ONLINE APPENDIX
"Implementation by vote-buying mechanisms"
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In this Appendix, we provide the following content:
1. A glossary of notation.
2. A rigorous formalization of our theory.
3. A proof of our main result.
4. An extension to non-neutral mechanisms.
5. Numerical examples.

1. Glossary of notation

In this subsection we provide a glossary of notation used in sections 2 and 3 of this Appendix.

We adopt the following notational conventions. We denote a random variable by $\tilde{x}$, and its realization by x. If the variable takes agent-specific values, x_i denotes the value of this variable for agent i, vector $x_{N^n} \equiv (x_1, ..., x_n)$ denote the profile of values for all agents in society N^n and $x_{-i} \equiv (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$ denotes the profile without the value for agent i. We use the symbol "$\equiv$" to denote that we are defining the left-hand-side object to be the right-hand-side object. A superscript k in x^k is always a label (often indicating the size of the society); if we want to indicate that $k \in \mathbb{R}$ is an exponent, we denote it by $(x)^k$. We denote an infinite sequence by $\{x^n\}_{n=1}^{\infty}$. Subindexes denote agents. In addition to these conventions, here is the glossary.

$N^n \equiv \{1, ..., n\}$: A society with $n \in \mathbb{N}$ agents, $n \geq 2$.

$n \in \mathbb{N}\backslash\{1\}$: The only agent who belongs to society N^n but not to society N^{n-1}. Also the size of society N^n.

i, j : Arbitrary agents.

A, B : Two abstract alternatives.

d : The social decision; the alternative collectively chosen.

$w_{\min} \in (0, 1)$: an exogenous parameter corresponding to the lower bound on the wealth endowment of any agent.

$w \in [0, 2]$: an arbitrary wealth level.

$w_i \in [w_{\min}, 1]$: Agent i's initial wealth or wealth endowment.

$\gamma_{\max} \in \mathbb{R}_{++}$: an exogenous parameter corresponding to the upper bound on the valuation of any agent.

$v \in [-\gamma_{\max}, \gamma_{\max}]$: a valuation for alternative A, indicating the willingness to trade wealth for A, at wealth level 1.

$\Theta \equiv [w_{\min}, 1] \times [-\gamma_{\max}, \gamma_{\max}]$: the set of all possible types.

$\theta \equiv (w, v) \in \Theta$: An arbitrary type, that is, a (wealth endowment, valuation) pair.

P and p : A probability measure on the Borel $\sigma-$algebra over Θ and its Radon-Nikodym derivative.
L : the Lebesgue measure.
$\gamma \in (0, \gamma_{\max}]$: The highest valuation to which P assigns positive density (depends on P).
$\Theta_P \subseteq \Theta$: The subset of types with positive density under P.
$\mathcal{P}$: The set of all probability measures satisfying our assumptions.
$w_i^O \in [0, 2]$: The final wealth outcome of agent i.
c : a function from $\mathbb{R}$ to $\mathbb{R}_+$ interpreted as a vote-buying mechanism.
$\mathcal{C}$ and $\mathcal{C}_A$: the set of all vote-buying mechanisms, and all admissible vote-buying mechanisms.
$a \in \mathbb{R}$: an arbitrary action, interpreted as the quantity of votes acquired.
$a(w) \in \mathbb{R}_{++}$: the maximum quantity affordable with wealth w (this varies with c).
G : a function from $\mathbb{R}$ to $[0, 1]$. Interpreted as an "outcome function" because $G\left(\sum_{i=1}^n a_i\right)$ is the probability that A is chosen given the vector of actions a_{N^n}.
g : the derivative of G, interpreted as "marginal pivotality."
$\mathcal{G}$: an arbitrary non-empty set of possible outcome functions.
$\kappa(c)$: the limit elasticity of cost function c as actions approach 0.
s : a function from Θ_P to $\mathbb{R}$, interpreted as a pure strategy.
S : the set of all feasible pure strategies.
r : a draw from a uniform distribution over the unit interval.
σ : a measurable function from $[0, 1] \times \Theta_P$ to $\mathbb{R}$, interpreted as a mixed strategy.
Σ : the set of all feasible mixed strategies.
$D \subseteq \Theta_P \times \mathbb{R}$: the set of all possible type-action pairs given distribution P.
μ : a probability measure over on the Borel $\sigma-$algebra over D, interpreted as a distributional strategy in the sense of Milgrom and Weber (1985).
$EU_i[(\theta, a, \sigma_{-i})]$: the expected utility of agent i with type $\theta_i = \theta$ taking action $a_i = a$ given that all other agents follow strategy σ_{-i}.
$\Gamma^{(n,P,c,G)}$: the game defined by the tuple (n, P, c, G).
$BR(\theta, \sigma_{-i})$: the set of best responses for agent i with type $\theta_i = \theta$ to the strategy profile σ_{-i} by other agents.
$BNE^{(n,P,c,G)}$: the set of interim Bayes-Nash equilibria of game $\Gamma^{(n,P,c,G)}$.
SC^n : a correspondence from $(\Theta)^n$ to $\{A, B\}$, interpreted as a social choice correspondence for society N^n.
$SC \equiv \{SC^n\}_{n=2}^{\infty}$: a sequence of social choice correspondences (one for each society size).
$\mathcal{SC}$: the set of all possible sequences of social choice correspondences.
$(\mathcal{P}, d_{TV})$: the metric space of probability distributions, with the "total variation" metric d_{TV}.
$Eq(SC) \subset \mathcal{SC}$: the subset of sequences of social choice correspondences that are generically asymptotically equivalent to SC, as defined in Definition 3.
$\tilde{d}_P^n(\sigma, \tilde{r}_{N^n}, \tilde{\theta}_{N^n})$: the social decision as a random variable whose realization depends on the realization of $\tilde{\theta}_{N^n}$, of $\tilde{r}_{N^n}$, and of the decision given probability $G\left(\sum_{k=1}^n \sigma(r_k, \theta_k)\right)$ of

choosing A.
$sgn(x)$: the sign function, $sgn(x) = 1$ if $x \geq 0$ and $sgn(x) = -1$ otherwise.
$SC_\rho \in \mathcal{SC}$: A sequence of social choice correspondences, as defined by Expression (12).
$\mathcal{SC}_{\mathbb{R}_{++}} \equiv \bigcup_{\rho \in (0,\infty)} Eq(SC_\rho) \subset \mathcal{SC}$. A class of sequences of social choice correspondences.
GBR and SBR : respectively the greatest and smallest best responses.
$\Pr[d = A|(a, \sigma_{-i})]$: Probability that A is chosen, given that agent i chooses $a_i = a$ and other agents follow strategy profile σ_{-i}.
s^σ : the pure strategy associated to mixed strategy σ, defined by Definition 6.
$Piv[a, s] \in \mathbb{R}_+$: the expected pivotality, or expectation of g, of an agent playing a given that all other agents play strategy s.
$\Theta^1_P(x^n, n)$, $\Theta^2_P(x^n, n)$ and $\Theta^3_P(x^n, n)$: Subset of types for which the equilibrium net sum of actions is strictly less than $-x^n$, in $[-x^n, x^n]$, and greater than x^n, respectively.
$E[\tilde{x}]$ and $Var[\tilde{x}]$: the expectation and variance of random variable $\tilde{x}$.
$q^n(\tilde{\theta})$: a random variable equal to $s^n(\tilde{\theta}) - E[s^n(\tilde{\theta})]$.
V^τ and $\hat{V}^\tau$: the cumulative distributions of two random variables defined in the proof of Lemma 8.
$N[0, 1]$: the standard Normal distribution function with mean zero and variance one.
z^θ : a function defined in Lemma 9.
J^c : an auxiliary function defined in Expression (36).
$\mathcal{P}^\rho \subset \mathcal{P}$: a subset of probability distributions defined in Expression (43).

2. The Formal Framework

Set of agents. Let $\mathbb{N}$ be a countably infinite set of agents. For each $n \in \mathbb{N}\backslash\{1\}$, let $N^n \equiv \{1, 2, ..., n\}$ with size n be the $n-th$ society in the sequence of societies $\{N^n\}_{n=2}^{\infty}$.

Social choice problem. For each $n \in \mathbb{N}\backslash\{1\}$, society N^n faces a binary collective choice over the set of alternatives $\{A, B\}$. Let the social decision $d \in \{A, B\}$ denote the alternative chosen. Let $i \in N^n$ denote an arbitrary agent in the society.

Individual types. Each agent $i \in N^n$ is characterized by her initial endowment of wealth $w_i \in [w_{\min}, 1]$, where $w_{\min} \in (0, 1)$ is a minimum wealth endowment, and by her valuation $v_i \in [-\gamma_{\max}, \gamma_{\max}]$ of alternative A, where $\gamma_{\max} \in \mathbb{R}_{++}$ is an arbitrarily large exogenous parameter. Define the type space $\Theta \equiv [w_{\min}, 1] \times [-\gamma_{\max}, \gamma_{\max}]$ and for each agent $i \in N^n$ let $\theta_i \equiv (w_i, v_i) \in \Theta$ denote agent i's type. Let $B(\Theta)$ be the Borel σ-algebra over Θ, given by the standard Euclidean metric on Θ and let $P : B(\Theta) \longrightarrow [0, 1]$ be a probability measure over $B(\Theta)$.

Assume that P is absolutely continuous with respect to the Lebesgue measure (denoted L), and that there exists an exogenous parameter $\gamma \in (0, \gamma_{\max}]$ representing the importance of the social decision, such that $P(O) \in (0, 1]$ for any open $O \subseteq [w_{\min}, 1] \times [-\gamma, \gamma]$ and

$P([w_{\min}, 1] \times [-\gamma, \gamma]) = 1$.[38] Let $\Theta_P \equiv [w_{\min}, 1] \times [-\gamma, \gamma]$ be the support of probability measure P. Let $p : \Theta \longrightarrow \mathbb{R}_+$ be the Radon–Nikodym derivative, or density, of P, so that $p \equiv \frac{dP}{dL}$. Note that $p(\theta) = 0$ for any $\theta \in \Theta \backslash \Theta_P$, and assume the restricted function $p|_{\Theta_P} : \Theta_P \longrightarrow \mathbb{R}_+$ defined by $p|_{\Theta_P}(\theta) = p(\theta)$ for any $\theta \in \Theta_P$ is continuous over Θ_P. Let $\mathcal{P}$ denote the set of all possible probability measures satisfying these conditions.

For each agent $i \in N^n$, type θ_i is a realization from the probability space $(\Theta, B(\Theta), P)$ for a given $P \in \mathcal{P}$. Assume P is common knowledge among agents, but types are privately observed, and independently drawn. The type of agent $i \in N^n$ ex-ante, or from the perspective of any other agent, is a random variable $\tilde{\theta}_i \equiv (\tilde{w}_i, \tilde{v}_i)$ distributed according to P over Θ.

Individual preferences. For each agent $i \in N^n$, let $w_i^O \in \mathbb{R}_+$ denote the final wealth outcome of agent i. Each agent i has von-Neumann Morgernstern preferences over lotteries over the pair (w_i^O, d), i.e., over her wealth and the social decision. We assume that agents' preferences are separable across final wealth and the social decision, and that they are strictly increasing and weakly risk-averse over their final wealth, with attitudes over risk that are common across agents. From these assumptions it follows that preferences over degenerate lotteries can be represented by

$$u(w_i^O) + 1_A(d)v_i, \tag{10}$$

where function $u : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is strictly increasing and weakly concave, $1_A(d)$ is the indicator function such that $1_A = 1$ if $d = A$ and $1_A = 0$ otherwise.

We assume that u is continuously differentiable and we normalize it so that $u(0) = 0$ and $u'(1) = 1$.

Valuation v_i is the marginal willingness to pay at wealth $w_i = 1$ for a marginal change in the probability of the decision from B to A. If agents are risk-averse, at any wealth level w, the marginal willingness to pay, or marginal rate of substitution between wealth and probability that alternative A is chosen, increases in magnitude with wealth, and is equal to the ratio of the valuation over the marginal utility of wealth, that is,

$$\frac{v_i}{u'(w)}.$$

Vote-buying mechanisms. A vote-buying mechanism is defined by a cost function $c : \mathbb{R} \longrightarrow \mathbb{R}_+$. Let $\mathcal{C}$ denote the set of all such cost functions from $\mathbb{R}$ to $\mathbb{R}_+$. A mechanism $c \in \mathcal{C}$ invites each agent $i \in N^n$ to choose any action in $\mathbb{R}$. Let $a_i \in \mathbb{R}$ denote the action chosen by agent i. For any agent $i \in N^n$ and for any $a \in \mathbb{R}$, if agent i chooses $a_i = a$, then agent i pays a cost $c(a)$. Given a mechanism $c \in \mathcal{C}$, and given any $w \in [w_{\min}, 1]$, we say that action a is affordable for agent i with initial wealth $w_i = w$ if $c(a) \in [0, w]$. For any $w \in [w_{\min}, 1]$, define $a(w) \equiv \{a \in \mathbb{R}_+ : c(a) = w\}$; then the set of affordable actions for agent i is $[-a(w_i), a(w_i)]$.

[38] Probability measure P is absolutely continuous (with respect to the Lebesgue measure L) over $B(\Theta)$ if, for any measurable set $A \subseteq B(\Theta)$, $L(A) = 0$ implies $P(A) = 0$. See Nielsen (1997) Definition 15.3.

All payments are redistributed equally among all other agents, so given a vector of actions $a_{N^n} \in \mathbb{R}^n$, each agent $i \in N^n$ obtains a net nominal wealth transfer equal to

$$-c(a_i) + \sum_{j \in N^n \setminus \{i\}} \frac{c(a_j)}{n-1}.$$

We assume that the execution of any mechanism entails some element of uncertainty, so that the mapping from actions to outcomes is stochastic: there exists an outcome function $G : \mathbb{R} \longrightarrow [0,1]$ such that for any $n \in \mathbb{N}\backslash\{1\}$ and any $a_{N^n} \in \mathbb{R}^n$, the probability that $d = A$ is $G\left(\sum_{j \in N^n} a_j\right)$. Let $\mathcal{G}$ be any non-empty class of strictly increasing, twice continuously differentiable functions from $\mathbb{R}$ to $[0,1]$ such that for any $\hat{G} \in \mathcal{G}$ with first derivative $\hat{g}$ and second derivative $\hat{g}'$:

i) $\hat{G}(x) - \frac{1}{2} = \frac{1}{2} - \hat{G}(-x)$ for any $x \in \mathbb{R}_{++}$;

ii) $\lim_{x \longrightarrow -\infty} \hat{G}(x) = 0$ and $\lim_{x \longrightarrow -\infty} \hat{g}(x) = 0$;

iii) $\exists \hat{\varepsilon} \in \mathbb{R}_{++}$ such that $\lim_{x \to \infty} \frac{\hat{g}'(x+\varepsilon)}{\hat{g}(x)} \in \mathbb{R}\ \forall \varepsilon \in (-\hat{\varepsilon}, \hat{\varepsilon})$.

Condition (i) is neutrality. Condition (ii) is a responsiveness condition: if the vote margin is sufficiently large, the outcome is the one with the vote advantage with probability arbitrarily close to one. Condition iii) requires the tails of the first derivative not to drop to zero too steeply. The set $\mathcal{G}$ can contain, among other functions, the cumulative distribution functions of all Logistic and Student-t distribution. We assume that $G \in \mathcal{G}$, and G is common knowledge among players. We propose mechanisms whose results are robust for any $G \in \mathcal{G}$, including those that are arbitrarily close to a step function with discontinuity at zero, as in Figure 1.[39,40]

Admissible vote-buying mechanisms. Let $\bar{\mathcal{C}} \subset \mathcal{C}$ be the set of continuously differentiable non-negative functions defined over $\mathbb{R}$ that are twice continuously differentiable over $\mathbb{R}\backslash\{0\}$. We define the set of admissible vote-buying mechanisms $\mathcal{C}_A \subset \mathcal{C}$ by $\mathcal{C}_A \equiv$

$$\left\{ \begin{array}{c} c \in \bar{\mathcal{C}} : c(0) = 0,\ c'(0) = 0,\ \lim_{a \longrightarrow 0} \frac{ac'(a)}{c(a)} \in (1, \infty),\ c'(a) > 0 \text{ for any } a \in \mathbb{R}_{++}, \\ \lim_{a \longrightarrow \infty} c(a) = \infty, \text{ and } c(a) = c(-a) \text{ for any } a \in \mathbb{R} \end{array} \right\}.$$

For any $c \in \mathcal{C}_A$, let $\kappa(c) \equiv \lim_{a \longrightarrow 0} \frac{ac'(a)}{c(a)}$ denote the limit of the elasticity of c at zero.

[39] Robustness across any $G \in \mathcal{G}$ implies that a mechanism designer who only knows that $G \in \mathcal{G}$, can infer the properties of the mechanism, regardless of the (unknown) G. The case in which G is known by the designer is a special case in which $\mathcal{G}$ is a singleton.

[40] This –minimally- stochastic element of the outcome as a function of the equilibrium strategies can be interpreted literally as a probabilistic outcome function given the vote tally. Alternatively, with a deterministic outcome function (the alternative with a greater tallied vote total is chosen with certainty), we can interpret G to capture some aggregate noise in agents' behavior, or in the tallying and recording of the votes cast so that a number of votes is assigned stochastically in addition to those cast by agents. In any of these cases, the objective function of a voter is identical, and hence the equilibrium behavior is identical as well.

Strategies. Each agent i in society N^n with size $n \in \mathbb{N}\backslash\{1\}$, facing a social choice problem to be decided according to mechanism $c \in \mathcal{C}_A$ under uncertainty $G \in \mathcal{G}$, and given the probability measure over types $P \in \mathcal{P}$, chooses an affordable action $a_i \in [-a(w_i), a(w_i)]$ as a function of the realization of her type $\theta_i \in \Theta_P$. We assume actions are taken simultaneously. Since the tuple (n, P, c, G) is common knowledge, and each type θ_i is private information to agent i, for any given tuple (n, P, c, G), a pure strategy is a mapping $s : \Theta_P \longrightarrow \mathbb{R}$ such that $s(\theta_i) \in [-a(w_i), a(w_i)]$ for any $\theta_i \in \Theta_P$, where $s(\theta_i)$ is the action taken given type θ_i according to strategy σ. Let S be the set of all feasible pure strategies. For each $s \in S$, for each $n \in \mathbb{N}\backslash\{1\}$, and for each $i \in N^n$, let $s_i = s$ denote that agent i chooses strategy s.

Following Aumann (1964), we define a mixed strategy as a measurable function $\sigma : [0,1] \times \Theta_P \longrightarrow \mathbb{R}$ such that $\sigma(r_i, \theta_i) \in [-a(w_i), a(w_i)]$ for any $\theta_i \in \Theta_P$, where r_i is an independent draw from a random variable $\tilde{r}$ with uniform distribution in the unit interval, used as a randomization device. Let Σ denote the set of all such mixed strategies. For each $i \in N^n$, let $\sigma_i \in \Sigma$ denote the mixed strategy used by agent i. Following Milgrom and Weber (1985) we define a distributional strategy as a probability measure μ_i on the Borel $\sigma-$algebra over $\Theta_P \times \mathbb{R}$ for which the marginal distribution over Θ_P is P and $\mu_i(\{\theta_i\} \times [-a(w_i), a(w_i)]) = 1$ for any $\theta_i \in \Theta_P$. That is, a distributional strategy captures the joint distribution over types and actions for each agent, with the constraint that the marginal distribution over types must coincide with the distribution of types, and that given a type, the distribution over actions must assign the entire probability to affordable actions.[41]

Definition 2 *A pure strategy $s \in S$ is **weakly monotone** if $s(w_i, v_i) \geq s(w_i, v_i')$ for any $v_i > v_i'$ and any w_i, and $|s(w_i, v_i)| \geq |s(w_i', v_i)|$ for any $w_i > w_i'$ and any v_i.*

That is, we say a strategy is weakly monotone if for a given wealth level, net contributions toward A are non-decreasing in the agents' valuation of A, and for any given valuation, the magnitude of the contribution is non-decreasing in wealth.

