0$, and $R_k$ in (A.24e), we have

$$\frac{\gamma}{1 + \gamma} \omega_{0,\text{irr}}^{CC} \left[ 1 + \frac{ \left( \beta A_0 \right)^{\frac{1}{\alpha - 1}} \left( X_0 \left( \omega_{0,\text{irr}}^{CC} \right) \right)^{-\frac{1}{\alpha}}} {1 - \delta} \right] = 1 - m,$$

in which $X_0^* (\cdot)$ is given in (A.29a). After some calculations, we can pin down the value of $X_0^* \left( \omega_{0,\text{irr}}^{CC} \right)$ as below:

$$X_0^* \left( \omega_{0,\text{irr}}^{CC} \right) = \beta A_0 A_1 X^* \left[ \frac{1 - \delta}{\beta} + \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) (1 - m) (1 - \delta) \frac{\alpha}{X^*} \right]^{\frac{1}{\alpha - 1}}. \quad (A.35j)$$

We find that $X_0^* \left( \omega_{0,\text{irr}}^{CC} \right) > X^* \text{ if and only if } \mathcal{R}^{CC} < 1$. Inserting its expression into the expression for leverage, we have

$$\omega_{0,\text{irr}}^{CC} = \frac{\frac{1 + \gamma}{\gamma} (1 - m) (1 - \delta)} {1 - \delta + \frac{1}{\beta} A_1 \left[ \frac{1 - \delta}{\beta} + \left( \frac{1}{\gamma} - \frac{1}{\beta} \right) (1 - m) (1 - \delta) \frac{\alpha}{X^*} \right]^{\frac{1}{\alpha - 1}}, \quad (A.35j)}$$

Again, the value of $k_{0,\text{irr}}^{CC}$ can be derived by $k_0^* \left( \omega_{0,\text{irr}}^{CC} \right)$ from equation (A.26):

$$k_{0,\text{irr}}^{CC} = \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\alpha - 1}} \frac{\alpha - 1}{X^* - 1 + m (1 - \delta)}. \quad (A.35k)$$

When $k_0 \geq k_{0,\text{irr}}^{CC}$ and $\omega_0 = \omega_{0,\text{irr}}^{CC}$, the leverage ratio equals to $m$.

**Putting the Pieces Together**

When $\mathcal{R}^{CC} \geq 1$, $k_{0,\text{irr}}^{CC}$ in equation (A.35k) is smaller than $\tilde{k}_0$ in (A.35e), thus as $k_0$ increases from 0, the curve $\omega_0^{CC} (k_0)$ intersects $k_0^* (\omega_0)$ in equation (A.25) before it reaches
Then the ZLB never binds at \( \omega_{0}^{CC} (k_{0}) \). When \( k_{0} \in \left[ 0, k_{0,IRR}^{CC} \right] \), \( \omega_{0}^{CC} (k_{0}) = \omega_{0, noZLB}^{CC} (k_{0}) \) in equation (A.35d); when \( k_{0} > k_{0,IRR}^{CC} \) in (A.35i), \( \omega_{0}^{CC} (k_{0}) = \omega_{0,IRR}^{CC} \) in equation (A.35h).

When \( R^{CC} < 1 \), \( k_{0,IRR}^{CC} \) in equation (A.35k) is larger than \( \tilde{k}_{0} \) in (A.35e). Thus when \( k_{0} \in \left[ 0, \tilde{k}_{0} \right] \), \( \omega_{0}^{CC} (k_{0}) = \omega_{0, noZLB}^{CC} (k_{0}) \) in equation (A.35d); when \( k_{0} \in \left[ \tilde{k}_{0}, k_{0,IRR}^{CC} \right] \), \( \omega_{0}^{CC} (k_{0}) = \omega_{0,ZLB}^{CC} (k_{0}) \) in equation (A.35g); and when \( k_{0} > k_{0,IRR}^{CC} \) in (A.35k), \( \omega_{0}^{CC} (k_{0}) = \omega_{0,IRR}^{CC} \) in equation (A.35j).

**G.2.2 Threshold for Binding Irreversibility Constraint**

Assume \( \omega_{0} \leq \frac{(1+\gamma)\gamma_{a} X^{*}}{1+\gamma X^{*}} \). Here we denote the threshold of \( k_{0} \) for a binding irreversibility constraint as \( k_{0}^{**} (\omega_{0}) \). As in the case for the natural borrowing limit in F.2, we give the expression of \( k_{0}^{**} (\omega_{0}) \) first and verify it later. \( k_{0}^{**} (\omega_{0}) \) is solved such that when \( k_{0} = k_{0}^{**} (\omega_{0}) \), \( q_{0} = 1 \), and \( k_{1} = (1-\delta) k_{0} \). \( k_{0}^{**} (\omega_{0}) \) also depends on whether the ZLB or the collateral constraint is binding or not at \( k_{0} = k_{0}^{**} (\omega_{0}) \). Eventually, we find its expression depends on the value of \( R^{CC} \) in equation (A.35a).

If \( R^{CC} \geq 1 \),

\[
k_{0}^{**} (\omega_{0}) = \begin{cases} k_{0}^{*} (\omega_{0}), & \omega_{0} > \omega_{0,IRR}^{CC}; \\ \frac{A_{1} \gamma_{a}^{CC}}{1-\delta} \left[ \frac{(1-\delta) m \frac{\gamma_{a}^{CC}}{X^{*}} + \frac{\gamma_{a}^{CC}}{R^{CC}} \alpha_{0} \frac{\gamma_{a}^{*}}{X^{*}} \left( \frac{1}{\gamma_{a}^{CC}} \frac{R^{CC}}{X^{*}} \right)^{\frac{\alpha_{0}-1}{\alpha_{0}}} \right] \frac{1}{\gamma_{a}} & , \omega_{0} \in \left[ \omega_{0,IRR}^{CC}, \omega_{0,CC}^{**} \right] \\ \left[ \frac{(1-\delta) m \frac{\gamma_{a}^{CC}}{X^{*}} + \frac{\gamma_{a}^{CC}}{R^{CC}} \alpha_{0} \frac{\gamma_{a}^{*}}{X^{*}} \left( \frac{1}{\gamma_{a}^{CC}} \frac{R^{CC}}{X^{*}} \right)^{\frac{\alpha_{0}-1}{\alpha_{0}}} \right] \frac{1}{\gamma_{a}} & , \omega_{0} < \omega_{0}^{**}. \end{cases}
\]

If \( R^{CC} < 1 \), then

\[
k_{0}^{**} (\omega_{0}) = \begin{cases} k_{0}^{*} (\omega_{0}), & \omega_{0} > \omega_{0,IRR}^{CC}; \\ \frac{A_{1} \gamma_{a}^{CC}}{1-\delta} \left[ \frac{(1-\delta) m \frac{\gamma_{a}^{CC}}{X^{*}} + \frac{\gamma_{a}^{CC}}{R^{CC}} \alpha_{0} \frac{\gamma_{a}^{*}}{X^{*}} \left( \frac{1}{\gamma_{a}^{CC}} \frac{R^{CC}}{X^{*}} \right)^{\frac{\alpha_{0}-1}{\alpha_{0}}} \right] \frac{1}{\gamma_{a}} & , \omega_{0} \in \left[ 0, \omega_{0,IRR}^{CC} \right]. \end{cases}
\]

See the blue solid line in Figure A.12 for an example. In the expressions above, \( L_{1}^{CC} \) is given in (A.33a). \( \omega_{0,IRR}^{CC} \) is given in equation (A.35b).

\[\text{With our calibrated parameters, this value is } 1.32.\]
\( R_{0}^{**}(\omega_0) \) is given as below:

\[
R_{0}^{**}(\omega_0) = \left[ \frac{\omega_0}{1 - \frac{\omega_0}{\gamma}} \cdot \frac{m}{X^*} + \frac{1}{\beta} \left( 1 - (1 - m) \frac{a}{X^*} \right) \right]^a.
\]  (A.36)

When \( \omega_0 \leq \frac{1+\gamma}{\gamma+X^*} \), \( R_{0}^{**}(\omega_0) \) is increasing in \( \omega_0 \). \( R_{0}^{**}(\omega_0) \geq 1 \) if and only if \( \omega_0 \geq \omega_{0}^{**} \) as follows:

\[
\omega_{0}^{**} = \frac{(1 + \gamma) \left( \beta A_0 A_1 \right)^{\frac{1-a}{a}} - (1 + \gamma) \frac{1-\delta}{\beta} \left( 1 - (1 - m) \frac{a}{X^*} \right)}{(\frac{a}{X^*} + \gamma) \left( \beta A_0 A_1 \right)^{\frac{1-a}{a}} + \left( m + \frac{\gamma}{\beta} (1 - m) \right) (1 - \delta) \frac{a}{X^*} - \frac{\gamma}{\beta} (1 - \delta)}.
\]  (A.37)

\( X_{0}^{**}(\omega_0) \) is given implicitly by the equation below:

\[
\frac{\left( \beta A_0 A_1 \right)^{\frac{1-a}{a}}}{1 - \delta} \left( 1 - \frac{\omega_0}{1 + \gamma} \omega_0 X^{**}(\omega_0) \right) \left( \frac{X_{0}^{**}(\omega_0)}{X^*} \right)^{\frac{a-1}{a}} = \frac{1 + \gamma}{1 + \gamma} \omega_0 - \frac{m}{X^*} + \frac{1}{\beta} \left( 1 - (1 - m) \frac{a}{X^*} \right).
\]  (A.38)

If \( \omega_0 < \frac{(1+\gamma)^{\frac{1-a}{a}} X^*}{1+\gamma \frac{a}{X^*}} \), \( X_{0}^{**}(\omega_0) \) is decreasing in \( \omega_0 \). At \( \omega_0 = \omega_{0}^{**} \), \( X_{0}^{**}(\omega_0) = X^* \).

G.3 Region with Non-binding Collateral Constraint

**Lemma 19.** With \( m < 1 \), the irreversibility constraint and

\[
\omega_0 \leq \frac{1 + \gamma}{\gamma} \left[ 1 - \frac{\alpha}{X^*} \min \left\{ \frac{X^*}{1 + \frac{1}{\gamma} \frac{a}{X^*}}, \frac{1}{1 - \frac{1}{\beta}} \right\} \right],
\]

there exists a unique equilibrium with non-binding collateral constraint if and only if \( \omega_0 > \omega_0^{CC}(k_0) \) given in Lemma 18.

**Proof.** Notice that an equilibrium with non-binding collateral constraint is equivalent to an equilibrium with natural borrowing limit analyzed in Proposition 3. By Lemma 17, given \( \omega_0 \leq \frac{1+\gamma}{\gamma} \frac{1-a}{a} X^* \min \left\{ \frac{X^*}{1 + \frac{1}{\gamma} \frac{a}{X^*}}, \frac{1}{1 - \frac{1}{\beta}} \right\} \), the leverage ratio \( \frac{b_1}{R_1 k_1} \) is decreasing in
\( \omega_0 \), and by Lemma 18, \( \omega_0^{CC} (k_0) \) is defined such that \( -\frac{b_1}{R_1^1 k_1} = m \) at \( \omega_0 = \omega_0^{CC} (k_0) \). When \( \omega_0 > \omega_0^{CC} (k_0) \), the leverage ratio is smaller than \( m \), and the collateral constraint is not binding. Then an equilibrium with non-binding collateral constraint exists and is unique by Proposition 3. When \( \omega_0 \leq \omega_0^{CC} (k_0) \), assuming a non-binding collateral constraint, the implied leverage ratio would be larger than \( m \) \( 17 \), which violates the collateral constraint. Thus there is no equilibrium with non-binding collateral constraint in that region.

