Projection of Private Values in Auctions: Online Appendix

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This Online Appendix is organized as follows. Section B presents proofs for the results with private and common values from Section V; Section C provides the analysis of asymmetric auctions underlying Section VI.A; and Section D considers an example with affiliated values.

B Proofs for Section V

Proof of Lemma 1. *Part 1.* Fix $\theta \in \Theta$. For any $t_i \in \mathcal{T}(\theta)$, let $\widehat{\mathcal{S}}(\theta|t_i)$ denote the set of realizations of S_j consistent with $\theta_j = \theta$. Note that if $\bar{t}(t_i) + \gamma \underline{s} < \underline{t}(t_i) + \gamma \bar{s}$, then

$$\widehat{\mathcal{S}}(\theta|t_i) = \begin{cases} [\underline{s}, (\theta - \underline{t}(t_i))/\gamma] & \text{if } \theta < \bar{t}(t_i) + \gamma \underline{s} \\ [(\theta - \bar{t}(t_i))/\gamma, (\theta - \underline{t}(t_i))/\gamma] & \text{if } \theta \in [\bar{t}(t_i) + \gamma \underline{s}, \underline{t}(t_i) + \gamma \bar{s}] \\ [(\theta - \bar{t}(t_i))/\gamma, \bar{s}] & \text{if } \theta > \underline{t}(t_i) + \gamma \bar{s}. \end{cases} \tag{B.1}$$

If instead $\bar{t}(t_i) + \gamma \underline{s} > \underline{t}(t_i) + \gamma \bar{s}$, then $\widehat{\mathcal{S}}(\theta|t_i)$ is identical to (B.1) except the middle region of θ has reversed bounds. Let $\hat{g}(s|\theta;t_i)$ denote Player i's perceived PDF of S_j conditional on $\theta_j = \theta$:

$$\hat{g}(s|\theta;t_i) = \frac{\hat{f}(\theta - \gamma s|t_i) g(s)}{\int_{\widehat{\mathcal{S}}(\theta|t_i)} \hat{f}(\theta - \gamma \tilde{s}|t_i) g(\tilde{s}) d\tilde{s}}. \tag{B.2}$$

Thus

$$\widehat{\mathbb{E}}[S_j|\theta_j = \theta; t_i] = \int_{\widehat{\mathcal{S}}(\theta|t_i)} s \hat{g}(s|\theta;t_i) ds. \tag{B.3}$$

Let $M_S(\theta) \equiv \widehat{\mathbb{E}}[S_j|\theta_j = \theta; \underline{t}]$ denote the expectation above according to a player with the lowest private value. We now show that the expectation according to any other player can be written in terms of M_S; namely, $\widehat{\mathbb{E}}[S_j|\theta_j = \theta; t_i] = M_S(\theta - \delta(t_i))$ where $\delta(t_i) \equiv \alpha(t_i - \underline{t})$. Using the relationship between the perceived and true PDF, notice that

$$M_S(\theta-\delta(t_i)) = \int_{\widehat{\mathcal{S}}(\theta-\delta(t_i)|\underline{t})} s\hat{g}(s|\theta-\delta(t_i);\underline{t})ds = \frac{\int_{\widehat{\mathcal{S}}(\theta-\delta(t_i)|\underline{t})} sf\left(\frac{\theta-\gamma s-\delta(t_i)-\alpha\underline{t}}{1-\alpha}\right)g(s)ds}{\int_{\widehat{\mathcal{S}}(\theta-\delta(t_i)|\underline{t})} f\left(\frac{\theta-\gamma s-\delta(t_i)-\alpha\underline{t}}{1-\alpha}\right)g(s)ds}$$
$$= \frac{\int_{\widehat{\mathcal{S}}(\theta-\delta(t_i)|\underline{t})} s\hat{f}\left(\theta-\gamma s|t_i\right)g(s)ds}{\int_{\widehat{\mathcal{S}}(\theta-\delta(t_i)|\underline{t})} \hat{f}\left(\theta-\gamma s|t_i\right)g(s)ds}. \tag{B.4}$$

From Equation B.1, notice that $\widehat{\mathcal{S}}(\theta-\delta(t_i)|\underline{t}) = \widehat{\mathcal{S}}(\theta|t_i)$. It thus follows from Equation (B.4) that

$$M_S(\theta-\delta(t_i)) = \frac{\int_{\widehat{\mathcal{S}}(\theta|t_i)} s\hat{f}\left(\theta-\gamma s|t_i\right)g(s)ds}{\int_{\widehat{\mathcal{S}}(\theta|t_i)} \hat{f}\left(\theta-\gamma s|t_i\right)g(s)ds} = \int_{\widehat{\mathcal{S}}(\theta|t_i)} s\hat{g}(s|\theta;t_i)ds = \widehat{\mathbb{E}}[S_j|\theta_j=\theta;t_i]. \tag{B.5}$$

Since log-concavity of f and g implies that M_S is increasing, it follows that $\widehat{\mathbb{E}}[S_j|\theta_j=\theta;t_i] = M_S(\theta-\alpha(t_i-\underline{t}))$ is decreasing in t_i.

Next, let $\hat{g}(s|\theta_j\leq\theta;t_i)$ denote Player i's perceived PDF of S_j conditional on $\theta_j\leq\theta$:

$$\hat{g}(s|\theta_j\leq\theta;t_i) = \frac{\widehat{F}(\theta-\gamma s|t_i)g(s)}{\widehat{H}(\theta|t_i)}, \tag{B.6}$$

where $\widehat{H}(\theta|t_i)$ is Player i's perceived CDF of θ. Hence, $\widehat{H}(\theta|t_i) = \int_{\underline{t}(t_i)+\gamma\underline{s}}^{\theta}\hat{h}(\tilde{\theta}|t_i)d\tilde{\theta}$, where $\hat{h}(\tilde{\theta}|t_i) = \int_{\widehat{\mathcal{S}}(\tilde{\theta}|t_i)}\hat{f}(\tilde{\theta}-\gamma s|t_i)g(s)ds$ is Player i's perceived PDF of θ. Notice that

$$\widehat{\mathbb{E}}[S_j|\theta_j\leq\theta;t_i] = \int_{\widehat{\mathcal{S}}(\theta|t_i)} s\hat{g}(s|\theta_j\leq\theta;t_i)ds. \tag{B.7}$$

