Online Appendix of "Market Fragmentation"

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Appendix A: Verification Theorem

Here, we prove a theorem that we later use to verify a candidate symmetric affine equilibrium. The theorem applies to the models of Section III, Section V, and Appendix H. In what follows, $\mathcal{F}_i$ denotes trader i's information set. In the models of Section III and Appendix H, $\mathcal{F}_i = \sigma(X_i)$ while in the model of Section V, $\mathcal{F}_i = \sigma(X_i, \sum_{j \in N} X_j)$. The set of admissible demand schedules, $\mathcal{M}_i$, is the set of all maps $h : \Omega \times \mathbb{R} \to \mathbb{R}$ that are $\mathcal{F}_i \times \mathcal{B}(\mathbb{R})$-measurable. We denote a candidate symmetric affine equilibrium by the associated triple of demand schedule coefficients, (Δ, α, ζ).

THEOREM 1: *Let (Δ, α, ζ) be a candidate symmetric affine equilibrium. For each $e \in E$ and $i \in N$ set*

$$r_{ie} := -\alpha \sum_{j \neq i} X_j + (N-1)\Delta - Q_e$$

and let f_{ie} be as in (2). Necessary conditions for (Δ, α, ζ) to be a symmetric affine equilibrium are that $\zeta \neq 0$ and

$$\text{(A1)} \qquad \mu_\pi - \mathbb{E}\left[2b\left(X_i + \sum_{e \in E} f_{ie}(\omega, p_e^f)\right) \mid \mathcal{F}_i, r_{ie}\right] = p_e^f + \frac{1}{\zeta(N-1)} f_{ie}(\omega, p_e^f)$$

holds almost surely for each $i \in N$ and $e \in E$. If $\zeta > 0$ then (A1) *is also sufficient.*

PROOF:

We first show that if $\zeta = 0$, then (Δ, α, ζ) can not be a symmetric affine equilibrium. In this candidate equilibrium, with probability 1, no trades are executed on any exchange.[1] Now suppose trader i deviates to submitting the demand schedule $-\epsilon p$ on a given exchange e. Then trader i will absorb $\alpha \sum_{j \neq i} X_j - (N-1)\Delta + Q_e$ units from the other traders. The market clearing price on exchange e will be

$$p_e = \frac{-\alpha \sum_{j \neq i} X_j + \Delta_E(N-1) - Q_e}{\epsilon}.$$

Thus the transfer to trader i is

$$-p_e q_{ie} = \frac{(\alpha \sum_{j \neq i} X_j - \Delta(N-1) + Q_e)^2}{\epsilon}.$$

[1] Recall from Section II that if a unique market clearing price does not exist no trades are executed.

That is, for $\epsilon > 0$ sufficiently small the deviation is profitable. Hence (α, ζ, Δ) can not be a symmetric affine equilibrium.

Going forward we assume $\zeta \neq 0$. Suppose that all traders $j \neq i$ submit demand schedules of the form in (2) to each exchange. Suppose trader i submits $g_{ie} \in \mathcal{M}$ to exchange e. Then if a market clearing price exists, it satisfies

$$p_e(\omega) = \frac{r_{ie}(\omega) + g_{ie}\left(\omega, p_e(\omega)\right)}{\zeta(N-1)}. \tag{A2}$$

For any given demand schedule g_{ie} which conditions the quantity purchased on the realization of p_e there is a function $\tilde{g}_{ie}$ which conditions the quantity purchased on the realization of r_{ie} such that

$$\tilde{g}_{ie}(\omega, r_{ie}(\omega)) = g_{ie}\left(\omega, p_e(\omega)\right)$$

for each $\omega \in \Omega$ for which a unique clearing price exists and

$$\tilde{g}_{ie}(\omega, r_{ie}(\omega)) = 0$$

for each $\omega \in \Omega$ such that there is no unique clearing price. To see this, define $\tilde{g}_{ie}$ as follows. For each $r \in \mathbb{R}$ let $p(r, \omega)$ denote the unique solution to

$$p = \frac{r + g_{ie}(\omega, p)}{\zeta(N-1)}$$

if such a solution exists. For all r such that $p(r, \omega)$ is well defined, we let

$$\tilde{g}_{ie}(\omega, r) = g_{ie}\left(\omega, p(r, \omega)\right).$$

Otherwise, set

$$\tilde{g}_{ie}(\omega, r) = 0.$$

Given $\{f_{ie}\}_{e \in E}$ as in the statement of theorem, define $\{\tilde{f}_{ie}\}_{e \in E}$ in this way. Then

$$\tilde{f}_{ie}(\omega, r) = -\alpha \frac{N-1}{N} X_i - \frac{r}{N} + \frac{N-1}{N} \Delta$$

for each $e \in E$.

It is convenient to relax trader i's optimization problem to

$$\sup_{\tilde{g}_i \in \mathcal{M}_i^E} \mathbb{E}\left[\pi \sum_{e \in E} \tilde{g}_{ie}(\omega, r_{ie}) - b\left(X_i + \sum_{e \in E} \tilde{g}_{ie}(\omega, r_{ie})\right)^2\right] - \mathbb{E}\left[\sum_{e \in E} \frac{r_{ie} + \tilde{g}_{ie}(\omega, r_{ie})}{\zeta(N-1)} \tilde{g}_{ie}(\omega, r_{ie})\right], \tag{A3}$$

where, as is standard, we suppress ω from the notation for X_i and r_{ie}. For some $\tilde{g}_i = (\tilde{g}_{i1}, ..., \tilde{g}_{iE}) \in \mathcal{M}_i^E$, the expectation may be infinite. As a result we first restrict to the subset $\hat{\mathcal{M}}^E$ where $\hat{\mathcal{M}}$ is the subset of $h \in \mathcal{M}$ such that $h(\,\cdot\,, r_{ie}(\,\cdot\,))$ is a finite-variance random variable. Later, we argue that this is without loss of generality because, if $\zeta > 0$, any profile of demand schedules outside of $\hat{\mathcal{M}}^E$ leads to a utility of $-\infty$.

To derive a first order condition, for each $e \in E$, we take the variation of $\tilde{f}_{ie}$ with an arbitrary $h_e \in \hat{\mathcal{M}}$ and substitute into the objective. This gives

$$\text{(A4)}\quad \mathbb{E}\left[\pi \sum_{e\in E} \big(\tilde{f}_{ie}(\omega, r_{ie}) + \nu h_e(\omega, r_{ie})\big) - b\big(X_i + \sum_{e\in E} \tilde{f}_{ie}(\omega, r_{ie}) + \nu h_e(\omega, r_{ie})\big)^2\right] - \mathbb{E}\left[\sum_{e\in E} \frac{r_{ie} + \tilde{f}_{ie}(\omega, r_{ie}) + \nu h_e(\omega, r_{ie})}{\zeta(N-1)} \big(\tilde{f}_{ie}(\omega, r_{ie}) + \nu h_e(\omega, r_{ie})\big)\right],$$

where ν is a constant in $\mathbb{R}$. Differentiating with respect to ν and evaluating at $\nu = 0$ gives:

$$\text{(A5)}\quad \mathbb{E}\left[\pi \sum_{e\in E} h_e(\omega, r_{ie}) - 2b\big(X_i + \sum_{e\in E} \tilde{f}_{ie}(\omega, r_{ie})\big) \sum_{e\in E} h_e(\omega, r_{ie})\right] - \mathbb{E}\left[\sum_{e\in E} \big(\frac{\tilde{f}_{ie}(\omega, r_{ie})}{\zeta(N-1)} + \frac{r_{ie} + \tilde{f}_{ie}(\omega, r_{ie})}{\zeta(N-1)}\big) h_e(\omega, r_{ie})\right] = 0.$$