By Proposition 3, we know that in this region, the irreversibility constraint is binding if and only if \( k_0 \geq k^*_0 (\omega_0) \) given in Subsection F.2. By Lemma 18, \( k_0^* (\omega_0) \) and \( \omega_0^{CC} (k_0) \) intersect at \( \{k_0^{CC,Irr}, \omega_0^{CC,Irr}\} \). In addition, by the construction of \( k_0^{**} (\omega_0) \) in Subsection G.2.2, \( k_0^{**} (\omega_0) = k_0^* (\omega_0) \) when \( \omega_0 > \omega_0^{CC,Irr} \). Thus we know that given \( \omega_0 > \omega_0^{CC} (k_0) \), the irreversibility constraint is binding if and only if \( k_0 \geq k_0^{**} (\omega_0) \).

\[ \square \]

### G.4 Region with Binding Collateral Constraint and Binding Irreversibility Constraint

**Lemma 20.** With \( m < 1 \), the irreversibility constraint and \( \omega_0 \leq \frac{(1+\gamma)\frac{1}{1+\gamma}\frac{\omega^*}{X^*}}{1+\gamma\frac{\omega^*}{X^*}} \), there exists a unique equilibrium with binding collateral constraint and binding irreversibility constraint if and only if \( \omega_0 \leq \omega_0^{CC,Irr} \) in (A.35b) and \( k_0 \geq k_0^{**} (\omega_0) \) given in Subsection G.2.2. In this equilibrium, \( q_0 \) is decreasing in \( k_0 \), and \( R_0, X_0 \) and the multiplier for the collateral constraint, \( \mu_0 \) are all independent of \( k_0 \).

**Proof.** Assuming the collateral constraint and the irreversibility constraint are binding. Then \( k_1 = (1-\delta) k_0 \) and \( -\frac{b_1}{R_1^1 k_1} = m \). We also have \( \omega_1 = 1-m \) and \( L_1 = L_1^{cc} \) as in (A.33a). In this case, we can express the system by two unknowns, \( (q_0, R_0) \) or \( (q_0, X_0) \) depending whether the ZLB is binding.

The first equation is derived by the entrepreneurs’ consumption choice \( c_0 = \frac{1}{1+\gamma} \omega_0 R_0^K k_0 \), the expression of \( R_0^K \) in (A.33b) and their budget constraint (10c) as below:

\[
 k_0 = \frac{A_1 L_1^{cc}}{1-\delta} \left[ \left( 1-\delta \right)^{\frac{\omega_0}{R_0 X^*}} \frac{q_0}{1+\gamma} + \frac{\gamma}{1+\gamma} \omega_0 \frac{A_1}{A_0} \left( 1+\gamma \right) R_0^1 \right]^{1-\frac{1}{\pi}} \frac{1}{q_0 (1-\delta) \left( 1-\frac{\gamma}{1+\gamma} \omega_0 \right)} . \tag{A.39a}
\]

The second equation is derived by the feasibility condition in period 0, \( c_0 + c'_0 = Y_0 \) with
expressions in Subsection G.1:

\[
\left( \frac{A_0 R_0 X^*}{X_0} \right)^{\frac{1-\alpha}{\alpha}} \left( \frac{(1-\delta) k_0}{A_1 L_{cc}^1} \right)^{\alpha-1} k_0
= \frac{1}{1+\gamma} \omega_0 k_0 \left[ q_0 (1-\delta) + \frac{\alpha}{X_0} \left( \beta R_0 A_0 X^* \right)^{\frac{1-\alpha}{\alpha}} \left( \frac{(1-\delta) k_0}{A_1 L_{cc}^1} \right)^{\alpha-1} \right] (A.39b)
+ \frac{1}{\beta R_0} 1 - \alpha \left( \frac{(1-\delta) k_0}{L_{cc}^1} \right)^{\alpha}.
\]

Combining these two equations together, we have one equation with one unknown, \( R_0 \) or \( X_0 \) depending on whether the ZLB is binding:

\[
\frac{1}{1-\delta} \left( 1 - \frac{1}{1+\gamma} \omega_0 \frac{\alpha}{X_0} \right) \left( \beta \frac{A_0 X^*}{A_1 X_0} \right)^{\frac{1-\alpha}{\alpha}} R_0^{\frac{1-\alpha}{\alpha}} = \frac{1}{1+\gamma} \omega_0 m \frac{\alpha}{X^*} + \frac{1}{\beta} \left( 1 - (1-m) \frac{\alpha}{X^*} \right). \tag{A.39c}
\]

Notice that the solution of (A.39c) is independent of \( k_0 \).

From (A.39a), \( q_0 \) can be expressed as below:

\[
q_0 = \frac{(1-\delta) \frac{m}{R_0} \frac{\alpha}{X^*} + \frac{\gamma}{1+\gamma} \omega_0 \frac{\alpha}{X_0} \left( \frac{1}{\beta} \frac{A_1 X_0}{X^*} \right)^{\frac{\alpha-1}{\alpha}} R_0^{\frac{1-\alpha}{\alpha}}}{(1-\delta) \left( 1 - \frac{\gamma}{1+\gamma} \omega_0 \right)} \left( \frac{(1-\delta) k_0}{A_1 L_{cc}^1} \right)^{\alpha-1}. \tag{A.39d}
\]

Notice that \( q_0 \) is decreasing in \( k_0 \).

Next, by equation (11), the multiplier for the collateral constraint, \( \mu_0 \) can be expressed as

\[
\mu_0 = \frac{1}{1-m} \left( \frac{1}{R_0} - \frac{q_0}{R_1^K} \right)
= \frac{1}{1-m} \left( \frac{1}{R_0} - \frac{\gamma}{1+\gamma} \omega_0 \frac{X^*}{X_0} \left( \frac{1}{\beta} \frac{A_1 X_0}{X^*} \right)^{\frac{\alpha-1}{\alpha}} R_0^{\frac{1-\alpha}{\alpha}} \right). \tag{A.39e}
\]

We find \( \mu_0 \) is also independent of \( k_0 \). For our assumption of binding irreversibility constraint and collateral constraint to be valid, the two equations above should imply \( q_0 \leq 1 \) and \( \mu_0 \geq 0 \).
Case I: Non-binding ZLB

First, we assume that the ZLB is not binding and impose $X_0 = X^*$ in (A.39c). The solution of $R_0 = R_0^{**}(\omega_0)$ is given by equation (A.36), which is increasing in $\omega_0$. $R_0^{**}(\omega_0) \geq 1$ if and only if $\omega_0 \geq \omega_0^{**}$ in (A.37).

The capital price becomes

$$q_0 = \frac{(1 - \delta) m^\frac{\alpha}{\gamma_0} + \frac{\gamma}{\gamma + \delta} \omega_0 \frac{\alpha}{\gamma} \left( \frac{1}{\beta} \frac{A_1}{A_0} \right)^{\frac{\alpha - 1}{\alpha}} [R_0^{**}(\omega_0)]^{\frac{1}{\alpha}}} {(1 - \delta) \left( 1 - \frac{\gamma}{\gamma + \delta} \omega_0 \right) R_0^{**}(\omega_0)} \left( 1 - \frac{\delta}{\beta} X^* \right)^{\frac{\alpha - 1}{\alpha}} \left( \frac{1 - \delta k_0}{A_1 L_1^{cc}} \right)^{\frac{1}{\alpha - 1}}.$$ 

In particular, when $k_0 = k_{0,noZLB}^{**}(\omega_0)$ as below:

$$k_{0,noZLB}^{**}(\omega_0) = \frac{A_1 L_1^{cc}}{1 - \delta} \left[ \frac{(1 - \delta) m^\frac{\alpha}{\gamma_0} + \frac{\gamma}{\gamma + \delta} \omega_0 \frac{\alpha}{\gamma} \left( \frac{1}{\beta} \frac{A_1}{A_0} \right)^{\frac{\alpha - 1}{\alpha}} [R_0^{**}(\omega_0)]^{\frac{1}{\alpha}}} {(1 - \delta) \left( 1 - \frac{\gamma}{\gamma + \delta} \omega_0 \right) R_0^{**}(\omega_0)} \right]^{\frac{1}{\alpha - 1}}, \quad (A.39f)$$

the implied $q_0 = 1$. Thus here we need $k_0 \geq k_{0,noZLB}^{**}(\omega_0)$. We also need to check whether $\mu_0$ implied by equation (A.39e) is positive. We find that $\mu_0 \geq 0$ if and only if $\omega_0 \leq \omega_0^{CC,Irr}$ in equation (A.35b).

To sum up, for an equilibrium with non-binding ZLB, binding collateral constraint and irreversibility constraint to exist, we should have $\omega_0^{**} < \omega_0 \leq \omega_0^{CC,Irr}$ and $k_0 \geq k_{0,noZLB}^{**}(\omega_0)$.

Case II: Binding ZLB

Assuming the ZLB is binding and impose $R_0 = 1$ in (A.39c) with $X_0$ as the only unknown. Denote its solution as $X_0^{**}(\omega_0)$, which is given in equation (A.38). If $\omega_0 < \frac{(1 + \gamma) \frac{1 - \delta}{\delta} X^*}{1 + \gamma \frac{1 - \delta}{\delta} X^*}$, $X_0^{**}(\omega_0)$ is decreasing in $\omega_0$. At $\omega_0 = \omega_0^{**}$, $X_0^{**}(\omega_0) = X^*$. Then for an equilibrium with binding ZLB to exist here, we need $\omega_0 \leq \omega_0^{**}$.

The capital price becomes

$$q_0 = \frac{(1 - \delta) m^\frac{\alpha}{\gamma_0} + \frac{\gamma}{\gamma + \delta} \omega_0 \frac{\alpha}{\gamma} X_0^{**}(\omega_0) \left( \frac{1}{\beta} \frac{A_1}{A_0} \frac{X_0^{**}(\omega_0)}{X^*} \right)^{\frac{\alpha - 1}{\alpha}}} {(1 - \delta) \left( 1 - \frac{\gamma}{\gamma + \delta} \omega_0 \right) R_0^{**}(\omega_0)} \left( 1 - \frac{\delta}{\beta} X^* \right)^{\frac{\alpha - 1}{\alpha}} \left( \frac{1 - \delta k_0}{A_1 L_1^{cc}} \right)^{\frac{1}{\alpha - 1}}.$$ 

A-62
In particular, when \( k_0 = k_{0,ZLB}^{**}(\omega_0) \) as below:

\[
k_{0,ZLB}^{**}(\omega_0) = \frac{A_1 L_{cc}^o}{1 - \delta} \left[ \frac{(1 - \delta) m X^\alpha + \frac{\gamma}{1+\gamma} \omega_0 X^\alpha(\omega_0) \left( \frac{1}{\beta} A_1 X_{0,ZLB}^{**}(\omega_0) \right)^{\frac{\alpha-1}{\alpha}}}{(1 - \delta) \left( 1 - \frac{\gamma}{1+\gamma} \omega_0 \right)} \right]^{\frac{1}{1-\alpha}}, \quad (A.39g)
\]

the implied \( q_0 = 1 \). Thus here we need \( k_0 \geq k_{0,ZLB}^{**}(\omega_0) \). We still need to check whether \( \mu_0 \) implied by equation (A.39e) is positive. We find that \( \mu_0 \geq 0 \) if and only if \( \omega_0 \leq \omega_{CC, Irr} \) in equation (A.35b).

To sum up, for an equilibrium with binding ZLB, collateral constraint and irreversibility constraint to exist, we should have \( \omega_0 \leq \min \{ \omega_{0,ZLB}^{**}, \omega_{CC, Irr} \} \) and \( k_0 \geq k_{0,ZLB}^{**}(\omega_0) \).