Payoffs. Given a society N^n with $(n, P, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{G}$ and given a mechanism $c \in \mathcal{C}_A$, for any agent $i \in N^n$, we compute the expected utility of agent i as a function of her type $\theta_i \in \Theta_P$, her action $a_i \in [-a(w_i), a(w_i)]$, and the strategy profile of every other player $\sigma_{-i} \in \Sigma^{n-1}$. Let $D \equiv \{(\theta, a) \in \Theta_P \times [-a(1), a(1)] : a \in [-a(w), a(w)]\}$ denote the set of possible (type, action) pairs (θ_i, a_i) for agent i, and let $EU_i : D \times \Sigma^n \longrightarrow \mathbb{R}$ denote the expected utility of agent i, where the expectation is over the realization of the random profile of types of other agents $\tilde{\theta}_{-i}$, the realization of their random draws $\tilde{r}_{-i}$ that determine the execution of their mixed strategies, and the resolution of uncertainty over the social decision d according to probability $G\left(\sum_{j \in N^n} a_j\right)$. For any $\theta \equiv (w, v) \in \Theta_P$ and any $a \in [-a(w), a(w)]$, $EU_i[(\theta, a, \sigma_{-i})]$ is equal to the expected utility from the social decision plus the wealth transfer.

[41] See Fudenberg and Tirole (1991) section 6.8 for a textbook treatment of mixed and distributional strategies over an uncountable action space.

$$EU_i[(\theta, a, \sigma_{-i})] = v \int\limits_{r_{-i}\in[0,1]^{n-1}} \int\limits_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod_{j\in N^n\setminus\{i\}} p(\theta_j) G\left(a + \sum_{j\in N^n\setminus\{i\}} \sigma_j(r_j, \theta_j)\right) d\theta_{-i} dr_{-i} +$$
$$\int\limits_{r_{-i}\in[0,1]^{n-1}} \int\limits_{\theta_{-i}\in(\Theta_P)^{n-1}} u\left(w - c(a) + \frac{1}{n-1} \sum_{j\in N^n\setminus\{i\}} p(\theta_j) c(\sigma_j(r_j, \theta_j))\right) d\theta_{-i} dr_{-i}, \qquad (11)$$

where the first term is the expectation over the utility derived from the uncertain social decision, and the second term is the expectation over the utility derived from the uncertain final wealth w_i^O, given that agent i has initial wealth endowment w and plays $a_i = a$, obtained by integrating over all possible realizations of the randomization device and type profile of other agents.

Game. For each tuple $(n, P, c, G) \in \mathbb{N}\setminus\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, let $\Gamma^{(n,P,c,G)}$ denote the game played by the n players in society N^n, with pure strategy set S for each agent, and expected utility given by EU_i in Expression (11) for each $n \in \mathbb{N}\setminus\{1\}$ and each $i \in N^n$.

Best responses. Given a tuple $(n, P, c, G) \in \mathbb{N}\setminus\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, and given game $\Gamma^{(n,P,c,G)}$, for each $i \in N^n$, for any $\theta \in \Theta_P$, for any $\sigma_{-i} \in \Sigma^{n-1}$, define the set of best response actions by agent i with type $\theta_i = \theta$ to the strategy profile σ_{-i} by

$$BR_i(\theta, \sigma_{-i}) \equiv \operatorname*{arg\,max}_{a\in[-a(w_i), a(w_i)]} EU_i[(\theta, a, \sigma_{-i}))].$$

Since the game is symmetric, for any symmetric strategy profile σ_{N^n}, the best response does not depend on the agent's identity, so for any symmetric σ_{N^n} and for any $\theta \in \Theta_P$, we can use $BR(\theta, \sigma_{-i})$ to denote $BR_i(\theta, \sigma_{-i})$.

Equilibria. For any tuple $(n, P, c, G) \in \mathbb{N}\setminus\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, a mixed strategy profile $\sigma^*_{N^n} \in \Sigma^n$ constitutes an **interim Bayes Nash equilibrium** of game $\Gamma^{(n,P,c,G)}$ if

$$EU_i[(\theta, a, \sigma^*_{-i})] \geq EU_i[(\theta, a', \sigma^*_{-i})]$$

for any $a \in [-a(w), a(w)]$ s.t. $L(\{r \in [0, 1] : \sigma^*_i(r, \theta) = a\}) > 0$, for any $a' \in [-a(w), a(w)]$, for any type realization $\theta = (w, v) \in \Theta_P$, for any $i \in N^n$.

That is, for any realization of agent i's type, mixed strategy σ^*_i assigns strictly positive probability only to actions that maximize the expected utility of agent i for that type.

For any tuple $(n, P, c, G) \in \mathbb{N}\setminus\{1\} \times \mathcal{P} \times C \times \mathcal{G}$, a mixed strategy profile $\sigma^*_{N^n} \in \Sigma^n$ constitutes an **ex-ante Bayes Nash equilibrium** of game $\Gamma^{(n,P,c,G)}$ if

$$E_{(\tilde{r},\tilde{\theta})}[EU_i[(\tilde{\theta}, \sigma^*_i(\tilde{r}, \tilde{\theta}), \sigma^*_{-i})]] \geq E_{(\tilde{r},\tilde{\theta})}[EU_i[(\tilde{\theta}, \sigma(\tilde{r}, \tilde{\theta}), \sigma^*_{-i})]]$$

for any $\sigma \in \Sigma$, for any $i \in N^n$, where the expectation is with respect to the realization of the randomization device for agent i and the realization of her type. That is, by playing $\sigma_i = \sigma^*_i$, each agent maximizes her expected utility evaluated before her type is drawn. An ex-ante Bayes Nash equilibrium allows for suboptimal play by types with probability measure zero.

Let $BNE^{(n,P,c,G)} \subseteq \Sigma^n$ denote the set of interim Bayes Nash Equilibria of game $\Gamma^{(n,P,c,G)}$. We are interested in the subset of symmetric interim BNE in which all agents play the same strategy $\sigma \in \Sigma$. Let $E^{(n,,P,c,G)} \subseteq \Sigma$ denote the set of mixed strategies that constitute a symmetric interim Bayes Nash equilibrium of game $\Gamma^{(n,P,c,G)}$. Hereafter, unless explicitly labeled otherwise, an "equilibrium" is always a strategy $\sigma \in E^{(n,P,c,G)}$.

Sequence of societies. We establish results for sufficiently large societies. For a given society size $n \in \mathbb{N}\backslash\{1\}$, the pair $(P, G) \in (\mathcal{P}, \mathcal{G})$ identifies the specific environment in which the social choice takes place. The pair (P, G) is common knowledge among the agents in the society, but is unobserved by the mechanism designer, who only knows $\mathcal{P}$ and $\mathcal{G}$. The institutional design problem is to design a mechanism that has desirable properties for any $(P, G) \in \mathcal{P} \times \mathcal{G}$, for any sufficiently large n.

Social Choice correspondences. For any $n \in \mathbb{N}$, a social choice correspondence $SC^n : (\Theta)^n \rightrightarrows \{A, B\}$ maps a profile of types θ_{N^n} into the subset of normatively desirable social decisions $SC^n(\theta_{N^n}) \subseteq \{A, B\}$. Let $SC \equiv \{SC^n\}_{n=1}^{\infty}$ denote a sequence of social choice correspondences, and let $\mathcal{SC}$ denote the set of all such sequences.

We say that two sequences of social choice correspondences $SC, SC' \in \mathcal{SC}$ are asymptotically equivalent if the probability that they select the same outcome converges to one, as n diverges to ∞. We say a property holds generically if it holds in an open dense subset of the set under consideration. To formally define generic asymptotic equivalence of SC and SC' over $\mathcal{P}$, we need to define more structure on $\mathcal{P}$.

We consider the metric space $(\mathcal{P}, d_{TV})$ with distance function $d_{TV} : \mathcal{P} \times \mathcal{P} \longrightarrow [0, 1]$ defined by $d_{TV}(P, P') \equiv \sup_{A \in B(\Theta)} |P(A) - P'(A)|$.[42] A subset $\mathcal{P}^D \subset \mathcal{P}$ is dense in $\mathcal{P}$ if the closure of $\mathcal{P}^D$ is equal to $\mathcal{P}$ (so any probability measure $P \in \mathcal{P}\backslash\mathcal{P}^D$ is the limit of a sequence of measures in $\mathcal{P}^D$). We can now precisely define the desired asymptotic equivalence notion.

Definition 3 *For any $P \in \mathcal{P}$, two sequences of social choice correspondences $SC \in \mathcal{SC}$ and $SC' \in \mathcal{SC}$ are **asymptotically equivalent** with respect to P if* $\lim_{n \longrightarrow \infty} \Pr\left[SC(\tilde{\theta}_{N^n}) = SC'(\tilde{\theta}_{N^n})\right] = 1$.

We say that SC and SC' are generically asymptotically equivalent if they are asymptotically equivalent for any P in an open dense set $\mathcal{P}^D \subseteq \mathcal{P}$, and for any $SC \in \mathcal{SC}$ we let $Eq(SC) \equiv \{SC' \in \mathcal{SC} : SC \text{ and } SC' \text{ are generically asymptotically equivalent}\}$ denote the set of sequences of social choice correspondences that are generically asymptotically equivalent to SC.

For ease of exposition, and since all our results are asymptotic, we refer to generically asymptotically equivalent sequences as "generically equivalent."

[42] This is the "total variation" metric over a set of probability measures.

Implementability. We say that a vote-buying mechanism $c \in \mathcal{C}_A$ asymptotically implements a sequence of social choice correspondences $SC \in \mathcal{SC}$ over a given subdomain $\hat{\mathcal{P}} \subseteq \mathcal{P}$ of probability measures over types if two conditions hold: i) an equilibrium exists for any sufficiently large society; and ii) in any sequence of equilibria, the probability that the social decision coincides with the alternative chosen by SC converges to one. For any subclass of vote-buying mechanisms $C \subseteq \mathcal{C}$, we say that a sequence $SC \in \mathcal{SC}$ is implementable by C over $\hat{\mathcal{P}}$ if there exists $c \in C$ that implements SC over $\hat{\mathcal{P}}$.

For any triple $(n, P, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{G}$, for any mixed strategy $\sigma \in \Sigma$, and given that each agent $i \in N^n$ plays $\sigma_i = \sigma$, let $\tilde{d}_P^n(\sigma, \tilde{r}_{N^n}, \tilde{\theta}_{N^n})$ be the social decision, which is a random variable that depends on the realization of the random type profile $\tilde{\theta}_{N^n}$, the realization of randomization device draws $\tilde{r}_{N^n}$, and on the realization of the outcome given probability $G\left(\sum_{i=1}^{n} a_i\right)$. The formal definition of implementation is then as follows.

Definition 4 *For any $\hat{\mathcal{P}} \subseteq \mathcal{P}$, a vote-buying mechanism $c \in \mathcal{C}_A$* ***asymptotically implements*** *a sequence of social choice correspondences $SC \in \mathcal{SC}$ over $\hat{\mathcal{P}}$ if for any $(P, G) \in \hat{\mathcal{P}} \times \mathcal{G}$,*
i) there is $n_{P,G} \in \mathbb{N}$ such that for any $n \geq n_{P,G}$, the set of equilibria $E^{(n,P,c,G)}$ is non empty, and
ii) for any $\varepsilon \in (0,1)$ and for any sequence of equilibria $\{\sigma^n\}_{n=\hat{n}}^{\infty}$, there exists $n_{\varepsilon,P,G} \in \mathbb{N}$ such that for any $n > n_{\varepsilon,P,G}$, $\Pr[\tilde{d}_P^n(\sigma^n, \tilde{r}_{N^n}, \tilde{\theta}_{N^n}) = SC^n(\tilde{\theta}_{N^n})] > 1 - \varepsilon$.

For any subset of vote-buying mechanisms $C \subseteq \mathcal{C}_A$, and for any sequence of social choice correspondences $SC \in \mathcal{SC}$, we say that the SC is asymptotically implementable by C over $\hat{\mathcal{P}}$ if there exists a mechanism $c \in C$ that asymptotically implements SC over $\hat{\mathcal{P}}$.

Since our implementation results are always asymptotic, if a mechanism c asymptotically implements SC, then we say simply that c "implements SC."

This implementation notion requires that, if the society is sufficiently large, the outcome in every equilibrium of the game induced by the mechanism must be the outcome desired by the social choice rule with probability arbitrarily close to one, for any probability measure over types. Depending on the domain of probability measures $\hat{\mathcal{P}}$ under consideration, such robustness across societies may not be attainable. We then seek, as a second best, a mechanism that works for most societies in the domain under consideration.

We define generic asymptotic implementability accordingly.

Definition 5 *A vote-buying mechanism $c \in \mathcal{C}_A$* ***asymptotically implements*** *a sequence of social choice correspondences SC* ***generically*** *if there exists an open $\mathcal{P}^D$ dense in $\mathcal{P}$ such that c implements SC over $\mathcal{P}^D$.*

For any $C \subseteq \mathcal{C}_A$, and for any $SC \in \mathcal{SC}$, we say that the sequence of social choice correspondences SC is generically asymptotically implementable by C if there exists a mechanism $c \in C$ that generically asymptotically implements SC.

If a mechanism c asymptotically implements a sequence of social choice correspondences SC generically, we say simply that c "implements SC generically."

We characterize the set of sequences of social choice correspondences that are generically implementable by vote-buying mechanisms.

Define the sequence of social choice correspondences $SC_\rho \equiv \{SC_\rho^n\}_{n=1}^\infty$ by

$$SC_\rho^n(\theta_{N^n}) = \begin{cases} B \text{ if } \sum\limits_{i\in N^n} sgn(v_i)\left|\frac{v_i}{u'(w_i)}\right|^\rho < 0, \\ \{A,B\} \text{ if } \sum\limits_{i\in N^n} sgn(v_i)\left|\frac{v_i}{u'(w_i)}\right|^\rho = 0, \\ A \text{ if } \sum\limits_{i\in N^n} sgn(v_i)\left|\frac{v_i}{u'(w_i)}\right|^\rho > 0. \end{cases} \tag{12}$$

Define the class of all sequences that are generically asymptotically equivalent to SC_ρ for some $\rho \in \mathbb{R}_{++}$ (Definition 3). Formally,

$$\mathcal{SC}_{\mathbb{R}_{++}} \equiv \bigcup_{\rho\in(0,\infty)} Eq(SC_\rho).$$

In our main result we show that class $\mathcal{SC}_{\mathbb{R}_{++}}$ is the class of sequences of social choice rules generically implementable by vote-buying mechanisms. We restate Theorem 1, now providing a complete characterization of the set of class of rules generically implementable by the class of admissible vote-buying mechanisms, and by each specific vote-buying mechanism.

Theorem 1 *Any $SC \in \mathcal{SC}$ is generically implementable by $\mathcal{C}_A$ if and only if $SC \in \mathcal{SC}_{\mathbb{R}_{++}}$. Further, any $c \in C_A$ generically implements $SC \in \mathcal{SC}$ if and only if SC is generically equivalent to $SC_{\frac{1}{\kappa(c)-1}}$.*

Our weaker statement in the main body of the paper only noted that each admissible vote-buying mechanism c with limit elasticity $\kappa(c)$ generically implements the sequence of rules $SC_{\frac{1}{\kappa(c)-1}}$. The restatement with a complete characterization here clarifies that in addition to these sequences of rules, only sequences of rules asymptotically equivalent to one of these are also generically implementable.

3. Proofs

We proceed to prove our main result in nine steps.

One - We first establish equilibrium existence, in two substeps: first, in Lemma 1 we establish existence of a symmetric ex-ante Bayes Nash equilibrium. It shortly follows that an interim Bayes Nash equilibrium, henceforth an "equilibrium", exists as well (Lemma 2).

Two - We show that any equilibrium strategy is weakly monotone in valuation (Lemma 3) and the magnitude of the greatest best response is weakly monotone in wealth (Lemma 4).

Three - We show that any equilibrium is "almost pure": a probability measure one of types play a pure strategy (Lemma 5). We "purify" any mixed equilibrium by identifying a pure equilibrium associated to it that coincides with it for a probability measure one of types (Lemma 6).

Steps four through six establish properties of these associated pure equilibria, before returning to all equilibria at step seven.

Four - We show that marginal pivotality (marginal probability of affecting the decision by changing one's action), and with it, equilibrium actions, converge to zero in the size of the society (Lemma 8).

Five - We prove that the ratio of marginal costs across two agents converges to the ratio of their marginal rates of substitution between the social decision and wealth (Lemma 10).

Six - After an auxiliary technical lemma (Lemma 11), we establish a key intermediate result: equilibrium actions are asymptotically linear in the power ρ of the marginal rate of substitution between the social decision and wealth (Lemmas 12 and Lemma 13).

Seven - After two technical lemmas (Lemmas 14 and 15) we establish a sufficient condition for a sequence of social choice correspondences to be implemented over a generic subset of probability measures over types (Proposition 1).

Eight - We find a necessary condition for such implementation (Proposition 2).

Nine - We show that the necessary condition is sufficient for generic implementability, establishing our main result (Theorem 1).

Lemma 1 *For any tuple $(n, P, c, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, a symmetric ex-ante Bayes Nash equilibrium of game $\Gamma^{(n,P,c,G)}$ exists.*

Proof. For any tuple $(n, P, c, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, and for each agent $i \in N^n$, consider the payoff function $\pi_i : \{((w,v), a_{N^n}) \in \Theta_P \times [-a(1), a(1)]^n : a_i \in [-a(w), a(w)]\} \longrightarrow \mathbb{R}$, where $\pi_i(\theta, a_{N^n})$ is the expected utility for agent i given that agent i's type is $\theta_i = \theta$ and that agents take the profile of actions a_{N^n}. Further, for any agent $i \in N^n$, and for any type realization $\theta \in \Theta_P$, define the function $\pi_i^\theta : [-a(w), a(w)] \times [-a(1), a(1)]^{n-1} \longrightarrow \mathbb{R}$ by $\pi_i^\theta(a_{N^n}) \equiv \pi_i(\theta, a_{N^n})$, and define the class of functions $\Pi_i \equiv \left\{\pi_i^\theta\right\}_{\theta \in \Theta_P}$. Since the range of possible wealth outcomes is bounded, and since the utility over wealth u is continuously differentiable, it follows that for each $i \in N^n$ and for each $\theta \in \Theta_P$, function π_i^θ and its derivatives $\frac{\partial \pi_i^\theta}{\partial a_k}$ for any $k \in N^n$, are all bounded as well. In consequence, the class of functions Π_i is equicontinuous: for any $a_{N^n} \in [-a(1), a(1)]^n$ and for any $\varepsilon \in \mathbb{R}_{++}$, there exists $\delta_\varepsilon \in \mathbb{R}_{++}$ such that for any $\theta \in \Theta_P$,

$$d_{\mathbb{R}^n}(a_{N^n}, a'_{N^n}) < \delta_\varepsilon \text{ implies } d_{\mathbb{R}^2}(\pi_i^\theta(a_{N^n}), \pi_i^\theta(a'_{N^n})) < \varepsilon,$$

where $d_{\mathbb{R}^n}$ and d_{R^2} are the standard Euclidean metric over, respectively, $\mathbb{R}^n$ and $\mathbb{R}^2$.[43]

[43] Let (X, d_X) and (Y, d_Y) be two metric spaces. A class of functions $\{h_\theta\}_{\theta \in \Theta}$ with domain X and codomain Y is equicontinuous if for every $x \in X$, and every $\varepsilon \in \mathbb{R}_{++}$, there exists $\delta_\varepsilon \in \mathbb{R}_{++}$ such that for any $\theta \in \Theta$, $d_X(x, x') < \delta_\varepsilon$ implies $d_Y(h_\theta(x), h_\theta(x')) < \varepsilon$.

Further, since the valuations are independently drawn, our game also satisfies Milgrom and Weber's R2 condition of "absolutely continuous information", by Proposition 3b of Milgrom and Weber (1985), and given that all agents' actions must be affordable, the action space is compact. Hence, the conditions of Theorem 1 in Milgrom and Weber (1985) apply and, in particular, the set of distributional strategies is compact in the weak topology.[44]

Now consider a normal form game $\hat{\Gamma}^{(n,P,c,G)}$ played by the set of players N^n, in which the set of pure strategies for each agent is the set of distributional strategies in $\Gamma^{(n,P,c,G)}$ and the payoff functions in game $\hat{\Gamma}^{(n,P,c,G)}$ are such that for each player $i \in N^n$, the payoff in game $\hat{\Gamma}^{n,P,c,G}$ given any pure strategy profile, is defined to be the expected payoff attained by the corresponding distributional strategy in game $\Gamma^{(n,P,c,G)}$. Compactness of the set of distributional strategies in $\Gamma^{(n,P,c,G)}$ implies compactness of the set of pure strategies in $\hat{\Gamma}^{(n,P,c,G)}$. Note that the payoff function in game $\hat{\Gamma}^{(n,P,c,G)}$ is linear and hence quasiconcave: for any two distributions μ and μ' over $\Theta_P \times [-a(1), a(1)]$ such that $\mu(\theta, a) = \mu'(\theta, a) = 0$ for any $a \in \mathbb{R}$ such that $c(a) > w_i$, and for any $\lambda \in (0, 1)$, if we define a distributional strategy $\mu_\lambda \equiv \lambda\mu + (1-\lambda)u'$, then the payoff from playing μ_λ is a convex combination of the payoff of μ and the payoff of μ', so it is weakly greater than the minimum of the payoff playing μ and the playoff paying μ'. Since payoffs in game $\hat{\Gamma}^{(n,P,c,G)}$ satisfy quasiconcavity, they satisfy Reny's (1999) weaker condition of "diagonal quasiconcavity." Further, by continuity of the utility function u, of the cost function c, and of the probability function G, the payoff function in game $\hat{\Gamma}^{(n,P,c,G)}$ is continuous in the strategy chosen by agent i, and thus it is upper semi-continuous and satisfies Reny's weaker condition of "diagonal payoff security." It then follows that Corollary 4.3 in Reny (1999) applies, and game $\hat{\Gamma}^{(n,P,c,G)}$ has a symmetric pure Nash equilibrium.