This holds if

$$\text{(A6)}\qquad \mathbb{E}\left[-2b\big(X_i + \sum_{k\in E} \tilde{f}_{ik}(\omega, r_{ik})\big) \,\middle|\, \mathcal{F}_i, r_{ie}\right] = \frac{r_{ie} + 2\tilde{f}_{ie}(\omega, r_{ie})}{\zeta(N-1)} - \mu_\pi$$

for each $e \in E$. We now show that (A6) is a sufficient condition for optimality within $\hat{\mathcal{M}}^E$. Differentiating (A4) with respect to ν twice we derive

$$\text{(A7)}\qquad \mathbb{E}\left[-2b\big(\sum_{e\in E} h_{ie}(\omega, r_{ie})\big)^2 - \frac{2}{\zeta(N-1)} \sum_{e\in E} h_{ie}(\omega, r_{ie})^2\right],$$

which is less than or equal to 0 for all $(h_1, ..., h_E) \in \hat{\mathcal{M}}^E$. The derivative is negative if one of $h_1, ..., h_N$ is nonzero on a set of positive measure provided $\zeta > 0$. Suppose for contradiction that $(\tilde{f}_{i1}, ..., \tilde{f}_{iE})$ satisfies (A6) but there exists $(h_1^*, ..., h_E^*) \in \hat{\mathcal{M}}^E$ which achieves a strictly higher value of (A3). Set $(h_1, ..., h_E) \equiv (h_1^* - \tilde{f}_{i1}, ..., h_E^* - \tilde{f}_{iE}) \in \hat{\mathcal{M}}^E$. Then the function (A4) achieves a higher value at $\nu = 1$ than at $\nu = 0$. However (A4) is a strictly concave function of ν and thus has a global maximum at $\nu = 0$. This is a contradiction.

To show that it is without loss of generality to restrict attention to optimality within $\hat{\mathcal{M}}^E$ we observe that the coefficient of $\tilde{g}_{ie}(\omega, r_e^i)^2$ in (A3) is negative if $\zeta > 0$. It is easy to see then that any $(\tilde{g}_{ie}, ..., \tilde{g}_{iE}) \notin \hat{\mathcal{M}}^E$ must result in $-\infty$ for the objective.

Using (A2), we see that (A6) is equivalent to (A1) which, if $\zeta > 0$, is therefore a sufficient condition for (Δ, α, ζ) to be a symmetric affine equilibrium. We now show that it is also a necessary condition (even if $\zeta < 0$). Suppose for some $e \in E$, (A1) does not hold and set

$$h_e(\omega, r_{ie}) = \mu_\pi + \mathbb{E}\left[-2b\big(X_i + \sum_{k\in E} \tilde{f}_{ik}(\omega, r_{ik})\big) \,\middle|\, \mathcal{F}_i, r_{ie}\right] - \frac{r_{ie} + 2\tilde{f}_{ie}(\omega, r_{ie})}{\zeta(N-1)}.$$

Note that h_e is an affine function of r_{ie} with a deterministic slope (does not depend on ω). This is because the conditional expectation above is affine in r_{ie} in each of the models of Section III, Section V, and Appendix H with a deterministic slope. Set $h_k(\omega, r_{ik}) = 0$ for $k \neq e$. Then (A5) is strictly positive. Thus for all $\nu > 0$ sufficiently small $(\tilde{f}_{i1}, ..., \tilde{f}_{ie} + \nu h_e, ..., \tilde{f}_{iE})$ achieves a higher value of the objective (A3) than does $(\tilde{f}_{i1}, ..., \tilde{f}_{iE})$. Define the demand schedule d_e such that for any given $p \in \mathbb{R}$ and $\omega \in \Omega$

$$d_e(\omega, p) = (\tilde{f}_{ie} + \nu h_e)(\omega, r(\omega, p))$$

where $r(\omega, p)$ is defined to be the r that solves

$$p = \frac{r + (\tilde{f}_{ie} + \nu h_e)(\omega, r)}{\zeta(N-1)}.$$

If $\nu > 0$ was chosen sufficiently small, $r(\omega, p)$ is well defined since the right hand side is an affine function of r with nonzero slope and so $d_e(\omega, p)$ is also well defined. Moreover

$$d_e(\omega, p_e(\omega)) = (\tilde{f}_{ie} + \nu h_e)(\omega, r_{ie}(\omega))$$

for each $\omega \in \Omega$. But then $(f_{i1}, ..., d_e, ..., f_{iE})$ gives higher expected utility to trader i than does $(f_{i1}, ..., f_{iE})$ which is a contradiction. Thus (A1) is a necessary condition.

Appendix B: Proofs for Section III

Here, we provide proofs for all results in Section III. We first prove Lemma 2 which states that an equilibrium is "more efficient" the closer is $E\alpha_E$ to 1. Lemma 2 will be used in the proof of Theorem 1.

LEMMA 2: *Let (Δ, α, ζ) be a symmetric affine equilibrium when there are E exchanges and $(\hat{\Delta}, \hat{\alpha}, \hat{\zeta})$ be a symmetric affine equilibrium when there are $\hat{E}$ exchanges. For each $\omega \in \Omega$, the sum of strategic traders' holding costs post trade is* *strictly lower in the equilibrium corresponding to* (α, ζ, Δ) *if and only if* $|1-E\alpha| < |1-\hat{E}\hat{\alpha}|$. *If the sum of strategic traders' holding costs post trade are equal across the two equilibria, then* $|1-E\alpha| = |1-\hat{E}\hat{\alpha}|$.

PROOF:

The sum of holding costs in the equilibrium (Δ, α, ζ) is

$$b \sum_{i\in N} \left((1-E\alpha)X_i + E\alpha \frac{1}{N} \sum_{j\in N} X_j + \frac{\sum_{e\in E} Q_e}{N} \right)^2 .$$

Expanding, rearranging, and combining like terms we obtain

$$b\left[(1-E\alpha)^2 \sum_{j\in N} X_j^2 + [1-(1-E\alpha)^2]\frac{1}{N}(\sum_{j\in N} X_j)^2\right] + b\left[2\frac{\sum_{e\in E} Q_e \sum_{j\in N} X_j}{N} + \frac{(\sum_{e\in E} Q_e)^2}{N}\right].$$

Thus the lemma is a result of Jensen's inequality.

B1. Proof of Theorem 1

We prove Theorem 1 in three steps. In step 1, we derive a system of equations and show that a necessary and sufficient condition for (Δ_E, α_E, ζ_E) to be a symmetric affine equilibrium is that they solve this system. In step 2 we prove that there is exists a unique solution to the system, thus establishing existence and uniqueness of a symmetric affine equilibrium. This proves the preamble in Theorem 1. In step 3, we prove Parts 1 through 7.

Step 1. Conjecture there exists a symmetric affine equilibrium $(\Delta_E, \alpha_E, \zeta_E)$. Under this conjecture, each agent $i \in N$ submits a demand schedule of the form in (2) to each $e \in E$ and $i \in N$. Market clearing in exchange e implies that the equilibrium price is

$$p_e^f = \frac{-\alpha_E(\sum_i X_i) + \Delta_E N - Q_e}{\zeta_E N}. \tag{B1}$$

Price impact can also be determined from the market clearing condition. If trader i purchases q units on exchange e when all other traders submit the equilibrium demand schedules then

$$-\alpha_E \sum_{j\neq i} X_j - \zeta_E(N-1)p_e + \Delta_E(N-1) + q = Q_e.$$

This implies that the inverse residual supply curve trader i faces is

$$p_e(q) = \frac{-\alpha_E \sum_{j \neq i} X_j + q + \Delta_E(N-1) - Q_e}{\zeta_E(N-1)}. \tag{B2}$$

