Online Appendix for "Money Mining and Price Dynamics"

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A Granger test

In this section we test whether the prices of gold and Bitcoin affect their production or mining intensity.

Gold: We use the historical mine production index and purchasing power of gold from Jastram (2009). This is an annual data covering 1870-1970, see Figure 1. Consider the two-variable VAR

$$\begin{bmatrix}\text{production}_t\\ \text{price}_t\end{bmatrix} = \mathbf{b_0} + \mathbf{B_1}\begin{bmatrix}\text{production}_{t-1}\\ \text{price}_{t-1}\end{bmatrix} + \cdots + \mathbf{B_k}\begin{bmatrix}\text{production}_{t-k}\\ \text{price}_{t-k}\end{bmatrix} + \begin{bmatrix}\varepsilon_{1,t}\\ \varepsilon_{2,t}\end{bmatrix}$$

where $\mathbf{b_0}$ is a vector of intercept terms and each of $\mathbf{B_1}$ to $\mathbf{B_k}$ is a matrix of coefficients. The lag length $k = 3$ is recommended by the likelihood ratio test, final prediction error and Akaike's information criterion.

We use the Granger test to test the null hypothesis that all coefficients on lags of the price in the production equation are equal to zero, against the alternative that at least one is not non-zero. The p-value is 0.02 and thus we conclude that the real price of gold Granger-causes the production at the 5% level.

Bitcoin: We use the monthly data on mining difficulty and Bitcoin price from the web site Bitcoinity, covering the period Aug 2010 to Oct 2018. We consider the following VAR model

$$\begin{bmatrix}\text{growth of diff level}_t\\ \text{growth of price}_t\end{bmatrix} = \mathbf{b_0} + \mathbf{B_1}\begin{bmatrix}\text{growth of diff level}_{t-1}\\ \text{growth of price}_{t-1}\end{bmatrix} + \cdots + \mathbf{B_k}\begin{bmatrix}\text{growth of diff level}_{t-k}\\ \text{growth of price}_{t-k}\end{bmatrix} + \begin{bmatrix}\varepsilon_{1,t}\\ \varepsilon_{2,t}\end{bmatrix}.$$

The recommended lag is $k = 2$ and the p-value of the causality test is 0.0004. Hence we conclude that the growth rate of prices Granger-causes the growth rate of the difficulty level at the 1% level.

B Search while mining

Now we allow agents to produce and mine at the same time, specifically we let miners produce with probability $\eta \in [0,1]$ when in contact with a buyer. The opportunity cost of mining becomes $C(1) = \alpha\sigma(1-\eta)A(-q + V_1 - V_0)$ and the probability that a random non-money holder can produce is then

$$\chi_t = \frac{1 - A - m(1-\eta)}{1-A}.$$

The gold mining model in Section 3 is the special case $\eta = 0$. Another polar case is $\eta = 1$ where agents can engage in mining without forgiving any trading opportunity. We will show that this possibility can change the dynamics of prices depending on the efficiency of the mining technology.

Proposition 9 ***(Search while mining)*** *There exists a steady-state monetary equilibrium iff*

$$r < \frac{\alpha\sigma}{1-\theta}\left[\frac{\alpha\sigma\theta(1-\eta) + \lambda\left(\theta - \bar{A}\right)}{\alpha\sigma(1-\eta)+\lambda}\right]. \tag{85}$$

The steady-state money supply, A^s, increases with η while the value of money, q^s, decreases with η.

Suppose $\eta = 1$. There exists a monetary equilibrium if $r < \alpha\sigma\left(\theta - \bar{A}\right)/(1-\theta)$ and it is such that A^s tends to $\bar{A} < \theta$. For all $A_0 < \bar{A}$ the unique equilibrium leading to the steady state is such that: A increases over time until it reaches $\bar{A}$; q increases over time if $\lambda > r/\left(\theta - \bar{A}\right)$, decreases if $\lambda < r/\left(\theta - \bar{A}\right)$, and remains constant if $\lambda = r/\left(\theta - \bar{A}\right)$.