The symmetric pure Nash equilibrium of game $\hat{\Gamma}^{(n,P,c,G)}$ is a symmetric equilibrium in distributional strategies of game $\Gamma^{(n,P,c,G)}$. Each distributional strategy corresponds to a class of mixed strategies (Milgrom and Weber 1985), and in particular the symmetric equilibrium in distributional strategies of game $\Gamma^{(n,P,c,G)}$ corresponds to a class of ex-ante Bayes Nash equilibria in mixed strategies, that are symmetric over probability measure one of type realizations, but allow free play for each player for a measure zero of types. Among this class of ex-ante Bayes Nash equilibria, the subclass in which all agents play the same actions also for the measure zero of types for which their actions are unconstrained by equilibrium requirements, constitute the desired symmetric ex-ante Bayes Nash equilibrium. ■

[44] Milgrom and Weber's argument for compactness of the set of distributional strategies relies on the tightness of the probability measure on the Borel sets of agent i's type space $[w_{\min}, 1] \times [-\gamma, \gamma]$, and on Prokhorov's theorem, which implies (among other results) that if the collection of measures is tight, then it is relatively compact in the topology of weak convergence (see Billingsley, Thm 5.1). Any probability measure on a complete, separable metric space is tight (Billingsley, Thm 1.3), so in particular the probability measure over types, defined on a subset of the (complete, separable) Euclidean space is tight, and thus relatively compact. A set is relatively compact if its closure is compact, and the set of distributional strategies is closed in the weak topology, so relative compactness equals compactness in this case.

Lemma 2 *For any tuple* $(n, P, c, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$*, a symmetric interim Bayes Nash equilibrium of game* $\Gamma^{(n,P,c,G)}$ *exists.*

Proof. For each $(n, F, c, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, by Lemma 1, a symmetric ex-ante Bayes Nash equilibrium of game $\Gamma^{(n,P,c,G)}$ exists. Take one such ex-ante BNE and denote it by ξ^*. Note that by playing $\sigma_i = \xi^*$, agent i optimizes at the interim stage for a measure one of types. Let $\Theta_{\xi^*} \subseteq \Theta_P$ be the subset of types (with probability measure one) over which ξ^* is interim optimal. Note that an optimal response to ξ^* is guaranteed to exist for every type by continuity of the payoff function and compactness of the action space. Define mixed strategy $\hat{\xi}$ by $\hat{\xi}((r, \theta) = \xi^*(r, \theta)$ for any $(r, \theta) \in [0, 1] \times \Theta_{\xi^*}$, letting $\hat{\xi}(r, (w, v))$ be an arbitrary best response to other players playing $\sigma_j = \xi^*$ for any $j \in N^n\backslash\{i\}$, for any $(r, \theta) \notin [0, 1] \times \Theta_{\xi^*}$. That is, $\hat{\xi}$ ties agents to optimal responses to ξ^*, even for the measure zero of types for which the ex-ante equilibrium ξ^* did not constraint their actions. Since the actions of a measure zero of types do not affect the expected payoff of other agents, a best response to ξ^* is also a best response to $\hat{\xi}$. Thus, a profile in which every agent plays $\hat{\xi}$ is a mutual best response and hence it is a symmetric pure interim BNE of game $\Gamma^{(n,P,c,G)}$. ■

For any tuple $(n, P, c, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, for any agent $i \in N^n$, for any strategy profile $\sigma_{-i} \in \Sigma^{n-1}$, and for any type $\theta \in \Theta_P$, define the greatest and smallest best responses of agent i with type $\theta_i = \theta$ to all other agents playing profile σ_{-i} respectively by

$$GBR_i(\theta_i, \sigma_{-i}) \equiv \max\{BR_i(\theta_i, \sigma_{-i})\} \text{ and } SBR_i(\theta, \sigma_{-i}) \equiv \min\{BR_i(\theta_i, \sigma_{-i})\}.$$

We drop the subindex to indicate greatest and smallest best responses (common across agents) to symmetric strategy profiles.

Note that because payoffs are continuous, and the set of affordable actions $[-a(w), a(w)]$ is compact for each type $\theta \equiv (w, v) \in \Theta_P$, for each $P \in \mathcal{P}$, the set of best responses $BR((w, v), \sigma_{-i})$ and $BR((w, v'), \sigma_{-i})$ are non-empty and compact (Aliprantis and Border 2013, theorem 2.43), so they have a maximum and a minimum, so GBR and SBR are well defined. Note as well that the greatest best response $GBR(\theta, \sigma_{-i})$ and the smallest best response $SBR(\theta, \sigma_{-i})$ are each unique.

We next establish that equilibria are weakly monotone. First, in Lemma 3 we establish weak monotonicity of the equilibrium action with response to changes in the valuation, holding wealth fixed; then in Lemma 4 we show weak monotonicity in the absolute value of the equilibrium action in response to changes in wealth, holding the valuation fixed.

Lemma 3 *For any tuple* $(n, P, c, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$*, for any equilibrium strategy* $\sigma \in E^{(n,P,c,G)}$*, for any* $w \in [w_{\min}, 1]$*, and for any* $v, v' \in [-\gamma, \gamma]$ *such that* $v' > v$*,*

$$GBR((w, v), \sigma_{-i}) \leq SBR((w, v'), \sigma_{-i}).$$

Proof. For any $(n, P, c, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, $\sigma \in E^{(n,P,c,G)}$; any $w \in [w_{\min}, 1]$; any $v, v' \in [-\gamma, \gamma]$ such that $v' > v$; and any $a \in [-a(w), GBR((w, v), \sigma_{-i})]$,

$$EU_i[((w, v), GBR((w, v), \sigma_{-i}), \sigma_{-i}))] \geq EU_i[((w, v), a, \sigma_{-i})]. \tag{13}$$

For each $j \in N^n \backslash \{i\}$, recall that $(\tilde{r}_j, (\tilde{w}_j, \tilde{v}_j))$ denotes agent j's random draw and type when considered ex-ante as a random variable as perceived by agent i, while $(r_j, (w_j, v_j))$ denotes the realization of this random draw and type as observed privately by agent j. Further, for any $a \in [-a(1), a(1)]$, and for any $\sigma_{-i} \in \Sigma^{n-1}$, let $\Pr[d = A|(a, \sigma_{-i})]$ denote the probability that the social decision is A, given that $a_i = a$ and that all other agents play σ_{-i}.

With this notation, then note that for any $a \in [-a(w), GBR((w, v), \sigma_{-i})]$, Inequality (13) holds if and only if

$$
\begin{aligned}
v \Pr[d &= A|(GBR((w,v),\sigma_{-i}),\sigma_{-i})] \\
&\quad + E_{(\tilde{w}_{-i},\tilde{v}_{-i})}\left[u\left(w + \frac{1}{n-1}\sum_{j\in N^n\backslash\{i\}} c(\sigma(\tilde{r}_j,(\tilde{w}_j,\tilde{v}_j))) - c(GBR((w,v),\sigma_{-i}))\right)\right] \\
&\geq v\Pr[d = A|(a,\sigma_{-i})] + E_{(\tilde{w}_{-i},\tilde{v}_{-i})}\left[u\left(w + \frac{1}{n-1}\sum_{j\in N^n\backslash\{i\}} c(\sigma(\tilde{r}_j,(\tilde{w}_j,\tilde{v}_j))) - c(a)\right)\right],
\end{aligned}
$$

or, rearranging terms

$$
\begin{aligned}
& v\left(\Pr[d = A|a_i = GBR((w,v),\sigma_{-i}),\sigma_{-i})] - \Pr[d = A|(a,\sigma_{-i})]\right) \qquad (14) \\
\geq\ & E_{(\tilde{w}_{-i},\tilde{v}_{-i})}\left[\begin{array}{c} u\left(w + \frac{1}{n-1}\sum_{j\in N^n\backslash\{i\}} c(\sigma(\tilde{r}_j,(\tilde{w}_j,\tilde{v}_j))) - c(a)\right) \\ -u\left(w + \frac{1}{n-1}\sum_{j\in N^n\backslash\{i\}} c(\sigma(\tilde{r}_j,(\tilde{w}_j,\tilde{v}_j))) - c(GBR((w,v),\sigma_{-i}))\right)\end{array}\right],
\end{aligned}
$$

Note that for any $a \in [-a(w), GBR((w, v), \sigma_{-i}))$,

$$\Pr[d = A|(GBR((w,v),\sigma_{-i}),\sigma_{-i})] > \Pr[d = A|(a,\sigma_{-i})]$$

and thus $v' > v$ implies

$$(v' - v)\left(\Pr[d = A|(GBR((w,v),\sigma_{-i}),\sigma_{-i})] - \Pr[d = A|(a;\sigma_{-i})]\right) > 0,$$

and since the right hand side of Inequality (14) is invariant in v, if we substitute in v' for v in Inequality (14), since $v' > v$, the inequality becomes strict, and thus,

$$EU_i[((w,v'), GBR((w,v),\sigma_{-i}),\sigma_{-i}))] > EU_i[((w,v'), a, \sigma_{-i})].$$

for any $a \in [-a(w), GBR((w, v), \sigma_{-i}))$, which implies

$$BR((w,v'),\sigma_{-i}) \subseteq [GBR((w,v),\sigma_{-i}), a(w)],$$

so in particular

$$SBR((w,v'),\sigma_{-i}) \geq GBR((w,v),\sigma_{-i})$$

as desired. ■

Lemma 4 *For any tuple $(n,P,c,G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$; for any equilibrium strategy $\sigma \in E^{(n,P,c,G)}$; for any $w, w' \in [w_{\min}, 1]$ such that $w' > w$; and for any $v \in [-\gamma, \gamma]$,*

$$|GBR((w',v),\sigma_{-i})| \geq |GBR((w,v),\sigma_{-i})|.$$

Proof. Consider any tuple $(n,P,c,G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, any equilibrium $\sigma \in E^{(n,P,c,G)}$, and any $v \in [-\gamma, \gamma]$. For any $w \in [w_{\min}, 1]$,

$$EU_i[((w,v), GBR((w,v),\sigma_{-i}), \sigma_{-i})] \geq EU_i[((w,v), a, \sigma_{-i})]$$

for any $a \in [-|GBR((w,v),\sigma_{-i})|, |GBR((w,v),\sigma_{-i})|]$, and thus,

$$\begin{aligned} & v \Pr[d = A | (GBR((w,v),\sigma_{-i}),\sigma_{-i})] \\ & + E_{(\tilde{w}_{-i},\tilde{v}_{-i})}\left[u\left(w + \frac{1}{n-1}\sum_{j \in N^n\backslash\{i\}} c(\sigma(\tilde{r}_j,(\tilde{w}_j,\tilde{v}_j))) - c(GBR((w,v),\sigma_{-i}))\right)\right] \geq \\ & v \Pr[d = A|(a,\sigma_{-i})] + E_{(\tilde{w}_{-i},\tilde{v}_{-i})}\left[u\left(w + \frac{1}{n-1}\sum_{j \in N^n\backslash\{i\}} c(\sigma(\tilde{r}_j,(\tilde{w}_j,\tilde{v}_j))) - c(a)\right)\right] \end{aligned} \tag{15}$$

for any $a \in [-|GBR((w,v),\sigma_{-i})|, |GBR((w,v),\sigma_{-i})|]$. Since u is weakly concave, for any $w' > w$, and for any $\kappa, \kappa' \in [-w, 1]$,

$$|u(w' + \kappa) - u(w' + \kappa')| \leq |u(w + \kappa) - u(w + \kappa')|,$$

so Inequality (15) implies

$$\begin{aligned} & v \Pr[d = A | (GBR((w,v),\sigma_{-i}),\sigma_{-i})] \\ & + E_{(\tilde{w}_{-i},\tilde{v}_{-i})}\left[u\left(w' + \frac{1}{n-1}\sum_{j \in N^n\backslash\{i\}} c(\sigma(\tilde{r}_j,(\tilde{w}_j,\tilde{v}_j))) - c(GBR((w,v),\sigma_{-i}))\right)\right] \geq \\ & v \Pr[d = A|(a,\sigma_{-i})] + E_{(\tilde{w}_{-i},\tilde{v}_{-i})}\left[u\left(w' + \frac{1}{n-1}\sum_{j \in N^n\backslash\{i\}} c(\sigma(\tilde{r}_j,(\tilde{w}_j,\tilde{v}_j))) - c(a)\right)\right] \end{aligned} \tag{16}$$

for any $a \in [-|GBR((w,v),\sigma_{-i})|, |GBR((w,v),\sigma_{-i})|]$. It follows from Inequality (16) that if

$$[-|GBR((w,v),\sigma_{-i})|, |GBR((w,v),\sigma_{-i})|] \bigcap BR((w',v),\sigma_{-i}) \neq \emptyset,$$

then $GBR((w,v),\sigma_{-i}) \in BR((w',v),\sigma_{-i})$, so $|GBR((w',v),\sigma_{-i})| \geq |GBR((w,v),\sigma_{-i})|$. ■

Lemma 5 *For any tuple $(n,P,c,G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, for any equilibrium strategy $\sigma \in E^{(n,P,c,G)}$, and for any player $i \in N^n$,*

$$P(\theta \in \Theta_P : BR(\theta, \sigma_{-i}) \text{ is a singleton}) = 1.$$

Proof. For any tuple $(n, P, c, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, for any agent $i \in N^n$, and for any equilibrium strategy $\sigma \in E^{n,P,c,G}$, for any $w \in [w_{\min}, 1]$, by Lemma 3, $GBR((w, v), \sigma_{-i})$ and $SBR((w, v), \sigma_{-i})$ are weakly increasing in $v \in [-\gamma, \gamma]$, and for any $v \in [-\gamma, \gamma]$, by Lemma 4, $|GBR((w, v), \sigma_{-i})|$ is weakly increasing in $w \in [w_{\min}, 1]$. Since $GBR(\theta, \sigma_{-i}) \geq 0$ for any $\theta \in [w_{\min}, 1] \times [0, \gamma]$, it follows that $GBR(\theta, \sigma_{-i})$ is weakly increasing in w for any for any $\theta \in [w_{\min}, 1] \times [0, \gamma]$. This monotonicity in both w and v over the compact space $[w_{\min}, 1] \times [0, \gamma]$ implies that GBR is continuous in θ almost everywhere with respect to the Lebesgue measure over the half space of types $[w_{\min}, 1] \times [0, \gamma]$ (Lavric 1996). An analogous argument establishes that GBR is also continuous in θ almost everywhere with respect to the Lebesgue measure over the half space of types $[w_{\min}, 1] \times [-\gamma, 0]$. Thus, for any $\sigma \in E^{n,P,c,G}$, GBR is continuous in θ almost everywhere with respect to the Lebesgue measure over Θ_P.

We want to show that $GBR(\theta, \sigma_{-i}) = SBR(\theta, \sigma_{-i}) = BR(\theta, \sigma_{-i})$ for any type $\theta \in (w_{\min}, 1) \times (-\gamma, \gamma)$ such that GBR is continuous in θ at (θ, σ_{-i}). Assume (absurd) not, so there exists $\theta \equiv (w, v) \in (w_{\min}, 1) \times (-\gamma, \gamma)$ such that GBR is continuous at (θ, σ_{-i}) and $GBR((w, v), \sigma_{-i}) \neq SBR((w, v), \sigma_{-i})$, and define $\varepsilon \equiv GBR((w, v), \sigma_{-i}) - SBR((w, v), \sigma_{-i})$, so $\varepsilon > 0$. Assume, without loss of generality, that $v \in (0, \gamma)$. Then, for any $v' \in [0, v)$, $GBR((w, v'), \sigma_{-i}) \leq SBR((w, v), \sigma_{-i})$ (by Lemma 3), which implies that $GBR((w, v), \sigma_{-i}) - GBR((w, v'), \sigma_{-i}) \geq \varepsilon$ for every $v' \in [0, v)$, and thus GBR is not continuous in v at $((w, v), \sigma_{-i})$, a contradiction, so for any (w, v) such that GBR is continuous at $(w, v) \in (w_{\min}, 1) \times (0, \gamma)$, it follows $GBR((w, v), \sigma_{-i}) = SBR((w, v), \sigma_{-i}) = BR((w, v), \sigma_{-i})$. An analogous proof establishes that $GBR((w, v), \sigma_{-i}) = SBR((w, v), \sigma_{-i})$, (so $BR((w, v), \sigma_{-i})$ is a singleton), for any $(w, v) \in (w_{\min}, 1) \times (-\gamma, 0)$ such that GBR is continuous in θ at $\theta = (w, v) \in (w_{\min}, 1) \times (-\gamma, 0)$. Thus, $BR(\theta, \sigma_{-i})$ is a singleton for any $\theta \in (w_{\min}, 1) \times (-\gamma, \gamma)$ such that GBR is continuous in θ at θ, and since (as proven above), GBR is continuous in θ almost everywhere with respect to the Lebesgue measure over Θ_P, it follows that $BR(\theta, \sigma_{-i})$ is a singleton except possibly over a set with of Lebesgue measure zero over Θ_P.

Since P is absolutely continuous in the Lebesgue measure over Θ_P, it follows that $P(\theta \in \Theta_P : BR(\theta, \sigma_{-i}) \text{ is a singleton}) = 1$, as desired. ■

By Lemma 5, the probability measure of types for which the best response to σ is not a singleton has measure zero. We define a specific purification of σ as follows.

Definition 6 *For any tuple $(n, P, c, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, and for any equilibrium strategy $\sigma \in E^{(n,P,c,G)}$, define the pure strategy $s^\sigma \in S^{n,P,c,G}$ associated to σ by $s^\sigma(\theta) \equiv GBR(\theta, \sigma_{-i})$ for each $\theta \in \Theta_P$.*

Lemma 6 *For any tuple $(n, P, c, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, and for any equilibrium strategy $\sigma \in E^{(n,P,c,G)}$, the associated pure strategy $s^\sigma \in S^{n,P,c,G}$ is weakly monotone and is an equilibrium.*

Proof. Since $\sigma_{N^n} = (\sigma, \sigma, ..., \sigma)$ is an equilibrium, by definition of equilibrium, $L(\{r \in [0, 1] : \sigma(r, \theta) \in BR(\theta, \sigma_{-i})\}) = 1$ for each $\theta \in \Theta_P$. Since for any $\theta \in \Theta_P$ such that $BR(\theta, \sigma_{-i})$ is a singleton, $BR(\theta, \sigma_{-i}) = GBR(\theta, \sigma_{-i}) = s^\sigma(\theta)$, by Lemma 5, $P(\theta \in \Theta_P : L(\{r \in [0, 1] :$

$\sigma(r,\theta) = s^{\sigma}(\theta)\}) = 1) = 1$, that is, the probability measure over the set of types for which σ plays s^{σ} with probability one is itself one. By construction, the pure strategy profile s_{N^n} such that each agent plays s^{σ} is such that every agent is best responding to σ. Because σ and s^{σ} play the same action with probability one for a measure one of types, and because the actions of a measure zero of types do not affect the expected utility of any other agent, a best response to σ is also a best response to s^{σ}, so s^{σ} is a mutual best response, and it constitutes a pure equilibrium with the same expected payoffs, and the same probability over actions, as σ. Note that s^{σ} is monotone by Lemma 3 and Lemma 4. ■

Equilibria σ and s^{σ} are indistinguishable to an outside observer, and in particular to the social planner, and they exhibit the same welfare properties. Thereafter, for any equilibrium σ, we work with its associated pure equilibrium s^{σ}.