**Putting the Two Pieces Together**

We can verify that when \( R_{CC} \) in equation (A.35a) is larger than one, \( \omega_{CC, Irr} < \omega_0^{**} \). Then the ZLB is binding when \( \omega_0 < \omega_0^{**} \) and \( k_0 \geq k_{0,ZLB}^{**}(\omega_0) \) in (A.39g), and not binding when \( \omega_0 \in (\omega_0^{**}, \omega_{CC, Irr}) \) and \( k_0 \geq k_{0, noZLB}^{**}(\omega_0) \) in (A.39f). By the construction of \( k_{0}^{**}(\omega_0) \) in Subsection G.2.2, \( k_{0}^{**}(\omega_0) = k_{0,ZLB}^{**}(\omega_0) \) when \( \omega_0 < \omega_0^{**} \); and \( k_{0}^{**}(\omega_0) = k_{0, noZLB}^{**}(\omega_0) \) when \( \omega_0 \in (\omega_0^{**}, \omega_{CC, Irr}) \).

When \( R_{CC} < 1 \), \( \omega_{CC, Irr} < \omega_0^{**} \). Then the ZLB is always binding when \( \omega_0 < \omega_{CC, Irr} \) and \( k_0 \geq k_{0,ZLB}^{**}(\omega_0) \). By the construction of \( k_{0}^{**}(\omega_0) \) in Subsection G.2.2, \( k_{0}^{**}(\omega_0) = k_{0,ZLB}^{**}(\omega_0) \) when \( \omega_0 \leq \omega_{CC, Irr} \).

To sum up, there exists a unique equilibrium with binding collateral constraint and binding irreversibility constraint if and only if \( \omega_0 \leq \omega_{CC, Irr} \) and \( k_0 \geq k_{0}^{**}(\omega_0) \).

\[\square\]

**G.5 Regions with Binding Collateral Constraint and Non-binding Irreversibility Constraint**

In this part, we assume the collateral constraint is binding and the irreversibility constraint is not binding. Then the equilibrium properties in this case is very similar to those in the simple two-period model in Subsection 2.4.

Here we have \( q_0 = 1 \), and \( -\frac{b_1}{R_1 k_1} = m \). By the definition of the wealth share in (4), we also have \( \omega_1 = 1 - m \) and \( L_1 = L_{cc}^o \) as in (A.33a). In this case, we can express the system by two unknowns, \( \{ r_1^K, R_0 \} \) or \( \{ r_1^K, X_0 \} \) depending whether the ZLB is binding.
The first equation is derived from the feasibility condition (3a) at \( t = 0 \):

\[
A_1 L^c_1 \left( \frac{X^*_1}{\alpha} r^K_1 \right)^{\frac{1}{\alpha-1}} = \left[ (1 - \delta) \left( 1 - \frac{\omega_0}{1 + \gamma} \right) + \left( \frac{X_0}{\alpha} - \frac{\omega_0}{1 + \gamma} \right) \left( \frac{X^*_1}{X_0} \right)^{\frac{1}{\alpha}} \left( \frac{\beta A_0 R_0}{A_1} \right)^{\frac{1}{\alpha-1}} \right] k_0 \tag{A.40a}
\]

\[
- \frac{1}{\beta R_0} \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*_1}{\alpha} r^K_1 \right)^{\frac{\alpha}{\alpha-1}}.
\]

The second equation is derived by the entrepreneurs’ consumption choice \( c_0 = \frac{1}{1 + \gamma} \omega_0 R^K_0 k_0 \), the expression of \( R^K_0 \) in (A.33b) and their budget constraint (10c) as below:

\[
\left( 1 - \frac{m r^K_1}{R_0} \right) \frac{1}{\alpha-1} A_1 L^c_1 \left( \frac{X^*_1}{\alpha} r^K_1 \right)^{\frac{1}{\alpha-1}} = \frac{\gamma}{1 + \gamma} \omega_0 k_0 \left[ 1 - \delta + \left( \frac{X^*_1}{X_0} \right)^{\frac{1}{\alpha}} \left( \frac{\beta A_0 R_0}{A_1} \right)^{\frac{1}{\alpha-1}} r^K_1 \right]. \tag{A.40b}
\]

A solution to the system of equations (A.40a) and (A.40b) corresponds to an equilibrium with binding collateral constraint if \( k_1 > (1 - \delta) k_0 \) and the multiplier \( \mu_0 \) implied by (11) is positive, i.e., if

\[
R_0 \leq R^K_0. \tag{A.40c}
\]

In the next subsection, we characterize the properties of the solution to (A.40a) and (A.40b), depending on whether the ZLB is binding. We temporarily ignore the requirements (A.40c) and \( k_1 > (1 - \delta) k_0 \), and will verify whether they hold or not later.

G.5.1 Equilibrium with Non-binding ZLB and Binding Collateral Constraint

**Lemma 21.** With \( \omega_0 < \frac{X^*_1}{\alpha} \), there exists a unique equilibrium with binding collateral constraint, non-binding irreversibility constraint and non-binding ZLB if and only if \( k_0 < k_0^{**} (\omega_0) \) given in Subsection G.2.2, \( \omega_0 \leq \omega_0^{CC} (k_0) \) in Lemma 18 and \( \omega_0 \) is larger than a cutoff value, \( \tilde{\omega}_0 (k_0) \). In this region, \( R_0 \) is increasing in \( \omega_0 \), and \( \frac{k_1}{k_0} \) is decreasing in \( k_0 \).

**Proof.** **Step 1: Equilibrium Existence**

Assuming that the collateral constraint is binding, the irreversibility constraint is non-binding, and ZLB is non-binding. Setting \( X_0 = X^*_1, r^K_1 \) can be expressed as functions of \( R_0 \) in both (A.40a) and (A.40b).
Equation (A.19c) becomes

\[
A_1 L_1^{cc} \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\alpha - 1}} + \frac{1}{\beta R_0} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\alpha - 1}} r_1^K = \left[ (1 - \delta) \left( 1 - \frac{\omega_0}{1 + \gamma} \right) + \left( \frac{X^*}{\alpha} - \frac{\omega_0}{1 + \gamma} \right) \left( \beta \frac{A_0}{A_1} R_0 \right) \right] \frac{1 - \frac{\alpha}{\gamma}}{\alpha} r_1^K \]

(A.41a)

in which \( r_1^K \) is a decreasing function of \( R_0 \). Denote this implicit function as \( r_1^K = h_1 (R_0) \). We easily verify that \( \lim_{R_0 \to 0} h_1 (R_0) \to +\infty \) and \( \lim_{R_0 \to +\infty} h_1 (R_0) \to 0 \).

We can write the equation above in the form of

\[
A_1 L_1^{cc} \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\alpha - 1}} = -\phi_1^1 (R_0) r_1^K + \phi_2^1 \left( r_1^K \right)^{\frac{1}{\alpha}} + \phi_3^1 (R_0) \left( r_1^K \right)^{1 + \frac{1}{\alpha}},
\]

(A.41b)

where \( \phi_1^1, \phi_2^1, \phi_3^1 > 0 \). Denote its right-hand side as \( H_1 (r_1^K, R_0) \).

Equation (A.40b) becomes

\[
1 = \frac{m r_1^K}{R_0} + \frac{1}{\gamma} \omega_0 k_0 + \frac{\gamma}{1 + \gamma} \frac{1 - \delta + \left( \frac{\beta R_0}{\alpha} \right) \frac{1 - \alpha}{\alpha} r_1^K}{A_1 L_1^{cc} \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\alpha - 1}} \left( r_1^K \right)^{\frac{1}{\alpha - 1}}},
\]

(A.41c)

which can be similarly written as

\[
A_1 L_1^{cc} \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\alpha - 1}} = \phi_1^2 (R_0) r_1^K + \phi_2^2 (R_0) \left( r_1^K \right)^{\frac{1}{\alpha}} + \phi_3^2 (R_0) \left( r_1^K \right)^{1 + \frac{1}{\alpha}},
\]

(A.41d)

where \( \phi_1^2, \phi_2^2, \phi_3^2 > 0 \). Denote its right-hand side as \( H_2 (r_1^K, R_0) \). Thus there exists a unique solution for \( r_1^K \) as a function of \( R_0 \). Denote this implicit function as \( r_1^K = h_2 (R_0) \). We can also easily verify that that \( \lim_{R_0 \to 0} h_2 (R_0) \to 0 \), and as \( \lim_{R_0 \to +\infty} h_2 (R_0) \to 0 \). Thus \( h_2 (R_0) \) is not monotone.

We show that, given \( \omega_0 < \frac{X^*}{\alpha} \), as \( R_0 \to +\infty \), \( h_2 (R_0) \) is asymptotically higher than \( h_1 (R_0) \).

As \( R_0 \to +\infty \), \( h_1 (R_0) \) and \( h_2 (R_0) \) both converge to zero. We can derive the following
asymptotic behaviors as $R_0 \to +\infty$:

$$[h_1 (R_0)]^{1+\frac{1}{1-\alpha}} \propto \frac{1}{\left(\frac{X^*}{\alpha} - \frac{\omega_0}{1+\gamma}\right)} \frac{A_1 L_1^{cc} \left(\frac{X^*}{\alpha}\right)^{\frac{1}{1-\alpha}}}{R_0^{\frac{\alpha-1}{\alpha}}} R_0^{\alpha-1},$$

$$[h_2 (R_0)]^{1+\frac{1}{1-\alpha}} \propto \frac{1+\gamma}{\omega_0} \frac{A_1 L_1^{cc} \left(\frac{X^*}{\alpha}\right)^{\frac{1}{1-\alpha}}}{R_0^{\frac{\alpha-1}{\alpha}}} R_0^{\alpha-1}.$$

If $\omega_0 < \frac{X^*}{\alpha}$, $h_2 (R_0)$ is asymptotically higher than $h_1 (R_0)$.

Then we obtain $h_1 (R_0) > h_2 (R_0)$ at $R_0 = 0$ and $h_1 (R_0) < h_2 (R_0)$ when $R_0$ is sufficiently high. By the Intermediate Value Theorem, the two functions will cross at least once. This guarantees the existence of a solution $(R_0, r_1^K)$ for the two equations (A.41a) and (A.41c).

**Step 2: Equilibrium Uniqueness**

We show at any intersection of $h_1$ and $h_2$, i.e. $h_1 (R_0) = h_2 (R_0)$, the slope of $h_2$ must be steeper than the one for $h_1$, i.e. $h_1' (R_0) < h_2' (R_0)$.

Since the proof for this statement is the same as Step 3 of Subsection D.2.1, we choose to omit this part. Combining the previous two steps together, we see that assuming a binding collateral constraint, non-binding irreversibility constraint and non-binding ZLB, a solution to (A.41a) and (A.41c) exists and is unique. (without checking whether the implied $R_0 \geq 1$.)

**Step 3: Comparative Statics**

In equation (A.41a), we see that fixing $r_1^K$, $R_0$ is increasing in $\omega_0$. In (A.41c), we see that fixing $R_0$, $r_1^K$ is decreasing in $\omega_0$. Since the slope of $h_2 (R_0)$ is steeper than the one for $h_2 (R_0)$, as $\omega_0$ increases both curves shift to the right, and the equilibrium $R_0$ increases. Thus $R_0$ is increasing in $\omega_0$.

To see how the ratio $\frac{k_1}{k_0}$ responds to $k_0$, define $\rho_k = \frac{k_1}{k_0}$, and equations (A.41a) and (A.41c) can be respectively written as below:

$$\rho_k^{1-\alpha} \left[ \rho_k - (1 - \delta) \left(1 - \frac{\omega_0}{1+\gamma}\right) \right] k_0^{1-\alpha} + \frac{1}{\beta R_0} \frac{1-\alpha}{X^*} A_1 (A_1 L_1^{cc})^{-\alpha} \rho_k$$

$$= \left(\frac{X^*}{\alpha} - \frac{\omega_0}{1+\gamma}\right) \left(\frac{\beta A_0}{A_1} R_0\right)^{\frac{1}{1-\alpha}} \frac{X^*}{\alpha} (A_1 L_1^{cc})^{1-\alpha},$$
and
\[ \rho_k^a \left[ \rho_k^{1-a} - \frac{m \alpha}{R_0 X} (A_1 L_1^{cc})^{1-a} k_0^{a-1} \right] \]
\[ = (1 - \delta) \frac{\gamma}{1 + \gamma} \omega_0 + \frac{\gamma}{1 + \gamma} \omega_0 \left( \beta R_0 A_0 \right) \frac{1}{1 - \alpha} (A_1 L_1^{cc})^{1-a} \rho_k^{a-1} k_0^{a-1}. \]

If we fix \( R_0 \), we see that \( \rho_k \) is decreasing in \( k_0 \) in both equations. As a result, \( \frac{k_1}{k_0} \) is decreasing in \( k_0 \).