Thus the price impact trader i faces in exchange e is $\Lambda := \frac{1}{\zeta_E(N-1)}$, which by symmetry, is the price impact each agent faces in all exchanges. Define $f_{ie}(X_i, p_e^f) := q_{ie}^f$. By Theorem 1 a necessary condition for $(\Delta_E, \alpha_E, \zeta_E)$ to be a symmetric affine equilibrium is that

$$-2b\left(X_i + q_{ie}^f + (E-1)\mathbb{E}\left[q_{ik}^f \,|\, p_e^f - \frac{q_{ie}^f}{\zeta_E(N-1)}, X_i\right]\right) = p_e^f + \Lambda_E q_{ie}^f - \mu_\pi \tag{B3}$$

holds almost surely for each $e \in E$ and trader $i \in N$. Moreover if $\zeta_E > 0$ it is also sufficient. In (B3), we have used symmetry of the exchanges. By the projection theorem,

$$\begin{aligned}\mathbb{E}\left[q_{ik}^f \;\middle|\; p_e^f - \frac{q_{ie}^f}{\zeta_E(N-1)}, X_i\right] = {} & -\alpha_E X_i \frac{N-1}{N} - \left(1 - \frac{N-1}{N}\gamma_E\right)\Delta_E \\ & - \frac{N-1}{N}\gamma_E \zeta_E p_e^f + \gamma_E \frac{q_{ie}^f}{N} + \Delta_E,\end{aligned} \tag{B4}$$

where

$$\gamma_E = \operatorname{corr}(p_e^f, p_k^f \,|\, X_i) = \frac{E\alpha_E^2(N-1)\sigma_X^2}{E\alpha_E^2(N-1)\sigma_X^2 + \sigma_Q^2}. \tag{B5}$$

Substituting (B4) and (2) into (B3) and matching coefficients we derive a system of three equations which characterize the three unknowns, Δ_E, α_E, and ζ_E. These equations are

$$\Delta_E = \frac{\mu_\pi}{2b\left(1 + \frac{\gamma_E(E-1)}{N} + \frac{(\gamma_E(E-1)+1)}{N-2} + (E-1)\frac{N-1}{N}\gamma_E\right)}, \tag{B6}$$

$$\alpha_E = \frac{1}{1 + \frac{\gamma_E(E-1)}{N} + \frac{(E-1)\gamma_E+1}{N-2} + (E-1)\frac{N-1}{N}}, \tag{B7}$$

and

$$\zeta_E = \frac{1}{2b((E-1)\gamma_E + 1)}\frac{N-2}{N-1}. \tag{B8}$$

Equations (B6), (B7), and (B8) are necessary and sufficient conditions for

$(\Delta_E, \alpha_E, \zeta_E)$ to be a symmetric affine equilibrium.

Step 2. We now prove existence of a symmetric affine equilibrium. When $E = 1$, there are closed form solutions to equations (B6), (B7), and (B8). When $E > 1$, by substituting (B5) into (B7) and rearranging we see that a cubic equation characterizes α_E. Since the equation is cubic, there exists at least one real root. Take this to be the value of α_E and compute ζ_E and Δ_E using equations (B5), (B8), and (B6). Thus a symmetric affine equilibrium exists.

To prove uniqueness, fix $E \in \mathbb{N}$ and define the function g by

$$g(a) = a - \frac{1}{E\gamma(\frac{1}{N} + \frac{1}{N-2}) + (1-\gamma)(\frac{1}{N} + \frac{1}{N-2}) + E\frac{N-1}{N}},$$

where γ is a function of a such that $\gamma(a)$ is equal to (B5) but with a in place of α_E. Since we have already shown existence there is an $a \in \mathbb{R}$ such that $g(a) = 0$. We observe that the second term in the above expression is strictly monotone decreasing in γ. By (B5) we see that γ is strictly monotone increasing in a. Thus g is strictly monotone increasing in a. Hence there can exist at most one value of $a \in \mathbb{R}$ such that $g(a) = 0$.

Step 3. Part 1 follows immediately from (B1) and the fact that $\Lambda_E = \frac{1}{(N-1)\zeta_E}$. Part 2 follows immediately from (B8). Part 4 follows by substituting (B1) into (2). Parts 5 and 6 can be seen from the fact that when $\sigma_Q^2 = 0$ or $E = 1$ there are closed form solutions to (B8), (B7), and (B6) for ζ_E, α_E, and Δ_E. Using these closed form solutions we find that $E\alpha_E$, by (B7), is equal to $\frac{N-2}{N-1}$ which is independent of E and also equal to $\frac{2b}{2b+\Lambda_1}$.

Finally, we prove part 7. By inspecting equations (B7) and (B5), $\gamma_E \to 0$. Using (B7), with some rearrangement we write

$$E\alpha_E = \frac{1}{\gamma_E(\frac{1}{N} + \frac{1}{N-2}) + (1-\gamma_E)\frac{1}{E}(\frac{1}{N} + \frac{1}{N-2}) + \frac{N-1}{N}}. \tag{B9}$$

Taking limits, $E\alpha_E \to \frac{N}{N-1}$. To prove that $E\alpha_E$ is strictly monotone increasing in E, suppose for contradiction that there exists $E \in \mathbb{N}$ such that $(E+1)\alpha_{E+1} \leq E\alpha_E$. Then by inspection it must be that $\gamma_{E+1} > \gamma_E$. But, inspecting (B5), this implies that $(E+1)\alpha_{E+1}^2 > E\alpha_E^2$ which in turn implies that $(E+1)\alpha_{E+1} > E\alpha_E$, a contradiction.

When E is equal to 1, $E\alpha_E$ is equal to $\frac{N-2}{N-1}$ by part 6. When $E \to \infty$, $E\alpha_E$ converges strictly monotonically to $\frac{N}{N-1}$. Thus for any $E > 1$ we have

$$\frac{1}{N-1} = |\,1 - \alpha_1\,| > |\,1 - E\alpha_E\,|.$$

That a fragmented market is always more efficient than a centralized market follows from Lemma 2.

Appendix C: Proofs for Section IV

C1. Proof of Proposition 1

That Λ_E is strictly monotone increasing and diverges to ∞ when $\sigma_Q^2 = 0$ is immediate from Theorem 1, where we showed that, in this case, $\Lambda_E = \frac{2bE}{N-2}$. For what follows assume $\sigma_Q^2 > 0$.

By Theorem 1 we have $\Lambda_E = \frac{2b(1+\gamma_E(E-1))}{N-2}$. To show Λ_E is strictly monotone increasing it suffices to show that $(E-1)\gamma_E$ is strictly monotone increasing. Fix an arbitrary $E \in \mathbb{N}$. If $\gamma_{E+1} > \gamma_E$, then it must be that $E\gamma_{E+1} > (E-1)\gamma_E$. Suppose $\gamma_{E+1} \leq \gamma_E$. Then to prove that $E\gamma_{E+1} > (E-1)\gamma_E$ it suffices to prove that $(E+1)\gamma_{E+1} > E\gamma_E$. Consider the equation for γ_n derived in the proof of Theorem 1 which holds for arbitrary $n \in \mathbb{N}$:

$$\frac{n\alpha_n^2(N-1)\sigma_X^2}{n\alpha_n^2(N-1)\sigma_X^2 + \sigma_Q^2}.$$

Denote the numerator, num_n so that

$$\gamma_n = \frac{num_n}{num_n + \sigma_Q^2}.$$

By Theorem 1, $(E+1)\alpha_{E+1} > E\alpha_E$ which implies that

$$(E+1)\gamma_{E+1} = \frac{(E+1)num_{E+1}}{num_{E+1} + \sigma_Q^2} > \frac{Enum_E}{num_E + \sigma_Q^2} = E\gamma_E$$

since $\gamma_{E+1} \leq \gamma_E$ implies that $num_{E+1} < num_E$.

We next prove that Λ_E converges and give an explicit expression for the limit point. We can, using the expression for γ_E, write Λ_E as

$$\frac{2b}{N-2}\left(1 + \frac{E^2\alpha_E^2(N-1)\sigma_X^2 - E\alpha_E^2(N-1)\sigma_X^2}{E\alpha_E^2(N-1)\sigma_X^2 + \sigma_Q^2}\right).$$

By Theorem 1, $E\alpha_E \to \frac{N}{N-1}$ which implies that $E\alpha_E^2 \to 0$. Taking limits of the right-hand side of the above equation we obtain $\Lambda_E \to \frac{2b}{N-2}(1 + \frac{N^2\sigma_X^2}{(N-1)\sigma_Q^2})$.