Note that for any $(n, P, G) \in \mathbb{N}\backslash\{1\} \times \mathcal{P} \times \mathcal{G}$, for any player $i \in N^n$ with wealth w_i, for any action $a \in [-a(w_i), a(w_i)]$, and for any strategy $s \in S$, the probability that the outcome is A given that i plays a and all other players follow strategy s is

$$\begin{aligned} \Pr[d &= A | a_i = a, s_j = s \ \forall j \in N^n\backslash\{i\}] \\ &= \int\limits_{\theta_{-i}\in(\Theta_P)^{n-1}} G\left(\sum_{j\in N^n\backslash\{i\}} s(\theta_j) + a\right) \prod_{j\in N^n\backslash\{i\}} p(\theta_j) d\theta_{-i}. \end{aligned}$$

Definition 7 *Define the marginal pivotality of agent i at $a_i = a$ and $s_j = s \forall j \in N^n\backslash\{i\}$, denoted by $Piv[a,s]$, as the derivative of* $\Pr[d = A|a_i = a, s_j = s \ \forall j \in N^n\backslash\{i\}]$ *with respect to a,*

$$Piv[a,s] \equiv \frac{d \Pr[d = A|a_i = a, s_j = s \ \forall j \in N^n\backslash\{i\}]}{da}.$$

As society grows arbitrarily large, in equilibrium the marginal pivotality and the equilibrium actions converge to zero. We use the following version of the Berry-Esseen theorem (Berry 1941; Esseen 1942). For any $z \in \mathbb{R}_{++}$ and any $x \in \mathbb{R}$, let $N[0,z](x)$ denote the value at x of the cumulative distribution of a normal distribution with mean zero and variance z.

Lemma 7 (Berry-Esseen Theorem) *For any finite set of independent, identically distributed random variables $\{\tilde{y}_1, ..., \tilde{y}_k\}$ with $E[\tilde{y}_1] = 0$, $E[(\tilde{y}_1)^2] > 0$ and $E[|\tilde{y}_1|^3] \in \mathbb{R}_{++}$, if we define F to be the cumulative distribution of $\frac{\sum_{i=1}^{k} \tilde{y}_i}{\sqrt{k}\sqrt{E[(\tilde{y}_1)^2]}}$, then for any $x \in \mathbb{R}$ and for any $k \in \mathbb{N}$,*

$$|F(x) - N[0,1](x)| \leq \frac{E[|\tilde{y}_1|^3]}{\sqrt{k}\,(E[(\tilde{y}_1)^2])^{\frac{3}{2}}}.$$

That is, given a sequence of independent and identically distributed random variables, with mean zero, positive variance, and finite third absolute moment, the cumulative distribution function of the mean multiplied by the square root of the number of draws, is close to the standard normal distribution. We use this result to approximate the shape of the distribution of equilibrium total votes.

Lemma 8 *For any tuple $(P, c, G) \in \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, and for any sequence $\{s^n\}_{n=1}^{\infty}$ of weakly monotone equilibrium strategies $s^n \in E^{(n,P,c,G)}$ for each $n \in \mathbb{N}\backslash\{1\}$, $\lim_{n\to\infty} Piv[a, s^n] = 0$ for each $a \in [-a(1), a(1)]$ and $\lim_{n\to\infty} s^n(\theta) = 0$ for each $\theta \in \Theta_P$.*

Proof. Note that since $s(\theta)$ does not vary with a, the total derivative of the probability $\Pr[d = A|a_i = a, s_j = s \; \forall j \in N^n\backslash\{i\}]$ with respect to a is equal to its partial derivative,

$$\begin{aligned} Piv[a, s] &= \frac{\partial \Pr[d = A|a_i = a, s_j = s \; \forall j \in N^n\backslash\{i\}]}{\partial a} \qquad (17) \\ &= \int_{\theta_{-i}\in(\Theta_P)^{n-1}} g\left(\sum_{j\in N^n\backslash\{i\}} s(\theta_j) + a\right) \prod_{j\in N^n\backslash\{i\}} p(\theta_j) d\theta_{-i}. \end{aligned}$$

For any infinite sequence $\{x^n\}_{n=1}^{\infty}$ such that $x^n \in \mathbb{R}_{++}$ for any $n \in \mathbb{N}$, for each sequence of equilibrium strategies $s^n \in E^{(n,P,c,G)}$ and for each $n \in \mathbb{N}\backslash\{1\}$, define $\Theta_P^1(x^n, n) \equiv \left\{\theta_{-i} \in (\Theta_P)^{n-1} : \sum_{j\in N^n\backslash\{i\}} s^n(\theta_j) < -x^n\right\}$, $\Theta_P^2(x^n, n) \equiv \left\{\theta_{-i} \in (\Theta_P)^{n-1} : \sum_{j\in N^n\backslash\{i\}} s^n(\theta_j) \in [-x^n, x^n]\right\}$ and $\Theta_P^3(x^n, n) \equiv \left\{\theta_{-i} \in (\Theta_P)^{n-1} : \sum_{j\in N^n\backslash\{i\}} s^n(\theta_j) > x^n\right\}$, where the superscripts $k \in \{1, 2, 3\}$ in $\Theta_P^k(x, n)$ are labels (not exponents). Then, for any $n \in \mathbb{N}\backslash\{1\}$, from Equality (17) we obtain

$$\begin{aligned} Piv[a, s^n] &= \int_{\theta_{-i}\in\Theta_P^1(x^n,n)} g\left(\sum_{j\in N^n\backslash\{i\}} s^n(\theta_j) + a\right) \prod_{j\in N^n\backslash\{i\}} p(\theta_j) d\theta_{-i} \\ &\quad + \int_{\theta_{-i}\in\Theta_P^2(x^n,n)} g\left(\sum_{j\in N^n\backslash\{i\}} s^n(\theta_j) + a\right) \prod_{j\in N^n\backslash\{i\}} p(\theta_j) d\theta_{-i} \\ &\quad + \int_{\theta_{-i}\in\Theta_P^3(x^n,n)} g\left(\sum_{j\in N^n\backslash\{i\}} s^n(\theta_j) + a\right) \prod_{j\in N^n\backslash\{i\}} p(\theta_j) d\theta_{-i}. \end{aligned}$$

Note that, by the definition of class $\mathcal{G}$, since $G \in \mathcal{G}$, $\lim_{x\to-\infty} G(x) = 0$ and $\lim_{x\to-\infty} G(x) = 1$, and thus, for any strictly increasing sequence $\{x_n\}_{n=1}^{\infty}$ that diverges to ∞, and for any $i \in N^n$

with wealth w_i and any $a \in [-a(w_i), a(w_i)]$

$$\lim_{n \longrightarrow \infty} \int\limits_{\theta_{-i} \in \Theta_P^1(x^n, n)} g\left(\sum_{j \in N^n \backslash \{i\}} s^n(\theta_j) + a\right) \prod_{j \in N^n \backslash \{i\}} p(\theta_j) d\theta_{-i}$$

$$= \lim_{n \to \infty} \int\limits_{\theta_{-i} \in \Theta_P^3(x^n, n)} g\left(\sum_{j \in N^n \backslash \{i\}} s^n(\theta_j) + a\right) \prod_{j \in N^n \backslash \{i\}} p(\theta_j) d\theta_{-i} = 0,$$

and thus

$$\lim_{n \longrightarrow \infty} Piv[a, s^n] = \lim_{n \longrightarrow \infty} \int\limits_{\theta_{-i} \in \Theta_P^2(x^n, n)} g\left(\sum_{j \in N^n \backslash \{i\}} s^n(\theta_j) + a\right) \prod_{j \in N^n \backslash \{i\}} p(\theta_j) d\theta_{-i} \tag{18}$$

We want to show that there exists a sequence $\{x^n\}_{n=2}^{\infty}$ with $x^n \in \mathbb{R}_{++}$ for any $n \in \mathbb{N} \backslash \{1\}$ and $\lim\limits_{n \longrightarrow \infty} x^n = \infty$ such that for any $n \in \mathbb{N} \backslash \{1\}$, and for any $a \in [-a(1), a(1)]$,

$$\int\limits_{\theta_{-i} \in \Theta_P^2(x^m, n)} g\left(\sum_{j \in N^m \backslash \{i\}} s^n(\theta_j) + a\right) \prod_{j \in N^m \backslash \{i\}} p(\theta_j) d\theta_{-i} \leq \frac{1}{x^n}.$$

Assume (absurd) not, that is, assume that for some $a \in [-a(1), a(1)]$, there exist $x \in \mathbb{R}_{++}$, and an infinite subsequence $\{s^{n(\tau)}\}_{\tau=1}^{\infty}$ of $\{s^n\}_{n=2}^{\infty}$ with $n : \mathbb{N} \backslash \{1\} \longrightarrow \mathbb{N}$ strictly increasing such that

$$\int\limits_{\theta_{-i} \in \Theta_P^2(x, n(\tau))} g\left(\sum_{j \in N^{n(\tau)} \backslash \{i\}} s^{n(\tau)}(\theta_j) + a\right) \prod_{j \in N^{n(\tau)} \backslash \{i\}} p(\theta_j) d\theta_{-i} > \frac{1}{x}$$

for any $\tau \in \mathbb{N}$.

Then it follows that for any $a' \in [-a(1), a(1)]$,

$$\int\limits_{\theta_{-i} \in \Theta_P^2(x + a(1), n(\tau))} g\left(\sum_{j \in N^{n(\tau)} \backslash \{i\}} s^{n(\tau)}(\theta_j) + a'\right) \prod_{j \in N^{n(\tau)} \backslash \{i\}} p(\theta_j) d\theta_{-i} > \frac{1}{x} \tag{19}$$

for any $\tau \in \mathbb{N}$.

For each $\tau \in \mathbb{N}$, given that all other agents play $s^{n(\tau)}$, the optimization problem of agent

i with type $\theta_i = (w, v) \in \Theta_P$ is

$$\max_{a \in [-a(w), a(w)]} v \int_{\theta_{-i} \in (\Theta_P)^{n(\tau)-1}} \prod_{j \in N^{n(\tau)} \backslash \{i\}} p(\theta_j) G\left(a + \sum_{j \in N^{n(\tau)} \backslash \{i\}} s^{n(\tau)}(\theta_j)\right) d\theta_{-i} + \tag{20}$$

$$\int_{\theta_{-i} \in (\Theta_P)^{n(\tau)-1}} u\left(w - c(a) + \frac{1}{n(\tau)-1} \sum_{j \in N^{n(\tau)} \backslash \{i\}} p(\theta_j) c(s^{n(\tau)}(\theta_j))\right) d\theta_{-i}.$$

For any $a \in [-a(1), a(1)]$, define the marginal benefit at a as the derivative with respect to action a of the first term of the summation in this objective function, evaluated at value a, and similarly define the marginal utility cost to be the derivative with respect to action a of the second term of the summation in this objective function, again evaluated at value a. This marginal benefit is $vPiv[a, s^{n(\tau)}]$, and by Inequality (19),

$$vPiv[a, s^{n(\tau)}] > v\frac{1}{x}. \tag{21}$$

By assumptions on the cost function, the marginal cost c' is continuous, and $c'(0) = 0$, so for any $\lambda \in \mathbb{R}_{++}$, there exists $\delta_c \in \mathbb{R}_{++}$ such that $c'(a) \leq \lambda$ for any $a \in [-\delta_c, \delta_c]$. Since u and u' are continuous and over any bounded interval contained in $\mathbb{R}_{++}$, it follows that the marginal expected utility cost is also zero at $a = 0$, and it is also continuous; therefore, for any $v \in [-\gamma, 0) \cup (0, \gamma]$, there exists $\delta(v) \in \mathbb{R}_{++}$ such that the absolute value of the marginal expected utility cost is less than $v\frac{1}{x}$ (and thus, by Inequality (21), less than the marginal benefit) for any $a \in [-\delta(v), \delta(v)]$.

So, for any $v \in [-\gamma, 0) \cup (0, \gamma]$, the solution to the individual maximization problem by agent i with wealth type $(w_{\min}, v)$ is such that, for any $\tau \in \mathbb{N}$, the equilibrium strategy $s^{n(\tau)}$ satisfies $|s^{n(\tau)}(w_{\min}, v)| \in [\delta(v), a(w_{\min})] \subset \mathbb{R}_{++}$, with $sgn(s^{n(\tau)}((w_{\min}, v))) = sgn(v)$, where sgn denotes the sign function with $sgn(y) = 1$ if $y \in \mathbb{R}_+$ and $sgn(y) = -1$ if $-y \in \mathbb{R}_{++}$.

We have thus far established that our (absurd) assumption that marginal pivotality does not converge to zero implies that any type with non-zero valuation takes an action bounded away from zero. We shall next show that this implies that marginal pivotality converges to zero.

Let $\tilde{\theta}$ be an arbitrary draw from Θ_P according to probability measure P. For each $n \in \mathbb{N}\backslash\{1\}$, let $E[s^n(\tilde{\theta})]$ denote the expectation of the random variable $s^n(\tilde{\theta})$; define

$$q^n(\tilde{\theta}) \equiv s^n(\tilde{\theta}) - E[s^n(\tilde{\theta})];$$

let $E[q^n(\tilde{\theta})]$ and $Var[q^n(\tilde{\theta})]$ respectively denote the expectation and variance of $q^n(\tilde{\theta})$; and note $E[q^n(\tilde{\theta})] = 0$. In addition, for each $n \in N\backslash\{1\}$ and for each $k \in \{1, .., n\}$, define as well the independent, identically distributed random variables $q^n(\tilde{\theta}_k) \equiv s^n(\tilde{\theta}_k) - E[s^n(\tilde{\theta})]$. Note that neither expectation nor the variance of $q^n(\tilde{\theta}_k)$ depend on k, and thus they are equal, respectively, to $E[q^n(\tilde{\theta}_k)] = E[q^n(\tilde{\theta})] = 0$ and $Var[q^n(\tilde{\theta}_k)] = Var[q^n(\tilde{\theta})]$.

Observe that by weak monotonicity of $s^{n(\tau)}$ (Lemma 6), it follows that for every $\tau \in \mathbb{N}$, for any $\hat{v} \in (-\gamma, 0)$ and for any $\theta \equiv (w, v) \in [w_{\min}, 1] \times [-\gamma, \hat{v}]$, it follows $s^{n(\tau)}(\theta) \in [-a(w), -\delta(\hat{v})]$; and for every $\tau \in \mathbb{N}$, for any $\hat{v} \in (0, \gamma)$ and any $\theta \equiv (w, v) \in [w_{\min}, 1] \times [\hat{v}, \gamma]$, it follows $s^{n(\tau)}(\theta) \in [\delta(\hat{v}), a(w]$. Therefore, there exists $\hat{\delta} \in \mathbb{R}_{++}$ such that $Var[q^{n(\tau)}(\tilde{\theta})] > \hat{\delta}$.

Note

$$Var[q^{n(\tau)}(\tilde{\theta})] \equiv E\left[\left(q^{n(\tau)}(\tilde{\theta})\right)^2\right] - \left(E[q^{n(\tau)}(\tilde{\theta})]\right)^2 = E\left[\left(q^{n(\tau)}(\tilde{\theta})\right)^2\right] = E\left[\left|q^{n(\tau)}(\tilde{\theta})\right|^2\right],$$

so $E\left[\left|q^{n(\tau)}(\tilde{\theta})\right|^2\right] > \hat{\delta}$, which implies $E\left[\left|q^{n(\tau)}(\tilde{\theta})\right|\right] > 0$ and $E\left[\left|q^{n(\tau)}(\tilde{\theta})\right|^3\right] > 0$. Since $\tilde{\theta}_k$ and $\tilde{\theta}$ are identically distributed for each $k \in \mathbb{N}$, it follows that $E\left[\left|q^{n(\tau)}(\tilde{\theta}_k)\right|\right] = E\left[\left|q^{n(\tau)}(\tilde{\theta})\right|\right]$ for each $\tau \in \mathbb{N}$ and for each $k \in \{1, ..., n(\tau)\}$, so $Var[q^{n(\tau)}(\tilde{\theta}_k)] = E\left[\left|q^{n(\tau)}(\tilde{\theta})\right|^2\right]$ and $E\left[\left|q^{n(\tau)}(\bar{\theta}_k)\right|^3\right] = E\left[\left|q^{n(\tau)}(\tilde{\theta})\right|^3\right]$.

For each $\tau \in \mathbb{N}$, let V^τ denote the cumulative distribution of the random variable $\sum_{k \in N^{n(\tau)} \setminus \{i\}} q^{n(\tau)}(\tilde{\theta}_k)$ and let $\hat{V}^\tau$ denote the cumulative distribution of $\frac{\sum_{k \in N^{n(\tau)} \setminus \{i\}} q^{n(\tau)}(\tilde{\theta}_k)}{\sqrt{n(\tau)-1}\sqrt{E\left[\left(q^{n(\tau)}(\tilde{\theta})\right)^2\right]}}$. By the Berry-Esseen theorem (Lemma 7), for any $\tau \in \mathbb{N}$ and any $x \in \mathbb{R}$,

$$\left|\hat{V}^\tau(y) - N[0,1](y)\right| \leq \frac{E\left[\left|q^{n(\tau)}(\tilde{\theta})\right|^3\right]}{\sqrt{n(\tau)-1}\left(E\left[\left(q^{n(\tau)}(\tilde{\theta})\right)^2\right]\right)^{\frac{3}{2}}}.$$

Then, multiplying both distributions on the left hand side by $\sqrt{n(\tau)-1}\sqrt{E\left[\left(q^{n(\tau)}(\tilde{\theta})\right)^2\right]}$, and recalling $E\left[\left|q^{n(\tau)}(\tilde{\theta})\right|^2\right] > \hat{\delta}$ on the right hand side, we obtain:

$$\left|V^\tau(y) - N\left[0, (n(\tau)-1)\, E\left[\left(q^{n(\tau)}(\tilde{\theta})\right)^2\right]\right](y)\right| < \frac{E\left[\left|q^{n(\tau)}(\tilde{\theta})\right|^3\right]}{\left(\sqrt{n(\tau)-1}\right)\left(\hat{\delta}\right)^{\frac{3}{2}}}, \tag{22}$$

Since S is bounded, $\{s^n(\tilde{\theta})\}_{n=1}^\infty$ is uniformly bounded, and thus $\{E[s^{n(\tau)}(\tilde{\theta})]\}_{\tau=1}^\infty$ and $\{q^{n(\tau)}(\tilde{\theta})\}_{\tau=1}^\infty$ are uniformly bounded as well, and hence $\left\{E\left[\left|q^{n(\tau)}(\tilde{\theta})\right|^3\right]\right\}_{\tau=1}^\infty$ is uniformly bounded. It follows that the right hand side of Inequality (22) converges to zero as τ diverges to infinity. Thus, the cumulative distribution V^τ converges as $\tau \longrightarrow \infty$ to a mean zero Normal distribution with variance $E\left[\left(q^{n(\tau)}(\tilde{\theta})\right)^2\right](n(\tau)-1)$. Since $E\left[\left(q^{n(\tau)}(\tilde{\theta})\right)^2\right] \geq \hat{\delta}$ for any $\tau \in \mathbb{N}$, it follows that $E\left[\left(q^{n(\tau)}(\tilde{\theta})\right)^2\right](n(\tau)-1)$ diverges to infinity as $\tau \longrightarrow \infty$. Therefore, for any $y \in \mathbb{R}_{++}$ and any $\{z^{n(\tau)}\}_{\tau=1}^{\infty}$ such that $z^{n(\tau)} \in \mathbb{R}$ for each $\tau \in \mathbb{N}$,

$$\lim_{\tau \longrightarrow \infty} P\left(\left\{\theta_{N^{n(\tau)}\backslash\{i\}} \in (\Theta_P)^{n-1} : \sum_{k \in N^{n(\tau)}\backslash\{i\}} q^{n(\tau)}(\theta_k) \in (z^{n(\tau)}-y, z^{n(\tau)}+y)\right\}\right) = 0. \quad (23)$$

In particular, for $z^{n(\tau)} = -E[s^n(\tilde{\theta})]$ for each $\tau \in \mathbb{N}$, Equality (23) implies that for any $y \in \mathbb{R}_{++}$,

$$\lim_{\tau \longrightarrow \infty} P\left(\left\{\theta_{N^{n(\tau)}\backslash\{i\}} \in (\Theta_P)^{n-1} : \sum_{k \in N^{n(\tau)}\backslash\{i\}} s^n(\tilde{\theta}) \in (-y, y)\right\}\right) = 0, \quad (24)$$

and thus

$$\lim_{\tau \longrightarrow \infty} P\left(\Theta_P^2(x, n(\tau))\right) = 0,$$

which contradicts Inequality (19). Therefore, our (absurd) assumption that *"for some $a \in [-a(1), a(1)]$, there exist $x \in \mathbb{R}_{++}$, and an infinite subsequence $\{s^{n(\tau)}\}_{\tau=1}^{\infty}$ of $\{s^n\}_{n=2}^{\infty}$ with $n : \mathbb{N}\backslash\{1\} \longrightarrow \mathbb{N}$ strictly increasing such that*

$$\int_{\theta_{-i} \in \Theta_P^2(x, n(\tau))} g\left(\sum_{k \in N^{n(\tau)}\backslash\{i\}} s^{n(\tau)}(\theta_k) + a\right) \prod_{k \in N^{n(\tau)}\backslash\{i\}} p(\theta_k) d\theta_{-i} > \frac{1}{x}$$

for any $\tau \in \mathbb{N}$" is false. Thus,

$$\lim_{\tau \longrightarrow \infty} Piv[a, s^{n(\tau)}] = 0 \text{ for each } a \in [-a(1), a(1)]. \quad (25)$$

Since the marginal benefit of an action a is $vPiv[a, s^n]$, the marginal benefit converges to zero, and thus the marginal utility cost must converge to zero, which, since marginal utility of wealth is strictly positive, implies that the marginal cost (in wealth terms) must also converge to zero, and since $c'(a) > 0$ for any $a \neq 0$, it follows that

$$\lim_{\tau \longrightarrow \infty} s^{n(\tau)}(\theta) = 0 \text{ for any } \theta \in \Theta_P.$$

■

The next lemma uses the First Order Condition of each agent's optimization problem to derive an equality that proves more convenient for subsequent results. For any $n \in \mathbb{N}\backslash\{1\}$, any action a, any pure strategy s, any agent $i \in \mathbb{N}^n$ and any initial wealth level w_i, let $E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right)|a_i = a, a_k = s(\tilde{\theta}_k)\ \forall k \in N^n \backslash \{i\}]$ denote the expected value of the marginal utility over wealth evaluated at the final outcome wealth $w_i^O = w_i - c(a) + \sum_{j \in N^n \backslash \{i\}} s^n(\theta_j)$ of agent i with initial wealth w_i, given that i takes action $a_i = a$ and other agents follow strategy s. The expectation is over the realization of other agents' types.