Let $\widetilde{M}_S(\theta)\equiv\widehat{\mathbb{E}}[S_j|\theta_j\leq\theta;\underline{t}]$ denote the expectation above according to a player with the lowest private value. We will show that $\widehat{\mathbb{E}}[S_j|\theta_j\leq\theta;t_i] = \widetilde{M}_S(\theta-\delta(t_i))$. Notice that

$$\widetilde{M}_S(\theta-\delta(t_i)) = \int_{\widehat{\mathcal{S}}(\theta-\delta(t_i)|\underline{t})} s\hat{g}(s|\theta_j\leq\theta-\delta(t_i);\underline{t})ds = \frac{\int_{\widehat{\mathcal{S}}(\theta-\delta(t_i)|\underline{t})} sF\left(\frac{\theta-\gamma s-\delta(t_i)-\alpha\underline{t}}{1-\alpha}\right)g(s)ds}{\widehat{H}(\theta-\delta(t_i)|\underline{t})}$$
$$= \frac{\int_{\widehat{\mathcal{S}}(\theta|t_i)} s\widehat{F}\left(\theta-\gamma s|t_i\right)g(s)ds}{\widehat{H}(\theta-\delta(t_i)|\underline{t})}, \tag{B.8}$$

where the final equality follows from the definition of $\widehat{F}(\cdot|t_i)$ and the fact that $\widehat{\mathcal{S}}(\theta-\delta(t_i)|\underline{t}) = \widehat{\mathcal{S}}(\theta|t_i)$ (as noted above). Furthermore,

$$\widehat{H}(\theta-\delta(t_i)|\underline{t}) = \int_{\underline{t}+\gamma\underline{s}}^{\theta-\delta(t_i)}\hat{h}(\tilde{\theta}|\underline{t})d\tilde{\theta} = \int_{\alpha t_i+(1-\alpha)\underline{t}+\gamma\underline{s}}^{\theta}\hat{h}(\tilde{\theta}-\delta(t_i)|\underline{t})d\tilde{\theta}, \tag{B.9}$$

and

$$\begin{aligned}\hat{h}(\tilde{\theta}-\delta(t_i)|\underline{t}) &= \int_{\widehat{\mathcal{S}}(\tilde{\theta}-\delta(t_i)|\underline{t})} \hat{f}(\tilde{\theta}-\gamma s-\delta(t_i)|\underline{t})g(s)ds \\ &= \int_{\widehat{\mathcal{S}}(\tilde{\theta}|t_i)} \hat{f}(\tilde{\theta}-\gamma s|t_i)g(s)ds = \hat{h}(\tilde{\theta}|t_i).\end{aligned} \tag{B.10}$$

Thus Equation (B.9) along with the fact that $\underline{t}(t_i) = \alpha t_i + (1-\alpha)\underline{t}$ implies that

$$\widehat{H}(\theta-\delta(t_i)|\underline{t}) = \int_{\underline{t}(t_i)+\gamma\underline{s}}^{\theta} \hat{h}(\tilde{\theta}|t_i)d\tilde{\theta} = \widehat{H}(\theta|t_i), \tag{B.11}$$

and Equation (B.8) then implies that

$$\widetilde{M}_S(\theta-\delta(t_i)) = \frac{\int_{\widehat{\mathcal{S}}(\theta|t_i)} s\widehat{F}\left(\theta-\gamma s|t_i\right)g(s)ds}{\widehat{H}(\theta|t_i)} = \widehat{\mathbb{E}}[S_j|\theta_j \le \theta; t_i]. \tag{B.12}$$

Since log-concavity of f and g implies that $\widetilde{M}_S$ is increasing, $\widehat{\mathbb{E}}[S_j|\theta_j \le \theta; t_i]$ is therefore decreasing in t_i.

Part 2. Notice that Player i believes the CDF of $\theta_{i,1}$ is $\widehat{H}(\theta|t_i)^{N-1}$, and hence

$$\widehat{\mathbb{E}}[\theta_{i,1}|\theta_{i,1} \le \theta; t_i] = (N-1)\int_{\underline{t}(t_i)+\gamma\underline{s}}^{\theta} \tilde{\theta}\frac{\hat{h}(\tilde{\theta}|t_i)\widehat{H}(\tilde{\theta}|t_i)^{N-2}}{\widehat{H}(\theta|t_i)^{N-1}}d\tilde{\theta}. \tag{B.13}$$

Let $M_\theta(\theta) \equiv \widehat{\mathbb{E}}[\theta_{i,1}|\theta_{i,1} \le \theta; \underline{t}]$ and note that Equation (B.13) along with (B.10) and (B.11) yields

$$\begin{aligned}M_\theta(\theta-\delta(t_i)) &= (N-1)\int_{\underline{t}+\gamma\underline{s}}^{\theta-\delta(t_i)} \tilde{\theta}\frac{\hat{h}(\tilde{\theta}|\underline{t})\widehat{H}(\tilde{\theta}|\underline{t})^{N-2}}{\widehat{H}(\theta-\delta(t_i)|\underline{t})^{N-1}}d\tilde{\theta} \\ &= (N-1)\int_{\alpha t_i+(1-\alpha)\underline{t}+\gamma\underline{s}}^{\theta} (\tilde{\theta}-\delta(t_i))\frac{\hat{h}(\tilde{\theta}-\delta(t_i)|\underline{t})\widehat{H}(\tilde{\theta}-\delta(t_i)|\underline{t})^{N-2}}{\widehat{H}(\theta-\delta(t_i)|\underline{t})^{N-1}}d\tilde{\theta} \\ &= (N-1)\int_{\underline{t}(t_i)+\gamma\underline{s}}^{\theta} (\tilde{\theta}-\delta(t_i))\frac{\hat{h}(\tilde{\theta}|t_i)\widehat{H}(\tilde{\theta}|t_i)^{N-2}}{\widehat{H}(\theta|t_i)^{N-1}}d\tilde{\theta} = \widehat{\mathbb{E}}[\theta_{i,1}|\theta_{i,1} \le \theta; t_i] - \delta(t_i),\end{aligned}$$

and thus

$$\widehat{\mathbb{E}}[\theta_{i,1}|\theta_{i,1} \le \theta; t_i] = M_\theta(\theta-\delta(t_i)) + \delta(t_i). \tag{B.14}$$

While Equation (B.14) will be useful in later proofs, it is not enough to establish that $\widehat{\mathbb{E}}[\theta_{i,1}|\theta_{i,1} \le \theta; t_i]$ is increasing in t_i. From Equation (B.13), this result follows if $\widehat{H}(\theta|t_i)^{N-1}$ conditionally stochastically dominates $\widehat{H}(\theta|t_i')^{N-1}$ for all $t_i' < t_i$; that is, for each $\theta \in \widehat{\Theta}(t_i) \cap \widehat{\Theta}(t_i')$, we have $\widehat{H}(\tilde{\theta}|t_i)/\widehat{H}(\theta|t_i) \le \widehat{H}(\tilde{\theta}|t_i')/\widehat{H}(\theta|t_i')$ for all $\tilde{\theta} \le \theta$ and strictly so for some $\tilde{\theta}$. It is well known that conditional stochastic dominance holds if and only if $\hat{h}(\theta|t_i)/\widehat{H}(\theta|t_i) \ge \hat{h}(\theta|t_i')/\widehat{H}(\theta|t_i')$ for all $\theta \in \widehat{\Theta}(t_i) \cap \widehat{\Theta}(t_i')$ and strictly so for some θ. From equations (B.10) and (B.11), the previous condition is equivalent to

$$\frac{\hat{h}(\tilde{\theta} - \delta(t_i)|\underline{t})}{\widehat{H}(\theta - \delta(t_i)|\underline{t})} \geq \frac{\hat{h}(\tilde{\theta} - \delta(t_i')|\underline{t})}{\widehat{H}(\theta - \delta(t_i')|\underline{t})}, \tag{B.15}$$

for all $\theta \in \widehat{\Theta}(t_i) \cap \widehat{\Theta}(t_i')$. Since $\delta(t_i) > \delta(t_i')$, Condition (B.15) holds for all such θ if $\hat{h}(x|\underline{t})/\widehat{H}(x|\underline{t})$ is decreasing in x. This is indeed the case since $\hat{h}(x|\underline{t})$ is log-concave given that it is the density of the convolution of two independent random variables that each have log-concave densities. ■