To prove that $\gamma_E \to 0$ we inspect (B7) to see that

$$\frac{1}{E(\frac{N-1}{N} + \frac{1}{N} + \frac{1}{N-2})} < \alpha_E < \frac{1}{E\frac{N-1}{N}}$$

for all E sufficiently large. Using this inequality, inspecting (B5), we see that for large E, the numerator is $O(\frac{1}{E})$. The denominator is roughly equal to σ_Q^2 for large E so it must be that $\gamma_E \to 0$. Note that the proof that $\gamma_E \to 0$ only makes use of (B7) and not the claim in Part 7 of Theorem 1 that $E\alpha_E \to \frac{N}{N-1}$.

Finally, we prove that γ_E is strictly monotone decreasing in E. Using (B9) and substituting into (B5) we derive a cubic equation which characterizes γ_E, in that

(C1)

$$\begin{aligned}\sigma_X^2(N-1) = {} & \gamma_E^3 E\sigma_Q^2\left(1-\frac{1}{E}\right)^2\left(\frac{1}{N}+\frac{1}{N-2}\right)^2 \\ & + \gamma_E E\sigma_Q^2\left(\frac{N-1}{N}+\frac{1}{E}\left(\frac{1}{N}+\frac{1}{N-2}\right)\right)^2 \\ & + 2\gamma_E^2\sigma_Q^2\left(1-\frac{1}{E}\right)\left(\frac{1}{N}+\frac{1}{N-2}\right)\left(\frac{N-1}{N}E+\frac{1}{N}+\frac{1}{N-2}\right) \\ & + \gamma_E\sigma_X^2(N-1).\end{aligned}$$

Each of the coefficients are unambiguously increasing in E except for possibly

$$E\sigma_Q^2\left(\frac{N-1}{N}+\frac{1}{E}\left(\frac{1}{N}+\frac{1}{N-2}\right)\right)^2.$$

Taking a derivative with respect to E we have

$$\begin{aligned}& \sigma_Q^2\left(\frac{N-1}{N}+\frac{1}{E}\left(\frac{1}{N}+\frac{1}{N-2}\right)\right)^2 \\ & \qquad - \frac{2}{E}\sigma_Q^2\left(\frac{N-1}{N}+\frac{1}{E}\left(\frac{1}{N}+\frac{1}{N-2}\right)\right)\left(\frac{1}{N}+\frac{1}{N-2}\right).\end{aligned}$$

This derivative is nonnegative if

$$E\frac{N-1}{N} \geq \frac{1}{N}+\frac{1}{N-2}.$$

The above holds for $E \geq 2$ since

$$2\frac{N-1}{N} \geq \frac{1}{N}+\frac{1}{N-2}$$

whenever $N \geq 3$. Since each of the coefficients of the powers of γ_E in (C1) are increasing in E and some are strictly increasing it must be that γ_E is strictly decreasing in E since the left hand side of (C1) is constant.

C2. Proof of Proposition 2

Substituting (B5) into (B7) and rearranging we obtain the following cubic equation which defines $E\alpha_E$ by

$$(E\alpha_E)^3\left(\sigma_X^2(N-1)(1+\frac{1}{N-2})\right)-(E\alpha_E)^2(N-1)\sigma_X^2 + E\alpha_E\sigma_Q^2\left(E-\frac{E}{N}+\frac{1}{N}+\frac{1}{N-2}\right)-E\sigma_Q^2=0. \tag{C2}$$

The efficient allocation is acheived at E^* such that $E^*\alpha_{E^*}=1$ provided E^* is in $\mathbb{N}$. Thus

$$\sigma_X^2(N-1)\left(1+\frac{1}{N-2}\right)-(N-1)\sigma_X^2+\sigma_Q^2\left(E^*-\frac{E^*}{N}+\frac{1}{N}+\frac{1}{N-2}\right)-E^*\sigma_Q^2=0.$$

Solving,

$$E^*=2+\frac{2}{N-2}+\frac{N-1}{N-2}\frac{N\sigma_X^2}{\sigma_Q^2}.$$

That the $E\in\mathbb{N}$ whose symmetric affine equilibrium allocation is most efficient is either $\lfloor E^*\rfloor$ or $\lceil E^*\rceil$ follows from the proof of Part 7 of Theorem 1 which shows that $E\alpha_E$ is strictly monotone increasing. By inspection, the proof did not rely upon E taking values in $\mathbb{N}$—the same method of proof can be adapted to show that if we increase E continuously, the corresponding α_E which simultaneously solves (B5) and (B7) is such that $E\alpha_E$ is strictly monotone increasing. Combining this observation with Lemma 2 gives the result.

C3. Proof of Proposition 3

We first prove part 1. Recall that

$$p_e^*=\frac{N-1}{N}\Lambda_E\left[\sum_{i\in N}-\alpha_E X_i+N\Delta_E-Q_e\right].$$

By the projection theorem

$$\text{var}\left(\sum_{i\in N}X_i\,|\,p_e^*\right)=\left(1-\frac{\alpha_E^2\text{var}\left(\sum_{i\in N}X_i)\right)}{\alpha_E^2\text{var}\left(\sum_{i\in N}X_i\right)+\frac{\sigma_Q^2}{E}}\right)\text{var}\left(\sum_{i\in N}X_i\right).$$

Since γ_E is strictly monotone decreasing to 0 as stated in Proposition 1, it follows that

$$\frac{\alpha_E^2\text{var}\left(\sum_{i\in N}X_i\right)}{\alpha_E^2\text{var}\left(\sum_{i\in N}X_i\right)+\frac{\sigma_Q^2}{E}}$$

also converges to 0 strictly monotonically as E diverges.

We now prove part 2. Since the price in each exchange consists of a common signal component and noise which is iid across exchanges, the sum of prices is a sufficient statistic for inference so that

$$\mathrm{var}\left(\sum_{i\in N} X_i \;\middle|\; \sum_{e\in E} p_e^*\right) = \mathrm{var}\left(\sum_{i\in N} X_i \;\middle|\; p_1^*, ..., p_E^*\right).$$

We have

$$\sum_{e\in E} p_e^* = \frac{N-1}{N}\Lambda_E\left(-E\alpha_E\sum_{i\in N} X_i - Q + EN\Delta_E\right).$$

By the projection theorem,

$$\mathrm{var}\left(\sum_{i\in N} X_i \;\middle|\; \sum_{e\in E} p_e^*\right) = \mathrm{var}\left(\sum_{i\in N} X_i\right) - \frac{(E\alpha_E)^2\mathrm{var}\left(\sum_{i\in N} X_i\right)}{(E\alpha_E)^2\mathrm{var}\left(\sum_{i\in N} X_i\right) + \sigma_Q^2}\mathrm{var}\left(\sum_{i\in N} X_i\right).$$

The result follows since

$$\frac{(E\alpha_E)^2\mathrm{var}\left(\sum_{i\in N} X_i\right)}{(E\alpha_E)^2\mathrm{var}\left(\sum_{i\in N} X_i\right) + \sigma_Q^2}$$

increases strictly monotonically, because $E\alpha_E$ increases strictly monotonically as seen from part 7 of Theorem 1.

PROPOSITION 1: *The expected payment of liquidity traders is* $\frac{N-1}{N}\Lambda_E\sigma_Q^2$ *and if* $\sigma_Q^2 > 0$ *is strictly monotone increasing in* E.

PROOF:

$$-\mathbb{E}\left(\sum_{e\in E} p_e^* Q_e\right) = -\frac{N-1}{N}\Lambda_E\mathbb{E}\left(\sum_{e\in E}\left(-\alpha_E\sum_{i\in N} X_i + N\Delta_E + Q_e\right) Q_e\right)$$
$$= \frac{N-1}{N}\Lambda_E\sigma_Q^2.$$

That the expected payment is strictly monotone increasing follows from the fact that Λ_E is strictly monotone increasing as stated in Proposition 1.

Appendix D: Proofs for Section V

D1. Proof of Theorem 2

We prove Theorem 2 in three steps. In step 1 we derive a candidate equilibrium. In step 2 we verify that the candidate equilibrium is in fact an equilibrium, and then establish that it is the unique symmetric affine equilibrium if for each $e \in E$, Q_e has full support over the real line. In step 3 we show that the derived equilibrium has properties 1 through 5 given in the statement of the theorem.

