Online Appendix for "Uncertainty and Business Cycles: Exogenous Impulse or Endogenous Response?"

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This Online Appendix provides the description of the sampling simulation used to construct confidence bands along with an additional result using 12-month uncertainty.

Sampling Simulation

In point-identified models, sampling uncertainty can be evaluated using frequentist confidence intervals or Bayesian credible regions, and they coincide asymptotically. Inference for set-identified SVARs is, however, more challenging because no consistent point estimate is available. As pointed out in Moon and Schorfheide (2012), the credible regions of Bayesian identified impulses responses will be distinctly different from the frequentist confidence sets, with the implication that Bayesian error bands cannot be interpreted as approximate frequentist error bands. Our analysis is frequentist, and while the two applications presented above illustrate how the dynamic responses vary across estimated models, where each model is evaluated at a solution in $\bar{\mathcal{B}}(\mathbf{B}; \bar{\mathbf{k}}, \bar{\tau}, \mathbf{S})$, we still need a way to assess the robustness of our procedure, especially since it is new to the literature.

Unfortunately, few methods are available to evaluate the sampling uncertainty of set identified SVARs from a frequentist perspective, and these tend to be specific to the imposition of particular identifying restrictions. Moon, Schorfheide and Granziera (2013) suggest a projections based method within a moment-inequality setup, but it is designed to study SVARs that only impose restrictions on one set of impulse response functions. Furthermore, the method is computationally intense, requiring a simulation of critical value for each rotation matrix. Gafarov, Meier and Olea (2015) suggest to collect parameters of the reduced form model in a $1 - \alpha$ Wald ellipsoid but the approach is conservative. For the method to get an exact coverage of $1 - \alpha$, the radius of the Wald-ellipsoid needs to be carefully calibrated. As discussed in Kilian and Lutkepohl (2016), even with these adjustments, existing frequentist confidence sets for set-identified models still tend to be too wide to be informative. It is fair to say that there exists no generally agreed upon method for conducting inference in set-identified SVARs.

We use a bootstrap/Monte Carlo procedure to assess the sampling error of our inequality restrictions when S_t and G_t are variables external to the three variable SVAR.

Let R be the number of replications in a repeated sampling experiment. Let "hats" denote estimated values from historical data, e.g., $\widehat{\mathbf{e}}_t$ denotes estimated structural shocks and $\widehat{\mathbf{B}}$ estimated structural covariance matrix. To denote simulated data, we use a "*", while to denote estimated values from simulated data, a "hat" is combined with a "*". To generate samples of the structural shocks from this solution in a way that ensures the events that appear in historical data also occur in our simulated samples, we draw randomly with replacement from the sample estimates of the shocks, $\widehat{\mathbf{e}}_t$, with the exception that we fix the values for these shocks in each replication in the periods $\bar{\tau}_1 \, \bar{\tau}_2$, $\bar{\tau}_3$, $\bar{\tau}_4$ and $\bar{\tau}_5$, where $\bar{\tau}_1$ is the period 1987:10 of the stock market crash, $\bar{\tau}_2$ is 1970:12, $\bar{\tau}_3 \in [2007:12, 2009:06]$, $\bar{\tau}_4$ is 1979:10 and $\bar{\tau}_5 \in [2011:07, 2011:08]$. Since we identify a set of estimated parameters $\widehat{\mathbf{B}}$ and therefore a set of estimated shocks $\widehat{\mathbf{e}}_t$, we generate R samples of data from each $\widehat{\mathbf{e}}_t$ in the set. This is then repeated for every solution/shock sequence in the identified set to obtain a confidence region for the identified set of impulse responses.