**Step 4: Checking the Assumptions of Binding Collateral Constraint and Non-binding Irreversibility**

Since the ratio \( \frac{k_1}{k_0} \) is decreasing in \( k_0 \), we find at \( k_0 = k_0^{**} (\omega_0) \) given in Subsection G.2.2, \( \rho_k = 1 - \delta \) in the two equations above. Thus there is no equilibrium with binding collateral constraint, Non-binding Irreversibility and non-binding ZLB when \( k_0 > k_0^{**} (\omega_0) \). Otherwise, the restriction \( \frac{k_1}{k_0} \geq 1 - \delta \) would be violated.

We can also check the assumption of a binding collateral constraint. Similar to Lemma 11, here we can show that the derivative \( R_1^K - R_0 \) is negative at \( \omega_0 = \omega_0^{CC} (k_0) \) given in Lemma 18. In addition, using the similar argument in Lemma 12, we can show the collateral constraint is violated if \( \omega_0 > \omega_0^{CC} (k_0) \), while it is satisfied when \( \omega_0 \leq \omega_0^{CC} (k_0) \).

**Step 5: Cutoff of \( \omega_0 \) for ZLB**

It remains to check whether the assumption of non-binding ZLB holds. Since \( R_0 \) is decreasing in \( \omega_0 \), we can identify the cutoff for binding ZLB, \( \hat{\omega}_0 (k_0) \), such that given \( k_0 \), \( R_0 = 1 \) at \( \omega_0 = \hat{\omega}_0 (k_0) \). The expression of \( \hat{\omega}_0 (k_0) \) can be solved implicitly by imposing \( R_0 = 1 \) in (A.41a) and (A.41c). To be specific, imposing \( R_0 = 1 \) in (A.41a), we have:
\[ \hat{\omega}_0 = \frac{\left( 1 - \delta + \frac{X^c}{\alpha} \left( \beta A_0 \right)^{\frac{1-a}{a}} r_1^K \right) k_0 - A_1 L_1^{cc} \left( \frac{X^c}{\alpha} \right)^{\frac{1}{1-a}} (r_1^K)^{\frac{1}{1-a}} - \frac{1}{p} \frac{1-a}{X^c} A_1 \left( \frac{X^c}{\alpha} r_1^K \right)^{\frac{a}{1-a}}}{\frac{1}{1+\gamma} \left[ 1 - \delta + \left( \beta A_0 \right)^{\frac{1-a}{a}} r_1^K \right] k_0}. \]

and imposing \( R_0 = 1 \) in (A.41c), we have:
\[ \hat{\omega}_0 = \frac{(1 - mr_1^K) A_1 L_1^{cc} \left( \frac{X^c}{\alpha} \right)^{\frac{1}{1-a}} (r_1^K)^{\frac{1}{1-a}}}{\frac{m}{1+\gamma} k_0 \left[ 1 - \delta + \left( \beta A_0 \right)^{\frac{1-a}{a}} r_1^K \right]}. \]
Combining both equations and after some calculation, we have

\[
\frac{1 + \gamma}{\gamma} A_1 L_1^{cc} \left( \frac{X^*}{\alpha} r_1^k \right)^{\frac{1}{\alpha}} + \left( \frac{1}{\beta} - \frac{m}{\gamma} \right) \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} r_1^k \right)^{\frac{m}{\gamma}} = \left( 1 - \delta + \frac{X^*}{\alpha} \left( \beta \frac{A_0}{A_1} \right)^{\frac{1}{\alpha}} r_1^k \right) k_0, \tag{A.41e}
\]

in which \( r_1^k \) is decreasing in \( k_0 \). For a given value of \( k_0 \), inserting the solved \( r_1^k \) from the equation above into either one of the expression for \( \hat{\omega}_0 \), we get the expression for \( \hat{\omega}_0 (k_0) \).

Since \( r_1^k \) is decreasing in \( k_0 \) in (A.41e), when \( k_0 = \tilde{k}_0 \) in equation (A.35e), \( r_1^k = 1 \) in equation (A.41e). Thus when \( k_0 > \tilde{k}_0 \), our assumption of a binding collateral constraint at \( \hat{\omega}_0 (k_0) \) does not hold anymore.

To see how the ratio \( \rho_k = \frac{k}{k_0} \) changes along \( \hat{\omega}_0 (k_0) \), we replace \( r_1^k \) by \( \rho_k \) in the equation above, which becomes

\[
k_0^{-1} = \frac{\left[ \frac{1 + \gamma}{\gamma} \rho_k - (1 - \delta) \right] (\rho_k)^{1 - \alpha}}{(A_1 L_1^{cc})^{1 - \alpha} \left( \left( \frac{A_0}{A_1} \right)^{\frac{1}{\alpha}} \frac{1 - \alpha}{X^*} - \left( \frac{1}{\beta} - \frac{\alpha}{X^*} \left( \frac{1 - m}{\gamma} + \frac{m}{\gamma} \right) \right) \rho_k \right).}
\]

Check carefully this equation, we see that \( \rho_k \) is decreasing in \( k_0 \) along \( \hat{\omega}_0 (k_0) \). In particular, when \( k_0 = \hat{k}_0^{**} \) as below:

\[
\hat{k}_0^{**} = \frac{A_1 L_1^{cc}}{1 - \delta} \left[ \frac{\gamma}{1 - \delta} \left( \beta \frac{A_0}{A_1} \right)^{\frac{1}{\alpha}} - \gamma \left( \frac{1}{\beta} - \frac{\alpha}{X^*} \left( \frac{1 - m}{\gamma} + \frac{m}{\gamma} \right) \right) \right]^{\frac{1}{\alpha}}, \tag{A.41f}
\]

\( \hat{\omega}_0 (\hat{k}_0^{**}) = \omega_0^{**} \) in equation (A.37). Thus suggests that when \( k_0 > \hat{k}_0^{**} \), our assumption of a non-binding irreversibility constraint at \( \hat{\omega}_0 (k_0) \) does not hold anymore.

When \( \mathcal{R}^{cc} \) in equation (A.35a) is larger than one, \( \hat{k}_0^{**} < \tilde{k}_0 \), and as \( k_0 \) increases, the curve \( \hat{\omega}_0 (k_0) \) will cross \( k_0^{**} (\omega_0) \) defined in Subsection G.2.2 at \( k_0 = \hat{k}_0^{**} \). When \( \mathcal{R}^{cc} < 1 \), \( \hat{k}_0^{**} > \tilde{k}_0 \), and as \( k_0 \) increases, the curve \( \hat{\omega}_0 (k_0) \) will cross \( \omega_0^{cc} (k_0) \) given in Lemma 18 at \( k_0 = \tilde{k}_0 \).

To sum up, when \( \omega_0 < \frac{X^*}{\alpha} \), there exists a unique equilibrium with binding collateral constraint, non-binding irreversibility constraint and non-binding ZLB if and only if \( k_0 < k_0^{**} (\omega_0) \), \( \omega_0 \leq \omega_0^{cc} (k_0) \) and \( \omega_0 > \hat{\omega}_0 (k_0) \). \( \square \)
G.5.2 Equilibrium with Binding ZLB and Binding Collateral Constraint

Lemma 22. Assume \( \omega_0 \) is smaller than\(^{39}\)

\[
\min \left\{ (1 + \gamma) \frac{1 - \alpha}{\alpha} X^*, \Xi (\alpha, m, X^*, \gamma, \beta) \right\},
\]

in which \( \Xi \) is a function defined in (A.42f), there exists a unique equilibrium with binding collateral constraint, non-binding irreversibility constraint and binding ZLB if and only if \( k_0 < k^{**}_0 (\omega_0) \) given in Subsection G.2.2, \( \omega_0 \leq \omega^{CC}_0 (k_0) \) in Lemma 18 and \( \omega_0 \leq \hat{\omega}_0 (k_0) \) given implicitly by (A.41e). In this region, \( X_0 \) is decreasing in \( \omega_0 \), and \( k_1 k_0 \) is decreasing in \( k_0 \).

Proof. As in the proof of Lemma 21, we firstassuming that the collateral constraint is binding, ZLB is binding, and the irreversibility constraint is non-binding. With these assumptions, we can represent the equilibrium by two equations. We will come back later to check whether the assumptions are valid.

**Step 1: Equilibrium Representation**

In this case, we represent the system as functions of \( \{ r^K_1, r^K_0 \} \). \( X_0 \) can be expressed as a function of \( \{ r^K_1, r^K_0 \} \) as below:

\[
X_0 = X^* \left( \beta \frac{A_0}{A_1} \right)^{1-\alpha} \left( \frac{r^K_1}{r^K_0} \right)^{\alpha}. \tag{A.42a}
\]

The counterpart for the restriction \( X_0 \geq X^* \) is

\[
r^K_0 \leq \left( \beta \frac{A_0}{A_1} \right)^{1-\alpha} r^K_1,
\]

and the irreversibility constraint sets a lower bound for \( r^K_1 \):

\[
r^K_1 \geq \frac{\alpha}{X^*} \left( \frac{(1 - \delta) k_0}{A_1 L^{cc}_1} \right)^{\alpha-1}. \tag{A.42b}
\]

Equation (A.40a) becomes

\[
A_1 L^{cc}_1 \left( \frac{X^*}{\alpha} r^K_1 \right)^{\frac{1}{\alpha}} + \frac{1}{\beta} \frac{1}{X^*} A_1 \left( \frac{X^*}{\alpha} r^K_1 \right)^{\frac{\alpha}{\alpha-1}} = (1 - \delta) \left( 1 - \frac{\omega_0}{1 + \gamma} \right) k_0 + \left( \frac{X^*}{\alpha} \left( \beta \frac{A_0}{A_1} \right)^{1-\alpha} \left( \frac{r^K_1}{r^K_0} \right)^{\alpha} \right) - \frac{\omega_0}{1 + \gamma} \right) r^K_0 k_0, \tag{A.42c}
\]

\(^{39}\)With our calibrated parameters, the value is 1.39.
If \( \omega_0 < (1 + \gamma) \frac{1 - \alpha}{\alpha} X^* \), we can show that \( r^K_0 \) is decreasing in \( r^K_1 \). Denote this implicit function as \( r^K_0 = u_1 \left( r^K_1 \right) \). Using implicit function theorem, we see that as \( k_0 \) increases, \( u_1 \left( r^K_1 \right) \) shifts to the left, i.e., holding \( r^K_0 \) unchanged, \( r^K_1 \) is decreasing in \( k_0 \).

Equation (A.40b) becomes

\[
\left( 1 - mr^K_1 \right) A_1 L^cc_1 \left( \frac{X^*}{\alpha} r^K_1 \right)^{\frac{1}{s-1}} = \frac{\gamma}{1 + \gamma} \omega_0 k_0 \left( 1 - \delta + r^K_0 \right) \quad (A.42d)
\]

in which \( r^K_0 \) is decreasing in \( r^K_1 \) as well. Denote this implicit function as \( r^K_0 = u_2 \left( r^K_1 \right) \). Using implicit function theorem, we can show that as \( k_0 \) increases, \( u_2 \left( r^K_1 \right) \) shifts to the left, i.e., holding \( r^K_0 \) unchanged, \( r^K_1 \) is increasing in \( k_0 \).