Proof of Proposition 9. Agents' value functions solve:

$$rV_1 = \alpha\sigma\left[1 - A - m(1-\rho)\right]\theta\left[u(q) - q\right] + \dot{V}_1 \tag{86}$$

$$rV_0 = \alpha\sigma A\rho(1-\theta)\left[u(q) - q\right] + \max\left\{\alpha\sigma A(1-\rho)(1-\theta)\left[u(q)-q\right], \lambda\left(\bar{A} - A\right)\omega(q)\right\} + \dot{V}_0. \tag{87}$$

The key novelty in (87) is that the opportunity cost of mining has been multiplied by $1-\rho$. In particular, if $\rho = 1$ there is no opportunity cost of mining and all agents without money mine. Subtracting (87) from (86) the value of money solves:

$$\begin{aligned} r\omega(q) = {} & \left\{1 - \left[1 + \rho\left(\frac{1-\theta}{\theta}\right)\right]A - m(1-\rho)\right\}\theta\alpha\sigma\left[u(q) - q\right] \\ & - \max\left\{\alpha\sigma A(1-\rho)(1-\theta)\left[u(q)-q\right], \lambda\left(\bar{A} - A\right)\omega(q)\right\} + \omega'(q)\dot{q}. \end{aligned} \tag{88}$$

The law of motion for A is:

$$\dot{A} = m\lambda\left(\bar{A} - A\right). \tag{89}$$

The locus of pairs (A, q) such that agents are indifferent between mining or not is given by:

$$A = \mu(q) \equiv \frac{\lambda \bar{A} \omega(q)}{\alpha\sigma(1-\rho)(1-\theta)\left[u(q)-q\right] + \lambda\omega(q)}$$

The μ-locus shifts to the right as ρ increases and it becomes vertical at $A = \bar{A}$ when $\rho = 1$.

By the same reasoning as in Section 3.1, q^s solves (18),

$$r\omega(q) = (\theta - A)\,\alpha\sigma\left[u(q) - q\right],$$

and A^s is the smallest root to

$$\lambda\left(\bar{A} - A\right)(\theta - A) - A(1-\eta)(1-\theta)r = 0. \tag{90}$$

It is easy to check that A^s increases with ρ while q^s decreases with η. Moreover, as η approaches to 1, A^s approaches to $\min\{\theta, \bar{A}\}$. By the same reasoning as in the proof of Proposition 1 there exists a steady-state monetary equilibrium iff

$$\lim_{q\to 0}\left\{r\omega(q) - \left[\theta - \mu(q)\right]\alpha\sigma\left[u(q)-q\right]\right\} < 0.$$

Dividing by $\omega(q) > 0$ this condition can be rewritten as:

$$\lim_{q\to 0}\left\{r - \alpha\sigma\frac{\left[\theta - \mu(q)\right]\left[u(q)-q\right]}{\omega(q)}\right\} < 0.$$

Using that $\lim_{q\to 0}\left\{\left[u(q)-q\right]/\omega(q)\right\} = 1/(1-\theta)$ and $\lim_{q\to 0}\mu(q) = \lambda\bar{A}/\left[\alpha\sigma(1-\eta)+\lambda\right]$ the condition above can be rewritten as (85). In particular, when $\eta = 1$,

$$r < \frac{\alpha\sigma}{1-\theta}\left(\theta - \bar{A}\right).$$

In that case a necessary condition for a steady-state monetary equilibrium is $\bar{A} < \theta$. Hence, $A^s = \theta < \bar{A}$.

The condition $\alpha\sigma(\theta - \bar{A}) > r(1-\theta)$ guarantees the existence of a steady-state monetary equilibrium when $\eta = 1$. The system of ODEs, (88) and (89), becomes:

$$\begin{aligned}
\omega'(q)\dot{q} &= \left[r + \lambda\left(\bar{A} - A\right)\right]\omega(q) - (\theta - A)\,\alpha\sigma\left[u(q)-q\right] \\
\dot{A} &= \lambda(1-A)\left(\bar{A} - A\right)
\end{aligned}$$

Linearizing the system around the steady state we obtain:

$$\begin{pmatrix} \dot{q} \\ \dot{A} \end{pmatrix} = \begin{pmatrix} \frac{r\omega'(q^s) - \left(\theta - \bar{A}\right)\alpha\sigma\left[u'(q^s)-1\right]}{\omega'(q^s)} & \frac{-\lambda\omega(q^s) + \alpha\sigma\left[u(q^s)-q^s\right]}{\omega'(q^s)} \\ 0 & -\lambda(1-\bar{A}) \end{pmatrix} \begin{pmatrix} q - q^s \\ A - A^s \end{pmatrix}.$$