Let M be the number of solutions in the identified set $\bar{\mathcal{B}}(\mathbf{B}; \bar{\mathbf{k}}, \bar{\boldsymbol{\tau}}, \mathbf{S})$ and let m index an arbitrary solution in the set. Index each draw from the estimated shocks with r and denote the rth draw from the mth solution as $\mathbf{e}_t^{mr}$. Each $\mathbf{e}_t^{mr}$ is combined with the $\mathbf{B}$ parameters of the mth solution, $\widehat{\mathbf{B}}^m$ to generate R samples of size T of $\boldsymbol{\eta}_t^{mr*} = \widehat{\mathbf{B}}^m \mathbf{e}_t^{mr}$. Next, R new samples of $\mathbf{X}_t$ are recursively generated for each replication $r = 1, ..., R$ using $\mathbf{X}_t = \sum_{j=1}^p \widehat{\mathbf{A}}_j \mathbf{X}_{t-j} + \boldsymbol{\eta}_t^{mr*}$, with initial conditions fixed at their sample values, $[\mathbf{X}_{-p+1}, ..., \mathbf{X}_0]$. Using each of these new samples of $\mathbf{X}_t$, we fit a VAR(p) model to obtain new least squares estimates $[\widehat{\boldsymbol{\eta}}_t^{mr*}, \widehat{\mathbf{A}}_1^{mr*}, ..., \widehat{\mathbf{A}}_p^{mr*}]$ and $\widehat{\boldsymbol{\Omega}}^{mr*} = \text{cov}(\widehat{\boldsymbol{\eta}}_t^{mr*}, \widehat{\boldsymbol{\eta}}_t^{mr*})$, and $\hat{\mathcal{B}}^{mr*} = \{\widehat{\mathbf{B}}^{mr*} = \hat{\mathbf{P}}^{mr*}\mathbf{Q} : \mathbf{Q} \in \mathbb{O}_n, \text{diag}(\widehat{\mathbf{B}}^{mr*}) \geq 0, \bar{g}_Z(\mathbf{B}) = \mathbf{0}\}$, where where $\mathbb{O}_n$ is the set of $n \times n$ orthonormal matrices and $\hat{\mathbf{P}}^{mr*}$ is the unique lower triangular Cholesky factor of $\widehat{\boldsymbol{\Omega}}^{mr*}$.

To generate samples of the external variables S_{1t} and S_{2t} from mth solution in a way that ensures that the correlations with the uncertainty shocks that appear in our historical data also appear in our simulated samples, we first generate historical idiosyncratic stock market shocks $e_{S_1t}^m$ and gold price shocks $e_{S_2t}^m$ as the fitted residuals from regressions of S_{1t} and S_{2t} on a single autoregressive lag and on $\widehat{\mathbf{e}}_t$, respectively. Next, we draw randomly with replacement from $e_{S_1t}^m$ and $e_{S_2t}^m$ with the exception that, as above, we fix the values for these shocks in each replication in the periods $\bar{\tau}_1 \, \bar{\tau}_2$, $\bar{\tau}_3$, $\bar{\tau}_4$ and $\bar{\tau}_5$, to obtain $r = 1, ..., R$ new values $e_{S_1t}^{mr}$ and $e_{S_2t}^{mr}$ and R new values of S_{1t} and S_{2t} by recursively iterating on

$$(A1) \qquad S_{1t}^{mr} = d_{01}^m + \widehat{\rho}_1 S_{1t-1}^{mr} + \mathbf{d}_1^{m\prime} \mathbf{e}_t^{mr} + e_{S_1t}^{mr}$$

$$(A2) \qquad S_{2t}^{mr} = d_{02}^m + \widehat{\rho}_2 S_{2t-1}^{mr} + \mathbf{d}_2^{m\prime} \mathbf{e}_t^{mr} + e_{S_2t}^{mr}$$

with initial conditions fixed at their initial sample values, $[S_{11}, S_{21}]$. The parameters $\widehat{\rho}_1$ and $\widehat{\rho}_2$ are the sample estimate slope coefficients from a first order autoregression of each

variable in historical data. The parameters $\mathbf{d}_1^{m\prime}$ and $\mathbf{d}_2^{m\prime}$ in (A1) and (A2) are calibrated to target the observed correlations $\text{corr}(S_{1t}, \widehat{\mathbf{e}}_t^m)$ and $\text{corr}(S_{2t}, \widehat{\mathbf{e}}_t^m)$ for the mth solution in historical data so that $\text{corr}(S_{1t}^{mr}, \mathbf{e}_t^{mr})$ and $\text{corr}(S_{2t}^{mr}, \mathbf{e}_t^{mr})$ equal the observed historical $\text{corr}(S_{1t}, \widehat{\mathbf{e}}_t^m)$ and $\text{corr}(S_{2t}, \widehat{\mathbf{e}}_t^m)$ on average across all replications R.