**Step 2: Equilibrium Existence**

We show that if \( \omega_0 \leq \hat{\omega}_0 \left( k_0 \right) \), defined implicitly in equation (A.41e), and given \( r^K_1 \leq \frac{\alpha}{X^*} \left( \frac{1 - \delta} {A_1 L^cc_1} \right)^{\alpha-1} \) due to the irreversibility constraint, there exists a unique solution \( \{r^K_1, r^K_0\} \) to equations (A.42c) and (A.42d).

The intuition of this result can be seen in Figure A.13. The black dashed line corresponds to \( x_0 = X^* \) below which we have \( x_0 \geq X^* \). When \( \omega_0 < \hat{\omega}_0 \left( k_0 \right) \), by equations (A.42c) and (A.42d), with \( R_0 = 1 \) and \( x_0 = X^* \), \( r^K_1 \) in (A.42c) is smaller than \( r^K_1 \) in (A.42d). Correspondingly, in Figure A.13, Point A, the intersection of \( u_1 \left( r^K_1 \right) \) and \( x_0 = X^* \) lies to the lower left of Point B, the intersection of \( u_2 \left( r^K_1 \right) \) and \( x_0 = X^* \). In other words, given \( r^K_1 = r^K_{1,B} \), the value at point B, \( u_1 \left( r^K_{1,B} \right) < u_2 \left( r^K_{1,B} \right) \).

Denote the horizontal intercept of \( u_1 \left( r^K_1 \right) \) as \( \hat{r}^K_a \), and the horizontal intercept of \( u_2 \left( r^K_1 \right) \) as \( \hat{r}^K_b \). The question is whether \( \hat{r}^K_a > \hat{r}^K_b \). Setting \( r^K_0 = 0 \) in (A.41a) and (A.41c) and applying the implicit function theorem, we have \( \frac{\partial r^K}{\partial x_0} < 0 \) and \( \frac{\partial r^K}{\partial k_0} > 0 \). Thus given \( \omega_0 \), there is a cutoff value of \( \bar{K}^{0}_{irr} \) such that \( \hat{r}^K_a \geq \hat{r}^K_b \) if and only if \( k_0 \leq \bar{K}^{0}_{irr} \). From (A.42c) and (A.42d), we can show that \( \bar{K}^{0}_{irr} \) can be solved by the following equation:

\[
2A_1 L^cc_1 \left( \frac{X^*}{\alpha} \hat{r}^K_1 \right)^{\frac{1}{s-1}} + \left( 1 - \frac{m}{\frac{X^*}{\alpha} - 1 + m} \right) \frac{1 - \alpha}{X^*} A_1 \left( \frac{X^*}{\alpha} \hat{r}^K_1 \right)^{\frac{s}{s-1}} = (1 - \delta) \bar{K}^{0}_{irr},
\]

in which \( L^cc_1 \) is from equation (A.33a). However, the restriction on \( r^K_1 \), (A.42b), is violated if \( k_0 \geq \bar{K}^{0}_{irr} \). Thus, for the current case, we must have \( \hat{r}^K_a > \hat{r}^K_b \).

Now with \( \hat{r}^K_a > \hat{r}^K_b \), we see that \( u_1 \left( \hat{r}^K_a \right) > u_2 \left( \hat{r}^K_b \right) \). Since both \( u_1 \left( r^K_1 \right) \) and \( u_2 \left( r^K_1 \right) \) are continuous, they should intersect at least once when \( r^K_1 \in \left[ r^K_{1,B}, \hat{r}^K_1 \right] \) with \( X_0 > X^* \). Thus there exists at least one solution to equations (A.42c) and (A.42d).

**Step 3: Equilibrium Uniqueness**

A-70
Figure A.13: Equilibria with Binding Collateral Constraint, ZLB and Non-binding Irreversibility

Note: This figure is generated by setting $\beta = 0.99$, $\gamma = 0.98$, $\alpha = 0.35$, $\delta = 0.025$, $A_0 = 1$, $A_1 = 1.005$, $m = 0.7$ and $e = 21$. $\omega_0 = 0.05$ and $k_0 = 0.3$.

When $\omega_0 \leq \hat{\omega}_0 (k_0)$ and $u_1 (r^K_1) = u_2 (r^K_1)$, the slope of $u_1 (r^K_1)$ is higher than the slope of $u_2 (r^K_1)$ when they intersect.

Using implicit function theorem, the derivatives of $u_1 (r^K_1)$ and $u_2 (r^K_1)$ are

\[
\frac{\partial u_1}{\partial r^K_1} (r^K_1) = -\frac{\frac{1}{1-\alpha} A_1 L^{cc}_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\gamma}} (r^K_1)^{\frac{2-\alpha}{\gamma}} + \frac{1}{\beta} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\gamma}} (r^K_1)^{\frac{1}{\gamma}} + X_0 \frac{r^K_0}{r^K_1} k_0}{\frac{1-\alpha}{\alpha} X_0 - \frac{\omega_0}{1+\gamma} k_0},
\]

\[
\frac{\partial u_2}{\partial r^K_1} (r^K_1) = -\frac{\frac{1}{1-\alpha} A_1 L^{cc}_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\gamma}} (r^K_1)^{\frac{2-\alpha}{\gamma}} + \frac{\alpha}{1-\alpha} m A_1 L^{cc}_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\gamma}} (r^K_1)^{\frac{1}{\gamma}}}{\frac{\gamma}{1+\gamma} \omega_0 k_0}.
\]

We will show that given $1 \leq r^K_1 \leq \frac{1}{m}$,

\[
\frac{\partial u_1}{\partial r^K_1} (r^K_1) > \frac{\partial u_2}{\partial r^K_1} (r^K_1).
\]

Indeed, the inequality can be rewritten as

\[
\left( \frac{1}{1-\alpha} A_1 L^{cc}_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\gamma}} (r^K_1)^{\frac{2-\alpha}{\gamma}} + \frac{1}{\beta} A_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\gamma}} (r^K_1)^{\frac{1}{\gamma}} \right) \frac{\gamma}{1+\gamma} \omega_0 + \frac{\gamma}{1+\gamma} \omega_0 X_0 \frac{r^K_0}{r^K_1} k_0 \\
\leq \left( \frac{1}{1-\alpha} A_1 L^{cc}_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\gamma}} (r^K_1)^{\frac{2-\alpha}{\gamma}} + \frac{\alpha}{1-\alpha} m A_1 L^{cc}_1 \left( \frac{X^*}{\alpha} \right)^{\frac{1}{\gamma}} (r^K_1)^{\frac{1}{\gamma}} \right) \left( \frac{1-\alpha}{\alpha} X_0 - \frac{\omega_0}{1+\gamma} \right).
\]
After some calculations, we can show a stronger result as below:

\[
\left( \frac{1}{1-\alpha} + \frac{1}{1+\gamma} \left( \frac{\gamma}{\beta} \frac{1}{L_{cc}^1} + \frac{\alpha}{1-\alpha} m \right) r_1^K \right) \omega_0 \\
\leq \left( \frac{1-\alpha}{\alpha} + 2mr_1^K \right) X_0.
\]  
(A.42e)

and this inequality holds if \(^{40}\)

\[
\omega_0 < \Xi (\alpha, m, X^*, \gamma, \beta) = \min \left\{ V \left( \frac{1}{1} \right), V \left( \frac{1}{m} \right) \right\},
\]  
(A.42f)

in which

\[
V \left( r_1^K \right) = \frac{\left( \frac{1-\alpha}{\alpha} + 2mr_1^K \right) X^*}{\frac{1}{1-\alpha} + \frac{1}{1+\gamma} \left( \frac{\gamma}{\beta} \frac{1-(1-m) \frac{\alpha}{1-\alpha} \beta}{X^*} + \frac{\alpha}{1-\alpha} m \right) r_1^K}.
\]  
(A.42g)

As a result, given \(k_0, \omega_0 \leq \hat{\omega}_0 (k_0)\) in (A.41e), a binding collateral constraint and a non-binding irreversibility constraint, an equilibrium with binding ZLB exists and is unique. Otherwise, if there are multiple equilibria in this region, \(u_1 (r_1^K)\) and \(u_2 (r_1^K)\) cross for multiple times, and then one of these equilibria features \(\frac{du_1}{dr_1^K} \leq \frac{du_2}{dr_1^K}\) which contradicts the slope comparison above.

**Step 4: Comparative Statics**

By checking equations (A.42c) and (A.42d) carefully, we see that as \(\omega_0\) increases, \(u_1 (r_1^K)\) shifts to the left, while \(u_2 (r_1^K)\) shifts to the right, making the equilibrium \(r_1^K\) lower and \(r_0^K\) higher. From (A.42a), \(X_0\) is also lower. Thus \(X_0\) is decreasing in \(\omega_0\).

To see how the ratio \(\rho_k = \frac{k_1}{k_0}\) changes with \(k_0\), we express (A.42c) and (A.42d) as functions of \(\{\rho_k, X_0\}\):

\[
\rho_k^{1-\alpha} \left[ \rho_k - (1-\delta) \left( \frac{\omega_0}{1+\gamma} \right) \right] k_0^{1-\alpha} + \frac{1}{\beta} \frac{\omega_0}{X^*} A_1 (A_1 L_{cc}^1)^{-\alpha} \rho_k \\
= \left( \frac{X_0}{X^*} - \frac{\omega_0}{1+\gamma} \frac{\alpha}{X^*} \right) \left( \frac{X^*}{X_0} \right)^{\frac{1}{2}} \left( \frac{A_0}{A_1} \right)^{\frac{1-\alpha}{2}} \left( A_1 L_{cc}^1 \right)^{1-\alpha}.
\]

\(^{40}\)With our calibrated parameters, the value of this upper bound is 1.39.
and
\[
\rho_k^a \left[ \rho_k^{1-a} - m \frac{\alpha}{X^*} (A_1 L_1^c)^{1-a} k_0^{a-1} \right] \\
= (1 - \delta) \frac{\gamma}{1 + \gamma} \omega_0 + \frac{\gamma}{1 + \gamma} \omega_0 \left( \frac{X^*}{X_0} \right) \frac{1}{2} \left( \beta \frac{A_0}{A_1} \right) \frac{1-a}{X^*} (A_1 L_1^c)^{1-a} \rho_k^{a-1} k_0^{a-1}.
\]

In both functions, given \( \omega_0 \) and fixing \( X_0 \), \( \rho_k \) is decreasing in \( k_0 \). Thus in equilibrium, \( \frac{k_1}{k_0} \) is decreasing in \( k_0 \).

Lastly, since \( r_1^K \) is decreasing in \( \omega_0 \), the excess return \( R_1^K - R_0 \) is also decreasing in \( \omega_0 \) since \( R_1^K = r_1^K \) and \( R_0 = 1 \).

Step 5: Checking the Assumptions of Binding Collateral Constraint and Non-binding Irreversibility

Since \( X_0 \) is decreasing at \( \omega_0 \), and at \( \omega_0 = \hat{\omega}_0(k_0) \) given in (A.41e), \( X_0 = X^* \), the assumption of a binding ZLB is violated if and only if \( \omega_0 > \hat{\omega}_0(k_0) \). In addition, since \( \frac{k_1}{k_0} \) is decreasing in \( k_0 \) and we can show that at \( k_0 = k_0^{**}(\omega_0) \) given in Subsection G.2.2, \( \frac{k_1}{k_0} = 1 - \delta \), the assumption of a non-binding irreversibility constraint is violated if and only if \( k_0 \geq k_0^{**}(\omega_0) \). Lastly, since \( R_1^K - R_0 \) is decreasing in \( \omega_0 \) and at \( \omega_0 = \omega_0^{CC}(k_0) \) given in Lemma 18, \( R_1^K - R_0 = 0 \), the assumption of a binding collateral constraint is violated if and only if \( \omega_0 > \omega_0^{CC}(k_0) \).