Proof of Proposition 7. Let $x = (t_1, s_1, t_2, s_2, \ldots, t_N, s_N) \in \mathcal{X} = (\mathcal{T} \times \mathcal{S})^N$ denote the vector of all players' private values and signals. Without loss of generality, normalize $\underline{t} = 0$ and let $t_1 > \max_{i \neq 1} t_i$—i.e., Player 1 is the efficient winner—and let $\mathcal{X}_1 \equiv \{x \in \mathcal{X} | t_1 > \max_{i \neq 1} t_i\}$.

Part 1. For all $\alpha \in [0, 1]$, we partition $\mathcal{X}_1$ into two non-empty subsets: $\mathcal{W}(\alpha) \equiv \{x \in \mathcal{X}_1 | \widehat{\beta}_{II}(\theta_1|t_1) > \max_{i \neq 1} \widehat{\beta}_{II}(\theta_i|t_i)\}$ and $\mathcal{L}(\alpha) \equiv \{x \in \mathcal{X}_1 | \widehat{\beta}_{II}(\theta_1|t_1) < \max_{i \neq 1} \widehat{\beta}_{II}(\theta_i|t_i)\}$. $\mathcal{W}(\alpha)$ contains all realizations where the SPA is efficient (because Player 1 wins), and $\mathcal{L}(\alpha)$ contains all those where it is not.

We first show that, in the SPA, projection preserves inefficient outcomes under rational bidding; that is, $\mathcal{L}(0) \subseteq \mathcal{L}(\alpha)$ whenever $\alpha > 0$. Let $x \in \mathcal{L}(0)$, which implies that there exists $j \neq 1$ such that $\theta_1 < \theta_j$. Fixing $\alpha > 0$, by Equation (12) Player i bids $\widehat{\beta}_{II}(\theta_i|t_i) = \theta_i + \gamma M_S(\theta_i - \delta(t_i)) + \gamma(N - 2)\widetilde{M}_S(\theta_i - \delta(t_i))$, where $\delta(t_i) \equiv \alpha(t_i - \underline{t})$ and $M_S(\cdot)$ and $\widetilde{M}_S(\cdot)$ are defined in Equations (B.5) and (B.12), respectively. Thus, since $\delta(t_i) = \alpha t_i$ given $\underline{t} = 0$, we have $\widehat{\beta}_{II}(\theta_1|t_1) < \widehat{\beta}_{II}(\theta_j|t_j) \Leftrightarrow$

$$\theta_1 - \theta_j < \gamma[M_S(\theta_j - \alpha t_j) - M_S(\theta_1 - \alpha t_1)] + \gamma(N-2)[\widetilde{M}_S(\theta_j - \alpha t_j) - \widetilde{M}_S(\theta_1 - \alpha t_1)]. \tag{B.16}$$

This condition holds because the left-hand side is negative, and the right-hand side is positive since M_S and $\widetilde{M}_S$ are increasing, $\theta_1 < \theta_j$, and $t_1 > t_j$. Thus, $x \in \mathcal{L}(\alpha)$ as desired.

We now show that an inefficient outcome in the SPA is more likely with projection because $\mathcal{W}(0) \cap \mathcal{L}(\alpha)$ has positive measure. Fix $\bar{x} = (\bar{t}_1, \bar{s}_1, \ldots, \bar{t}_N, \bar{s}_N)$ such that: (i) $\bar{x} \in \mathcal{X}_1$; (ii) for some j, $\bar{\theta}_1 \equiv \bar{t}_1 + \gamma \bar{s}_1 = \bar{t}_j + \gamma \bar{s}_j \equiv \bar{\theta}_j$; and (iii) $\bar{\theta}_k < \bar{\theta}_1$ for all $k \neq 1, j$. Let $\bar{x}(\varepsilon)$ be a vector of types identical to $\bar{x}$ except Player j's signal is $s_j = \bar{s}_j - \varepsilon/\gamma$ for some $\varepsilon \geq 0$. At $\bar{x}(\varepsilon)$, Player j's aggregate type is $\bar{\theta}_1 - \varepsilon$, and thus $\bar{x}(\varepsilon) \in \mathcal{W}(0)$. Furthermore, $\bar{x}(\varepsilon) \in \mathcal{L}(\alpha)$ if Player j outbids Player 1 at $\bar{x}(\varepsilon)$. From (B.16), this happens if and only if

$$\varepsilon < \gamma \left[M_S\left(\bar{\theta}_1 - \varepsilon - \alpha \bar{t}_j\right) - M_S\left(\bar{\theta}_1 - \alpha \bar{t}_1\right)\right] + \gamma(N-2)\left[\widetilde{M}_S\left(\bar{\theta}_1 - \varepsilon - \alpha \bar{t}_j\right) - \widetilde{M}_S\left(\bar{\theta}_1 - \alpha \bar{t}_1\right)\right]. \tag{B.17}$$

When $\varepsilon = 0$, this inequality holds since $\bar{t}_j < \bar{t}_1$. Furthermore, since the right-hand side of (B.17) is continuously decreasing in ε, it is immediate that there is an open set $\mathcal{E}$ of $\varepsilon > 0$ sufficiently small such that Condition (B.17) holds at $\bar{x}(\varepsilon)$ for all $\varepsilon \in \mathcal{E}$. Hence, $\bar{x}(\varepsilon) \in \mathcal{W}(0) \cap \mathcal{L}(\alpha)$ for $\varepsilon \in \mathcal{E}$. Furthermore, for $\varepsilon \in \mathcal{E}$, all perturbations of $\bar{x}(\varepsilon)$ that change the signals and tastes of Players $k \neq 1, j$, yet preserve the assumption that Player 1 has the highest taste and aggregate type, are also in $\mathcal{W}(0) \cap \mathcal{L}(\alpha)$. Thus, $\mathcal{W}(0) \cap \mathcal{L}(\alpha)$ has positive measure.