Lemma 9 *For any tuple $(P, c, G) \in \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, for any sequence $\{s^n\}_{n=1}^{\infty}$ such that $s^n \in E^{(n,P,c,G)}$ is weakly monotone for each $n \in \mathbb{N}\backslash\{1\}$, for any $n \in \mathbb{N}\backslash\{1\}$, and for each $\theta \equiv (w, v) \in \Theta_P$, there exists a function $z^\theta : [-(n-1)a(1), (n-1)a(1)] \longrightarrow [s^n(\theta), 0) \cup (0, s^n(\theta)]$ such that $sgn(z^\theta(x)) = sgn(v)$ for any $x \in [-(n-1)a(1), (n-1)a(1)]$, and*

$$\begin{aligned} & c'(s^n(\theta))E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right)|a_i = s^n(\theta), a_j = s^n(\tilde{\theta}_j)\ \forall j \in N^n \backslash \{i\}] \qquad (26) \\ & = v \int_{\theta_{-i} \in (\Theta_P)^{n-1}} \prod_{j \in N^n \backslash \{i\}} p(\theta_j) g\left(\sum_{j \in N^n \backslash \{i\}} s^n(\theta_j)\right) d\theta_{-i} \\ & + v s^n(\theta) \int_{\theta_{-i} \in (\Theta_P)^{n-1}} \prod_{j \in N^n \backslash \{i\}} p(\theta_j) g'\left(\sum_{j \in N^n \backslash \{i\}} s^n(\theta_j) + z^\theta\left(\sum_{j \in N^n \backslash \{i\}} s^n(\theta_j)\right)\right) d\theta_{-i}. \end{aligned}$$

Proof. For any given $n \in \mathbb{N}\backslash\{1\}$, only a compact subset of the domain of G, namely $[-na(1), na(1)]$ is relevant, since $ns^n(\theta) \in [-na(1), na(1)]$ for any $\theta \in \Theta_P$. And G is twice continuously differentiable. Note that given that all other agents play s^n, the optimization problem of agent i with type $\theta_i = (w, v) \in \Theta_P$ is to maximize

$$\begin{aligned} & \max_{a \in [-a(w), a(w)]} v \int_{\theta_{-i} \in (\Theta_P)^{n-1}} \prod_{j \in N^n \backslash \{i\}} p(\theta_j) G\left(a + \sum_{j \in N^n \backslash \{i\}} s^n(\theta_j)\right) d\theta_{-i} + \qquad (27) \\ & \int_{\theta_{-i} \in (\Theta_P)^{n-1}} \prod_{j \in N^n \backslash \{i\}} p(\theta_j) u\left(w - c(a) + \frac{1}{n-1} \sum_{j \in N^n \backslash \{i\}} p(\theta_j) c(s^n(\theta_j))\right) d\theta_{-i}. \end{aligned}$$

Since equilibrium actions converge to zero for any $\theta \equiv (w, v) \in \Theta_P$ for sufficiently large n (Lemma 8), the solution is interior, namely any maximizer, and in particular the equilibrium action $s^n(\theta)$, is in $(-a(w), a(w))$ and thus $s^n(\theta)$ satisfies the first order condition

$$\begin{aligned} & v \int_{\theta_{-i} \in (\Theta_P)^{n-1}} \prod_{j \in N^n \backslash \{i\}} p(\theta_j) g\left(s^n(\theta) + \sum_{j \in N^n \backslash \{i\}} s^n(\theta_j)\right) d\theta_{-i} \qquad (28) \\ = \;& c'(s^n(\theta)) \int_{\theta_{-i} \in (\Theta_P)^{n-1}} \prod_{j \in N^n \backslash \{i\}} p(\theta_j) u'\left(w - c(s^n(\theta)) + \frac{1}{n-1} \sum_{j \in N^n \backslash \{i\}} p(\theta_j) c(s^n(\theta_j))\right) d\theta_{-i} \end{aligned}$$

where the right hand side of this equality is

$$c'(s^n(\theta))E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right)|a_i = s^n(\theta), a_j = s^n(\tilde{\theta}_j)\ \forall j \in N^n\backslash\{i\}].$$

We want to show that for any $x \in [-(n-1)a(1), (n-1)a(1)]$, and any $a \in (0, a(w))$, there exists a $z^a(x) \in (0, a)$ such that

$$g(x+a) = g(x) + ag'(x + z^a(x)). \tag{29}$$

For each $x \in [-(n-1)a(1), (n-1)a(1)]$, define $y_{\min} \equiv \arg \min_{y\in[x,x+a]} g'(y)$ and $y_{\max} \equiv \arg \max_{y\in[x,x+a]} g'(y)$. Then note

$$ag'(y_{\min}) \leq g(x+a) - g(x) \leq ag'(y_{\max})$$

Since g is continuous, by the Intermediate Value Theorem, there exists some value $y(x) \in [x, x+a]$ such that

$$ag'(y(x)) = g(x+a) - g(x).$$

Then, define $z^a(x) \equiv y(x) - x$ and we obtain Equality (29). An analogous argument, in this instance with $y(x) \in [x+a, x]$, establishes that for any $a \in (-a(w), 0)$, there exists a $z^a(x) \in [a, 0]$ such that Equality (29) holds. Then, defining $x \equiv \sum_{j\in N^n\backslash\{i\}} s^n(\theta_j)$, we obtain Equation (26) by using Equality (29) for $a = s^n(\theta)$ to restate the right-hand side of Equality (28). ■

The next lemma uses Lemma 9 to establish that for any pair of agents, the ratio of their respective marginal costs of contributing is equal to their respective marginal rates of substitution between choosing alternative A and wealth, at their initial endowment wealth level.

Lemma 10 *For any tuple $(P, c, G) \in P \times \mathcal{C}_A \times \mathcal{G}$; for any sequence $\{s^n\}_{n=2}^{\infty}$ such that for each $n \in \mathbb{N}\backslash\{1\}$, $s^n \in E^{(n,P,c,G)}$ and s^n is weakly monotone; for any $\theta \in \Theta_P$; and for any $\hat{\theta} \equiv (\hat{w}, \hat{v}) \in \Theta_P$ such that $\hat{v} \in [-\gamma, 0) \cup (0, \gamma]$,*

$$\lim_{n\to\infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \frac{\frac{v}{u'(w)}}{\frac{\hat{v}}{u'(\hat{w})}}.$$

Proof. For any tuple $(P, c, G) \in P \times \mathcal{C}_A \times \mathcal{G}$, let $\{s^n\}_{n=2}^{\infty}$ be a sequence of equilibria, that is, $s^n \in E^{(n,P,c,G)}$ for each $n \in \mathbb{N}\backslash\{1\}$. From Lemma 9, for each $\theta \in \Theta_P$,

$$c'(s^n(\theta))E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right)|a_i = s^n(\theta), a_j = s^n(\tilde{\theta}_j)\ \forall j \in N^n\backslash\{i\}]$$

$$= v \int_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod_{j\in N\backslash\{i\}} p(\theta_j) g\left(\sum_{j\in N\backslash\{i\}} s^n(\theta_j)\right) d\theta_{-i}$$

$$+ vs^n(\theta) \int_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod_{j\in N\backslash\{i\}} p(\theta_j) g'\left(\sum_{j\in N\backslash\{i\}} s^n(\theta_j) + z^\theta\left(\sum_{j\in N\backslash\{i\}} s^n(\theta_j)\right)\right) d\theta_{-i}.$$

Notice that since g is strictly positive and continuous, and g' is continuous, for any $x, y \in \mathbb{R}$, $\frac{g'(y)}{g(x)}$ is continuous, and over any closed interval of $\mathbb{R}$, it is bounded. Further, by Condition (iii) of the definition of $\mathcal{G}$, $\exists \hat{\varepsilon} \in \mathbb{R}_{++}$ such that for any $\varepsilon \in (0, \hat{\varepsilon})$,

$$\lim_{x \to -\infty} \frac{g'(x+\varepsilon)}{g(x)} \in \mathbb{R} \text{ and } \lim_{x \to \infty} \frac{g'(x+\varepsilon)}{g(x)} \in \mathbb{R}. \tag{30}$$

Therefore, there exists $\lambda \in \mathbb{R}_{++}$ such that $\frac{g'(x+\varepsilon)}{g(x)} \in [-\lambda, \lambda]$, for any $\varepsilon \in (0, \hat{\varepsilon})$ and for any $x \in \mathbb{R}$. Equivalently,

$$-\lambda g(x) \leq g'(x+\varepsilon) \leq \lambda g(x) \ \forall \varepsilon \in (0, \hat{\varepsilon}), \ \forall x \in \mathbb{R}. \tag{31}$$

Since for any sequence $\{s^n\}_{n=1}^{\infty}$ of equilibria $\lim_{n \to \infty} s^n(\theta) = 0$ for each $\theta \in \Theta_P$ (Lemma 8), and since $z^{\theta}(x)$ defined in Lemma 9 satisfies $z^{\theta}(x) \in (0, s^n(\theta))$, it follows $\lim_{n \to \infty} z^{\theta}(x) = 0$ for each $\theta \in \Theta_P$ and for each $x \in [-(n-1)a(1), (n-1)a(1)]$. Then, it follows from Expression (31), that there exists $\hat{n} \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ satisfying $n > \hat{n}$, for each $x \in [-(n-1)a(1), (n-1)a(1)]$, for any $\theta \equiv (w, v) \in \Theta_P$ such that $v \in [-\gamma, 0) \cup (0, \gamma]$, and for any equilibrium strategy s^n, we have:

$$-\lambda g(x) < g'(x + z^{\theta}(x)) < \lambda g(x).$$

Therefore, for any $v \in (0, \gamma]$, multiplying each side by $s^n(\theta)$, adding $g(x)$ to each side, and then multiplying each side by v,

$$(1 - \lambda s^n(\theta))g(x)v < (g(x) + s^n(\theta)g'(x + z^{\theta}(x)))v < (1 + \lambda s^n(\theta))g(x)v, \tag{32}$$

with the inequalities reversed for $v \in [-\gamma, 0)$.

Once again since $\lim_{n \to \infty} s^n(\theta) = 0$ for each $\theta \in \Theta_P$ (Lemma 8), there exists $\hat{n}$ such that $1 - \lambda s^n(\theta) > 0$ for every $n > \hat{n}$. From the pair of inequalities (32), substituting $\sum_{j \in N \setminus \{i\}} s^n(\tilde{\theta}_j)$

for x, and taking expectations over $\tilde{\theta}_{-i}$, for any $v \in (0, \gamma]$,

$$\begin{aligned}
& v(1-\lambda s^n(\theta)) \int\limits_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod_{j\in N\backslash\{i\}} p(\theta_j) g\left(\sum_{j\in N\backslash\{i\}} s^n(\theta_j)\right) d\theta_{-i} \\
< \; & v \int\limits_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod_{j\in N\backslash\{i\}} p(\theta_j) g\left(\sum_{j\in N\backslash\{i\}} s^n(\theta_j)\right) d\theta_{-i} \\
& + v s^n(\theta) \int\limits_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod_{j\in N\backslash\{i\}} p(\theta_j) g'\left(\sum_{j\in N\backslash\{i\}} s^n(\theta_j) + z^\theta\left(\sum_{j\in N\backslash\{i\}} s^n(\theta_j)\right)\right) d\theta_{-i} \\
< \; & v(1+\lambda s^n(\theta)) \int\limits_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod_{j\in N\backslash\{i\}} p(\theta_j) g\left(\sum_{j\in N\backslash\{i\}} s^n(\theta_j)\right) d\theta_{-i},
\end{aligned}$$

with the inequality signs reversed for $v \in [-\gamma, 0)$.

Since the middle term of these inequalities is the right hand side of Equality (26), substituting in the left hand side instead, we find that for any $v \in [-\gamma, 0)$,

$$\begin{aligned}
& c'(s^n(\theta)) E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right) | a_i = s^n(\theta), a_j = s^n(\tilde{\theta}_j) \; \forall j \in N^n\backslash\{i\}] \\
\in & \begin{pmatrix} v(1+\lambda s^n(\theta)) \int_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod_{j\in N\backslash\{i\}} p(\theta_j) g\left(\sum_{j\in N\backslash\{i\}} s^n(\theta_j)\right) d\theta_{-i}, \\ v(1-\lambda s^n(\theta)) \int_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod_{j\in N\backslash\{i\}} p(\theta_j) g\left(\sum_{j\in N\backslash\{i\}} s^n(\theta_j)\right) d\theta_{-i} \end{pmatrix} \text{ for any } v \in [-\gamma, 0), \\
\in & \begin{pmatrix} v(1-\lambda s^n(\theta)) \int_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod_{j\in N\backslash\{i\}} p(\theta_j) g\left(\sum_{j\in N\backslash\{i\}} s^n(\theta_j)\right) d\theta_{-i}, \\ v(1+\lambda s^n(\theta)) \int_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod_{j\in N\backslash\{i\}} p(\theta_j) g\left(\sum_{j\in N\backslash\{i\}} s^n(\theta_j)\right) d\theta_{-i} \end{pmatrix} \text{ for any } v \in (0, \gamma].
\end{aligned} \tag{33}$$

Then, for any $\theta, \hat{\theta} \in \Theta_P$ such that $\hat{v} \in [-\gamma, 0) \cup (0, \gamma]$,

$$\frac{c'(|s^n(\theta)|)}{c'(|s^n(\hat{\theta})|)} \frac{E_{\tilde{\theta}_{-i}}[u'\left(w_i^O\right)|a_i = s^n(\theta), a_j = s^n(\tilde{\theta}_j)\forall j \in N^n \backslash \{i\}]}{E_{\tilde{\theta}_{-i}}[u'\left(w_i^O\right)|a_i = s^n(\hat{\theta}), a_j = s^n(\tilde{\theta}_j)\forall j \in N^n \backslash \{i\}]} \tag{34}$$

$$\in \left(\frac{|v|(1-sgn(v)\lambda s^n(\theta)) \int\limits_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod\limits_{j\in N^n\backslash\{i\}} p(\theta_j) g\left(\sum\limits_{j\in N^n\backslash\{i\}} s^n(\theta_j)\right) d\theta_{-i}}{|\hat{v}|(1+sgn(\hat{v})\lambda s^n(\hat{\theta})) \int\limits_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod\limits_{j\in N^n\backslash\{i\}} p(\theta_j) g\left(\sum\limits_{j\in N^n\backslash\{i\}} s^n(\theta_j)\right) d\theta_{-i}}, \frac{|v|(1+sgn(v)\lambda s^n(\theta)) \int\limits_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod\limits_{j\in N^n\backslash\{i\}} p(\theta_j) g\left(\sum\limits_{j\in N^n\backslash\{i\}} s^n(\theta_j)\right) d\theta_{-i}}{|\hat{v}|(1-sgn(\hat{v})\lambda s^n(\hat{\theta})) \int\limits_{\theta_{-i}\in(\Theta_P)^{n-1}} \prod\limits_{j\in N^n\backslash\{i\}} p(\theta_j) g\left(\sum\limits_{j\in N^n\backslash\{i\}} s^n(\theta_j)\right) d\theta_{-i}} \right)$$

$$= \left(\frac{|v|(1 - sgn(v)\lambda s^n(\theta))}{|\hat{v}|(1 + sgn(\hat{v})\lambda s^n(\hat{\theta}))}, \frac{|v|(1 + sgn(v)\lambda s^n(\theta))}{|\hat{v}|(1 - sgn(\hat{v})\lambda s^n(\hat{\theta}))} \right).$$

Rearranging terms

$$\frac{c'(|s^n(\theta)|)}{c'(|s^n(\hat{\theta})|)} \in \left(\frac{\frac{|v|(1-sgn(v)\lambda s^n(\theta))}{E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right)|a_i=s^n(\theta),\, a_j=s^n(\tilde{\theta}_j)\ \forall j\in N^n\backslash\{i\}]}}{\frac{|\hat{v}|(1+sgn(\hat{v})\lambda s^n(\hat{\theta}))}{E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right)|a_i=s^n(\hat{\theta}),\, a_j=s^n(\tilde{\theta}_j)\ \forall j\in N^n\backslash\{i\}]}}, \frac{\frac{|v|(1+sgn(v)\lambda s^n(\theta))}{E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right)|a_i=s^n(\theta),\, a_j=s^n(\tilde{\theta}_j)\ \forall j\in N^n\backslash\{i\}]}}{\frac{|\hat{v}|(1-sgn(\hat{v})\lambda s^n(\hat{\theta}))}{E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right)|a_i=s^n(\hat{\theta}),\, a_j=s^n(\tilde{\theta}_j)\ \forall j\in N^n\backslash\{i\}]}} \right) \tag{35}$$

Because $\lim\limits_{n\longrightarrow\infty} s^n(\check{\theta}) = 0$ for any $\check{\theta} \in \Theta_P$ (Lemma 8),

$$\lim_{n\longrightarrow\infty} \left(\left(\frac{\frac{|v|(1-sgn(v)\lambda s^n(\theta))}{E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right)|a_i=s^n(\theta),\, a_j=s^n(\tilde{\theta}_j)\ \forall j\in N^n\backslash\{i\}]}}{\frac{|\hat{v}|(1+sgn(\hat{v})\lambda s^n(\hat{\theta}))}{E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right)|a_i=s^n(\hat{\theta}),\, a_j=s^n(\tilde{\theta}_j)\ \forall j\in N^n\backslash\{i\}]}}, \frac{\frac{|v|(1+sgn(v)\lambda s^n(\theta))}{E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right)|a_i=s^n(\theta),\, a_j=s^n(\tilde{\theta}_j)\ \forall j\in N^n\backslash\{i\}]}}{\frac{|\hat{v}|(1-sgn(\hat{v})\lambda s^n(\hat{\theta}))}{E_{\tilde{\theta}_{-i}}[u'\left(\tilde{w}_i^O\right)|a_i=s^n(\hat{\theta}),\, a_j=s^n(\tilde{\theta}_j)\ \forall j\in N^n\backslash\{i\}]}} \right) \right)$$

$$= \left\{ \frac{\frac{|v|}{u'(w)}}{\frac{|\hat{v}|}{u'(\hat{w})}} \right\}$$

so taking the limit as n diverges to infinity on both sides of Expression (35),

$$\lim_{n\longrightarrow\infty} \frac{c'(|s^n(\theta)|)}{c'(|s^n(\hat{\theta})|)} = \frac{\frac{|v|}{u'(w)}}{\frac{|\hat{v}|}{u'(\hat{w})}}.$$

Since for any $n \in \mathbb{N}\backslash\{1\}$, $s^n(\check{\theta}) < 0$ for any $\check{v} \in [-\gamma, 0)$ and $s^n(\check{\theta}) > 0$ for any $\check{v} \in (0, \gamma]$, and since $c'(a) < 0$ for any $a < 0$ and $c'(a) > 0$ for any $a > 0$, it follows that for any