If $\left(\theta-\bar{A}\right)\alpha\sigma > r(1-\theta)$ then $r\omega'(q^s) > \left(\theta-\bar{A}\right)\alpha\sigma\left[u'(q^s)-1\right]$. It follows that the determinant of the Jacobian matrix is negative, i.e., the steady state is a saddle point. The negative eigenvalue is $e_1 = -\lambda(1-\bar{A})$ and the associated eigenvector is

$$\overrightarrow{v}_1 = \begin{pmatrix} \frac{\left[\lambda - r/\left(\theta-\bar{A}\right)\right]\omega(q^s)}{\left[r+\lambda(1-\bar{A})\right]\omega'(q^s)-\left(\theta-\bar{A}\right)\alpha\sigma\left[u'(q^s)-1\right]} \\ 1 \end{pmatrix}$$

where we used that $r\omega(q^s) = \left(\theta-\bar{A}\right)\alpha\sigma\left[u(q^s)-q^s\right]$. The first component of $\overrightarrow{v}_1$ is of the same sign as $\lambda - r/\left(\theta-\bar{A}\right)$. The solution to the linearized system is

$$\begin{pmatrix} q-q^s \\ A-A^s \end{pmatrix} = Ce^{-\lambda(1-\bar{A})t}\overrightarrow{v}_1,$$

where C is some constant. Hence, in the neighborhood of the steady state,

$$\frac{\partial q}{\partial A} = \frac{\left[\lambda - r/\left(\theta-\bar{A}\right)\right]\omega(q^s)}{\left[r+\lambda(1-\bar{A})\right]\omega'(q^s)-\left(\theta-\bar{A}\right)\alpha\sigma\left[u'(q^s)-1\right]},$$

which is of the same sign as $\lambda - r/\left(\theta-\bar{A}\right)$. If $\lambda > r/\left(\theta-\bar{A}\right)$, then the saddle path in the neighborhood of the steady state is upward sloping, i.e., q and A increase over time. We can show that this result holds globally since the equation of the q-isocline is:

$$\frac{\omega(q)}{u(q)-q} = \frac{\left(\theta-A\right)\alpha\sigma}{r+\lambda\left(\bar{A}-A\right)}.$$

The q-isocline is upward sloping when $\lambda > r/\left(\theta-\bar{A}\right)$. See left panel of Figure 15. By the same reasoning, if $\lambda < r/\left(\theta-\bar{A}\right)$, then the saddle path is downward sloping and along the equilibrium path, q decreases while A increases. See middle panel of Figure 15. Finally, if $\lambda = r/\left(\theta-\bar{A}\right)$, then the q-isocline is horizontal. In that case q is constant over time. See right panel of Figure 15. ■

According to (85) the set of parameter values for which a steady-state monetary equilibrium exists shrinks as η increases. If agents can meet trading partners more frequently while mining, then the opportunity cost of mining is lower and the incentives to mine are greater, which leads to a higher supply of money. But for a monetary equilibrium to exist, the money supply cannot be too large. A higher η also reduces the value of money. In the limiting case where $\eta = 1$, there is no opportunity cost to engage in mining and all agents without money mine, $m = 1 - A$. At the steady state the money supply is equal to the maximum stock of money that could be mined, $\bar{A}$. We now turn to the transition dynamics for this special case.

Proposition 9 shows that when there is no opportunity cost of mining, the correlation between the value of money and the money stock along the transitional path depends on the efficiency of the mining technology.[22]

[22] While Proposition 9 focuses on the unique equilibrium leading to the steady state, there is also a continuum of equilibria where the value of money vanishes asymptotically. In the left panel of Figure 15, when λ is high, the value of money increases