To sum up, when \( \omega_0 \) is smaller than
\[
\min \left\{ (1 + \gamma) \frac{1 - \alpha}{\alpha} X^*, \Xi(\alpha, m, X^*, \gamma, \beta) \right\},
\]
there exists a unique equilibrium with binding collateral constraint, non-binding irreversibility constraint and binding ZLB if and only if \( k_0 < k_0^{**}(\omega_0) \), \( \omega_0 \leq \omega_0^{CC}(k_0) \) and \( \omega_0 \leq \hat{\omega}_0(k_0) \).

G.6 AS-AD Representation

If the collateral constraint is not binding, the equilibrium properties and the AS-AD curves are the same as the model with natural borrowing limit analyzed in Appendix F.5. On the other hand, if the irreversibility constraint is not binding, the equilibrium properties are similar to the simple two-period model in Subsection 2.4. So here we focus on the AS-AD curves with both binding collateral constraint and binding irreversibility constraint.

When the collateral constraint is binding, labor supply at \( t = 1 \) is constant and given
by $L^c_1$ in equation (A.33a). Together with the labor-leisure choice of the households at $t = 0, 1$, the households’ Euler equation for bond holding, as well as $k_1 = (1 - \delta) k_0$ given a binding irreversibility constraint, the AS curve can be written as

$$Y_0^{AS} = \left( \beta R_0 \frac{A_0 X^*}{A_1 X_0} \right)^{\frac{1-\delta}{\alpha}} \left( \frac{(1 - \delta) k_0}{A_0 L^c_1} \right)^{\alpha-1} k_0.$$  

(A.43a)

As $k_1 = (1 - \delta) k_0$, the AD curve is given by summing up $c_0$ and $c'_0$ in Subsection G.1:

$$Y_0^{AD} = \frac{1}{1+\gamma} \omega_0 k_0 \left[ q_0 (1 - \delta) + \frac{\alpha}{X_0} \left( \beta R_0 \frac{A_0 X^*}{A_1 X_0} \right)^{\frac{1-\delta}{\alpha}} \left( \frac{(1 - \delta) k_0}{A_0 L^c_1} \right)^{\alpha-1} \right] + \frac{1}{\beta R_0} \frac{1-\alpha}{X^*} A_1^{1-\alpha} \left( \frac{(1 - \delta) k_0}{L^c_1} \right)^{\alpha}. $$

(A.43b)

When the irreversibility constraint is binding, Proposition 4 shows that both $R_0$ and $X_0$ remain independent to $k_0$ and are functions of $\omega_0$ only. Thus given $\{k_0, \omega_0\}$, we can express the AS-AD curves as functions of $q_0$ while replacing $R_0$ and $X_0$ by their equilibrium values.

As an example, the AS-AD curves are plotted in Figure A.14 when all three constraints: ZLB, collateral constraint and the irreversibility constraint are all binding. The AS curve is inelastic to $q_0$, and the AD curve is positively sloped. As $k_0$ increases, both AS and AD curves shift to the right leading $Y_0$ to increase. The effect on $q_0$ might be ambiguous but by equation (A.39d), $q_0$ decreases. Similarly, as $\omega_0$ increases, both AS and AD curves shift to the right leading $Y_0$ to increase. By equation (A.39d), $q_0$ increases.

H More Details from the Quantitative Model

In Appendix H.1, we present the full quantitative model. Appendix H.2 describes our global solution method and Appendix H.3 provides the computed policy functions from the benchmark quantitative model. Appendix H.5 carries out analyses of numerical errors from our global solution.

H.1 Complete Setup

Section 3.1 describes the essential ingredients of the quantitative model. We now describe the remaining setup of the model and refer to the common components shared with the two-period model when necessary. Time is discrete, starts from 0 and goes to infinity.
The aggregate shocks consist of a productivity shock and a credit shock, as specified in Section 3.1.

The households The representative households supply labor endogenously and make saving and borrowing decisions to maximize the expected lifetime utility

$$E \sum_{t=0}^{\infty} \beta^t \left[ \log c_t - \frac{1}{\eta} (L_t')^\eta \right]$$

where $c_t'$ is the consumption and $L_t'$ is the labor supply. $\beta > 0$ is the households’ common discount factor. The households are subject to the following sequential budget constraint

$$P_t c_t' + \frac{B_t'}{R_t} \leq B_{t-1}' + P_t w_t L_t' + P_t \int_0^1 \Xi_t(z)dz,$$

taking $P_t, R_t, w_t$ and $\int_0^1 \Xi_t(z)dz$ as given, where $B_{t-1}'$ is the nominal bond accumulated in the previous period, $R_t$ is the nominal interest rate, $w_t$ is the market real wage, and $\int_0^1 \Xi_t(z)dz$ is the profits transferred from intermediate good retailers in real terms.

The entrepreneurs The representative entrepreneurs maximize the following expected utility

$$E \sum_{t=0}^{\infty} \gamma^t \log c_t$$
subject to the sequential budget constraint

\[ P_t c_t + P_t q_t^K K_{t+1} + \frac{B_t}{R_t} \leq B_{t-1} + P_t q_t^K k_t + P_t^e Y_t^e - P_t w_t L_t, \]

production technology

\[ Y_t^e = k_t^\alpha (A_t L_t)^{1-\alpha}, \]

and the collateral constraint

\[ \min \left\{ m_t P_{t+1} [q_{t+1}^K + r_{t+1}^K] k_{t+1} + B_t \right\} \geq 0, \]

taking \( P_t, q_t^K, q_t^{K'}, R_t, \pi_t, P_t^e, w_t \) as given, where \( q_t^{K'} \) and \( q_t^K \) are market prices for new capital and existing capital, respectively. \( P_t^e \) is the price of intermediate goods.

**Equilibrium** The problem of the final-good producers is the same as the one in the two-period model. The retailers’ problem, capital producing firm’s problem and monetary policy rules are specified in section 3.1. We define the real value of debt \( b_t = \frac{B_t}{P_t} \) and the markup charged by retailers \( X_t = \frac{P_t}{P_t^e} \). Then the budget constraints of the entrepreneurs and the households can be written in real variables.

We adopt the standard notation of uncertainty. Time is discrete and runs from 0 to infinity. In each period, an aggregate shock \( s_t = (g_t, m_t, \chi_t^m) \) is realized. \( s_t \) follows a finite-state Markov chain described in Subsection 3.1. Let \( s^t = (s_0, s_1, \ldots, s_t) \) denote the history of realizations of shocks until date \( t \). Assume \( A_{-1}, g_{-1}, m_{-1}, \chi_{-1}^m \) are given. To simplify notations, for each variable \( x \), we use \( x_t \) as a shortcut for \( x_t(s_t) \).

**Definition 3.** A competitive equilibrium is sequences, which depend on time \( t \) and the history of shocks \( s^t \), of inflation and markup \( \{\pi_t, X_t\}_{t,s^t} \), prices \( \{w_t, r_t^K, q_t^K, q_t^{K'}, R_t\}_{t,s^t} \), retailer real profits \( \left\{ \int_0^1 \Xi_t(z) dz \right\}_{t,s^t} \) and allocations \( \{c_t, c_t', k_{t+1}, Y_t, Y_t^e, b_t, b_t', L_t, L_t'\}_{t,s^t} \) such that given the initial conditions \( k_0 > 0, b_{-1} \) and \( b_{-1}' = -b_{-1} \):

(i) The allocations solve the entrepreneurs and the households’ decision problems.

(ii) Markets for labor, bond, intermediate good and final good clear:

\[ L_t = L_t', \]
\[ b_t + b_t' = 0, \]
\[ Y_t = Y_t^e, \]
\[ c_t + c_t' + \Omega (k_t, k_{t+1}) + \theta \phi (\pi_t) Y_t = Y_t, \]
in which $\Omega (k_t, k_{t+1})$ and $\theta \phi (\pi_t) Y_t$ are the capital adjustment cost and price adjustment cost, respectively, as specified in Section 3.1.

(iii) Retailers’ profits satisfy equation (16). Capital prices satisfy equations (17).

As in the two-period model in Section 2, we focus on sequential competitive equilibria with the wealth share of the entrepreneurs as an endogenous state variable. Their wealth share is defined as

$$\omega_t = \frac{(r^K_t + q^K_t) k_t + \frac{b_{t-1}}{1+\pi_t}}{(r^K_t + q^K_t) k_t}.$$  \hspace{1cm} (A.44)

Correspondingly, the households’ wealth share is

$$\omega'_t = \frac{b'_{t-1}}{(r^K_t + q^K_t) k_t}.$$  \hspace{1cm} (A.45)

From the bond market clearing condition, $\omega'_t = 1 - \omega_t$ in any competitive equilibrium.

**Definition 4.** A wealth-recursive equilibrium in the infinite-horizon economy is a sequential competitive equilibrium in which allocations $\{c_t, c'_t, k_{t+1}, Y_t, b_t, L_t\}$, prices $\{w_t, r^K_t, q^K_t, q'K_t, R_t\}$ and inflation and markup $\{\pi_t, X_t\}$ are functions of $\{k_t, \omega_t, s_t, A_t\}$.

### H.2 Global Solution Method

To calculate wealth-recursive equilibria, we de-trend the retailers’ real profits $\{\int_0^1 \Xi_t(z) dz\}$ and allocations $\{c_t, c'_t, k_{t+1}, Y_t, b_t\}$ by the aggregate TFP shock $A_t$ and remove $A_t$ from the list of exogenous state variables. With some abuse of notations, we use the same symbols here to denote their corresponding de-trended values.

Given $\{k_t, \omega_t, s_t\}$, we have $7 + S$ unknown variables: $c_t, c'_t, k_{t+1}, b_t, \mu_t, \pi_t, X_t, \{\omega_{t+1}(s_{t+1})\}_{s_{t+1}}$ in which $S$ is the number of states in period $t + 1$, and $\omega_{t+1}(s_{t+1})$ is the wealth share in period $t + 1$ when the state in the next period is $s_{t+1}$. \hspace{1cm} (A.46a)

We use the following system with $7 + S$ equations to pin down the values of the variables:

1. Feasibility constraint:

$$c_t + c'_t + \Omega (k_t, k_{t+1}) + \theta \phi (\pi_t) Y_t = Y_t,$$  \hspace{1cm} (A.45a)

\[\text{From equation (A.44), given } \{k_{t+1}, b_t\}, \omega_{t+1} \text{ is endogenous to state } s_{t+1} \text{ since } \{r^K_{t+1}, q^K_{t+1}, \pi_{t+1}\} \text{ are affected by } s_{t+1}.\]
2. FOC for households’ bond holding:

\[-1 + \beta \mathbb{E}_t \left[ \frac{1}{1 + \pi_{t+1}} R_t \frac{c_t'}{1 + \pi_{t+1} c_t' + 1} \right] = 0, \quad (A.45b)\]

3. Entrepreneurs’ budget:

\[c_t + q_t^K k_{t+1} + \frac{b_t}{R_t} = \left( r_t^K + q_t^K \right) k_t + g_t \omega_t, \quad (A.45c)\]

4. FOC for entrepreneurs’ capital holding:

\[-1 + \kappa_t \mu_t q_t^K + \gamma \mathbb{E}_t \left[ \frac{1}{1 + \pi_{t+1}} \left( \frac{r_{t+1}^K + q_{t+1}^K}{q_t^K} \right) \frac{c_t}{c_{t+1}} \right] = 0, \quad (A.45d)\]

5. Complementary-slackness condition for the collateral constraint:

\[\mu_t \left[ b_t + \kappa_t k_{t+1} \right] = 0, \quad (A.45e)\]

with \( \mu_t \geq 0, \)