Part 2. In Part 1, the proof that the SPA is efficient less often under projection than under rational bidding follows entirely from the fact that $\widehat{\beta}_{II}(\theta_i|t_i)$ is decreasing in t_i holding θ_i fixed. Analogously, if $\widehat{\beta}_I(\theta_i|t_i)$ is *increasing* in t_i holding θ_i fixed, then a symmetric argument (with the appropriate swapping of signs) implies that the FPA is efficient *more often* under projection than under rational bidding. By Equations (13), (B.14), and (B.12), in the FPA Player i bids $\widehat{\beta}_I(\theta_i|t_i) = M_\theta(\theta_i - \alpha t_i) + \alpha t_i + \gamma(N-1)\widetilde{M}_S(\theta_i - \alpha t_i)$ and

$$\frac{\partial}{\partial t_i}\widehat{\beta}_I(\theta_i|t_i) = -\alpha[M'_\theta(\theta_i - \alpha t_i) + \gamma(N-1)\widetilde{M}'_S(\theta_i - \alpha t_i)] + \alpha.$$

Since M_θ and $\widetilde{M}_S$ are increasing, this derivative is positive if and only if

$$\gamma < \frac{1 - M'_\theta(\theta_i - \alpha t_i)}{(N-1)\widetilde{M}'_S(\theta_i - \alpha t_i)}. \tag{B.18}$$

Using Equation (B.14), notice that $M'_\theta(\theta - \alpha t_i) < 1$ for all θ since $\frac{\partial}{\partial t_i}\widehat{\mathbb{E}}[\theta_{i,1}|\theta_{i,1} \leq \theta; t_i] > 0$ by Lemma 1. Thus, the right-hand side of Condition (B.18) is positive. Let

$$\bar{\gamma} \equiv \min_{\theta \in \widehat{\Theta}(\underline{t})} \frac{1 - M'_\theta(\theta)}{(N-1)\widetilde{M}'_S(\theta)} > 0. \tag{B.19}$$

It thus follows that, if $\gamma < \bar{\gamma}$, then $\widehat{\beta}_I(\theta_i, t_i)$ is increasing in t_i at all $(t_i, s_i) \in T \times S$ (when holding θ_i fixed), and hence the FPA is more efficient under projection than under rational bidding.

Part 3. Adopting the notation from the proof of Part 1, we first show that if the FPA is inefficient at $x \in \mathcal{X}_1$, then the SPA is also inefficient at x. From Equation (13), Player j outbids Player 1 in the FPA if and only if

$$\alpha(t_1 - t_j) - [M_\theta(\theta_j - \alpha t_j) - M_\theta(\theta_1 - \alpha t_1)] < \gamma(N-1)[\widetilde{M}_S(\theta_j - \alpha t_j) - \widetilde{M}_S(\theta_1 - \alpha t_1)]. \tag{B.20}$$

Hence, since M_θ and $\widetilde{M}_S$ are increasing, a necessary condition for the FPA to be inefficient is that, for some $j \neq 1$,

$$\theta_1 - \alpha t_1 < \theta_j - \alpha t_j. \tag{B.21}$$

Furthermore, if $\theta_j > \theta_1$ for some $j \neq 1$, then the SPA is inefficient (because in this case the SPA is inefficient with rational bidders and, hence, it is also inefficient with projection by Part 1). Therefore, it suffices to show that if $x \in \mathcal{X}_1$ and Condition (B.21) holds for some $j \neq 1$ with

$\theta_1 > \theta_j$, then inefficiency in the FPA implies inefficiency in the SPA.

Since $M'_\theta(\theta) < 1$ for all $\theta \in \widehat{\Theta}(\underline{t})$ (as noted in the proof of Part 2), $M_\theta(\theta_j - \alpha t_j) - M_\theta(\theta_1 - \alpha t_1) < (\theta_j - \alpha t_j) - (\theta_1 - \alpha t_1)$. Applying this bound to the left-hand side of Condition (B.20) implies that a necessary condition for inefficiency in the FPA is

$$\theta_1 - \theta_j < \gamma(N-1)[\widetilde{M}_S(\theta_j - \alpha t_j) - \widetilde{M}_S(\theta_1 - \alpha t_1)]. \tag{B.22}$$

Moreover, from Equation (12), the SPA is inefficient if and only if, for some $j \neq 1$,

$$\theta_1 - \theta_j < \gamma[M_S(\theta_j - \alpha t_j) - M_S(\theta_1 - \alpha t_1)] + \gamma(N-2)[\widetilde{M}_S(\theta_j - \alpha t_j) - \widetilde{M}_S(\theta_1 - \alpha t_1)]. \tag{B.23}$$

Thus, the SPA is necessarily inefficient at any $x \in \mathcal{X}_1$ where the FPA is inefficient if $M_S(\theta_j - \alpha t_j) - M_S(\theta_1 - \alpha t_1) > \widetilde{M}_S(\theta_j - \alpha t_j) - \widetilde{M}_S(\theta_1 - \alpha t_1)$. This condition holds because (i) we are considering $\theta_j - \alpha t_j > \theta_1 - \alpha t_1$ (since B.21 must hold) and (ii) $M_S(y) - \widetilde{M}_S(y)$ is increasing (by the assumption that $\mu(x) \equiv \mathbb{E}[S_j|\theta_j = \theta] - \mathbb{E}[S_j|\theta_j \leq \theta]$ is increasing). Hence, the necessary condition for inefficiency in the FPA, Condition (B.22), implies inefficiency in the SPA. Finally, since Condition (B.22) is not generically sufficient for inefficiency in the FPA, the FPA strictly outperforms the SPA in terms of efficiency. ■

Proof of Lemma 2. From Equation (B.5), $\widehat{\mathbb{E}}[S_d|\theta_d = \tilde{\theta}^i_d; t_i] = M_S(\tilde{\theta}^i_d - \delta(t_i))$. Therefore, we will show that, fixing $(p_1, \ldots, p_d)$, $t_j < t_i$ implies that $M_S(\tilde{\theta}^j_d - \delta(t_j)) > M_S(\tilde{\theta}^i_d - \delta(t_i))$. Recall that, for each $d \in \{1, \ldots, N-1\}$, $\tilde{\theta}^i_d$ is defined recursively as follows: initially, $\tilde{\theta}^i_1$ solves

$$p_1 = \widehat{\beta}_0(\tilde{\theta}^i_1|t_i) = \tilde{\theta}^i_1 + \gamma(N-1)M_S(\tilde{\theta}^i_1 - \delta(t_i)), \tag{B.24}$$

and then for $d > 1$, $\tilde{\theta}^i_d$ solves

$$p_d = \widehat{\beta}_{d-1}(\tilde{\theta}^i_d; p_1, \ldots, p_{d-1}|t_i) = \tilde{\theta}^i_d + \gamma(N-d)M_S(\tilde{\theta}^i_d - \delta(t_i)) + \gamma \textstyle\sum_{d'=1}^{d-1} M_S(\tilde{\theta}^i_{d'} - \delta(t_i)). \tag{B.25}$$

For any integer $d \geq 1$, define the function $m_d(x) \equiv x + \gamma(N-d)M_S(x)$, which is strictly increasing in x and hence invertible. This implies that (B.25) can be written as

$$\begin{aligned} & p_d = m_d(\tilde{\theta}^i_d - \delta(t_i)) + \delta(t_i) + \gamma \textstyle\sum_{d'=1}^{d-1} M_S(\tilde{\theta}^i_{d'} - \delta(t_i)) \\ \Leftrightarrow \quad & \tilde{\theta}^i_d = m_d^{-1}\left(p_d - \delta(t_i) - \gamma \textstyle\sum_{d'=1}^{d-1} M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right) + \delta(t_i). \end{aligned}$$