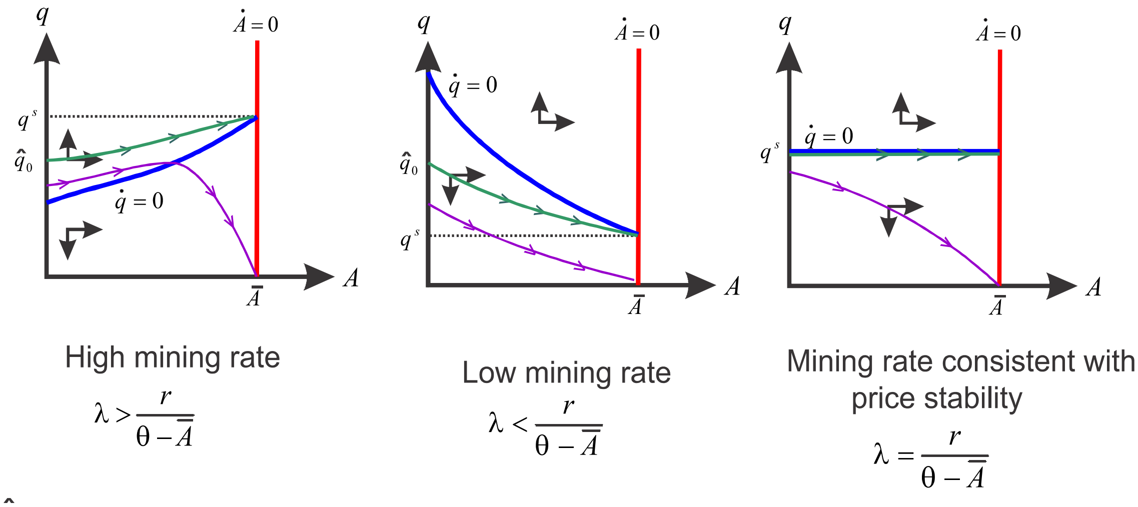

Figure 15: Phase diagrams when agents can mine while searching for trading partners ($\eta = 1$).

If the mining intensity is high, the value of money increases with the money supply. If the mining intensity is low, then the opposite correlation prevails and the value of money decreases as the money supply increases. Finally, there is a mining rate such that the price level is constant, the value of money is independent of the money stock.

first and then decreases. In the middle and right panels, when λ is low, the value of money is monotone decreasing in time.

C General matching function

Consider the gold mining model in Section 3. But suppose now that only buyers (money holders) and producers participate in the matching process according to a constant returns to scale matching function. The matching probability of a buyer is $\alpha(\tau)$ where $\tau = (1 - A - m)/A$ is market tightness expressed as the ratio of sellers to buyers. As is standard, we assume that $\alpha' > 0$, $\alpha'' < 0$, $\alpha'(0) = +\infty$, $\alpha'(+\infty) = 0$. A matching function that satisfies these properties is the Cobb-Douglas matching function.

The HJB equations of agents with and without money are:

$$rV_1 = \alpha(\tau)\sigma\theta \left[u(q) - q\right] + \dot{V}_1 \tag{91}$$

$$rV_0 = \max\left\{\frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\left[u(q) - q\right], \lambda\left(\bar{A} - A\right)\omega(q)\right\} + \dot{V}_0. \tag{92}$$

The novelty is that the matching rate of a buyer is $\alpha(\tau)$ while the matching rate of a seller is $\alpha(\tau)/\tau$. Using that $\lim_{\tau \to 0} \alpha(\tau)/\tau = +\infty$, it follows that $\tau > 0$ in equilibrium, i.e., $m < 1 - A$. The goods market is always active and

$$\max\left\{\frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\left[u(q) - q\right], \lambda\left(\bar{A} - A\right)\omega(q)\right\} = \frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\left[u(q) - q\right]. \tag{93}$$

Subtracting (92) from (91) the value of money solves:

$$r\omega(q) = \left[\alpha(\tau)\sigma\theta - \frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\right]\left[u(q) - q\right] + \omega'(q)\dot{q}. \tag{94}$$

From (93) market tightness in the goods market solves:

$$\frac{\alpha(\tau)}{\tau}\sigma(1-\theta)\left[u(q) - q\right] \geq \lambda\left(\bar{A} - A\right)\omega(q), \text{ “} = \text{" if } \tau < \frac{1-A}{A}.$$

Solving for τ we obtain:

$$\tau(\omega, A) = \min\left\{g^{-1}\left[\frac{\lambda\left(\bar{A} - A\right)\omega}{\sigma(1-\theta)S(\omega)}\right], \frac{1-A}{A}\right\}. \tag{95}$$

where $S(\omega) \equiv u\left[q(\omega)\right] - q(\omega)$ and $g(\tau) \equiv \alpha(\tau)/\tau$. For all (ω, A) such that $\frac{\lambda(\bar{A}-A)\omega}{\sigma(1-\theta)S(\omega)} \geq g\left(\frac{1-A}{A}\right)$, $m > 0$ and $\tau(\omega, A)$ is decreasing in ω and increasing in A. Moreover, $\tau(+\infty, A) = 0$ and $\tau(0, A) > 0$. The money supply evolves according to

$$\dot{A} = \left[1 - A\left(1 + \tau\right)\right]\lambda\left(\bar{A} - A\right), \tag{96}$$

where we used that $1 - A\left(1 + \tau\right) = m$.