6. F.O.C. for entrepreneurs’ bond holding:

\[-1 + R_t \mu_t + \gamma \mathbb{E}_t \left[ \frac{1}{1 + \pi_{t+1}} R_t \frac{c_t'}{1 + \pi_{t+1} c_t' + 1} \right] = 0, \quad (A.45f)\]

7. New-Keynesian Phillips curve:

\[(1 + \pi_t) \phi' (\pi_t) = \frac{\epsilon - 1}{\theta} \left( \frac{1}{X_t} - \frac{\epsilon - 1}{\epsilon} \right) + \beta \mathbb{E}_t \left[ \frac{1}{1 + \pi_{t+1}} Y_t + 1 + \pi_{t+1} \phi' (\pi_{t+1}) Y_{t+1} \right], \quad (A.45g)\]

8. Consistency condition:

\[\omega_{t+1} = \frac{(r_{t+1}^K + q_{t+1}^K) k_{t+1} + b_t}{(r_{t+1}^K + q_{t+1}^K) k_{t+1}}, \quad \forall s_{t+1}, \quad (A.45h)\]

in which \( r_{t+1}^K, q_{t+1}^K \) and \( \pi_{t+1} \) are functions of \( \{k_{t+1}, \omega_{t+1}, s_{t+1}\} \).
The auxiliary variables in the equations above are given as follows:

\[
\Omega(kt, kt+1) = kt+1 - \frac{1 - \delta}{1 + g_t}kt + \frac{\zeta}{2} \left( \frac{kt+1 - \frac{kt}{1+g_t}}{kt+1 - \frac{kt}{1+g_t}} \right)^2,
\]

\[
\kappa_t = m_t \min_{s_t+1} \left\{ \left( r^K_{t+1} + q^K_{t+1} \right) \pi_{t+1} \right\},
\]

\[
q^K_t = 1 + \frac{\zeta}{1+g_t} \left( \frac{kt+1 - \frac{kt}{1+g_t}}{kt+1 - \frac{kt}{1+g_t}} \right),
\]

\[
q^K_t = (1 - \delta) - \frac{\zeta}{2} \left[ 1 - \left( \frac{(1+g_t)kt+1}{kt} \right)^2 \right],
\]

\[
\phi(\pi_t) = \frac{\pi_t - \bar{\pi}}{\sqrt{\pi_t - \bar{\pi}}} - 2 \sqrt{\pi_t - \bar{\pi}} + 2 \sqrt{\bar{\pi} - \pi},
\]

\[
R_t = \max \left\{ \bar{R} \left( \frac{1 + \pi_t}{1 + \bar{\pi}} \right)^{\phi_k} \left( \frac{Y_t}{Y} \right)^{\phi_y}, 1 \right\},
\]

\[
L_t = \left[ 1 - \alpha \frac{k^a_t}{X_t (1+g_t) c^\prime_t} \right]^{\frac{1}{a+\eta-1}},
\]

\[
r^K_t = \frac{\alpha}{X_t} \left[ \frac{kt}{(1+g_t)L_t} \right]^{\alpha-1},
\]

\[
Y_t = \left( \frac{kt}{1+g_t} \right)^{\alpha} L_t^{1-\alpha}.
\]

In addition, we can invert equation (A.44) to get \( b_{t-1} \):

\[
b_{t-1} = (\omega_t - 1) (1 + \pi_t) \left( r^K_t + q^K_t \right) k_t.
\]

We solve for the recursive equilibrium in this economy using the algorithm in Cao and Nie (2017) and Cao (2018). The original algorithm in Cao (2018) uses wealth share as endogenous state variables. Cao and Nie (2017) add labor choice as well as housing consumption decisions. In the current paper, we show that the original algorithm works similarly when we add imperfect price stickiness, capital and capital adjustment cost, and Taylor-rule based monetary policy with two occasional binding constraints, ZLB and the collateral constraint.\footnote{We also solved another version of the infinite-horizon economy with a third occasional binding constraint: investment irreversibility as discussed in Section E.} Here we present the details of this algorithm.

Our algorithm looks for a recursive equilibrium mapping from \( \{kt, \omega_t \} \) and the exogenous aggregate shock, \( s_t = \{g_t, m_t, \chi^m_t \} \), to the allocations \( \{c_t, c^\prime_t, k_{t+1}, Y_t, b_t, L_t \} \), prices
\{w_t, r^K_t, q^K_t, q'_t, R_t\}, inflation and markup \{\pi_t, X_t\}, as well as the future financial wealth distributions, \(\omega_{t+1}(s_{t+1})\), depending on the realization of future aggregate shocks, \(s_{t+1}\). Indeed, given the mapping from \(\{k_{t+1}, \omega_{t+1}, s_{t+1}\}\) to \(\{r^K_{t+1}, q^K_{t+1}, \pi_{t+1}, c_{t+1}, c'_{t+1}, Y_{t+1}\}\), for a given set of \(\{k_t, \omega_t, s_t\}\), we can solve for the other variables using equations (A.45a) to (A.45h). In particular, we follow Cao (2018) in solving for \(\omega_{t+1}\) simultaneously with other unknowns. The additional equations needed to solve for \(\omega_{t+1}\) are equation (A.45h) applied to each of the future state \(s_{t+1}\) in which the mapping from \(\{k_{t+1}, \omega_{t+1}, s_{t+1}\}\) to \(r^K_{t+1}, q^K_{t+1}\) and \(\pi_{t+1}\) are given by the mapping obtained in the previous iteration of the algorithm.

We solve for the recursive equilibrium using backward induction. The algorithm starts by solving for the equilibrium mapping for 1-period economy. Then given the mapping from \(t = 0\) to \(t = 1\) for \(T\)-period economy, we can solve for the mapping for \((T+1)\)-period economy following the procedure described above. The algorithm converges when the mappings for \(T\)-period economy and \((T+1)\)-period economy are sufficiently close to each other.\(^{43}\)

\(^{43}\)For the model without the credit shock, i.e., \(m_t \equiv m\), based on the definition of \(\omega_t\) in (A.44) and the borrowing constraint (15), we can easily see that the lower bound of \(\omega_t\) is \(1 - m\). With shocks to \(m_t\), the expression of the lower bound of \(\omega_t\) is unknown ex ante, and we show the model can also be solved using \(c_t\) as an endogenous state variable instead of \(\omega_t\).
H.3 Policy Functions and ZLB duration from the Quantitative Model

The policy functions in the full quantitative model carry all the intuitions we have learned from the two-period model. Figure A.15 plots the policy functions for several key equilibrium variables over the endogenous states variables \((k, \omega)\). The policy functions are evaluated with productivity growth rate equal to its unconditional mean. The two surfaces correspond to policy functions with different levels of leverage constraint \(m\) (the warm-color surfaces correspond to \(m = 0.45\) and the cold-colored ones correspond to \(m = 0.1\)). As shown in the figure, given the productivity, the ZLB tends to bind when capital stock is high or the entrepreneur wealth share is low. The collateral constraint tends to bind when the entrepreneur wealth share is low. The inverse of markup drops substantially when the ZLB binds, and even more so when the ZLB and the collateral constraint both bind. Both capital price and output are substantially lower in the regions where zero lower bound binds, and more so when both constraints bind.

What is new in the quantitative model with imperfect price stickiness compared to the two-period model is that now inflation is allowed to be different from one. Inflation changes in the same direction along with the inverse of markup over the state space since it is associated with the inverse of markup through the New-Keynesian Phillips Curve. It

Figure A.15: Policy Functions over \((k, \omega)\) Space

Note: The policy functions are evaluated with productivity growth rate equal to its unconditional mean. Surfaces with warm colors correspond to \(m\) High \((m = 0.45)\). Surfaces with cold colors correspond to \(m\) Low \((m = 0.1)\).
changes more smoothly due to the forward-looking nature of the Phillips Curve. Introducing imperfect price stickiness allows the movement of inflation to feedback into the collateral constraint, through a traditional Fisherian “nominal debt-deflation channel”. However, from both the policy functions and the crisis episode we study in Section 3, the movement in inflation is usually small and is not likely to play a quantitatively important role in determining the severity of the crisis compared to other channels.

Figure A.16 and A.17 plot the regions for binding ZLB and collateral constraints, and the policy functions projecting onto either $k$ or $\omega$ space. Both figures resemble their counterparts for the two-period model and we refer the reader to the main text for the analytical characterizations and the discussions.

---

**Figure A.16: Regions for ZLB and Binding Collateral Constraint**

*Note: The region is based on policy functions at $m = \bar{m}$ and $g = \bar{g}$.***
Figure A.17: Policy Functions Varying Capital (four upper panels) and Wealth Share (four lower panels)

Note: The policy functions are at $g = g$. $m_{\text{High}}$ corresponds to $m = 0.45$. $m_{\text{Low}}$ corresponds to $m = 0.1$.

The average duration of a ZLB episode in the ergodic set is around 2 quarters. Nevertheless, the histogram of ZLB durations, shown in Figure A.18, exhibits a long right tail, and the model can produce lengthy ZLB episodes with the appropriate choice of realized shock series, albeit with low probability.
H.4 Asset Prices in the Data and in the Model

In this appendix, we discuss the dynamics of asset prices implied by the quantitative model during the Great Recession and compare them to the data. As we described in Subsection 3.1, capital in the model stands in for a combination of housing and non-housing capital. Therefore, in the left panel of Figure A.19, we plot the model capital price against the price indices of both housing and stock market from the data. Stock prices dropped significantly more than housing prices but recovered more quickly. Overall, our model captures relatively well the timing and magnitude of the average dynamics of prices in these time series. The magnitude of the drops in the model is slightly smaller because we leave out other important factors influencing prices during the Great Recession, such as changes in risk premium and liquidity, and deteriorated balance sheet of the financial sector.

The right panel of Figure A.19 plots the model implied excess returns against the credit spread constructed by Gilchrist and Zakrajsek (2012) (GZ spread). The dynamics of excess return in the model tracks the overall timing of the rise and fall of the GZ spread. However, the magnitude of the rise of the excess return is significantly larger than that of the GZ spread. This is partly because the size of the credit shock in the model is calibrated to match the overall drop in bank loans and the excess returns correspond to excess returns on a broad range of assets, whereas the GZ spread measures the spread on bonds issued by publicly listed firms which have better access to external financing.
For each state \( x_t = (k_t, \omega_t, g_t, m_t, \chi_t^m) \), following Judd, Maliar, and Maliar (2011) and Guerreri and Iacoviello (2015), we define the unit-free Euler equation errors for the bond and capital choices of the households and the entrepreneurs as below:

\[
\mathcal{E}^b(x_t) = -1 + \beta \mathbb{E}_t \left[ \frac{1}{1 + g_{t+1}} \frac{1}{1 + \pi_{t+1}} \frac{1}{1 + c_{t+1}} \right] R_t \left( 1 + \frac{c_t}{R_t} \right),
\]

\[
\mathcal{E}^b(x_t) = -1 + \mu_t + \gamma \mathbb{E}_t \left[ \frac{1}{1 + g_{t+1}} \frac{1}{1 + \pi_{t+1}} \frac{1}{1 + c_{t+1}} \right] \frac{R_t}{1 + \frac{c_t}{R_t}}.
\]

\[
\mathcal{E}^k(x_t) = -1 + \frac{\kappa_t \mu_t}{q_t^T} + \gamma \mathbb{E}_t \left[ \frac{1}{1 + g_{t+1}} \left( \frac{r^K_{t+1} + q^K_{t+1}}{q_t^K} \right) \frac{c_t}{c_{t+1}} \right].
\]

\( (A.46) \)

We evaluate the errors for a sample of 120000 \( x_t \) drawn from the model’s ergodic set.\(^{44}\)

| Table A.1: Euler Equation Errors |
|-------------------|-------------------|-------------------|-------------------|
|                   | mean | | mean | | mean | |
| All Samples       | 2.9E-04 | 2.5E-05 | 4.1E-04 |
| ZLB Binding       | 4.3E-04 | 4.1E-05 | 3.4E-04 |
| ZLB & CC Binding  | 4.2E-04 | 4.0E-05 | 3.1E-04 |

\(^{44}\)To draw samples from the ergodic set, we simulate 24 paths of 6000 periods. Notice by the ergodic theory, the long run distribution of samples across time of a single simulation path converges to the ergodic distribution; we choose multiple paths to utilize parallel computation. We drop the first 1000 periods and keep the remaining 5000 periods of the 24 paths, which give us 120000 observations in total.
Table A.1 reports the mean absolute errors across $x_t$ in the full samples, the samples with binding ZLB, and the samples with both binding ZLB and collateral constraints. As shown, the mean absolute errors across all samples and subsamples are below $5E - 4$. The accuracy is slightly lower for states with binding ZLB due to the nonlinear dynamics in these regions of the state space. The numerical errors are of similar magnitude as the errors from Guerrieri and Iacoviello’s OccBin for their model without ZLB Guerrieri and Iacoviello (2015, Figure 4) and are lower than the OccBin errors for their model with ZLB Guerrieri and Iacoviello (2015, Figure 6).