$(\theta, \hat{\theta}) \in (\Theta_P)^2$ such that $\hat{v} \in [-\gamma, 0) \cup (0, \gamma]$,

$$\lim_{n \longrightarrow \infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \frac{\frac{v}{u'(w)}}{\frac{\hat{v}}{u'(\hat{w})}}.$$

■

We next define an auxiliary function and prove a lemma related to it. For any cost function $c \in \mathcal{C}_A$, recall that the elasticity of the cost function at zero is $\kappa(c) \equiv \lim_{a \longrightarrow 0} \frac{ac'(a)}{c(a)}$, and define $J^c : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}_+$ by

$$J^c(x, y) = \begin{cases} \frac{yc''(y)}{c'(y)} & \text{if } x = y, \\ \frac{\ln c'(x) - \ln c'(y)}{\ln x - \ln y} & \text{otherwise.} \end{cases} \tag{36}$$

Lemma 11 *Let $\{x_n\}_{n=1}^{\infty} \in \mathbb{R}^{\infty}_{++}$ and $\{y_n\}_{n=1}^{\infty} \in \mathbb{R}^{\infty}_{++}$ be two converging sequences with $\lim_{n \longrightarrow \infty} x_n = \lim_{n \longrightarrow \infty} y_n = 0$. Then for any $c \in \mathcal{C}_A$,*

$$\lim_{n \longrightarrow \infty} J^c(x_n, y_n) = \kappa(c) - 1.$$

Proof. Note that for any $c \in \mathcal{C}_A$, and for any $y \in \mathbb{R}_{++}$,

$$\lim_{x \longrightarrow 0} J^c(x, y) = \frac{\ln c'(0) - \ln c'(y)}{\ln 0 - \ln y} = \frac{-\infty}{-\infty},$$

and applying L'Hopital rule,

$$\lim_{x \longrightarrow 0} J^c(x, y) = \lim_{x \longrightarrow 0} \frac{\frac{c''(x)}{c'(x)}}{\frac{1}{x}} = \lim_{x \longrightarrow 0} \frac{xc''(x)}{c'(x)}.$$

Since $\kappa(c) \equiv \lim_{x \longrightarrow 0} \frac{xc'(x)}{c(x)}$ and $\lim_{x \longrightarrow 0} \frac{xc'(x)}{c(x)} = \frac{0}{0}$, applying L'Hopital rule,

$$\kappa(c) = \lim_{x \longrightarrow 0} \frac{c'(x) + xc''(x)}{c'(x)} = 1 + \lim_{x \longrightarrow 0} \frac{xc''(x)}{c'(x)}$$

$$\kappa(c) - 1 = \lim_{x \longrightarrow 0} \frac{xc''(x)}{c'(x)}, \tag{37}$$

so $\lim_{x \longrightarrow 0} J^c(x, y) = \kappa(c) - 1$. Note as well that, using L'Hopital rule,

$$\lim_{\varepsilon \longrightarrow 0} J^c(x, x + \varepsilon) = \frac{-\frac{c''(x)}{c'(x)}}{-\frac{1}{x}} = \frac{xc''(x)}{c'(x)}$$

so J^c is continuous.

Define the function $f^c : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by

$$f^c(x) = \begin{cases} \kappa(c) - 1 & \text{if } x = 0, \\ \frac{xc''(x)}{c'(x)} & \text{if } x \in \mathbb{R}_{++}. \end{cases}$$

By Equality (37), $\lim_{x \longrightarrow 0} \frac{xc''(x)}{c'(x)} = \kappa(c) - 1$ and hence $\lim_{x \longrightarrow 0} f^c(x) = \kappa(c) - 1$ and f^c is continuous.

Define the function $f^c_{\max} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $f^c_{\max}(w) = \max_{x \in [0,w]} f^c(x)$ for each $w \in \mathbb{R}_+$ and the function $f^c_{\min} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by $f^c_{\min}(w) \equiv \min_{x \in [0,w]} f^c(x)$ for each $w \in \mathbb{R}_+$. Since f^c is continuous, $\max_{x \in [0,w]} f^c(x)$ and $\min_{x \in [0,w]} f^c(x)$ are non-empty for each $w \in \mathbb{R}_+$ and $f^c_{\max}$ and $f^c_{\min}$ are continuous (Berge's maximum theorem). Further, note that $f^c_{\max}$ is non-decreasing and $f^c_{\min}$ is non-increasing.

Construct two sequences $\{x_t\}_{t=1}^{\infty} \in \mathbb{R}_+^{\infty}$ and $\{y_t\}_{t=1}^{\infty} \in \mathbb{R}_+^{\infty}$ such that $\lim_{t \longrightarrow \infty} x_t = \lim_{t \longrightarrow \infty} y_t = 0$. Then by Equality (53),

$$\lim_{t \longrightarrow 0} \frac{x_t c''(x_t)}{c'(x_t)} = \lim_{t \longrightarrow 0} \frac{y_t c''(y_t)}{c'(y_t)} = \kappa(c) - 1.$$

Note that for any $y \in \mathbb{R}_{++}$, and for any $x \in (0, y)$, J^c is differentiable and

$$\begin{aligned} \frac{\partial J^c}{\partial x}(x,y) &= \frac{\frac{c''(x)}{c'(x)}(\ln x - \ln y) - (\ln c'(x) - \ln(c'(y))\frac{1}{x}}{(\ln x - \ln y)^2} \\ &= \frac{xc''(x)(\ln x - \ln y) - c'(x)(\ln c'(x) - \ln(c'(y))}{xc'(x)\,(\ln x - \ln y)^2}. \end{aligned}$$

Hence $\frac{\partial J^c}{\partial x}(x,y) = 0$ if and only if

$$\begin{aligned} xc''(x)(\ln x - \ln y) &= c'(x)(\ln c'(x) - \ln(c'(y)) \\ \frac{xc''(x)}{c'(x)} &= \frac{\ln c'(x) - \ln c'(y)}{\ln x - \ln y}, \end{aligned}$$

that is, $\frac{\partial J^c}{\partial x}(x,y) = 0$ if and only if $J^c(x,y) = \frac{xc''(x)}{c'(x)}$.

Since $x \in \arg\max_{x \in (0,y)} J^c(x,y)$ implies $\frac{\partial J^c}{\partial x}(x,y) = 0$, it follows that for any $y \in \mathbb{R}_{++}$ and any $x \in \arg\max_{x \in (0,y)} J^c(x,y)$, $J^c(x,y) = f^c(x)$, so $J^c(x,y) \leq f^c_{\max}(x)$. Since $f^c_{\max}$ is non-decreasing, it follows that if $\arg\max_{x \in (0,y)} J^c(x,y) \neq \varnothing$, then $\max_{x \in (0,y)} J^c(x,y) \leq f^c_{\max}(y)$. If $\arg\max_{x \in (0,y)} J^c(x,y) = \varnothing$, then $\sup_{x \in (0,y)} J^c(x,y) \in \left\{\lim_{x \longrightarrow 0} J^c(x,y), J^c(y,y)\right\} = \{\kappa(c) - 1, f^c(y)\} \leq f^c_{\max}(y)$. So $\sup_{x \in (0,y)} J^c(x,y) \leq f^c_{\max}(y)$ for any $y \in \mathbb{R}_{++}$. Similarly, it can be shown that $\sup_{y \in (0,x)} J^c(x,y) \leq f^c_{\max}(x)$ for any $x \in \mathbb{R}_{++}$.

Moreover, since $x \in \arg\min_{x\in(0,y)} J^c(x,y)$ implies $\frac{\partial J^c}{\partial x}(x,y) = 0$, it follows that for any $y \in \mathbb{R}_{++}$ and any $x \in \arg\min_{x\in(0,y)} J^c(x,y)$, $J^c(x,y) = f^c(x)$, so $J^c(x,y) \geq f^c_{\min}(x)$. Since $f^c_{\min}$ is non-increasing, it follows that if $\arg\min_{x\in(0,y)} J^c(x,y) \neq \varnothing$, then $\min_{x\in(0,y)} J^c(x,y) \geq f^c_{\min}(y)$. If $\arg\min_{x\in(0,y)} J^c(x,y) = \varnothing$, then $\inf_{x\in(0,y)} J^c(x,y) \in \left\{\lim_{x\longrightarrow 0} J^c(x,y), J^c(y,y)\right\} = \{\kappa(c) - 1, f^c(y)\} \geq f^c_{\min}(y)$. So $\inf_{x\in(0,y)} J^c(x,y) \geq f^c_{\min}(y)$ for any $y \in \mathbb{R}_{++}$. Similarly, it can be shown that $\inf_{y\in(0,x)} J^c(x,y) \geq f^c_{\min}(y)$ for any $x \in \mathbb{R}_{++}$.

From all the above it follows that for any $t \in \mathbb{N}$, $J^c(x_t, y_t) \in [f^c_{\min}(\max\{x_t, y_t\}), f^c_{\max}(\max\{x_t, y_t\})]$. Notice that $\lim_{t\longrightarrow\infty} \max\{x_t, y_t\} = 0$, and thus

$$\lim_{t\longrightarrow 0} f^c_{\min}(\max\{x_t, y_t\}) = \kappa(c) - 1 \text{ and } \lim_{t\longrightarrow 0} f^c_{\max}(\max\{x_t, y_t\}) = \kappa(c) - 1,$$

and hence $\lim_{n\longrightarrow\infty} J^c(x_n, y_n) = \kappa(c) - 1$. ■

We next establish a key intermediary result: equilibrium actions are asymptotically piecewise linear (within each half space of positive and negative valuations) in the marginal rate of substitution $\frac{v}{u'(w)}$ taken to the power $\frac{1}{\kappa(c)-1}$.

Lemma 12 *For any tuple $(P, c, G) \in \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, for any $\{s^n\}_{n=1}^{\infty}$ such that $s^n \in E^{(n,P,c,G)}$ and s^n is weakly monotone for each $n \in \mathbb{N}\backslash\{1\}$, and for any $(\theta, \hat{\theta}) \in ([w_{\min}, 1] \times [-\gamma, 0))^2 \cup ([w_{\min}, 1] \times (0, \gamma])^2$,*

$$\lim_{n\to\infty} \frac{s^n(\theta)}{s^n(\hat{\theta})} = \left(\frac{\frac{v}{u'(w)}}{\frac{\hat{v}}{u'(\hat{w})}}\right)^{\frac{1}{\kappa(c)-1}}.$$

Proof. For any $(\theta, \hat{\theta}) \in ([w_{\min}, 1] \times [-\gamma, 0))^2 \cup ([w_{\min}, 1] \times (0, \gamma])^2$, by Lemma 10,

$$\lim_{n\to\infty} \frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))} = \frac{\frac{v}{u'(w)}}{\frac{\hat{v}}{u'(\hat{w})}}.$$

Taking logarithms on both sides,

$$\lim_{n\to\infty} (\ln c'(s^n(\theta)) - \ln c'(s^n(\hat{\theta})) = \ln\left(\frac{\frac{v}{u'(w)}}{\frac{\hat{v}}{u'(\hat{w})}}\right). \tag{38}$$

By Lemma 11, for any $\{x_n\}_{n=1}^{\infty} \in \mathbb{R}^{\infty}_{++}$ and $\{y_n\}_{n=1}^{\infty} \in \mathbb{R}^{\infty}_{++}$ with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$,

$$\lim_{n\to\infty} \frac{\ln c'(x_n) - \ln c'(y_n)}{\ln \frac{x_n}{y_n}} = \kappa(c) - 1,$$

thus, in particular,

$$\lim_{n\to\infty} \frac{\ln c'(s^n(\theta)) - \ln c'(s^n(\hat{\theta}))}{\ln \frac{s^n(\theta)}{s^n(\hat{\theta})}} = \kappa(c) - 1,$$

so, since (because $\hat{v} \neq 0$) the denominator of the left-hand side is strictly positive, we can rearrange terms to obtain

$$\lim_{n\to\infty} \left(\ln c'(s^n(\theta)) - \ln c'(s^n(\hat{\theta}))\right) = \lim_{n\to\infty} \ln \left(\frac{s^n(\theta)}{s^n(\hat{\theta})}\right)^{\kappa(c)-1},$$

and thus, substituting the left hand side according to Equality (38), we obtain

$$\begin{aligned} \ln\left(\frac{\frac{v}{u'(w)}}{\frac{\hat{v}}{u'(\hat{w})}}\right) &= \lim_{n\to\infty} \ln\left(\frac{s^n(\theta)}{s^n(\hat{\theta})}\right)^{\kappa(c)-1}, \\ \lim_{n\to\infty} \frac{s^n(\theta)}{s^n(\hat{\theta})} &= \left(\frac{\frac{v}{u'(w)}}{\frac{\hat{v}}{u'(\hat{w})}}\right)^{\frac{1}{\kappa(c)-1}}. \end{aligned} \tag{39}$$

■

Further, we can strengthen this result, to obtain linearity in a power of the marginal rate of substitution, across agents with valuations of different signs.

Lemma 13 *For any tuple $(P, c, G) \in \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, for any $\{s^n\}_{n=1}^{\infty}$ such that $s^n \in E^{(n,P,c,G)}$ and s^n is weakly monotone for each $n \in \mathbb{N}\backslash\{1\}$, and for any $(\theta, \hat{\theta}) \in \Theta_P \times ([w_{\min}, 1) \times ([-\gamma, 0) \cup (0, \gamma]))$,*

$$\lim_{n\to\infty} \frac{s^n(\theta)}{s^n(\hat{\theta})} = sgn\left(\frac{v}{\hat{v}}\right) \left|\frac{\frac{v}{u'(w)}}{\frac{\hat{v}}{u'(\hat{w})}}\right|^{\frac{1}{\kappa(c)-1}}. \tag{40}$$

Proof. For any $(\theta, \hat{\theta}) \in ([w_{\min}, 1] \times [-\gamma, 0))^2 \cup ([w_{\min}, 1] \times (0 - \gamma, \gamma])^2\, [-1, 0]^2 \cup [0, 1]^2$, Equality (40) reduces to Equality (39), which holds by Lemma 12. We want to show that Equality (40) holds as well for any $(\theta, \hat{\theta})$ such that $sgn(v) \neq sgn(\hat{v})$, that is, v and $\hat{v}$ have opposite sign. For any $\theta \equiv (w, v) \in [w_{\min}, 1] \times [-\gamma, 0) \cup (0, \gamma]$, by Lemma 10,

$$\lim_{n\to\infty} \frac{c'(s^n((w, v)))}{c'(s^n((w, -v))} = \frac{\frac{v}{u'(w)}}{\frac{-v}{u'(w)}} = -1.$$

Hence, for any $\theta, \hat{\theta} \in \Theta_P$ such that $v\hat{v} < 0$,

$$\lim_{n\to\infty} \frac{c'(s^n((w, v)))}{c'(s^n((\hat{w}, \hat{v}))} = \lim_{n\to\infty} \frac{-c'(s^n((w, |v|)))}{c'(s^n((\hat{w}, |\hat{v}|)))} = -\frac{\frac{|v|}{u'(w)}}{\frac{|\hat{v}|}{u'(\hat{w})}},$$

Thus,

$$-\lim_{n\to\infty}\frac{c'(s^n(\theta))}{c'(s^n(\hat{\theta}))}=\frac{\frac{|v|}{u'(w)}}{\frac{|\hat{v}|}{u'(\hat{w})}}. \tag{41}$$

Note that the left hand side of Expression (41) is equal to $\lim_{n\to\infty}\frac{c'(|s^n(\theta)|)}{c'(|s^n(\hat{\theta})|)}\in\mathbb{R}_+$, so we can take logarithms on both side, and obtain

$$\lim_{n\to\infty}\left(\ln c'(|s^n(\theta)|)-\ln c'(|s^n(\hat{\theta})|)\right)=\ln\left(\frac{\frac{|v|}{u'(w)}}{\frac{|\hat{v}|}{u'(\hat{w})}}\right). \tag{42}$$

By Lemma 11, for any $\{x_n\}_{n=1}^{\infty}\in\mathbb{R}_{++}^{\infty}$ with $\lim_{n\to\infty}x_n=0$ and $\{y_n\}_{n=1}^{\infty}\in\mathbb{R}_{++}^{\infty}$ with $\lim_{n\to\infty}y_n=0$,

$$\lim_{n\to\infty}\frac{\ln c'(x_n)-\ln c'(y_n)}{\ln\frac{x_n}{y_n}}=\kappa(c)-1,$$

thus, in particular,

$$\begin{aligned}\lim_{n\to\infty}\frac{\ln c'(|s^n(\theta)|)-\ln c'(|s^n(\hat{\theta})|)}{\ln\frac{|s^n(\theta)|}{|s^n(\hat{\theta})|}} &= \kappa(c)-1,\\ \lim_{n\to\infty}\left(\ln c'(|s^n(\theta)|)-\ln c'(|s^n(\hat{\theta})|)\right) &= \lim_{n\to\infty}\ln\left|\frac{s^n(\theta)}{s^n(\hat{\theta})}\right|^{\kappa(c)-1},\end{aligned}$$

and thus substituting the left hand side according to Equality (42), we obtain

$$\begin{aligned}\ln\left(\frac{\frac{|v|}{u'(w)}}{\frac{|\hat{v}|}{u'(\hat{w})}}\right) &= \lim_{n\to\infty}\ln\left|\frac{s^n(\theta)}{s^n(\hat{\theta})}\right|^{\kappa(c)-1}, \text{ so } \lim_{n\to\infty}\left|\frac{s^n(\theta)}{s^n(\hat{\theta})}\right|=\left|\frac{\frac{v}{u'(w)}}{\frac{\hat{v}}{u'(\hat{w})}}\right|^{\frac{1}{\kappa(c)-1}}, \text{ and}\\ \lim_{n\to\infty}\frac{s^n(\theta)}{s^n(\hat{\theta})} &= sgn\left(\frac{v}{\hat{v}}\right)\left|\frac{\frac{v}{u'(w)}}{\frac{\hat{v}}{u'(\hat{w})}}\right|^{\frac{1}{\kappa(c)-1}}.\end{aligned}$$

■

So acquisitions of votes converge to linear in a power of valuations.

For any $\rho\in\mathbb{R}_{++}$, define the subset of probability measures $\mathcal{P}^{\rho}\subset\mathcal{P}$ by

$$\mathcal{P}^{\rho}\equiv\left\{P\in\mathcal{P}:\int_{\theta\in\Theta_P}p(\theta)sgn(v)\left|\frac{v}{u'(w)}\right|^{\rho}d\theta\in\mathbb{R}\backslash\{0\}\right\}. \tag{43}$$

Lemma 14 *For any $\rho\in\mathbb{R}_{++}$, $\mathcal{P}^{\rho}$ is open and dense in the metric space $(\mathcal{P},d_{TV})$.*

Proof. Consider an arbitrary $P \in \mathcal{P}^{\rho}$, so $\int\limits_{\theta\in\Theta_P} p(\theta)sgn(v)\left|\frac{v}{u'(w)}\right|^{\rho} d\theta \in \mathbb{R}\backslash\{0\}$, and without loss of generality (up to relabeling of alternatives), assume that there exists $\delta \in \mathbb{R}_{++}$ such that

$$\int_{\theta\in\Theta_P} p(\theta)sgn(v)\left|\frac{v}{u'(w)}\right|^{\rho} d\theta = \delta.$$

For any $\varepsilon \in \mathbb{R}_{++}$, let $N_{\varepsilon}(P)$ be the open $\varepsilon-$neighborhood around P, in the metric space $(\mathcal{P}, d_{TV})$. For any $\varepsilon \in \mathbb{R}_{++}$, and for any $P' \in N_{\varepsilon}(P)$,

$$\sup_{D\in B(\Theta)} \{|P(D) - P'(D)|\} < \varepsilon.$$

Let F_P be the cumulative distribution of $sgn(\tilde{v})\left|\frac{\tilde{v}}{u'(\tilde{w})}\right|^{\rho}$ given that $\tilde{\theta}$ is distributed over Θ_P according to P. Since $P(O) \in (0,1]$ for any open $O \subseteq [w_{\min}, 1] \times [-\gamma, \gamma]$, F_P is strictly increasing in $sgn(\tilde{v})\left|\frac{\tilde{v}}{u'(\tilde{w})}\right|^{\rho}$ from $F_P\left(-\left(\frac{\gamma}{u'(1)}\right)^{\rho}\right) = 0$ to $F_P\left(\left(\frac{\gamma}{u'(1)}\right)^{\rho}\right) = 1$ so we can define the inverse $F_P^{-1} : [0,1] \longrightarrow \left[-\left(\frac{\gamma}{u'(1)}\right)^{\rho}, \left(\frac{\gamma}{u'(1)}\right)^{\rho}\right]$ by $F_P^{-1}(y) = x \in \left[-\left(\frac{\gamma}{u'(1)}\right)^{\rho}, \left(\frac{\gamma}{u'(1)}\right)^{\rho}\right]$ such that $F_P(x) = y$.