We construct confidence sets for the set of IRFs in repeated samples as follows. The number of replications is set to $R = 1,000$. In each replication of each solution, $K = 1.5$ million rotation matrices $\mathbf{Q}$ are entertained, but only $K_{mr} \leq K$ rotations will generate solutions that are admitted into the identified set for that replication, $\bar{\mathcal{B}}^{mr*}(\cdot)$. Let $\Theta_{i,j,s}^{m,r,k}$ be the s-period ahead response of the ith variable to a standard deviation change in shock j at the k-th rotation of K_{mr}, for replication r and solution m.[1] Let $\underline{\Theta}_{i,j,s}^{m,r} = \min_{k \in [1,K_{mr}]} \Theta_{i,j,s}^{m,r,k}$ and $\overline{\Theta}_{i,j,s}^{m,r} = \max_{k \in [1,K_{mr}]} \Theta_{i,j,s}^{m,r,k}$. Each $(\underline{\Theta}_{i,j,s}^{m,r}, \overline{\Theta}_{i,j,s}^{m,r})$ pair represents the extreme (highest and lowest) dynamic responses in replication r of solution m. From the quantiles of the set $\left\{\underline{\Theta}_{i,j,s}^{m,r}\right\}_{m=1,r=1}^{M,R}$ that includes all replications for all solutions we can obtain the $\alpha/2$ critical point $\underline{\Theta}_{i,j,s}(\alpha/2)$. Similarly, from the quantiles of $\left\{\overline{\Theta}_{i,j,s}^{m,r}\right\}_{m=1,r=1}^{M,R}$, we have the $1 - \alpha/2$ critical point $\overline{\Theta}_{i,j,s}(1 - \alpha/2)$. Eliminating the lowest and highest $\alpha/2$ percent of the samples gives a $(1 - \alpha)\%$ percentile-based confidence interval defined by

$$CI_{\alpha,g} = \left[\underline{\Theta}_{i,j,s}(\alpha/2), \ \ \overline{\Theta}_{i,j,s}(1 - \alpha/2)\right].$$

$CI_{\alpha,g}$ denotes the confidence intervals for sets of solutions that satisfy all constraints, including the event and external variable constraints: $\bar{g}_Z(\mathbf{B}) = 0$, $\bar{g}_E(\mathbf{B}; \bar{\tau}, \bar{\mathbf{k}}) \geq 0$, $\bar{g}_C(\mathbf{B}; \mathbf{S}) \geq 0$. We use CI_{α,g_Z} to denote the confidence intervals for sets of solutions that satisfy only the reduced form covariance restrictions $\bar{g}_Z(\mathbf{B}) = 0$.

Longer Horizon Uncertainty

We examine longer horizon uncertainty. Figure A1 presents the IRFs when we use a system with $h = 12$ month-ahead macro and financial uncertainty, along with ip_t. This system is denoted $\mathbf{X}_t^{(12)} = (U_{Mt}(12), ip_t, U_{Ft}(12))'$, where $U_{Mt}(12)$ denotes twelve-month-ahead macro uncertainty, and likewise for $U_{Ft}(12)$. The same identifying restrictions are used as for the $h = 1$ month-ahead base case system. These results are similar to those for the cases that use $h = 1$ month ahead uncertainty.