This inverse is well-defined given our assumption of full-support signals. Thus

$$M_S(\tilde{\theta}^j_d - \delta(t_j)) - M_S(\tilde{\theta}^i_d - \delta(t_i)) = M_S\left(m_d^{-1}\left(p_d - \delta(t_j) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}^j_{d'} - \delta(t_j))\right)\right)$$
$$-M_S\left(m_d^{-1}\left(p_d - \delta(t_i) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right)\right). \tag{B.26}$$

Since $M_S \circ m_d^{-1}$ is increasing, the difference above is positive if and only if

$$p_d - \delta(t_j) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}^j_{d'} - \delta(t_j)) > p_d - \delta(t_i) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}^i_{d'} - \delta(t_i))$$
$$\Leftrightarrow \quad \delta(t_i) - \delta(t_j) > \sum_{d'=1}^{d-1} \gamma\left(M_S(\tilde{\theta}^j_{d'} - \delta(t_j)) - M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right). \tag{B.27}$$

When $d = 1$, Condition (B.27) trivially holds if $t_i > t_j$, because the sum terms vanish and $\delta(t_i) - \delta(t_j) > 0$. Hence, to complete the proof we need to show that Condition (B.27) holds for $d \in \{2, \ldots, N-1\}$ given $t_i > t_j$. To do this, we prove by induction that, for $d \geq 2$,

$$\sum_{d'=1}^{d-1} \gamma\left(M_S(\tilde{\theta}^j_{d'} - \delta(t_j)) - M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right) < \frac{d-1}{N-1}(\delta(t_i) - \delta(t_j)), \tag{$*$}$$

which implies Condition (B.27).

Base Case: $d = 2$. We will show that $\gamma M_S(\tilde{\theta}^j_1 - \delta(t_j)) - \gamma M_S(\tilde{\theta}^i_1 - \delta(t_i)) < \frac{1}{N-1}(\delta(t_i) - \delta(t_j))$. Define the function $Z_d(x) \equiv \gamma M_S(m_d^{-1}(x))$. Hence,

$$\begin{aligned} \frac{d}{dx} Z_d(x) &= \gamma M_S'(m_d^{-1}(x)) \frac{d}{dx} m_d^{-1}(x) = \frac{\gamma M_S'(m_d^{-1}(x))}{1 + \gamma(N-d) M_S'(m_d^{-1}(x))} \\ &= \frac{1}{N - d + \left(\gamma M_S'(m_d^{-1}(x))\right)^{-1}}, \end{aligned} \tag{B.28}$$

where we have used $\frac{d}{dx} m_d^{-1}(x) = \left(m_d'(m_d^{-1}(x))\right)^{-1}$ and $m_d'(x) = 1 + \gamma(N-d) M_S'(x)$. Note that $\left(\gamma M_S'(m_d^{-1}(x))\right)^{-1} > 0$ since M_S' is positive. Thus (B.28) implies that $Z_d'(x) < \frac{1}{N-d}$. Therefore, from Equation (B.26), we have

$$\gamma M_S(\tilde{\theta}^j_1 - \delta(t_j)) - \gamma M_S(\tilde{\theta}^i_1 - \delta(t_i)) =$$
$$\gamma M_S\left(m_d^{-1}\left(p_1 - \delta(t_j)\right)\right) - \gamma M_S\left(m_d^{-1}\left(p_1 - \delta(t_i)\right)\right) = Z_1\left(p_1 - \delta(t_j)\right) - Z_1\left(p_1 - \delta(t_i)\right)$$
$$< \tfrac{1}{N-1}\left((p_1 - \delta(t_j)) - (p_1 - \delta(t_i))\right) = \tfrac{1}{N-1}(\delta(t_i) - \delta(t_j)).$$

Induction Step: We show that if Condition ($*$) holds for $d > 2$, then it holds for $d+1$. Note that

$$\sum_{d'=1}^{d} \gamma \left(M_S(\tilde{\theta}^j_{d'} - \delta(t_j)) - M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right) = \sum_{d'=1}^{d-1} \gamma \left(M_S(\tilde{\theta}^j_{d'} - \delta(t_j)) - M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right) + \gamma \left(M_S(\tilde{\theta}^j_{d} - \delta(t_j)) - M_S(\tilde{\theta}^i_{d} - \delta(t_i))\right). \tag{B.29}$$

Following the same approach as in the base case and using Equation (B.26), we can write the second term on the right-hand side of Equation (B.29) as:

$$\begin{aligned}\gamma \left(M_S(\tilde{\theta}^j_{d} - \delta(t_j)) - M_S(\tilde{\theta}^i_{d} - \delta(t_i))\right) &= \gamma M_S \left(m_d^{-1}\left(p_d - \delta(t_j) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}^j_{d'} - \delta(t_j))\right)\right) \\ &\quad - \gamma M_S \left(m_d^{-1}\left(p_d - \delta(t_i) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right)\right) \\ &= Z_d\left(p_d - \delta(t_j) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}^j_{d'} - \delta(t_j))\right) - Z_d\left(p_d - \delta(t_i) - \gamma \sum_{d'=1}^{d-1} M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right) \\ &< \frac{1}{N-d}\left(\delta(t_i) - \delta(t_j) - \sum_{d'=1}^{d-1} \gamma \left(M_S(\tilde{\theta}^j_{d'} - \delta(t_j)) - M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right)\right).\end{aligned}$$

Applying this bound to Equation (B.29) reveals that

$$\begin{aligned}\sum_{d'=1}^{d} \gamma \left(M_S(\tilde{\theta}^j_{d'} - \delta(t_j)) - M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right) &< \sum_{d'=1}^{d-1} \gamma \left(M_S(\tilde{\theta}^j_{d'} - \delta(t_j)) - M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right) \\ &\quad + \frac{1}{N-d}\left(\delta(t_i) - \delta(t_j) - \sum_{d'=1}^{d-1} \gamma \left(M_S(\tilde{\theta}^j_{d'} - \delta(t_j)) - M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right)\right) \\ &= \frac{1}{N-d}(\delta(t_i) - \delta(t_j)) + \frac{N-d-1}{N-d} \sum_{d'=1}^{d-1} \gamma \left(M_S(\tilde{\theta}^j_{d'} - \delta(t_j)) - M_S(\tilde{\theta}^i_{d'} - \delta(t_i))\right) \\ &< \frac{1}{N-d}(\delta(t_i) - \delta(t_j)) + \frac{N-d-1}{N-d}\left(\frac{d-1}{N-1}\right)(\delta(t_i) - \delta(t_j)) = \frac{d}{N-1}(\delta(t_i) - \delta(t_j)),\end{aligned}$$

where the final inequality follows from the induction assumption (i.e., Condition ($*$) holds for d). ■

Proof of Proposition 8. Fixing $\alpha > 0$, we will show that the English auction is less efficient than the SPA by proving that (i) for any realization of bidders' types where the SPA is inefficient, the English auction is also inefficient, and (ii) there is a positive measure of realizations such that the English auction is inefficient but the SPA is not. Given Parts 1 and 3 of Proposition 7, this will additionally imply that (i) the English auction with projection is less efficient than the English auction with rational bidders, and (ii) under projection, the English auction is less efficient than the FPA.