We summarize the equilibrium by a system of two ODEs in ω and A:

$$\dot{\omega} = r\omega - \{\alpha [\tau(\omega, A)] \sigma\theta - g[\tau(\omega, A)] \sigma(1-\theta)\} S(\omega) \tag{97}$$

$$\dot{A} = \{1 - A[1 + \tau(\omega, A)]\} \lambda (\bar{A} - A). \tag{98}$$

The locus of the points such that $\dot{A} = 0$ corresponds to all pairs (ω, A) such that $\tau(\omega, A) = (1-A)/A$. From (95) it is given by:

$$\frac{\lambda (\bar{A} - A) \omega}{\sigma(1-\theta)S(\omega)} \leq g\left(\frac{1-A}{A}\right). \tag{99}$$

Condition (99) at equality gives a positive relationship between ω and A. As ω approaches 0, A tends to the solution to $\lambda (\bar{A} - A) = \sigma g\left(\frac{1-A}{A}\right)$. As ω tends to $+\infty$, A tends to $\bar{A}$. This locus is represented by a red upward-sloping curve in Figure 16.

The locus of the points such that $\dot{\omega} = 0$ and $\dot{A} > 0$ is such that

$$r\frac{\omega}{S(\omega)} = \{\alpha [\tau(\omega, A)] \sigma\theta - g[\tau(\omega, A)] \sigma(1-\theta)\}. \tag{100}$$

The left side is increasing in ω while the right side is decreasing in ω but increasing in A. For given A there is a unique ω solution to (100) provided that

$$r(1-\theta) < \{\alpha [\tau(0, A)] \sigma\theta - g[\tau(0, A)] \sigma(1-\theta)\},$$

where $\tau(0, A)$ is the solution to $g(\tau) = \lambda (\bar{A} - A)/\sigma$. If this condition holds for $A = 0$, then it holds for all A. Hence, we assume

$$r(1-\theta) < [\alpha(\tau_0)\sigma\theta - g(\tau_0)\sigma(1-\theta)] \text{ where } \tau_0 = g^{-1}[\lambda (\bar{A} - A)/\sigma]. \tag{101}$$

Assuming this condition is satisfied, the ω-isocline is upward sloping as illustrated in Figure 16. As A goes to zero, ω tends to a positive value.

There is a unique steady state such that agents are indifferent between mining or not and it solves

$$g(\tau) = g\left(\frac{1-A}{A}\right) = \frac{\lambda (\bar{A} - A) \omega}{\sigma(1-\theta)S(\omega)} \tag{102}$$

$$r\frac{\omega}{S(\omega)} = [\alpha(\tau)\sigma\theta - g(\tau)\sigma(1-\theta)]. \tag{103}$$

Equation (102) specifies the market tightness such that agents are indifferent between mining or participating in the goods market. Equation (103) gives the value of money given market tightness. Combining (102) and

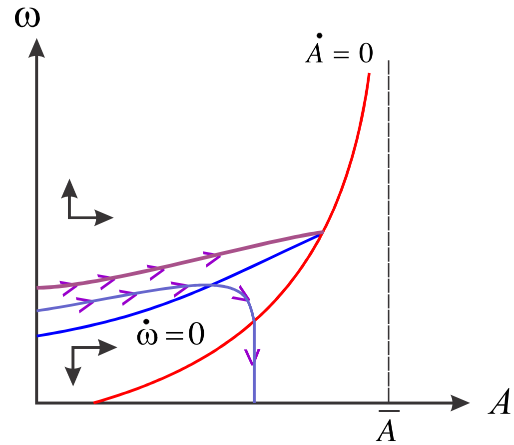

Figure 16: Phase diagram under matching function satisfying Inada conditions.

(103), steady-state market tightness solves:

$$\sigma(1-\theta)\left[\frac{r}{\lambda\left(\bar{A}-\frac{1}{1+\tau}\right)}+1\right]g(\tau)=\alpha(\tau)\sigma\theta. \tag{104}$$

It is easy to check that there is a unique $\tau^s \in \left(0, \frac{1-\bar{A}}{\bar{A}}\right)$ solution to this equation. The supply of money at the steady state is then $A^s = 1/(1+\tau^s)$. The equilibrium is monetary if (101) holds. The existence of a unique steady state guarantees that the A-isocline and ω-isocline only intersect once, i.e., the ω-isocline is located above the A-isocline as illustrated in Figure 16.