For any $\varepsilon \in \mathbb{R}_{++}$, define the set of types $\Theta_{P,1-\varepsilon} \equiv \left\{\theta \in \Theta_P : sgn(v)\left|\frac{v}{u'(w)}\right|^{\rho} \leq F_P^{-1}(1-\varepsilon)\right\}$ and let P_{ε} be probability measure over Θ constructed by assigning $P_{\varepsilon}(\Theta_P \backslash \Theta_{P,1-\varepsilon}) = 0$, $P_{\varepsilon}(\{(1, -\gamma_{\max})\}) = \varepsilon$, and $P_{\varepsilon}(D) = P(D)$ for any $D \subseteq \Theta_{P,1-\varepsilon}\backslash\{(1, -\gamma_{\max})\}$. That is, P_{ε} transforms P by assigning no probability to the set with probability measure ε under P of types with greatest value of $sgn(v)\left|\frac{v}{u'(w)}\right|^{\rho}$, and assigning that ε probability to the type that minimizes $sgn(v)\left|\frac{v}{u'(w)}\right|^{\rho}$ across the space of possible types Θ; namely, to type $(-1, \gamma_{\max})$.[45] Note that for any $P' \in N_{\varepsilon}(P)$,

$$\int_{\theta\in\Theta} p'(\theta)sgn(v)\left|\frac{v}{u'(w)}\right|^{\rho} d\theta > \varepsilon\left(sgn(v)\left|\frac{-\gamma_{\max}}{u'(1)}\right|^{\rho}\right) + \int_{\theta\in\Theta_{P,1-\varepsilon}} p(\theta)sgn(v)\left|\frac{v}{u'(w)}\right|^{\rho} d\theta.$$

Further, note that

$$\begin{aligned}
&\lim_{\varepsilon\longrightarrow 0}\left(\varepsilon\left(sgn(v)\left|\frac{-\gamma_{\max}}{u'(1)}\right|^{\rho}\right) + \int_{\theta\in\Theta_{P,1-\varepsilon}} p_{\varepsilon}(\theta)sgn(v)\left|\frac{v}{u'(w)}\right|^{\rho} d\theta\right) \\
= &\int_{\theta\in\Theta_P} p(\theta)sgn(v)\left|\frac{v}{u'(w)}\right|^{\rho} d\theta = \delta \in \mathbb{R}_{++}.
\end{aligned}$$

[45] Note that the probability measure P_{ε} is not absolutely continuous, so it does not belong to the class of measures $\mathcal{P}$, but this observation is irrelevant to our argument.

Thus, there exists $\varepsilon \in \mathbb{R}_{++}$ such that

$$\int_{\theta \in \Theta} p'(\theta) sgn(v) \left| \frac{v}{u'(w)} \right|^{\rho} d\theta \in \mathbb{R}_{++}$$

for any $P' \in N_{\varepsilon}(P)$ with density p', so $N_{\varepsilon}(P) \subseteq \mathcal{P}^{\rho}$ so $\mathcal{P}^{\rho}$ is open in $(\mathcal{P}, d_{TV})$.

To show that $\mathcal{P}^{\rho}$ is dense in $(\mathcal{P}, d_{TV})$, let $P \in \mathcal{P}$ be such that

$$\int_{\theta \in \Theta_P} p(\theta) sgn(v) \left| \frac{v}{u'(w)} \right|^{\rho} d\theta = 0,$$

and, for each $\delta \in \mathbb{R}_{++}$, construct a probability measure $P_{\delta} \in N_{\delta}(P)$ with density p_{δ} by

$$P_{\delta} \left(\theta \in \Theta_P : sgn(v) \left| \frac{v}{u'(w)} \right|^{\rho} \leq F_P^{-1} (1 - \delta) \right) = 1,$$

and

$$p_{\delta}(\theta) = \frac{p(\theta)}{1 - \delta} \text{ for any } \theta \in \Theta_P \text{ such that } sgn(v) \left| \frac{v}{u'(w)} \right|^{\rho} \leq F_P^{-1} (1 - \delta).$$

Then for each $\delta \in \mathbb{R}_{++}$,

$$\int_{\theta \in \Theta_P} p_{\delta}(\theta) sgn(v) \left| \frac{v}{u'(w)} \right|^{\rho} d\theta < 0,$$

and thus $P_{\delta} \in \mathcal{P}^{\rho}$, and a sequence $\{P_{\delta}\}$ with $\delta \longrightarrow 0$ converges to P. Hence, $\mathcal{P}^{\rho}$ is dense in $\mathcal{P}$. ■

Lemma 15 below generalizes to two dimensions the one dimensional version from Pólya, Exercise 127 in Part II, Chapter 3 of Pólya and Szegő (1978). Lemma 15 uses the following –standard– definition of "*weakly monotone*" functions.

Definition 8 *Given an ordered set $(X, \geq_X)$, a function $f : X \longrightarrow \mathbb{R}$ is weakly monotone if $f(x) \geq f(y)$ for any $x, y \in X$ such that $x \geq y$.*

Notice that Definition 8 of "*weakly monotone*" functions differs from our ad hoc Definition 2 of "*weakly monotone*" strategies. We use Definition 8 only in the context Lemma 15 and Corollary 2, and in this context, we explicitly mention which definition of "weak monotonicity" we use each time we invoke the term. Elsewhere in the paper, "weak monotonicity" is always the property of a strategy defined by Definition 2.

Lemma 15 *If a sequence of real valued weakly monotone (continuous or discontinuous) functions converges on a closed rectangle to a continuous function, it converges uniformly.*

Proof. Consider the rectangle $R \equiv [x_1, x_2] \times [y_1, y_2] \subseteq \mathbb{R}^2$, a continuous function $f : R \to \mathbb{R}$, and a sequence of weakly monotone functions $\{f^n\}_{n=1}^{\infty}$ with $f^n : S \to \mathbb{R}$ such that $\lim_{n\to\infty} f^n(x) = f$ for each $x \in R$ (point-wise convergence).

For each $m \in \mathbb{N}$, let L_m be the finite lattice

$$\left\{x_1, \frac{(m-1)x_1 + x_2}{m}, ..., \frac{(m-k)x_1 + kx_2}{m}, ..., x_2\right\} \times \left\{y_1, \frac{(m-1)y_1 + y_2}{m}, ..., \frac{(m-k)y_1 + ky_2}{m}, ..., x_2\right\}.$$

Note that since f is continuous and R is compact, f is uniformly continuous in R (by the Heine-Cantor theorem). Note as well that since for each $n \in \mathbb{N}$, f^n is a weakly monotone function, and $\{f^n\}_{n=1}^{\infty}$ converges to f, it follows that f is also a weakly monotone function. Since f is uniformly continuous, and weakly monotone in R, for any $\varepsilon \in \mathbb{R}_{++}$ there exists $m_\varepsilon \in \mathbb{N}$ such that for any $m > m_\varepsilon$, and for any $x \in R$, there exists $x' \in L_m$ and $x'' \in L_m$ such that

$$f(x' \wedge x'') \leq f(x) \leq f(x' \vee x'') \text{ and } f(x' \vee x'') - f(x' \wedge x'') \leq \varepsilon, \tag{44}$$

where $x' \wedge x''$ is the "meet" of x' and x'' and $x' \vee x''$ is the "join" of x' and x''.[46]

For any $m \in \mathbb{N}$, by the finiteness of L_m and by the fact that the sequence $\{f^n\}_{n=1}^{\infty}$ converges point-wise to f it follows that there exists a function $n : \mathbb{R}_{++} \longrightarrow \mathbb{N}$ such that for any $n > n(\varepsilon)$, $|f^n(x) - f(x)| < \varepsilon$ for every $x \in L_m$, and in particular,

$$f(x' \wedge x'') - \varepsilon \leq f^n(x' \wedge x'') \text{ and } f^n(x' \vee x'') \leq f(x' \vee x'') + \varepsilon. \tag{45}$$

Since f^n is weakly monotone in the order $\geq$, and since $x' \wedge x'' \leq x' \vee x''$, it follows $f^n(x' \wedge x'') \leq f^n(x) \leq f^n(x' \vee x'')$, so from Expression (45), for every $n > n(\varepsilon)$, and for every $x \in R$,

$$f(x' \wedge x'') - \varepsilon \leq f^n(x) \leq f(x' \vee x'') + \varepsilon. \tag{46}$$

From Expression (44), for every $m > m_\varepsilon$, $f(x' \wedge x'') \geq f(x) - \varepsilon$ and $f(x' \vee x'') \leq f(x) + \varepsilon$, so from Expression (46), for every $n > n(\varepsilon)$ and every $m > m_\varepsilon$

$$f(x) - 2\varepsilon \leq f^n(x) \leq f(x) + 2\varepsilon,$$

so $\{f^n\}_{n=1}^{\infty}$ converges uniformly to f in R. ■

The relevant implication from Lemma 15 for our proof is the following corollary.

Corollary 2 *For any tuple $(P, c, G) \in \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, for any sequence $\{\sigma^n\}_{n=2}^{\infty}$ such that $\sigma^n \in E^{(n,P,c,G)}$, and for each $n \in \mathbb{N}\backslash\{1\}$, let $s^{\sigma^n} \in S \cap E^{(n,P,c,G)}$ be the associated pure equilibrium strategy (Definition 6). Then $\left\{\frac{s^{\sigma^n}((w,v))}{s^{\sigma^n}((1,\gamma))}\right\}_{n=2}^{\infty}$ converges uniformly to $sgn(v)\left|\frac{\frac{v}{u'(w)}}{\gamma}\right|^{\rho}$.*

[46] The meet of $x \equiv (x_1, x_2) \in \mathbb{R}^2$ and $y \equiv (y_1, y_2) \in \mathbb{R}^2$ is $(\min\{x_1, y_1\}, \min\{x_2, y_2\})$ and their join is $(\max\{x_1, y_1\}, \max\{x_2, y_2\})$.

Proof. For any tuple $(P, c, G) \in \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, for any sequence $\{\sigma^n\}_{n=2}^{\infty}$ such that $\sigma^n \in E^{(n,P,c,G)}$, and for each $n \in \mathbb{N} \backslash \{1\}$, s^{σ^n} is weakly monotone in the sense of Definition 2 (by Lemma 6), and thus, since $sgn(s^{\sigma^n}((w,v)) = sgn(v)$ for any $(w,v) \in \Theta_P$, it follows that for each $n \in \mathbb{N} \backslash \{1\}$ equilibrium strategy s^{σ^n} is also weakly monotone in the sense of Definition 8 on $[w_{\min}, 1] \times [0, \gamma]$, so since $\left\{\frac{s^{\sigma^n}((w,v))}{s^{\sigma^n}((1,\gamma))}\right\}_{n=2}^{\infty}$ converges pointwise to the continuous function $\left(\frac{\frac{v}{u'(w)}}{\gamma}\right)^{\rho}$ (Lemma 13), it follows by Lemma 15 that it converges uniformly. Further, for each $n \in \mathbb{N} \backslash \{1\}$, the function $f^n : [w_{\min}, 1] \times [0, \gamma]$ defined by $f^n((w,v)) = \frac{-s^{\sigma^n}((w,-v))}{s^{\sigma^n}((1,\gamma))}$ is also weakly monotone in the sense of Definition 8, and it converges pointwise to the continuous function $\left(\frac{\frac{v}{u'(w)}}{\gamma}\right)^{\rho}$ (Lemma 13), thus it converges uniformly (Lemma 15). Since for any $(w,v) \in [w_{\min}, 1] \times [-\gamma, 0]$, $\frac{s^{\sigma^n}((w,v))}{s^{\sigma^n}((1,\gamma))} = -f^n((w,-v))$, it follows that $\left\{\frac{s^{\sigma^n}((w,v))}{s^{\sigma^n}((1,\gamma))}\right\}_{n=2}^{\infty}$ converges uniformly to $-\left(\frac{\frac{-v}{u'(w)}}{\gamma}\right)^{\rho}$ for any $(w,v) \in [w_{\min}, 1] \times [-\gamma, 0]$.

Therefore, $\left\{\frac{s^{\sigma^n}((w,v))}{s^{\sigma^n}((1,\gamma))}\right\}_{n=2}^{\infty}$ converges uniformly to $sgn\,(v) \left|\frac{\frac{v}{u'(w)}}{\gamma}\right|^{\rho}$ for any $\theta \in \Theta_P$. ■

We use this uniform convergence to prove a main proposition.

Proposition 1 *For any $\rho \in \mathbb{R}_{++}$, the sequence of social choice correspondence SC_ρ is implementable over $\mathcal{P}^\rho$ by any vote-buying mechanism $c \in \mathcal{C}_A$ such that $\kappa(c) = \frac{1+\rho}{\rho}$.*

Proof. Let $c \in \mathcal{C}_A$ be such that $\kappa(c) = \frac{1+\rho}{\rho}$. For any tuple $(P, c, G) \in \mathcal{P} \times \mathcal{C}_A \times \mathcal{G}$, and for any $\{\sigma^n\}_{n=1}^{\infty}$ such that $\sigma^n \in E^{(n,P,c,G)}$ for each $n \in \mathbb{N} \backslash \{1\}$, where for each $n \in \mathbb{N} \backslash \{1\}$, $s^{\sigma^n} \in S \cap E^{(n,P,c,G)}$ is the pure strategy associated to σ^n (Definition 6), by Corollary 2, the sequence $\left\{\frac{s^{\sigma^n}(\theta)}{s^{\sigma^n}((1,\gamma))}\right\}_{n=2}^{\infty}$ converges uniformly to $sgn\,(v) \left|\frac{\frac{v}{u'(w)}}{\gamma}\right|^{\rho}$. That is, for any $\varepsilon \in \mathbb{R}_{++}$, there exists $n(\varepsilon) \in \mathbb{N}$ such that for any $\theta \in \Theta_P$, and for any $n > n(\varepsilon)$,

$$\left|\frac{s^{\sigma^n}(\theta)}{s^{\sigma^n}((1,\gamma))} - sgn\,(v) \left|\frac{\frac{v}{u'(w)}}{\gamma}\right|^{\rho}\right| < \varepsilon. \tag{47}$$

Take any $P \in \mathcal{P}^\rho$ such that $\int_{\theta \in \Theta_P} p(\theta) sgn(v) \left|\frac{v}{u'(w)}\right|^{\rho} d\theta \neq 0$, and without loss generality (up to relabeling of alternatives), assume $\int_{\theta \in \Theta_P} p(\theta) sgn(v) \left|\frac{v}{u'(w)}\right|^{\rho} d\theta > 0$, and take any

$$\hat{\varepsilon} \in \left(0, \frac{1}{(\gamma)^{\rho}} \int_{\theta \in \Theta_P} p(\theta) sgn(v) \left|\frac{v}{u'(w)}\right|^{\rho} d\theta\right).$$

By the weak law of large numbers, the random variable

$$\frac{1}{(n)(\gamma)^{\rho}} \sum_{k=1}^{n} sgn\,(\tilde{v}_k) \left|\frac{\tilde{v}_k}{u'(\tilde{w}_k)}\right|^{\rho} - \hat{\varepsilon},$$

where for each $k \in \{1, ..., n\}$, random variable $\tilde{\theta}_k$ is distributed according to P, converges to its expectation

$$\frac{1}{(\gamma)^\rho} \int_{\theta \in \Theta_P} p(\theta) sgn(v) \left| \frac{v}{u'(w)} \right|^\rho d\theta - \hat{\varepsilon} > 0;$$

and therefore,

$$\lim_{n \longrightarrow \infty} \Pr \left[\frac{1}{(n)(\gamma)^\rho} \sum_{k=1}^{n} sgn\left(\tilde{v}_k\right) \left| \frac{\tilde{v}}{u'(\tilde{w})} \right|^\rho - \hat{\varepsilon} > 0 \right] = 1. \tag{48}$$

Since, by Inequality (47), for any $n > n(\hat{\varepsilon})$, and for any $\theta \in \Theta_P$,

$$\frac{s^{\sigma^n}((w, v))}{s^{\sigma^n}((1, \gamma))} > \frac{1}{(\gamma)^\rho} sgn\left(v\right) \left| \frac{v}{u'(w)} \right|^\rho - \hat{\varepsilon},$$

it follows that ex-ante,

$$\Pr \left[\frac{s^{\sigma^n}((\tilde{w}, \tilde{v}))}{s^{\sigma^n}((1, \gamma))} > \frac{1}{(\gamma)^\rho} sgn\left(\tilde{v}\right) \left| \frac{\tilde{v}}{u'(\tilde{w})} \right|^\rho - \hat{\varepsilon} \right] = 1,$$

so aggregating and averaging across agents,

$$\Pr \left[\frac{1}{n} \sum_{k=1}^{n} \frac{s^{\sigma^n}((\tilde{w}_k, \tilde{v}_k))}{s^{\sigma^n}((1, \gamma))} > \frac{1}{(n)(\gamma)^\rho} \sum_{k=1}^{n} sgn\left(\tilde{v}_k\right) \left| \frac{\tilde{v}_k}{u'(\tilde{w}_k)} \right|^\rho - \hat{\varepsilon} \right] = 1,$$

and then from Equality (48),

$$\lim_{n \longrightarrow \infty} \Pr \left[\frac{1}{n} \sum_{k=1}^{n} \frac{s^{\sigma^n}((\tilde{w}_k, \tilde{v}_k))}{s^{\sigma^n}((1, \gamma))} > 0 \right] = 1,$$

and thus since $n > 0$ and $s^{\sigma^n}((1, \gamma)) > 0$,

$$\lim_{n \longrightarrow \infty} \Pr \left[\sum_{k=1}^{n} s^{\sigma^n}((\tilde{w}_k, \tilde{v}_k)) > 0 \right] = 1. \tag{49}$$

Since for any $x \in \mathbb{R}_{++}$, by Equality (24),

$$\lim_{n \longrightarrow \infty} \Pr \left[\sum_{k=1}^{n} s^{\sigma^n}((\tilde{w}_k, \tilde{v}_k)) \in [-x, x] \right] = 0,$$

it then follows that for any $x \in \mathbb{R}_{++}$,

$$\lim_{n \longrightarrow \infty} \Pr \left[\sum_{k=1}^{n} s^{\sigma^n}((\tilde{w}_k, \tilde{v}_k)) > x \right] = 1. \tag{50}$$

Since for each $n \in \mathbb{N}^n$, by definition of s^{σ^n},

$$P\left(\{\theta \in \Theta_P : L(\{r \in [0,1] : \sigma^n(r,\theta\} = s^{\sigma^n}(\theta)\}) = 1\right) = 1,$$

(that is, since σ and s^{σ^n} coincide generically across the space of types), it follows that

$$\Pr\left[\sum_{k=1}^{n} s^{\sigma^n}(\tilde{\theta}_k) = \sum_{k=1}^{n} \sigma^n(\tilde{r}_k, \tilde{\theta}_k)\right] = 1,$$

so Equality (50) implies that for any $x \in \mathbb{R}_{++}$,

$$\lim_{n \longrightarrow \infty} \Pr\left[\sum_{k=1}^{n} \sigma^n(\tilde{r}_k, \tilde{\theta}_k) > x\right] = 1. \tag{51}$$

Construct a sequence $\{x^t\}_{t=1}^{\infty}$ such that $x^t \in \mathbb{R}_{++}$ and $G(x^t) \in \left(1 - \frac{1}{t}, 1\right)$ for each $t \in \mathbb{N}$. From Equality (51), there exists a strictly increasing function $m : \mathbb{N} \longrightarrow \mathbb{N}$ such that for any $t \in \mathbb{N}$, and for any $n > m(t)$,

$$\Pr\left[\sum_{k=1}^{n} \sigma^n(\tilde{r}_k, \tilde{\theta}_k) > x^t\right] > 1 - \frac{1}{t},$$

so

$$\Pr\left[G\left(\sum_{k=1}^{n} \sigma^n(\tilde{r}_k, \tilde{\theta}_k)\right) > 1 - \frac{1}{t}\right] > 1 - \frac{1}{t},$$

so

$$\Pr\left[\tilde{d}_P^n(\sigma^n, \tilde{r}_{N^n}, \tilde{\theta}_{N^n}) = A\right] > \left(1 - \frac{1}{t}\right)^2, \tag{52}$$

from which it follows that

$$\lim_{n \longrightarrow \infty} \Pr\left[\tilde{d}_P^n(\sigma^n, \tilde{r}_{N^n}, \tilde{\theta}_{N^n}) = A\right] = 1 \tag{53}$$

for any $P \in \mathcal{P}^\rho$ such that

$$\int_{\theta \in \Theta_P} p(\theta) sgn(v) \left|\frac{v}{u'(w)}\right|^\rho d\theta > 0. \tag{54}$$

Note that for any $P \in \mathcal{P}^\rho$ satisfying Inequality (54), since by the weak law of large numbers,

$$\lim_{n \longrightarrow \infty} \Pr\left[\sum_{k=1}^{n} sgn(\tilde{v}) \left|\frac{\tilde{v}}{u'(\tilde{w})}\right|^\rho d\theta > 0\right] = 1,$$

and since $SC^n_\rho(\theta_{N^n}) = A$ if and only if

$$\sum_{k=1}^{n} sgn(v_k) \left| \frac{v_k}{u'(w_k)} \right|^\rho > 0,$$

it follows that

$$\lim_{n \longrightarrow \infty} \Pr\left[SC^n_\rho(\tilde{\theta}_{N^n}) = A \right] = 1. \tag{55}$$

Therefore, by equalities (53) and (55), for any $P \in \mathcal{P}^\rho$ satisfying Inequality (54) we obtain

$$\lim_{n \longrightarrow \infty} \Pr\left[\tilde{d}^n_P(\sigma^n, \tilde{r}_{N^n}, \tilde{\theta}_{N^n}) = SC^n_\rho(\tilde{\theta}_{N^n}) \right] = 1. \tag{56}$$

An analogous proof shows that for any $P \in \mathcal{P}^\rho$ such that $\int_{\theta \in \Theta_P} p(\theta) sgn(v) \left| \frac{v}{u'(w)} \right|^\rho d\theta < 0$,

$$\lim_{n \longrightarrow \infty} \Pr\left[\tilde{d}^n_P(\sigma^n, \tilde{r}_{N^n}, \tilde{\theta}_{N^n}) = SC^n_\rho(\tilde{\theta}_{N^n}) = B \right] = 1,$$

so we conclude that Equality (56) holds for any $P \in \mathcal{P}^\rho$, and mechanism c asymptotically implements the sequence of choice rules SC_ρ over the set of probability measures $\mathcal{P}^\rho$. ■

After having detailed sufficient conditions for generic implementability in Proposition 1, we next prove that these conditions are (almost) also necessary.