Step 1. To begin the first step, we conjecture that there exists a symmetric affine equilibrium, denoted (Δ, α, ζ) in which each trader submits demand schedules of the form in (2). Note that in this case Δ may depend on the aggregate endowment $\sum_{j\in N} X_j$. Under this conjecture, by market clearing, the residual supply curve trader i faces in exchange e is

$$p_e(q) = \frac{(\sum_{j\neq i} -\alpha X_j) + (N-1)\Delta + q - Q_e}{(N-1)\zeta},$$

which implies that $\Lambda = \frac{1}{(N-1)\zeta}$. Also by market clearing we have

$$p_e^f = \frac{(\sum_{j\in N} -\alpha X_j) + N\Delta - Q_e}{N\zeta} \tag{D1}$$

for each $e \in E$. By observing p_e^f trader i can infer the realization of Q_e but this is uninformative of p_k^f for $k \neq e$. By Theorem 1 a necessary and sufficient condition for (Δ, α, ζ) to be a symmetric affine equilibrium is that

$$-2b(X_i + q_{ie}^f + (E-1)\mathbb{E}[q_{ik}^f \,|\, \mathcal{F}_i]) = p_e^f + q_{ie}^f\Lambda - \mu, \tag{D2}$$

where we have used symmetry. Rearranging, we have

$$q_{ie}^f = \frac{-2bX_i - 2b(E-1)\mathbb{E}[q_{ik}^f \,|\, \mathcal{F}_i] - p_e^f + \mu_\pi}{\Lambda + 2b}.$$

Substituting p_k^f into the conjectured equilibrium demand schedule, we have

$$q_{ik}^f = -\alpha X_i + \frac{(\sum_{j\in N} \alpha X_j) + Q_k}{N}$$

so that

$$\mathbb{E}[q_{ik}^f \,|\, \mathcal{F}_i] = -\alpha X_i + \frac{(\sum_{j\in N} \alpha X_j)}{N}.$$

We therefore have

$$q_{ie}^f = \frac{(-2b + 2b(E-1)\alpha)X_i - 2b(E-1)\frac{(\sum_{j\in N}\alpha X_j)}{N} - p_e^f + \mu_\pi}{\frac{1}{(N-1)\zeta} + 2b}.$$

We now match coefficients with our conjecture that $q_{ie}^f = -\alpha X_i - \zeta p_e^f + \Delta$ to determine that

$$\zeta = \frac{N-2}{N-1}\frac{1}{2b}, \tag{D3}$$

$$\Lambda = \frac{2b}{N-2}, \tag{D4}$$

$$\alpha = \frac{2b}{\Lambda + 2bE}, \tag{D5}$$

and

$$\Delta = \frac{-2b(E-1)\frac{2b}{\Lambda+2bE}\frac{\sum_{i\in N} X_i}{N} + \mu_\pi}{2b + \Lambda}. \tag{D6}$$

Step 2. To complete step 2 we appeal to Theorem 1 which can be applied since (D2) holds. To see that the symmetric affine equilibrium is unique when each Q_e has full support over the real line suppose that there exists a symmetric affine equilibrium such that at least one of the equations (D3), (D5), and (D6) are not satisfied. Then equation (D2) is violated for some realization of the price in some exchange $e \in E$ for some agent $i \in N$. Continuity implies that (D2) must be violated for realizations of p_e^f in an open neighborhood of positive Lebesgue measure. Since each Q_e has full support over the real line and is independent of $\mathcal{F}_i$ (D2) is violated on a set of positive $\mathbb{P}$-measure. This contradicts Theorem 1.

Step 3. Part 1 was shown in equation (D4). Part 2a follows from substituting equations (D3), (D5), and (D6) into (D1). Part 2b follows from substituting the equation for price in part 2 in to the equilibrium demand schedule. Part 3 follows from part 2b and taking the limit as E tends to infinity. To prove part 4, we have

$$-\mathbb{E}\left[\sum_{e\in E} p_e^* Q_e\right] = -\frac{2b(N-1)}{N(N-2)}\mathbb{E}\left[\sum_{e\in E}\left(-\alpha\sum_{i\in N} X_i + N\Delta - Q_e\right) Q_e\right]$$
$$= \frac{2b(N-1)}{N(N-2)}\text{var}\left[\sum_{e\in E} Q_e\right].$$

Appendix E: Proofs for Section VI

This Appendix provides a proof of Theorem 3, characterizing an efficient equilibrium for the dynamic version of the model.

E1. Proof of Theorem 3

The proof proceeds in six steps. In Step 1 we derive the Bellman equation for the dynamic programming problem of trader i. In Step 2 we conjecture a continuation value function V as a solution to the Bellman and we derive a first order condition characterizing the optimal demand schedules of trader i in a restricted domain of demand schedules. In Step 3, we use the first order condition to compute the necessary number E of exchanges and the demand-schedule coefficients ρ, ζ, and χ. In Step 4 we relax the domain restriction on demand schedules. In Step 5, we verify a transversality condition on the value function. In Step 6 we verify that the strategy of submitting demand schedules with coefficients derived in Step 3 from the Bellman equation is in fact optimal. In what follows, for notation, we use σ_ϵ^2 in place of σ_X^2.

Step 1. For a given date t, let

$$H_t := (\{q_{ie}\}_{e\in E, s<t}, \{p_e\}_{e\in E, s<t}, \{X_{is}\}_{s\leq t})$$

denote the history of past quantities purchased by trader i, prices on each of the exchanges, and inventory levels. An admissible demand schedule submitted to an exchange e is a function f specifying the quantity $f(H_t, p)$ that trader i will purchase for any given realization $p \in \mathbb{R}$ of the price in the exchange following the history H_t. By inspecting (16) and following a similar argument to that given in the proof of Theorem 1, we see that for any such demand function f there exists a corresponding function $\hat{f}$ that instead specifies the quantity purchased by trader i as a function of H_t and

$$W_{et} := \frac{1}{E}\sum_{j\in N} X_{jt} + Q_{et}.$$

For instance, in the conjectured equilibrium, on exchange e, trader i makes the socially efficient purchase

$$\hat{f}_{iet}\left(H_t, W_{et}\right) = -\frac{1}{E}X_{it} + \frac{W_{et}}{N}, \tag{E1}$$

as can be seen by substituting (16) into (14).

We first relax the dynamic programming problem by allowing trader i to select demand functions of the type $\hat{f}$. Let κ_{et} denote (X_{it}, B_t, W_{et}). Under the relaxation, the Bellman equation characterizing trader i's continuation value function $V(\,\cdot\,)$ is

$$V(X_{it}, B_t) = \sup_{\{g_{i1t},\cdots,g_{iEt}\}} \mathbb{E}_{it}\left[u_{it} + e^{-r\Delta}V(X_{i,t+1}, B_{t+1})\right], \tag{E2}$$

where

$$u_{it} = \mu_\pi \Delta \left(X_{it} + \sum_{e\in E} g_{iet}(\kappa_{et})\right) - b\left(X_{it} + \sum_{e\in E} g_{iet}(\kappa_{et})\right)^2 - \sum_{e\in E} p_{et}g_{iet}(\kappa_{et}).$$

Above, each $g_{iet} : \mathbb{R}^3 \to \mathbb{R}$ is an arbitrary measurable function. We will assume for now that each g_{iet} is such that $g_{iet}(\kappa_{et})$ is of finite variance conditional on X_{it} and B_t. Call the set of all such measurable functions with this property $\tilde{\mathcal{M}}$. We will show in step 4 that the finite variance assumption is without loss of generality. Note that $\hat{f}_{iet}$ is in $\tilde{\mathcal{M}}$. The operator $\mathbb{E}_{it}$ is the conditional expectation given the state variables, X_{it} and B_t. These are the relevant state variables because, at date t, trader i infers that $\frac{N-1}{N}B_t$ is the total inventory held by the other traders following trade at date $t-1$. Thus X_{it} and B_t are sufficient statistics for trader i to conduct inference on the residual supply curves on each exchange at each future trading date. The law of motion for (X_{it}, B_t) is given by (12) and (15).