In Figure 16 we represent the phase diagram of the dynamic system (97)-(98) and its arrows of motion. It can be checked that the steady state is a saddle path and given the initial condition $A_0 = 0$ there is a unique path leading to it. Along that path the value of money rises over time. There is also a continuum of other equilibria where the value of money vanishes asymptotically.

In order to characterize the path for market tightness, we can rewrite (100) as

$$r\frac{\omega(\tau, A)}{S[\omega(\tau, A)]}=[\alpha(\tau)\sigma\theta-g(\tau)\sigma(1-\theta)],$$

where $\omega(\tau, A)$ is defined implicitly by $\tau = \tau(\omega, A)$. Assuming $m > 0$, ω is a decreasing function of τ and an increasing function of A. Hence, the τ-isocline is upward sloping. The A-isocline becomes $A = 1/(1+\tau)$. By the same reasoning as above, the saddle path is upward sloping, which means that τ increases over time.

D Interest-bearing/commodity monies

Suppose money is a commodity that provides some direct utility, e.g., gold or silver, or a financial asset that pays dividends. We denote $d>0$ the dividend flow enjoyed by each money holder. The Bellman equation of a money holder is:

$$rV_1 = d + \alpha\sigma\left(1 - A - m\right)\theta\left[u(q) - q\right] + \dot{V}_1. \tag{105}$$

The only novelty is the first term on the right side representing the dividend flow. The Bellman equation for an agent without money is unchanged. It follows that the dynamic equation for the value of money is:

$$\begin{aligned} r\omega(q) &= d + \alpha\sigma\left(1 - A - m\right)\theta\left[u(q) - q\right] \\ &\quad - \max\left\{\alpha\sigma A(1-\theta)\left[u(q) - q\right], \lambda\left(\bar{A} - A\right)\omega(q)\right\} + \omega'(q)\dot{q}. \end{aligned} \tag{106}$$

A steady-state equilibrium, (q^s, A^s), solves:

$$\begin{aligned} r\omega(q) &= d + \alpha\sigma\left(\theta - A\right)\left[u(q) - q\right] \\ A &= \frac{\lambda\bar{A}\omega(q)}{\alpha\sigma(1-\theta)\left[u(q) - q\right] + \lambda\omega(q)} \end{aligned} \tag{107}$$

The first equation gives a negative relationship between q and A while the second equation gives a positive relationship between A and q. So there is a unique steady state and $\partial q^s/\partial d > 0$ and $\partial A^s/\partial d > 0$.

Out of steady state, if $m<1-A$, then the trajectory is $A=\mu(q)$ as in the baseline. If $m=1-A$, then:

$$\dot{q} = \frac{\left[r + \lambda\left(\bar{A} - A\right)\right]\omega(q) - d}{\omega'(q)} \tag{108}$$

$$\dot{A} = (1 - A)\lambda\left(\bar{A} - A\right). \tag{109}$$

The slope $\partial q/\partial A = \dot{q}/\dot{A}$ falls in d for any given (A, q), but one can show that $\dot{q}>0$ at all time. If $\dot{q}=0$ at some time t, then $\dot{q}<0$ after t by (108). The trajectory cannot change regime after t as a regime switch requires both trajectories to have the same slope but the locus $A=\mu(q)$ is always upward sloping. In the regime $m=1-A$, $\dot{q}=\alpha\sigma(1-A)\theta\left[u(q)-q\right]/\omega'(q)>0$ when $q\approx q^s$ by (106) and (107). Hence $\dot{q}\neq 0$ at all t.

By the proof of Proposition 2, mining and production co-exist near the steady state if only if

$$\left.\frac{\partial q}{\partial A}\right|_{m=1-A} > \left.\frac{\partial q}{\partial A}\right|_{m\in(0,1-A)} \iff \frac{\mu'(q^s)/\mu(q^s)}{\omega'(q^s)/\omega(q^s)} > \frac{1-\theta}{\theta}.$$

As d increases there are two opposing effects. Since $\partial q/\partial A|_{m=1-A}$ falls in d for any given (A, q), it is more likely that $m = 1 - A$ near the steady state when d is large. On the other hand q^s and A^s increase in d and therefore agents have less incentive to mine around the steady state. The net effect is ambiguous in general.

E Divisible assets

We now study price dynamics when all assets are perfectly divisible and individual asset holdings are unrestricted, $a \in \mathbb{R}_+$. This model, which is a continuous-time version of the New-Monetarist model of Lagos and Wright (2005), will be useful to check the robustness of our earlier results. Choi and Rocheteau (2019b) provide a detailed description of the New Monetarist model in continuous time and its solution methods.