Assume $S_i \sim N(\mu, \rho^2)$ and $T_i \sim N(0, \sigma^2)$. It readily follows that, under projection, Player i's expectation of γS_j conditional on $\theta_j = \theta$ is

$$\widehat{\mathbb{E}}[\gamma S_j | \theta_j = \theta; t_i] = \lambda\left(\theta - \alpha t_i\right) + (1-\lambda)\gamma\mu, \tag{B.30}$$

where

$$\lambda \equiv \frac{\gamma^2\rho^2}{\gamma^2\rho^2 + (1-\alpha)^2\sigma^2} \in (0,1). \tag{B.31}$$

Substituting Equation (B.30) into the bidding strategies in (14) yields:

$$\begin{aligned}\widehat{\beta}_D(\theta_i;\cdot|t_i) &= \theta_i + (N-1-D)\left[\lambda\left(\theta_i - \alpha t_i\right) + (1-\lambda)\gamma\mu\right] + \sum_{d=1}^{D}\left[\lambda\left(\tilde{\theta}_d^i - \alpha t_i\right) + (1-\lambda)\gamma\mu\right] \\ &= \theta_i\left[1 + \lambda(N-1-D)\right] + (N-1)\left[(1-\lambda)\gamma\mu - \lambda\alpha t_i\right] + \sum_{d=1}^{D}\lambda\tilde{\theta}_d^i, \end{aligned} \tag{B.32}$$

where each $\tilde{\theta}_d^i$ solves $p_d = \widehat{\beta}_{d-1}(\tilde{\theta}_d^i;\cdot|t_i)$ and hence

$$\tilde{\theta}_d^i = \frac{p_d + (N-1)\left[\lambda\alpha t_i - (1-\lambda)\gamma\mu\right] - \sum_{d'=1}^{d-1}\lambda\tilde{\theta}_{d'}^i}{1+\lambda(N-d)}. \tag{B.33}$$

We first show that, in the English auction, the sum of the differences between two players' inferences about the aggregate types of competitors who have dropped out is equal to the initial difference in their inferences about this type scaled by the number of players who have dropped out; i.e.,

$$\sum_{d=1}^{D}\left(\tilde{\theta}_d^j - \tilde{\theta}_d^i\right) = D\left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right), \tag{B.34}$$

for all N and $D \leq N-1$. The proof follows from induction on D.

Base Case: $D=2$. We will show that $\sum_{d=1}^{2}\left(\tilde{\theta}_d^j - \tilde{\theta}_d^i\right) = 2\left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right)$. From Equation (B.33),

$$\tilde{\theta}_d^i - \tilde{\theta}_d^j = \frac{(N-1)\lambda\alpha\left(t_i - t_j\right) - \sum_{d'=1}^{d-1}\lambda\left(\tilde{\theta}_{d'}^i - \tilde{\theta}_{d'}^j\right)}{1+\lambda(N-d)}. \tag{B.35}$$

Hence,

$$\tilde{\theta}_1^j - \tilde{\theta}_1^i = \frac{(N-1)\lambda\alpha\left(t_j - t_i\right)}{1+\lambda(N-1)}, \tag{B.36}$$

and

$$\begin{aligned}\sum_{d=1}^{2}\left(\tilde{\theta}_d^j - \tilde{\theta}_d^i\right) &= \frac{(N-1)\lambda\alpha\left(t_j - t_i\right) - \lambda\left(\tilde{\theta}_1^i - \tilde{\theta}_1^j\right)}{1+\lambda(N-2)} + \left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right) \\ &= \frac{1+\lambda(N-1)}{1+\lambda(N-2)}\left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right) + \frac{1+\lambda(N-3)}{1+\lambda(N-2)}\left(\tilde{\theta}_1^i - \tilde{\theta}_1^j\right) = 2\left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right).\end{aligned}$$

Induction Step: Suppose that $\sum_{d=1}^{D}\left(\tilde{\theta}_d^j - \tilde{\theta}_d^i\right) = D\left(\tilde{\theta}_1^j - \tilde{\theta}_1^i\right)$. Using Equations (B.35) and

(B.36), we have that

$$\begin{aligned}\sum_{d=1}^{D+1}\left(\tilde{\theta}_d^j-\tilde{\theta}_d^i\right) &= D\left(\tilde{\theta}_1^j-\tilde{\theta}_1^i\right)+\left(\tilde{\theta}_{D+1}^j-\tilde{\theta}_{D+1}^i\right)\\ &= D\left(\tilde{\theta}_1^j-\tilde{\theta}_1^i\right)+\frac{(N-1)\lambda\alpha\left(t_j-t_i\right)-\lambda D\left(\tilde{\theta}_1^j-\tilde{\theta}_1^i\right)}{1+\lambda(N-D-1)}\\ &= \frac{D+1+(D+1)\lambda(N-D-1)}{1+\lambda(N-D-1)}\left(\tilde{\theta}_1^j-\tilde{\theta}_1^i\right)=(D+1)\left(\tilde{\theta}_1^j-\tilde{\theta}_1^i\right),\end{aligned}$$

which completes the induction step.

We now prove that, in the English auction, the ranking of bidders' drop-out prices remains fixed as the auction unfolds; i.e., for all $D<N-1$, we have $\widehat{\beta}_0(\theta_j|t_j)>\widehat{\beta}_0(\theta_i|t_i)$ if and only if $\widehat{\beta}_D(\theta_j;p_1,\ldots,p_D|t_j)>\widehat{\beta}_D(\theta_i;p_1,\ldots,p_D|t_i)>0$. This implies that the final winner of the auction is the bidder who plans to bid higher at the beginning of the auction, before any bidder drops out.

From Equation (B.32), notice that $\widehat{\beta}_0(\theta_j|t_j)>\widehat{\beta}_0(\theta_i|t_i)$ if and only if

$$(1+\lambda(N-1))\left(\theta_j-\theta_i\right)+\lambda(N-1)\alpha\left(t_i-t_j\right)>0. \tag{B.37}$$

Now consider $0<D<N-1$. Using Equations (B.32), (B.34), and (B.36), we have $\widehat{\beta}_D(\theta_j;p_1,\ldots,p_D|t_j)>\widehat{\beta}_D(\theta_i;p_1,\ldots,p_D|t_i)$ if and only if

$$\begin{aligned}&(1+\lambda(N-D-1))(\theta_j-\theta_i)+\lambda(N-1)\alpha\left(t_i-t_j\right)+\lambda\underbrace{\sum_{d=1}^{D}\left(\tilde{\theta}_d^j-\tilde{\theta}_d^i\right)}_{=D\left(\tilde{\theta}_1^j-\tilde{\theta}_1^i\right)}>0\\ \Leftrightarrow\quad &(1+\lambda(N-D-1))(\theta_j-\theta_i)+\lambda(N-1)\alpha\left(t_i-t_j\right)+\lambda D\left(\frac{\lambda(N-1)\alpha\left(t_j-t_i\right)}{1+\lambda(N-1)}\right)>0\\ \Leftrightarrow\quad &(1+\lambda(N-D-1))\left[(\theta_j-\theta_i)+\frac{\lambda(N-1)\alpha\left(t_i-t_j\right)}{1+\lambda(N-1)}\right]>0,\end{aligned}$$

which holds if and only if condition (B.37) is satisfied.