A standard verification argument implies that if V satisfies the Bellman equation, and for every feasible strategy, the transversality condition

$$\lim_{t\to\infty} e^{-r\Delta t}\mathbb{E}_{i0}\left[V(X_{it}, B_t)\right] = 0, \tag{E3}$$

then V is indeed the value function and the strategy achieving the supremum in (E2) determines the optimal policy.

Step 2. We conjecture the value function V defined by

$$V(X_{it}, B_t) = \sum_{s=t}^{\infty} e^{-r\Delta(s-t)}M_s, \tag{E4}$$

where

$$M_s = \mathbb{E}_{it}\left[\mu_\pi \Delta\left(X_{is}^{\hat{f}} + \sum_{e\in E} q_{ies}^{\hat{f}}\right) - b\left(X_{is}^{\hat{f}} + \sum_{e\in E} q_{ies}^{\hat{f}}\right)^2 - \sum_{e\in E} p_{es}^{\hat{f}} q_{ies}^{\hat{f}}\right].$$

and where the superscript $\hat{f}$ implies that the inventories, quantities, and prices are those induced by the conjectured equilibrium strategy in which any given trader i selects (E1) for any given exchange e. Substituting (E4) into the right hand side of the Bellman and using the law of iterated expectations, we can write the objective function in the Bellman equation as

$$\mathbb{E}_{it}\left[\sum_{s=t}^{\infty} e^{-r\Delta(s-t)}\mu_\pi\Delta\left(X_{is}^{g} + \sum_{e\in E} q_{ies}^{g}\right)\right] - \mathbb{E}_{it}\left[\sum_{s=t}^{\infty} e^{-r\Delta(s-t)}\left(b\left(X_{is}^{g} + \sum_{e\in E} q_{ies}^{g}\right)^2 + \sum_{e\in E} p_{es}^{g} q_{ies}^{g}\right)\right],$$

where the superscript g indicates that inventories, quantities, and prices are those generated by a strategy that selects at date t demands according to the functions $g_{i1t}, ..., g_{iEt}$, and then reverts back to the conjectured equilibrium strategy at date $t+1$. We now derive the E, ρ, ζ, and χ such that the optimal choice of $g_{i1t}, ..., g_{iEt}$ coincides with (E1), thus verifying the conjecture (E4).

To simplify the objective further, we recognize that for any choice of the deviating demands $g_{i1t}, ..., g_{iEt}$, following trade at date $t+1$, the inventory of trader i returns to the efficient level, so all inventories, prices, and quantities at dates $s > t+1$ would be the same as if trader i had never deviated and therefore do not depend on the chosen $g_{i1t}, ..., g_{iEt}$. Thus, it suffices to consider the objective

(E5)
$$\mathbb{E}_{it}\left[\mu_\pi\Delta\sum_{e\in E} q_{iet}^{g} - b\left(X_{it}^{g} + \sum_{e\in E} q_{iet}^{g}\right)^2 - \sum_{e\in E} p_{et}^{g} q_{iet}^{g} - e^{-r\Delta}\sum_{e\in E} p_{e,t+1}^{g} q_{ie,t+1}^{g}\right].$$

Let $\eta_{et} \equiv -\frac{1}{E}\sum_{j\neq i} X_{jt} - Q_{et}$. Then

$$\sum_{e\in E} p_{et}^{g} q_{iet}^{g} = \sum_{e\in E} \frac{\eta_{et} + (N-1)\rho B_t + (N-1)\chi + q_{iet}^{g}}{\zeta(N-1)} q_{iet}^{g}$$

(E6)
$$= \frac{1}{\zeta(N-1)}\left[\sum_{e\in E}\big(\eta_{et} + (N-1)\rho B_t + (N-1)\chi\big) q_{iet}^{g} + \sum_{e\in E}(q_{iet}^{g})^2\right].$$

From (12), (14), and (16),

$$q^g_{ie,t+1} = -\frac{1}{E}\left(X_{it} + \sum_{e\in E} q^g_{iet} + \epsilon_{i,t+1}\right) + \frac{1}{NE}\sum_{j\in N} X^g_{j,t+1} + \frac{Q_{e,t+1}}{N}$$

and

$$p^g_{e,t+1} = \frac{-\frac{1}{E}\sum_{j\in N} X^g_{j,t+1} + N\rho\left(NE\rho B_t + EN\chi - \zeta N\sum_{e\in E} p^g_{et}\right) - Q_{e,t+1} + N\chi}{\zeta N}.$$

From the above two equations,

$$\text{(E7)}\quad p^g_{e,t+1}q^g_{ie,t+1} = \left(\frac{1}{\zeta N}\frac{1}{E^2}\sum_{j\in N} X^g_{j,t+1} - \frac{N\rho^2}{\zeta}B_t - \frac{N\rho\chi}{\zeta} - \frac{1}{E}\frac{\chi}{\zeta}\right)\sum_{e\in E} q^g_{ie,t} - N\rho\left(-\frac{1}{E}X_{it} + \frac{1}{NE}\sum_{j\in N} X^g_{j,t+1}\right)\sum_{e\in E} p^g_{et} + N\rho\frac{1}{E}\sum_{e\in E} p^g_{et}\sum_{e\in E} q^g_{iet} + O_e,$$

where O_e is a term whose conditional expectation does not depend on the choice of $\{g_{iet}\}_{e\in E}$. Above we have used the fact that the aggregate endowment of strategic traders is exogenous. Equivalently, we can express (E7) as

$$\text{(E8)}\quad p^g_{e,t+1}q^g_{ie,t+1} = \left(\frac{1}{\zeta N}\frac{1}{E^2}\sum_{j\in N} X^g_{j,t+1} - \frac{N\rho^2}{\zeta}B_t - \frac{N\rho\chi}{\zeta} - \frac{1}{E}\frac{\chi}{\zeta}\right)\sum_{e\in E} q^g_{iet} - N\rho\left(-\frac{1}{E}X_{it} + \frac{1}{NE}\sum_{j\in N} X^g_{j,t+1}\right)\sum_{e\in E}\frac{\eta_{et} + (N-1)\left(\rho B_t + \chi\right) + q^g_{iet}}{\zeta(N-1)} + N\rho\frac{1}{E}\sum_{e\in E}\frac{\eta_{et} + (N-1)\rho B_t + (N-1)\chi + q^g_{iet}}{\zeta(N-1)}\sum_{e\in E} q^g_{iet} + O_e.$$

By substituting (E6) and (E8) into (E5), recalling that by definition $q^g_{iet} = g_{iet}\left(\kappa_{et}\right)$, and ignoring terms whose conditional expectation does not depend on the choice $\{g_{iet}\}_{e\in E}$ we have transformed the objective function in the Bellman equation into

$$\text{(E9)}\qquad \mathbb{E}_{it}\left[A\left(\sum_{e\in E} g_{iet}(\kappa_{et})\right)^2 + B\sum_{e\in E} g_{iet}(\kappa_{et}) + C\delta_{it},\right],$$

where

$$\delta_{it} = \sum_{e \in E} (\eta_{et} + (N-1)\rho B_t + (N-1)\chi) g_{iet}(\kappa_{et}) + g_{iet}(\kappa_{et})^2,$$

for coefficients

$$\text{(E10)} \quad A = -b - N\rho \frac{1}{\zeta(N-1)} e^{-r\Delta}$$

$$B = \mu_\pi \Delta - 2bX_{it} - e^{-r\Delta} \left[\frac{1}{\zeta N} \frac{1}{E} \sum_{j \in N} X^g_{j,t+1} - \frac{NE\rho^2}{\zeta} B_t - \frac{NE\rho\chi}{\zeta} - \frac{\chi}{\zeta} \right]$$
$$+ \frac{e^{-r\Delta} N\rho}{\zeta(N-1)} \left(-X_{it} + \frac{1}{N} \sum_{j \in N} X^g_{j,t+1} \right)$$
$$- \frac{e^{-r\Delta} N\rho}{\zeta(N-1)} \sum_{e \in E} \big(\eta_{et} + (N-1)\rho B_t + (N-1)\chi \big)$$