Consider the gold mining model in Section 3. We add a centralized market (CM) where price-taking agents can trade continuously a good, distinct from the one traded in pairwise meetings, for money. The purpose of these CMs is to allow agents to readjust their money holdings to some targeted level in-between pairwise meetings, so as to keep the distribution of money holdings degenerate. In reality, the CMs could correspond to the several exchanges where individuals trade crypto-currencies for different government-supplied currencies using credit or debit cards (e.g., Coinbase, Coinmama, Luno...). In the following we take the CM good as the numéraire. Agents have the technology to produce h units of the numéraire good at a linear cost h ($h < 0$ is interpreted as consumption). Hence, agents' discounted lifetime utility in-between pairwise meetings is $-\int_0^{+\infty} e^{-rt} dH(t)$ where $H(t)$ is a measure of the cumulative production of the numéraire good (net of its consumption) up to t. This formulation allows agents to produce or consume the numéraire good in flows (in which case $H(t)$ admits a density $h(t)$) or in discrete amounts (in which case $H(t^+) - H(t^-) \neq 0$). Preferences during pairwise meetings are as before. Money is a Lucas tree that pays a dividend flow $d \geq 0$. The case $d = 0$ corresponds to fiat money. The CM price of the asset is denoted ϕ_t.

Let $V(a)$ be the value function of an agent with a units of assets expressed in terms of the numéraire. At any point in time between pairwise meetings, an agent can readjust her asset holdings by consuming or producing the numéraire good. Formally,

$$V(a) = \max_h \{-h + V(a+h)\} = a + \max_{a^* \geq 0} \{-a^* + V(a^*)\},$$

where h is the production of the numéraire and a^* is the agent's targeted asset holdings (expressed in terms of the numéraire). The value function, $V(a)$, is linear in a.

We now consider the bargaining problem in a pairwise meeting between a buyer holding a^b units of assets and a seller holding a^s units of asset. The outcome of the negotiation is a pair $(q, p) \in \mathbb{R}_+ \times [-a^s, a^b]$ where q is the amount of goods produced by the seller for the buyer and p is the transfer of assets from the buyer to the seller. Feasibility requires that $-a^s \leq p \leq a^b$. By the linearity of $V(a)$ the buyer's surplus is

$u(q) + V(a^b - p) - V(a^b) = u(q) - p$ and the seller's surplus is $-q + V(a^s + p) - V(a^s) = -q + p$. According to the Kalai proportional solution, the buyer's surplus is equal to a fraction θ of the total surplus of the match, i.e., $u(q) - p = \theta\,[u(q) - q]$. Moreover, the solution is pairwise Pareto efficient, which implies that $q \leq q^*$ with an equality if $p \leq a^b$ does not bind. Using the notation $\omega(q)$ from (8), the buyer's consumption as a function of her asset holdings, $q(a^b)$, is such that $q(a^b) = q^*$ if $a^b \geq \omega(q^*)$ and $\omega(q) = a^b$ otherwise.

Consider the lifetime expected utility of the agent holding her targeted asset holdings, $V(a^*)$. Choi and Rocheteau (2019b) show that it solves the following HJB equation that is similar to (9) and (10) combined:

$$rV(a^*) = \rho a^* + \alpha(1-m)\sigma\theta\left\{u\left[q(a^*)\right] - q(a^*)\right\}$$
$$+ \max\left\{\alpha\sigma(1-\theta)\left\{u\left[q(\bar{a})\right] - q(\bar{a})\right\}, \lambda(\bar{A} - A)\phi\right\} + \dot{V}(a^*), \tag{110}$$

where the rate of return of assets is

$$\rho = \frac{d + \dot{\phi}}{\phi}. \tag{111}$$

The first term on the right side of (110) is the flow return of the asset. The second term is analogous to the first term on the right side of the HJB equation for V_1, (9). The agent receives an opportunity to consume at Poisson arrival rate $\alpha\sigma$. The partner can produce if she is not a miner, with probability $1 - m$. The third term on the right side of (110) is analogous to the right side of the HJB equation for V_0 in (10). It corresponds to the occupational choice according to which agents can choose to be producers and enjoy the flow payoff $\alpha\sigma(1-\theta)\left\{u\left[q(\bar{a})\right] - q(\bar{a})\right\}$ or miners and enjoy $\lambda(\bar{A} - A)\phi$. The term $\bar{a}$ represents asset holdings of other agents in the economy. The expected gain from mining describes the assumption that at Poisson arrival rate $\lambda(\bar{A} - A)$ the miner digs a unit of money which is worth ϕ units of numéraire. The last term is the change in the value function for a given asset position, $\dot{V}(a) = \partial V_t(a)/\partial t$.