To complete the proof, we show that if Bidder i is the efficient winner and $\widehat{\beta}_{II}(\theta_j|t_j)>\widehat{\beta}_{II}(\theta_i|t_i)$, then $\widehat{\beta}_0(\theta_j|t_j)>\widehat{\beta}_0(\theta_i|t_i)$; i.e., if the efficient bidder loses a second-price auction, then he does not have the highest drop-out price at the beginning of an English auction and, given the result above, he thus loses the English auction as well. From Equation (12), and defining $M_S(\cdot)$ and $\widetilde{M}_S(\cdot)$ as in the proof of Lemma 1, we have that $\widehat{\beta}_{II}(\theta_j|t_j)>\widehat{\beta}_{II}(\theta_i|t_i)$ if and only if

$$\theta_i-\theta_j-\gamma[M_S(\theta_j-\delta(t_j))-M_S(\theta_i-\delta(t_i))]<\gamma(N-2)[\widetilde{M}_S(\theta_j-\delta(t_j))-\widetilde{M}_S(\theta_i-\delta(t_i))]. \tag{B.38}$$

Similarly, from Equation (14), $\widehat{\beta}_0(\theta_j|t_j) > \widehat{\beta}_0(\theta_i|t_i)$ if and only if

$$\theta_i - \theta_j - \gamma[M_S(\theta_j - \delta(t_j)) - M_S(\theta_i - \delta(t_i))] < \gamma(N-2)[M_S(\theta_j - \delta(t_j)) - M_S(\theta_i - \delta(t_i))].$$

Hence, the former condition (B.38) implies the latter whenever

$$M_S(\theta_j - \delta(t_j)) - M_S(\theta_i - \delta(t_i)) > \widetilde{M}_S(\theta_j - \delta(t_j)) - \widetilde{M}_S(\theta_i - \delta(t_i)), \tag{B.39}$$

which holds whenever $\theta_j - \delta(t_j) > \theta_i - \delta(t_i)$ due to our assumption that $\mu(x) \equiv \mathbb{E}[S_j|\theta_j = \theta] - \mathbb{E}[S_j|\theta_j \leq \theta]$ is increasing. Given that inefficiency in the SPA requires $\theta_j - \delta(t_j) > \theta_i - \delta(t_i)$, we have thus established that, with projection, the English auction is always inefficient when the SPA is. Finally, the English auction is strictly less efficient than the SPA since inefficiency in the SPA (Condition B.38) is sufficient for inefficiency in the English auction but not necessary given that (B.39) strictly holds. ■

C Asymmetric Auctions

In this section, we derive the bidding strategies reported in Section VI.A.

Example 1. Given his perception of the strong bidder's strategy, a weak bidder with value t expects to always lose the auction and, therefore, it is a best response for him to bid his value; that is, $\widehat{\beta}_W(t) = t$. Hence, we only need to prove that, when the weak bidder bids his value, a strong bidder with value t is willing to bid $\widehat{\beta}_S(t) = (1-\alpha)(\underline{t}+k) + \alpha t$—his perception of the highest possible value of a weak bidder—in order to always win. The strong bidder solves:

$$\max_{b_S} \frac{b_S - [(1-\alpha)\underline{t} + \alpha t]}{k(1-\alpha)}(t - b_S).$$

The FOC yields

$$b_S = \frac{t(1+\alpha) + (1-\alpha)\underline{t}}{2}.$$

This is weakly higher than $(1-\alpha)(\underline{t}+k) + \alpha t$, for any $t \geq \underline{t} + 2k$.

Example 2. Changing notation for convenience, let the weak and strong bidder's valuations be distributed uniformly on $[\underline{\omega}, \omega_W]$ and $[\underline{\omega}, \omega_S]$, respectively, where $\omega_S > \omega_W$. We first derive the BNE bidding strategies for these generic supports, and then modify them to obtain the NBE strategies. Following Maskin and Riley (2000), the equilibrium bidding functions are the solutions of the following system of differential equations

$$\phi_i'(b) = \frac{\frac{\phi_i(b)-\underline{\omega}}{\omega_i-\underline{\omega}}}{\frac{1}{\omega_i-\underline{\omega}}} \frac{1}{\phi_j(b)-b}, \quad i,j = W,S, \quad i \neq j, \tag{C.1}$$

where ϕ denotes the inverse bidding function. Simplifying and re-arranging yields

$$(\phi_i'(b)-1)(\phi_j(b)-b) = \phi_i(b) - \phi_j(b) + b - \underline{\omega}, \quad i,j = W,S, \quad i \neq j.$$

Adding these two differential equations and re-arranging yields

$$\frac{d}{db}\left\{(\phi_j(b)-b)(\phi_i(b)-b)\right\} = 2(b-\underline{\omega}),$$

and, integrating both sides, we obtain

$$(\phi_j(b)-b)(\phi_i(b)-b) = (b-\underline{\omega})^2. \tag{C.2}$$

(The constant of integration is zero since $\phi_i(\underline{\omega}) = \underline{\omega}$.) Now, substituting (C.2) into (C.1) yields

$$\phi_i'(b) = \frac{(\phi_i(b)-\underline{\omega})(\phi_i(b)-b)}{(b-\underline{\omega})^2}, \quad i = W,S. \tag{C.3}$$

In order to solve the differential equation (C.3), we use a change of variables. Let $\psi_i(b)$ be implicitly defined by

$$\phi_i(b) = b + \psi_i(b)(b-\underline{\omega}) \tag{C.4}$$

so that

$$\phi_i'(b) = \psi_i'(b)(b-\underline{\omega}) + \psi_i(b) + 1.$$

It then follows that the differential equation (C.3) can be re-written as

$$\psi_i'(b)(b-\underline{\omega}) + \psi_i(b) + 1 = \frac{(b-\underline{\omega})(\psi_i(b)+1)\psi_i(b)(b-\underline{\omega})}{(b-\underline{\omega})^2}$$

$$\Leftrightarrow \quad \psi_i'(b)(b-\underline{\omega}) = (\psi_i(b)+1)(\psi_i(b)-1) \quad \Leftrightarrow \quad \frac{\psi_i'(b)}{\psi_i(b)^2-1} = \frac{1}{b-\underline{\omega}},$$

whose solution can be easily verified to be

$$\psi_i(b) = \frac{1-k_i(b-\underline{\omega})^2}{1+k_i(b-\underline{\omega})^2},$$

where k_i is a constant of integration.[1]

Substituting $\psi_i(b)$ into (C.4) yields

$$\phi_i(b) = b + \frac{1 - k_i(b-\underline{\omega})^2}{1 + k_i(b-\underline{\omega})^2}(b-\underline{\omega}) = \frac{2b - \underline{\omega} + \underline{\omega}k_i(b-\underline{\omega})^2}{1 + k_i(b-\underline{\omega})^2}. \tag{C.5}$$