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[1] The s-period ahead dynamic responses to one-standard deviation shocks in the jth variable are defined as

$$\frac{\partial \mathbf{X}_{t+s}}{\partial e_{jt}} = \widehat{\Psi}_s^{mr*} \widehat{\mathbf{b}}^{mrkj*},$$

where $\widehat{\mathbf{b}}^{mrkj*}$ is the jth column of $\widehat{\mathbf{B}}^{mrk*}$ and the coefficient matrixes $\widehat{\Psi}_s^{mr*}$ are given by $\widehat{\Psi}^{mr*}(L) = \widehat{\Psi}_0^{mr*} + \widehat{\Psi}_1^{mr*}L + \widehat{\Psi}_2^{mr*}L^2 + \ldots = \widehat{\mathbf{A}}^{mr*}(L)^{-1}$.
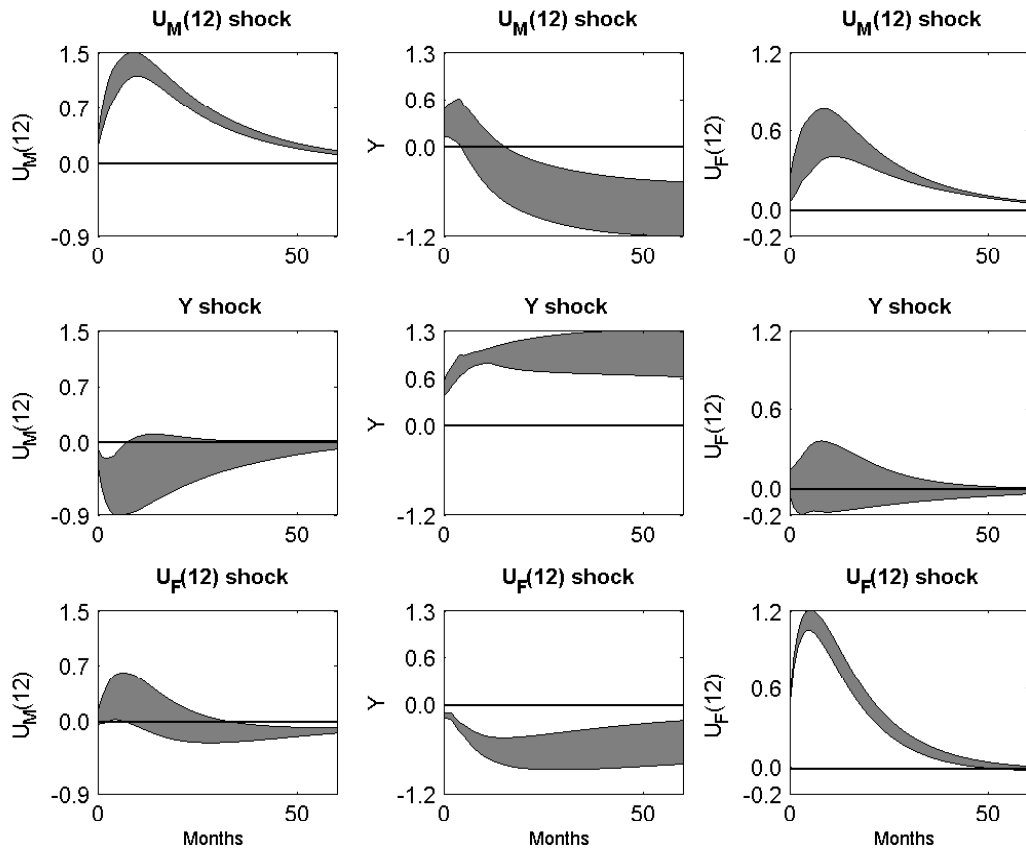
Figure A1. 12 Month-Ahead Uncertainty. The figure shows results from the identified set for system $\mathbf{X}_t = (U_{Mt}(12), ip_t, U_{Ft}(12))'$ using 12 month-ahead uncertainty and the full set of constraints with each argument of $\bar{\mathbf{k}}$ set to their 75th-percentile values of the unconstrained set. It reports the identified set of impulse response to positive, one standard deviation shocks in units of percentage points. The sample spans the period 1960:07 to 2015:04.

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