The envelope condition associated with (110) together with $V'(a^*) = 1$ gives

$$r - \rho = \alpha(1-m)\sigma\theta\left\{\frac{u'\left[q(a^*)\right] - 1}{(1-\theta)u'\left[q(a^*)\right] + \theta}\right\}, \tag{112}$$

where we used $q'(a) = 1/\omega'(q)$ if $a < \omega(q^*)$ and $\partial^2 V(a)/\partial a \partial t = 0$ as $V'(a^*) = 1$ at all t. The opportunity cost of holding the asset on the left side is the difference between the rate of time preference and the real rate of return of the asset. The right side is the marginal value of an asset if a consumption opportunity arises.

Since now agents can carry money and mine at the same time, the measure of miners solves

$$m \left\{ \begin{array}{c} = 1 \\ \in [0,1] \\ = 0 \end{array} \right. \quad \text{if} \quad \lambda(\bar{A} - A)\phi \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right. \alpha\sigma(1-\theta)\left[u\left(q\right) - q\right]. \tag{113}$$

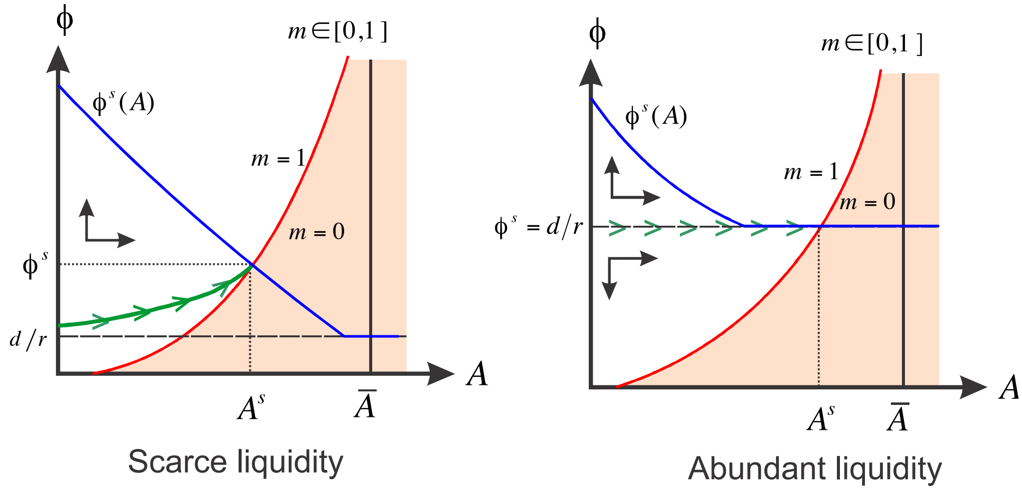

Figure 17: Phase diagram with divisible assets.

By market clearing:

$$a^* = \phi A. \tag{114}$$

The supply of assets evolves according to:

$$\dot{A} = \lambda m(\bar{A} - A). \tag{115}$$

An equilibrium is a list, $\langle a_t^*, m_t, \phi_t, A_t \rangle$, that solves (112), (113), (114), and (115).

We represent the phase diagram and equilibrium trajectory in Figure 17. There are two main insights relative to the model with indivisible money and no centralized exchanges. First, there is a regime where the asset supply at the steady state is abundant enough to satiate agents' liquidity needs and to allow agents to trade q^* in all matches. In such equilibria, the asset is priced at its fundamental value at all dates, see the right panel of Figure 17. A necessary (but not sufficient) condition is that the potential asset supply when valued at its fundamental price, $\bar{A}d/r$, is larger than agents' liquidity needs, $\omega(q^*)$. It is the standard condition in the literature for abundant liquidity since Geromichalos et al. (2007), except that it applies to the potential asset supply, $\bar{A}$, and not the actual asset supply, A, which is endogenous. The second insight is that there is a regime with scarce liquidity that is qualitatively similar to the equilibria of the model with indivisible money. The price of the asset is above its fundamental value at all dates and it keeps increasing over time until it reaches a steady state as shown in the left panel of Figure 17.