Since $\phi_i(b) = t$, solving for b yields the following equilibrium bidding functions:

$$\beta_i^*(t) = \underline{\omega} + \frac{1}{k_i(t-\underline{\omega})}\left(1 - \sqrt{1 - k_i(t-\underline{\omega})^2}\right), \quad i = W, S. \tag{C.6}$$

To find k_i, let $\bar{b}$ be the bid of the highest-value bidder. Since $\phi_i\left(\bar{b}\right) = \omega_i$, Equation (C.2) yields

$$\left(\omega_j - \bar{b}\right)\left(\omega_i - \bar{b}\right) = \left(\bar{b} - \underline{\omega}\right)^2 \quad \Leftrightarrow \quad \bar{b} = \frac{\omega_i\omega_j - \underline{\omega}^2}{\omega_i + \omega_j - 2\underline{\omega}}.$$

Hence, for $b = \bar{b}$, Equation (C.5) becomes

$$\begin{aligned}
\omega_i &= \frac{2\left(\frac{\omega_i\omega_j - \underline{\omega}^2}{\omega_i+\omega_j-2\underline{\omega}}\right) - \underline{\omega} + \underline{\omega}k_i\left(\frac{\omega_i\omega_j - \underline{\omega}^2}{\omega_i+\omega_j-2\underline{\omega}} - \underline{\omega}\right)^2}{1 + k_i\left(\frac{\omega_i\omega_j - \underline{\omega}^2}{\omega_i+\omega_j-2\underline{\omega}} - \underline{\omega}\right)^2} \\
\Leftrightarrow \quad k_i\left(\frac{\omega_i\omega_j - \underline{\omega}^2}{\omega_i + \omega_j - 2\underline{\omega}} - \underline{\omega}\right)^2(\omega_i - \underline{\omega}) &= \frac{\omega_j(\omega_i - \underline{\omega}) - \omega_i(\omega_i - \underline{\omega})}{\omega_i + \omega_j - 2\underline{\omega}} \\
\Leftrightarrow \quad k_i &= \frac{(\omega_j - \omega_i)(\omega_i + \omega_j - 2\underline{\omega})}{\left[\underline{\omega}(\omega_j + \omega_i) - (\omega_j\omega_i + \underline{\omega}^2)\right]^2}. \qquad \text{(C.7)}
\end{aligned}$$

From these BNE bidding functions, we can obtain the NBE bidding functions by replacing $\underline{\omega}$, ω_i, and ω_j with the appropriate expressions. Namely, replacing $\underline{\omega}$ with $\widehat{\underline{\omega}} = \alpha t$ and replacing ω_i and ω_j with $\widehat{\omega}_i = \alpha t + (1-\alpha)\omega_i$ and $\widehat{\omega}_j = \alpha t + (1-\alpha)\omega_j$, respectively, yields

$$\widehat{\beta}_i(t) = \alpha t + \frac{(1-\alpha)(\omega_i\omega_j)^2}{t\left(\omega_j^2 - \omega_i^2\right)}\left(1 - \sqrt{1 - \frac{\left(\omega_j^2 - \omega_i^2\right)t^2}{(\omega_i\omega_j)^2}}\right),$$

[1] Indeed, it is easy to verify that

$$\psi_i'(b) = -\frac{4k_i(b-\underline{\omega})}{\left[1 + k_i(b-\underline{\omega})^2\right]^2}$$

and

$$\frac{\psi_i'(b)}{\psi_i(b)^2 - 1} = \frac{-4k_i(b-\underline{\omega})}{\left[1 - k_i(b-\underline{\omega})^2\right]^2 - \left[1 + k_i(b-\underline{\omega})^2\right]^2} = \frac{1}{(b-\underline{\omega})}.$$

and

$$\widehat{\beta}_j(t) = \alpha t + \frac{(1-\alpha)(\omega_i\omega_j)^2}{t\left(\omega_i^2 - \omega_j^2\right)}\left(1 - \sqrt{1 - \frac{\left(\omega_i^2 - \omega_j^2\right)t^2}{(\omega_i\omega_j)^2}}\right).$$

For $\omega_i = \frac{1}{1-z}$ and $\omega_j = \frac{1}{1+z}$, we obtain the bidding strategies in the text.

D Example with Affiliated Private Values

The following example illustrates claims from Section VI.B. Namely, bidding under projection with IPV leads all types to overbid (relative to rational IPV benchmark), whereas rational bidding with APV leads high types to overbid and low types to under bid (again, relative to the rational IPV benchmark).

Suppose $N = 2$. Private values for each bidder have a marginal distribution $F(t) = .5t(t+1)$ over $\mathcal{T} = [0,1]$; hence, $f(t) = .5 + t$. First, consider the case where private values are independent across bidders. Under projection, bidder i with type t_i perceives the CDF of valuations as

$$\widehat{F}(t|t_i) = F\left(\frac{t-\alpha t_i}{1-\alpha}\right) = \frac{(t-\alpha t_i)(t-\alpha t_i + 1 - \alpha)}{2(1-\alpha)^2}. \tag{D.1}$$

Using Proposition 2, the NBE bidding function is

$$\widehat{\beta}_{IPV}(t_i) = \beta^*_{IPV}(t_i) + \alpha\left[\frac{2t_i^2 + 3t_i}{6(t_i+1)}\right], \tag{D.2}$$

where $\beta^*_{IPV}(t_i) = \frac{t_i(4t_i+3)}{6(t_i+1)}$ is the rational bidding function. It is immediate that $\widehat{\beta}_{IPV}(t) > \beta^*_{IPV}(t)$ for all $t > 0$ whenever $\alpha > 0$.

This "uniform overbidding" relative to the rational IPV benchmark does not emerge when valuations are affiliated and bidders are rational. To see this, now suppose that the joint distribution of valuations (consistent with the marginal distribution above) is $F(t_1,t_2) = \frac{1}{2}t_1t_2(t_1t_2+1)$. Then the posterior CDF of an opponent's valuation is $F(x|t) = \frac{x}{2t+1}(2xt+1)$, and the rational bidding function is

$$\beta^*_{APV}(t) = \int_0^t y\,d\left(e^{-\int_y^t \frac{1+4z^2}{z+2z^3}dz}\right) = \frac{(4t^2-1)\sqrt{2t^2+1}+1}{6t\sqrt{2t^2+1}}. \tag{D.3}$$

Importantly, one can show that $\beta^*_{APV}(t)$ crosses $\beta^*_{IPV}(t)$ only once and from below: there exists a $\bar{t} \in (0,1)$ such that $\beta^*_{APV}(t) < \beta^*_{IPV}(t)$ for $t \in (0,\bar{t})$ and $\beta^*_{APV}(t) > \beta^*_{IPV}(t)$ for $t \in (\bar{t},1)$.

References

Maskin, Eric, and John Riley. 2000. "Asymmetric Auctions." *Review of Economic Studies*, 67(3): 413–438.