$$C = -\frac{1}{\zeta(N-1)}.$$

Next, for each $e \in E$, we let

$$g_{iet}(\kappa_{et}) = \hat{f}_{iet}(\kappa_{et}) + \nu h_{iet}(\kappa_{et}),$$

for an arbitrary measurable deviation h_{iet} in $\tilde{\mathcal{M}}$ from the conjectured optimal $\hat{f}_{iet}$, and for some arbitrary constant ν. Substituting into (E9) leaves

$$\text{(E11)} \quad \mathbb{E}_{it} \Bigg[A \left(\sum_{e \in E} \hat{f}_{iet} + \nu \sum_{e \in E} h_{iet} \right)^2 + B \sum_{e \in E} (\hat{f}_{iet} + \nu h_{iet})$$
$$+ C \sum_{e \in E} \big(\eta_{et} + (N-1)\rho B_t + (N-1)\chi \big) \big(\hat{f}_{iet} + \nu h_{iet} \big) + (\hat{f}_{iet} + \nu h_{iet})^2 \Bigg],$$

where we have suppressed the argument κ_{et} from the notation, and will continue to do so whenever convenient. Taking a derivative with respect to ν, evaluating the derivative at $\nu = 0$, and setting the derivative equal to 0 gives the necessary optimality condition

$$\mathbb{E}_{it} \left[2A \sum_{e \in E} \hat{f}_{iet} \sum_{e \in E} h_{iet} + B \sum_{e \in E} h_{iet} \right]$$
$$+ \mathbb{E}_{it} \left[C \sum_{e \in E} (\eta_{et} + (N-1)\rho B_t + (N-1)\chi) h_{iet} + 2\hat{f}_{iet} h_{iet} \right] = 0,$$

which holds if, for each $k \in E$,
(E12)

$$\mathbb{E}_{it}\left[2A\sum_{e\in E}\hat{f}_{iet} + B + C\big(\eta_{kt} + (N-1)\rho B_t + (N-1)\chi\big) \;\Big|\; \eta_{kt}\right] = -2C\hat{f}_{ikt}.$$

The necessary condition (E12) is also sufficient for optimality if the second derivative of (E11) with respect to ν is negative, that is,

$$\mathbb{E}_{it}\left[A\left(\sum_{e\in E} h_{iet}\right)^2 + C\sum_{e\in E} h_{iet}^2\right] < 0. \tag{E13}$$

To see why, suppose for contradiction that there exists a candidate $(L_{i1t}, ..., L_{iEt})$ in $\tilde{\mathcal{M}}$ satisfying the first order condition (E12) that achieves a strictly higher value of the objective than $(\hat{f}_{i1t}, \ldots, \hat{f}_{iEt})$. In that case, let $h_{iet} = L_{iet} - \hat{f}_{iet}$ for each $e \in E$. Then (E11) achieves a higher value at $\nu = 1$ than at $\nu = 0$. This is a contradiction since (E13) ensures that (E11) is maximized at $\nu = 0$.

Step 3. We derive the E, ζ, ρ, and χ such that (E12) holds and then show that (E13) is satisfied. This implies that we have found a solution to the Bellman equation. We first derive the moments in (E12). By (E1),

$$\mathbb{E}_{it}\left[\sum_{e\in E}\hat{f}_{iet} \;\Big|\; \eta_{kt}\right] = -X_{it} + \frac{1}{N}\mathbb{E}_{it}\left[\sum_{j\in N} X_{jt} + Q_{kt} \;\Big|\; \eta_{kt}\right]$$

and

$$\mathbb{E}_{it}[B \mid \eta_{kt}] = -\left(2b + \frac{2e^{-r\Delta}N\rho}{\zeta(N-1)}\right)X_{it} + e^{-r\Delta}\frac{\chi}{\zeta} + \mu_\pi\Delta$$
$$+\left(-\frac{e^{-r\Delta}}{\zeta N E} + \frac{e^{-r\Delta}\rho(N+1)}{\zeta(N-1)}\right)\mathbb{E}_{it}\left[\sum_{j\in N} X_{j,t} + Q_{kt} \;\Big|\; \eta_{kt}\right].$$

By the projection theorem,

$$\mathbb{E}_{it}\left[\sum_{j\in N} X_{jt} + Q_{kt} \;\Big|\; \eta_{kt}\right] = \left(\frac{N-1}{N}B_t + X_{it}\right)(1-\Gamma)\frac{E-1}{E}$$
$$-\left(1+\Gamma(E-1)\right)\left(\eta_{kt} - \frac{1}{E}X_{it}\right),$$

where

$$\Gamma = \frac{(N-1)\sigma_\epsilon^2}{(N-1)\sigma_\epsilon^2 + E\sigma_Q^2}.$$

Finally, we use the fact that

$$\hat{f}_{ikt} = -\frac{1}{E}X_{it} - \frac{1}{N}\left(\eta_{kt} - \frac{1}{E}X_{it}\right)$$

and match coefficients in (E12). Matching the coefficient on X_{it} gives

$$\text{(E14)}\quad \frac{2C}{E} = 2A\frac{1}{N}(1-\Gamma)\frac{E-1}{E} + e^{-r\Delta}\left(\frac{-1}{\zeta NE} + \frac{\rho(N+1)}{\zeta(N-1)}\right)(1-\Gamma)\frac{E-1}{E} + \frac{C}{E}.$$

Matching the coefficient on $\eta_{kt} - \frac{1}{E}X_{it}$ gives

$$\text{(E15)}\quad \frac{2C}{N} = -2A\frac{1}{N}(1+\Gamma(E-1)) - e^{-r\Delta}\left(\frac{-1}{\zeta NE} + \frac{\rho(N+1)}{\zeta(N-1)}\right)(1+\Gamma(E-1)) + C.$$

Matching the coefficient on B_t gives

$$\text{(E16)}\quad (1-N)\rho C = 2A\frac{N-1}{N^2}(1-\Gamma)\frac{E-1}{E} + e^{-r\Delta}\left(\frac{-1}{\zeta NE} + \frac{\rho(N+1)}{\zeta(N-1)}\right)\frac{N-1}{N}(1-\Gamma)\frac{E-1}{E}.$$

Matching the constant coefficient gives

$$\text{(E17)}\quad 0 = C(N-1)\chi + e^{-r\Delta}\frac{\chi}{\zeta} + \mu_\pi\Delta.$$

Using (E14) and (E15), we have

$$\frac{N-2}{N} = \frac{1+\Gamma(E-1)}{(1-\Gamma)(E-1)}.$$

Rearranging gives

$$\text{(E18)}\quad E = \frac{2N-2}{N-2-N\frac{\Gamma}{1-\Gamma}}.$$

As an aside, this expression is useful in so far as it characterizes the efficient number of exchanges in a partial equilibrium model in which strategic traders percieve the correlation in exchange prices to be Γ. Taking Γ as given, the analysis does not depend on σ_ϵ^2 or σ_Q^2.

We deduce from (E18) that

$$E = 2 + \frac{2}{N-2} + \frac{N(N-1)}{N-2}\frac{\sigma_\epsilon^2}{\sigma_Q^2}. \tag{E19}$$

Thus the number of exchanges achieving the efficient allocation is precisely that of the static case, as stated by the Theorem.