Proposition 2 *Any $SC \in \mathcal{SC}$ that is implementable generically over $\mathcal{P}$ by $\mathcal{C}_A$ is generically equivalent to SC_ρ for some $\rho \in \mathbb{R}_{++}$.*

Proof. Assume mechanism $c \in \mathcal{C}_A$ implements SC generically. We want to show that there exists $\rho \in \mathbb{R}_{++}$ such that SC is generically equivalent to SC_ρ.

Recall that for any vote-buying mechanism $c \in \mathcal{C}_A$, $\kappa(c) \in (1, \infty)$. Then note that from Proposition 1, for any $\rho \in \mathbb{R}_{++}$, any vote-buying mechanism mechanism $c \in \mathcal{C}_A$ with $\kappa(c) = \frac{1+\rho}{\rho}$ implements SC_ρ over $\mathcal{P}^\rho$, so defining $z \equiv \frac{1+\rho}{\rho}$, and hence $\rho = \frac{1}{z-1}$ for any $z \in (1, \infty)$, any vote-buying mechanism $c \in \mathcal{C}_A$ with $\kappa(c) = z$ implements $SC_{\frac{1}{z-1}} = SC_\rho$ over $\mathcal{P}^\rho$. Since $\bigcup_{z \in (1,\infty)} \{c \in \mathcal{C}_A : \kappa(c) = z\} = \mathcal{C}_A$, it follows that for any $c \in \mathcal{C}_A$, $\exists \rho \in \mathbb{R}_{++}$ such that c implements SC_ρ over $\mathcal{P}^\rho$ (in particular, $\rho = \frac{1}{\kappa(c)-1}$), so for any $P \in \mathcal{P}^\rho$,

$$\lim_{n \longrightarrow \infty} \Pr\left[\tilde{d}^n_P(\sigma^n, \tilde{r}_{N^n}, \tilde{\theta}_{N^n}) = SC^n_\rho(\tilde{\theta}_{N^n}) \right] = 1. \tag{57}$$

Since c also implements SC generically over $\mathcal{P}$, there exists an open subset of measures $\mathcal{P}^D \subseteq \mathcal{P}$ dense in $\mathcal{P}$ such that c implements SC over $\mathcal{P}^D$, so

$$\lim_{n \longrightarrow \infty} \Pr\left[\tilde{d}^n_P(\sigma^n, \tilde{r}_{N^n}, \tilde{\theta}_{N^n}) = SC^n(\tilde{\theta}_{N^n}) \right] = 1. \tag{58}$$

It follows from equalities (57) and (58) that for any $P \in \mathcal{P}^\rho \cap \mathcal{P}^D$,

$$\lim_{n \longrightarrow \infty} \Pr\left[SC^n_\rho(\tilde{\theta}_{N^n}) = SC^n(\tilde{\theta}_{N^n})\right] = 1.$$

Since $\mathcal{P}^\rho$ and $\mathcal{P}^D$ are open and dense in $\mathcal{P}$ (Lemma 14), and since the intersection of two open dense sets is dense by Baire's (1899) Category Theorem, $\mathcal{P}^\rho \cap \mathcal{P}^D$ is itself an open set that is dense in $\mathcal{P}$, and SC is generically equivalent to SC_ρ. ■

Propositions 1 and 2 together lead to our main result: the characterization of generically implementable sequences of social choice correspondences in Theorem 1. We restate the theorem here.

Theorem 1. Any $SC \in \mathcal{SC}$ is generically implementable by $\mathcal{C}_A$ if and only if $SC \in \mathcal{SC}_{\mathbb{R}_{++}}$. *Further, any $c \in \mathcal{C}_A$ generically implements $SC \in \mathcal{SC}$ if and only if SC is generically equivalent to $SC_{\frac{1}{\kappa(c)-1}}$.*

Proof of Theorem 1. By Proposition 1, for any $\rho \in \mathbb{R}_{++}$, any vote-buying mechanism $c \in \mathcal{C}_A$ with $\kappa(c) = \frac{1+\rho}{\rho}$ (so $\rho = \frac{1}{\kappa(c)-1}$) implements SC_ρ over $\mathcal{P}^\rho$, and by Lemma 14, $\mathcal{P}^\rho$ is an open dense subset of $\mathcal{P}$, hence c implements $SC_\rho = SC_{\frac{1}{\kappa(c)-1}}$ generically, and thus, it also implements any $SC' \in Eq(SC_\rho)$ generically.

For any $SC \in \mathcal{SC}_{\mathbb{R}_{++}}$, by definition of $\mathcal{SC}_{\mathbb{R}_{++}}$, there exists $\rho \in \mathbb{R}_{++}$ such that $SC \in Eq(SC_\rho)$. Since SC is generically equivalent to SC_ρ, there exists an open dense set $\mathcal{P}^D \subseteq \mathcal{P}$ such that for any $P \in \mathcal{P}^D$,

$$\lim_{n \longrightarrow \infty} \Pr\left[SC^n(\tilde{\theta}_{N^n}) = SC^n_\rho(\tilde{\theta}_{N^n})\right] = 1.$$

Since SC and SC_ρ are generically equivalent over $\mathcal{P}^\rho \cap \mathcal{P}^D$,

$$\lim_{n \longrightarrow \infty} \Pr\left[SC^n(\tilde{\theta}_{N^n}) = SC^n_\rho(\tilde{\theta}_{N^n})\right] = 1 \text{ for any } P \in \mathcal{P}^\rho \cap \mathcal{P}^D, \tag{59}$$

and, since vote-buying mechanism c with $\kappa(c) = \frac{1+\rho}{\rho}$ implements SC_ρ over $\mathcal{P}^\rho$,

$$\lim_{n \longrightarrow \infty} \Pr\left[\tilde{d}^n_P(\sigma^n, \tilde{r}_{N^n}, \tilde{\theta}_{N^n}) = SC^n_\rho(\tilde{\theta}_{N^n})\right] = 1 \text{ for any } P \in \mathcal{P}^\rho \cap \mathcal{P}^D. \tag{60}$$

It follows from equations (59) and (60) that

$$\lim_{n \longrightarrow \infty} \Pr\left[\tilde{d}^n_P(\sigma^n, \tilde{r}_{N^n}, \tilde{\theta}_{N^n}) = SC^n(\tilde{\theta}_{N^n})\right] = 1 \text{ for any } P \in \mathcal{P}^\rho \cap \mathcal{P}^D.$$

Since $\mathcal{P}^\rho$ is open and dense in $\mathcal{P}$ (Lemma 14), and since the intersection of two open dense sets is open dense (as noted above, by the Category Theorem by Baire (1899)), it follows that $\mathcal{P}^\rho \cap \mathcal{P}^D$ is itself an open dense set in $\mathcal{P}$, and thus c with $\kappa(c) = \frac{1+\rho}{\rho}$ implements SC generically. Therefore, the class of mechanisms $\mathcal{C}_A$ generically implements $\mathcal{SC}_{\mathbb{R}_{++}}$.

For any sequence $SC \notin \mathcal{SC}_{\mathbb{R}_{++}}$, sequence SC is not generically equivalent to SC_ρ for any $\rho \in \mathbb{R}_{++}$, and thus SC is not implementable generically by $\mathcal{C}_A$ over $\mathcal{P}$, by Proposition 2. ■

4. An extension to non neutral mechanisms

We explore whether vote-buying mechanisms can implement non-neutral collective choice rules that are "biased" toward one alternative. Consider, for instance, a choice between a status quo and a reform proposal. In such instances, perhaps it may be deemed normatively desirable that the reform be enacted only if it enjoys great support. As we saw in our main result, the choice rules that are implementable by admissible vote-buying mechanisms are all neutral by construction, in the sense that a reversal in the sign of the valuation for each agent reverses the order of the alternatives too.

Here we note that if the election designer can charge $c(x)$ for x votes for A (reform) and $\beta \times c(x)$ for x votes for B (status-quo), for some $\beta \in (0, 1]$ and some admissible vote-buying mechanism c, then it can implement non-neutral rules that require the society to be substantially in favor of the reform in order for the status-quo to be defeated. Let $\mathcal{C}_\beta$ denote the class of non-neutral vote-buying mechanisms defined by a pair (β, c), where $\beta \in (0, 1)$ and c is an admissible vote-buying mechanism, that is, $c \in \mathcal{C}_A$

Claim 1 *Any non-neutral vote-buying mechanism (β, c) such that c has limit elasticity $\kappa(c)$ generically implements rule the following social choice rule:*

$$\begin{cases} B \text{ if } \sum_{k \in N_A} \left(\frac{v_k}{u'(w_k)}\right)^{\frac{1}{\kappa(c)-1}} < \sum_{k \in N_B} \left(\frac{v_k}{\beta u'(w_k)}\right)^{\frac{1}{\kappa(c)-1}} \\ \{A, B\} \text{ if } \sum_{k \in N_A} \left(\frac{v_k}{u'(w_k)}\right)^{\frac{1}{\kappa(c)-1}} = \sum_{k \in N_B} \left(\frac{v_k}{\beta u'(w_k)}\right)^{\frac{1}{\kappa(c)-1}} \\ A \text{ if } \sum_{k \in N_A} \left(\frac{v_k}{u'(w_k)}\right)^{\frac{1}{\kappa(c)-1}} > \sum_{k \in N_B} \left(\frac{v_k}{\beta u'(w_k)}\right)^{\frac{1}{\kappa(c)-1}} . \end{cases}$$

Proof. The proof follows the logic of the proof of Theorem 1. With cost function c_β defined by $c_\beta(a) = c(a)$ for any $a \in \mathbb{R}_+$ and $c_\beta(a) = \beta c(a)$ for any $a < 0$, the marginal utility cost for negative actions is $\beta u'(w)c'(a)$, instead of $u'(w)c'(a)$ as in the proof of Theorem 1; therefore replace $u'(w)c'(a)$ with $\beta u'(w)c'(a)$ and follow each step of the proof of Theorem 1 to obtain this generalization, with Theorem 1 then as the special case for $\beta = 1$. ■

5. A numerical example

We illustrate Theorem 1 with a numerical example. Consider a society of agents with quasilinear preferences over wealth and over the decision on whether or not to pass policy A. Assume that if A passes, each agent i, with independently drawn probability $\frac{1}{4}$, receives a benefit $v > 0$, and with probability $\frac{3}{4}$, the agent incurs a disutility of one (if A does not pass, a status quo B remains in place, with utilities normalized to zero). At the constitutional stage in which the procedure to make the collective decision is instituted, v is not known. Suppose that v can take either a high value of 4, or a low value of 2. We consider, for this example, three possible social choice rules that society might wish to implement:

i. Choose the alternative that maximizes the sum of the square root of valuations (i.e., $\rho = \frac{1}{2}$);
ii. Choose the alternative that maximizes the sum of valuations (i.e., $\rho = 1$ utilitarianism); or
iii. Choose the alternative that maximizes the sum of squares of valuations, preserving the sign (i.e., $\rho = 2$).

ρ	$v = 2$	$v = 4$
$\frac{1}{2}$	B	B
1	B	A
2	A	A

Table 1: Asymptotically desired choice, as a function of ρ.

Table 1 shows the alternative that, with probability converging to one, society would like to choose according to its choice rule, as a function of the value of v. As a benchmark, note that simple majority voting with full turnout and sincere voting asymptotically results in B with probability one, so in this example simple majority works well if society wants to implement the rule with parameter $\rho = \frac{1}{2}$, but not if it wants to implement the rules with $\rho = 1$ or $\rho = 2$. Since a vote-buying mechanism with $c(a) = |a|^{\frac{1+\rho}{\rho}}$ asymptotically implements the choice rule with parameter ρ (Theorem 1), let's show how the mechanisms $c(a) = |a|^3$, $c(a) = |a|^2$ and $c(a) = |a|^{\frac{3}{2}}$ perform, respectively, for $\rho = \frac{1}{2}$, $\rho = 1$ or $\rho = 2$. In all cases, we assume $G(x) = \frac{e^x}{1+e^x}$ with derivative $g(x) = \frac{e^x}{(1+e^x)^2}$, we compute results for societies of sizes 200, 800, 2000 and 4000 citizens, and we tract the speed of convergence of equilibrium actions to zero, pivotality ratios to one, and probability of choosing the desired outcome to one.[47]

5.1. Maximize the sum of square roots of valuations ($\rho = \frac{1}{2}$)

In Table 2 we show the equilibrium actions of agents who favor (a_A) and oppose (a_B) adopting A; the ratio of their respective marginal pivotality ratios piv_A and piv_B relative to the marginal pivotality of a null action; and under the row labelled "*success*", the probability that the alternative chosen by the vote-buying mechanism $c(a) = |a|^3$ is the one desired by society's choice rule, for the environment in which $v = 2$. Since $\kappa(c) = 3$ and $\rho = \frac{1}{2}$, the desired condition, $\rho = \frac{1}{\kappa(c)-1}$, holds, and the given vote-buying mechanism should implement society's choice rule.

Notice an underdog effect: the choice is almost always B, so the pivotality is always higher for the underdog (supporters of A), but the magnitude of this effect asymptotically vanishes.

[47] We obtain similar convergence results if we assume instead that $u(w) = \ln w$, that $v = 1$, and that the wealth (and thus the marginal willingness to pay) of the type who likes A is two or four times as large as the wealth (assumed to be one) of the type who dislikes A.

n	200	800	2000	4000
a_A	0.0989	0.0317	0.0149	0.084
a_B	-0.0648	-0.0218	-0.0104	-0.0059
piv_A	1.0961	1.0320	1.0150	1.0084
piv_B	0.9413	0.9785	0.9897	0.9941
$Success$	98.64%	99.86%	99.97%	99.99%

Table 2: Results if $v = 2$; asympotically B is the desired choice.

Table 3 shows analogous results, now for the case in which $v = 4$.

n	200	800	2000	4000
a_A	0.2464	0.0762	0.0328	0.0179
a_B	-0.11184	-0.0361	-0.0160	-0.0088
piv_A	1.1409	1.0753	1.0332	1.0181
piv_B	0.9399	0.9661	0.9842	0.9912
$Success$	94.44%	99.59%	99.92%	99.98%

Table 3: Results if $v = 4$; asymptotically, B is the desired choice.

We observe the following intuitive comparative static across the two tables: if the underdogs (A supporters) care more ($v = 4$ instead of $v = 2$), they try harder, outcomes become more competitive, and convergence is not as fast. Vote-buying mechanism will also asymptotically deliver outcome B (with probability converging to one) for any $v < 9$, with convergence ever slower as v approaches 9; whereas, for any $v > 9$, the mechanism will asymptotically deliver A instead, again in accordance to society's desired rule.

5.2. Maximize the sum of valuations ($\rho = 1$)

Next we consider a society that wishes to implement utilitarianism. If v is low ($v = 2$), the expected per capita value of A is $\frac{-1}{4}$, and asymptotically, the realized per capita value converges to its expectation, so with probability converging to one, the utilitarian choice is B. Table 4 shows the results under the vote-buying mechanism $c(a) = |a|^2$ if $v = 2$, where "success" is now the probability of choosing the utilitarian alternative. We observe that $\rho = \frac{1}{\kappa(c)-1}$, and hence, according to Theorem 1 the posited mechanism should deliver outcomes that align with utilitarianism.

If v is high ($v = 4$), the expected per capita value of A is $\frac{1}{4}$, and asymptotically, the realized per capita value converges to its expectation, so with probability converging to one, the utilitarian choice is A. Table 4 shows the results under the vote-buying mechanism $c(a) = |a|^2$ if $v = 4$.

Convergence is even faster if the expected valuation is more lopsided; as indicated in the body of the paper. For instance, if $v = 10$, the probability that the outcome is the utilitarian

n	200	800	2000	4000
a_A	0.1160	0.03986	0.0178	0.0098
a_B	-0.0528	-0.0189	-0.0087	-0.0048
piv_A	1.0667	1.0354	1.0172	1.0096
piv_B	0.9701	0.9835	0.9917	0.9952
$Success$	85.84%	95.91%	98.21%	99.04%

Table 4: Results if $v = 2$, so asymptotically, B is the desired choice.

n	200	800	2000	4000
a_A	0.2223	0.0951	0.0397	0.0207
a_B	-0.0613	-0.0258	-0.0104	-0.0053
piv_A	0.9247	0.9372	0.9645	0.9802
piv_B	1.0203	1.0175	1.0095	1.0051
$Success$	82.29%	93.71%	97.84%	98.93%

Table 5: Results if $v = 4$, so asymptotically, A is the desired choice.

one is over 0.993 if $n = 1,000$, and over 0.999 if $n = 10,000$.

5.3. Maximize the sum of squares of valuations ($\rho = 2$)

Finally, suppose society wants to implement the rule with $\rho = 2$. In this case, whether $v = 2$ or $v = 4$, asymptotically the desired choice is A. Table 6 shows the results using mechanism $c(a) = |a|^{\frac{3}{2}}$ if $v = 2$, and Table 6 shows the results if $v = 4$, under this same mechanism. Again, the relevant condition described in Theorem 1 holds, and the vote-buying mechanism should implement the choice rule characterized by $\rho = 2$.

n	200	800	2000	4000
a_A	0.0733	0.0368	0.0185	0.0104
a_B	-0.0194	-0.0097	-0.0048	-0.0027
piv_A	0.9775	0.0972	0.9869	0.9916
piv_B	1.0057	1.0055	1.0035	1.0022
$Success$	68.73%	80.96%	88.02%	91.50%

Table 6: Results if $v = 2$; asymptotically A is the desired choice.

Similarly, the mechanism asymptotically yields A with probability converging to one for any $v > \sqrt{3}$, and it (asymptotically, probabilistically) yields B if $v < \sqrt{3}$. Competition is weaker and convergence is faster if v is further away from this cutoff, as illustrated by the comparison between the two tables.

n	200	800	2000	4000
a_A	0.0557	0.0179	0.0082	0.0045
a_B	-0.0038	-0.0012	-0.0005	-0.0003
piv_A	0.9576	0.9842	0.9924	0.9957
piv_B	1.0029	1.0010	1.0005	1.0003
$Success$	89.67%	94.61%	96.45%	97.41%

Table 7: Results if $v = 4$; asymptotically A is the desired choice.

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