H.6 Comparisons with Piecewise-linear Solutions

The global nonlinear solutions provide a full characterization of the economy in and out of normal times, and capture agents’ precautionary motives facing severe although infrequent crises. An alternative approach, popularized by the toolbox OccBin (Guerrieri and Iacoviello, 2015), approximates the nonlinear solutions with piecewise linear functions. This section compares the OccBin solution with the global nonlinear solution, highlighting the non-linearity of the current model and the importance of capturing agents’ precautionary motives in understanding the crisis dynamics.

To do so, we need to modify the benchmark model in several ways. First, the benchmark collateral constraint is specified as

$$m_t \cdot \min \left[ P_{t+1} \left( q_{t+1}^{K} + r_{t+1}^{K} \right) k_{t+1} + B_t \right] \geq 0.$$  

This constraint corresponds to a condition that the entrepreneurs will not default under any realization of future exogenous states, and hence equips the lenders with a strong precautionary motive. Since the local solution cannot handle the min operator, we modify the collateral constraint to

$$m_t \cdot \mathbb{E}_t \left[ P_{t+1} \left( q_{t+1}^{K} + r_{t+1}^{K} \right) k_{t+1} + B_t \right] \geq 0,$$

where $\mathbb{E}_t$ is the expectation operator conditional on the current state. Second, in the benchmark model we specify the credit shock, $m_t$, to have innovations with asymmetric distributions, aiming at capturing the infrequent nature of financial crises. The piecewise linear solution method cannot handle this asymmetry, so we modify the process of the
credit shock to an AR(1) process:

\[ m_{t+1} = (1 - \rho^m)\bar{m} + \rho^m m_t + \epsilon^m, \]

where \( \epsilon^m \) satisfies the normal distribution with mean zero and standard deviation \( \sigma^m \). We choose \( \bar{m} \) to be the same as in the benchmark model, \( \rho^m = 0.99 \) and \( \sigma^m = 0.01 \), so the process is close to the one used in the benchmark model. Third, after the two modifications above, we recalibrate the discount factor of the entrepreneurs, \( \beta \), so that the average nominal interest rate is 5% in the ergodic set based on the nonlinear solution.

---

45 For the collateral constraint to be well-defined, we need to truncate \( m_t \) to be within \([0,1]\), but in the simulated ergodic set, \( m_t \) never hits the upper or lower bounds with the chosen standard deviation of the shock.

46 Notice for the event study interested in this subsection, the choice of \( \sigma^m \) matters for the global nonlinear solution but not for the piecewise linear solution. Setting \( \sigma^m = 0.01 \) allows us to discretize the innovation to be \([-0.01, 0.01]\) based on a two-point Gaussian quadrature, and together with a high \( \rho^m \), brings the process of \( m_t \) close to the one in the benchmark model.

47 This procedure is mainly to ensure comparability across models. The recalibrated \( \beta = 0.9991 \), close to the calibrated value 0.9993 in the benchmark model.
Figure A.20: The ZLB Episode, Global Nonlinear Solutions v.s. Piecewise Linear Solutions

Note: Output, investment, capital price, and the volume of credit are ratios to their 2007Q3 values. “Benchmark” corresponds to the benchmark result in Section 3.3. “Alternative Spec” corresponds to the alternative collateral constraint specification that associates the borrowing limit with the expected value of future capital. “Alternative Spec, OccBin” corresponds to the piecewise linear solution obtained using toolbox OccBin. The initial capital and debt levels are calibrated so that the nominal interest rate is 5% and the debt-to-asset ratio is 35%, based on the nonlinear solution for each specification.

Since the piecewise linear solution method is not designed to produce policy functions over a global domain such as those in Figure A.15, we focus on the comparison for the ZLB episode studied in Section 3.3. To do so, we compare the dynamics implied by different solution methods by starting from the same initial capital and debt levels, and feeding in the same sequences of $g_t$ and $m_t$ as described in Section 3.3. Figure A.20 plots the aggregate dynamics in models with different specifications and solved with different solution methods. The solid lines correspond to the benchmark results in Section 3.3. The dash-dotted lines correspond to the results under the alternative collateral constraint specification, based on the global nonlinear solution. As shown, the effects of the negative productivity growth shock and credit shock are already much larger under the alternative specification than the benchmark. This is because in the benchmark model...

---

48 The initial capital and debt levels are calibrated so that the nominal interest rate is 5% and the debt-to-asset ratio is 35% based on the nonlinear solution, the same targets as in the benchmark experiment in Section 3.3.

49 One intermediate model specification between the benchmark and the current alternative model is to
the lenders assign full weight to the worst scenario when evaluating the collateral value of future capital, and thus are well prepared for the crisis state by allowing lower leverages ex ante. Whereas with the alternative specification, the lenders are less precautionary by allowing the entrepreneurs to borrow against the expected value of future capital, putting less weight on the rare but severe crisis state. Consistent with this intuition, the bottom right panel shows that the volume of credits grows much faster before the crisis hits under the alternative specification than under the benchmark specification (the volume of credit in 2007Q3 is normalized to 1). Similarly, the dotted lines correspond to the piecewise log-linear solution for the alternative specification, obtained using OccBin. As shown, although the piecewise log-linear solution captures the overall shape of the dynamics, it overstates the severity of the crisis even more than the global solution for the alternative specification: output and capital price drop by more than 40%, whereas both drops implied by the benchmark model are modest and well align with the data.

In summary, for the model to produce crises with magnitude in line with data, it is important to model agents taking precautionary measures against the rare but severe crisis state, and to capture the high nonlinearity of the model when the crisis hits. The global nonlinear solution is able to appropriately take into account both features.

I Representative Agent Model with Exogenous Wedges

The representative agent model shares all the ingredients with the full model, except that the households and entrepreneurs are combined into representative households, who solve the following problem:

\[
\max_{c_t, L_t, k_{t+1}, B_t} \mathbb{E}_0 \left[ \log c_t - \frac{1}{\eta} (L_t)^{\eta} \right]
\]

subject to:

\[
P_t c_t + \frac{1}{1 - \Delta^b_t} q^{K^r}_t k_{t+1} + \frac{1}{1 + \Delta^k_t} \frac{B_t}{R_t} \leq B_{t-1} + P_t (r^K_t + q^K_t) k_t + P_t w_t L_t + P_t \int_0^1 \Xi_t(z) dz,
\]

where, to remind readers, \( P_t \) is the price level, \( R_t \) is the bond nominal interest rate, \( r^K_t \) is the real return on capital, \( q^{K^r}_t \) and \( q^K_t \) are the market prices for new and existing capital, \( w_t \) is the real wage, and \( \int_0^1 \Xi_t(z) dz \) is the profits transferred from the retailers in real terms. The terms \( \Delta^b_t \) and \( \Delta^k_t \) are exogenous wedges and correspond to errors in the Euler equations.

use the benchmark credit shock process and the alternative collateral constraint specification. The responses in this intermediate model are also significantly larger than those in the benchmark model, suggesting that it is the collateral constraint specification rather than the credit shock process that drives the main difference. Results from the intermediate model are available upon requests.
for bond and capital holdings of the representative households. The Euler equations with these wedges are given in (19).

Following a tradition in the literature (e.g., Smets and Wouters (2007), Coibion et al. (2012), Christiano et al. (2015), Gust et al. (2017)), we use this representative agent model to interpret the data generated from the full model. To do so, we treat $\Delta_b^t$ and $\Delta_k^t$ as exogenous consumption and financial wedges and back out them from the simulated time series data from the full model. We treat the nominal interest rate, $R_t$, expected inflation rate $E_t[\pi_{t+1}]$, and the expected real return to capital $E_t[r_{K_{t+1}} + q_{K_{t+1}}]$ as observables and, following Christiano et al. (2015), we set $\frac{M_{t+1}}{M_t} = \beta$ when constructing the two wedges.51

We allow households to correctly forecast future prices and allocations taking into account the effects of their expectation errors, and to understand that $\Delta_b^t$ and $\Delta_k^t$ are recurrent exogenous shocks. To do so, we estimate an AR(1) process for each of $\Delta_b^t$ and $\Delta_k^t$:

$$\Delta_{t+1}^x = \mu^x + \rho^x \Delta_t^x + \varepsilon_t^x, \quad \varepsilon_t^x \sim \text{Normal}(0, \sigma_{\varepsilon}^2)$$

for $x = b, k$. In implementation, we draw a 50000-period time series from the full model, drop the first 10000 period observations and estimate the AR(1) processes with the remaining sample. Table A.2 reports the point estimates for the coefficients.

<table>
<thead>
<tr>
<th></th>
<th>$\mu$</th>
<th>$\rho$</th>
<th>$\sigma_\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_b^t$</td>
<td>-0.0015</td>
<td>0.678</td>
<td>0.0032</td>
</tr>
<tr>
<td>$\Delta_k^t$</td>
<td>-0.0068</td>
<td>0.470</td>
<td>0.0123</td>
</tr>
</tbody>
</table>

We embed the processes of the two wedges, as well as the productivity growth process that is the same as in the full model, to the representative agent model. This leaves four continuous state variables (three for exogenous shocks and one for capital). We solve the model using the global solution method described in Appendix H.2. We keep all param-

---

50 To estimate $\Delta_b^t$, Christiano et al. (2015) use the federal funds rate for $R_t$ and the core CPI-inflation forecasts from the Survey of Professional Forecasters for $E_t[\pi_{t+1}]$. Then the consumption wedge is calculated from $1 + \Delta_b^t = (1 + E_t[\pi_{t+1}])/(\beta R_t)$ by ignoring the covariance terms. They estimate $\Delta_k^t$ using the credit spread constructed in Gilchrist and Zakrajsek (2012). The nominal interest rate, expected inflation rate, and capital return in our model, which we treat as observables to the econometrician, contain the same set of information as the empirical counterparts used in Christiano et al. (2015).

51 The choice of the deterministic discount factor affects the estimated mean of the consumption wedge and the financial wedge, but not the time variations. As long as the discount factor is recalibrated to match the same bond interest rate, which we do, the choice of the deterministic discount factor when backing out the wedges does not matter.
ters the same as in the full model, except for the households’ discount factor $\beta$, which we recalibrate to match the same target as in the full model—an annualized average nominal bond interest rate of 5%.

We then take the representative agent model to analyze the ZLB episode. We set the initial capital stock (in 2015Q1) such that the annualized nominal bond interest rate is at 5%. For the “Benchmark” experiment presented in Figure 8, we construct the consumption and financial wedges from the ZLB episode in the full model that is described in Section 3.3. In the experiment labeled “No ZLB,” we keep the consumption and financial wedges constructed from the full model. In the experiment labeled “Shutdown ZLB wedge,” we construct the wedges from the full model with the ZLB constraint relaxed.