Next, using (E14) and (E16), we can solve for

$$\rho = -\frac{1}{NE}.$$

Now, in order to solve for ζ, we use (E23) with (E15) to get

$$\frac{1}{N} = \frac{N-1}{N} + e^{-r\Delta}\frac{N-1}{N}\left(N\rho - \frac{1}{E}\right)(1+\Gamma(E-1)) - 2b\zeta(N-1)\frac{1}{N}(1+\Gamma(E-1)),$$

which rearranges to

$$2b\zeta(N-1)(1+\Gamma(E-1)) = N - 2 - e^{-r\Delta}(N-1)\frac{2}{E}(1+\Gamma(E-1)).$$

Thus

$$\zeta = \frac{N-2}{2b(N-1)(1+\Gamma(E-1))} - e^{-r\Delta}\frac{2N-2}{(N-1)2bE}.$$

Using (E17) we find

$$\chi = \frac{\mu_\pi \Delta}{1 - e^{-r\Delta}}\zeta. \tag{E20}$$

The within-period price impact is

$$\frac{1}{\zeta(N-1)} = \frac{2b(1+\Gamma(E-1))}{N - 2 - e^{-r\Delta}\frac{2N-2}{E}(1+\Gamma(E-1))},$$

as stated in the Theorem. Comparing with the static model, we see that price impact is higher in the dynamic model. We now verify that $\zeta > 0$ by showing that

$$N - 2 > e^{-r\Delta}\frac{2N-2}{E}(1+\Gamma(E-1)).$$

The above equality holds since (E18) implies that

$$(N-2)E = (2N-2)(1+\Gamma(E-1)).$$

Using (19), the cross-period cross-exchange price impact is

$$\frac{dp_{e,t+1}}{dq_{kt}} = -N\rho\frac{1}{(N-1)\zeta} = \frac{1}{E}\frac{2b(1+\Gamma(E-1))}{N-2-e^{-r\Delta}\frac{2N-2}{E}(1+\Gamma(E-1))},$$

as stipulated by the Theorem. Finally, we verify the sufficient condition for optimality (E13) is negative by showing that

$$A\left(\sum_{e\in E} h_{iet}\right)^2 + C\sum_{e\in E} h_{iet}^2 < 0.$$

Using (E10) and (E23), this is equivalent to

$$\left(-b+\frac{1}{E}\frac{1}{\zeta(N-1)}e^{-r\Delta}\right)\left(\sum_{e\in E} h_{iet}\right)^2 - \frac{1}{\zeta(N-1)}\sum_{e\in E} h_{iet}^2 < 0,$$

which holds by Jensen's inequality because $\zeta > 0$. Thus, (E4) solves the Bellman equation when the domain of admissible demand functions is restricted to $\tilde{\mathcal{M}}$.

Step 4. In this step, we show that if any measurable $g_{iet}: \mathbb{R}^3 \to \mathbb{R}$ outside of $\tilde{\mathcal{M}}$ is chosen, the objective associated with the Bellman equation is $-\infty$. Towards this end, consider the terms in (E9) involving $(\sum_{e\in E} g_{iet})^2$ and $\sum_{e\in E} g_{iet}^2$, which sum to

$$\left[-b+\frac{1}{E\zeta(N-1)}e^{-r\Delta}\right]\left(\sum_{e\in E} g_{iet}\right)^2 - \frac{1}{\zeta(N-1)}\sum_{e\in E} g_{iet}^2.$$

By Jensen's inequality, the above expression is less than

$$-b\left(\sum_{e\in E} g_{iet}\right)^2 - (1-e^{-r\Delta})\frac{1}{\zeta(N-1)}\sum_{e\in E} g_{iet}^2. \tag{E21}$$

The other terms in (E9) are $B\sum_{e\in E} g_{iet}$, which is only linear in $\sum_{e\in E} g_{iet}$, with B having finite variance, and $C\sum_{e\in E}(\eta_{et} + (N-1)\rho B_t + (N-1)\chi)g_{iet}$, where each η_{et} is of finite conditional variance. We define J_e by

$$Bg_{iet} + C\left(\eta_{et} + (N-1)\rho B_t + (N-1)\chi\right)g_{iet} = J_e g_{iet}.$$

Note that each J_e is of finite conditional variance.

Then

$$\mathbb{E}_{it}\left[-\frac{1-e^{-r\Delta}}{\zeta(N-1)}g_{iet}^2 + J_e g_{iet}\right] = \int_{\left\{\omega\in\Omega:|J_e|>|\frac{1-e^{-r\Delta}}{2\zeta(N-1)}g_{iet}|\right\}} \left(-\frac{1-e^{-r\Delta}}{\zeta(N-1)}g_{iet}^2 + J_e g_{iet}\right) d\mathbb{P}(\omega) + \int_{\left\{\omega\in\Omega:|J_e|\leq|\frac{1-e^{-r\Delta}}{2\zeta(N-1)}g_{iet}|\right\}} \left(-\frac{1-e^{-r\Delta}}{\zeta(N-1)}g_{iet}^2 + J_e g_{iet}\right) d\mathbb{P}(\omega).$$

The first integral must be finite since J_e is a finite-variance random variable and the integrand satisfies

$$\left| -\frac{1-e^{-r\Delta}}{\zeta(N-1)}g_{iet}^2 + J_e g_{iet} \right| \leq K J_e^2,$$

for some constant $K \in \mathbb{R}$. The second integral must be $-\infty$ since the integrand satisfies

$$-\frac{1-e^{-r\Delta}}{\zeta(N-1)}g_{iet}^2 + J_e g_{iet} \leq -\frac{1-e^{-r\Delta}}{2\zeta(N-1)}g_{iet}^2.$$

Thus, if g_{iet} is of infinite variance then the second integral must be $-\infty$. Hence, in this case,

$$\mathbb{E}_{it}\left[-\frac{1-e^{-r\Delta}}{\zeta(N-1)}g_{iet}^2 + J_e g_{iet}\right] = -\infty.$$

With this observation and inspecting (E21) and (E9) we see that if a chosen g_{iet} is not in $\tilde{\mathcal{M}}$, the objective function would equal to $-\infty$.

Step 5. We now check that the transversality condition (E3) holds. We compute the moments involved in the terms M_s defining the candidate value function V of (E4). For $s \geq t$,

$$\mathbb{E}_{it}\left[X_{is} + \sum_{e\in E} q_{es}^{\hat{f}}\right] = \frac{1}{N}X_{it} + \frac{N-1}{N^2}B_t$$

and

$$\mathbb{E}_{it}\left[\left(X_{is} + \sum_{e\in E} q_{es}^{\hat{f}}\right)^2\right] = \frac{1}{N^2}\left[(X_{it} + \frac{N-1}{N}B_t)^2 + \sigma_\epsilon^2\big(N(s-t)+N-1\big) + \sigma_Q^2(s-t+1)\right].$$

For $s \geq t+1$ and $e \in E$,

$$\begin{aligned}
&\mathbb{E}_{it}[p^{\hat{f}}_{es} q^{\hat{f}}_{ies}] = \\
&\quad \mathbb{E}_{it}\left[\frac{-\frac{1}{E}\sum_{j\in N} X_{js} - \frac{1}{E}B_s + N\chi - Q_{es}}{\zeta N} \left(-\frac{1}{E}X_{is} + \frac{1}{NE}\sum_{j\in N} X_{js} + \frac{Q_{es}}{N} \right) \right] \\
&\quad = \mathbb{E}_{it}\left[\frac{-\frac{1}{E}\sum_{j\in N} X_{js} - \frac{1}{E}B_s + N\chi - Q_{es}}{\zeta N} \left(\frac{1}{NE}\sum_{j\in N} \epsilon_{js} - \frac{1}{E}\epsilon_{is} + \frac{Q_{es}}{N} \right) \right] \\
&\quad = -\frac{\sigma_Q^2}{E\zeta N^2}.
\end{aligned}$$

Above, we have used the results that $\rho = \frac{-1}{NE}$ and $B_s = \frac{N-1}{N}(\sum_{j\in N} X_{j,s-1} + \sum_{e\in E} Q_{e,s-1})$ for $s \geq t+1$. Next,

$$\begin{aligned}
&\mathbb{E}_{it}[p^{\hat{f}}_{et} q^{\hat{f}}_{iet}] = \\
&\quad \mathbb{E}_{it}\left[\frac{-\frac{1}{E}\sum_{j\in N} X_{jt} - \frac{1}{E}B_t + N\chi - Q_{et}}{\zeta N} \left(-\frac{1}{E}X_{it} + \frac{1}{NE}\sum_{j\in N} X_{jt} + \frac{Q_{et}}{N} \right) \right] \\
&\quad = \frac{N-1}{\zeta N^2 E^2} X_{it}^2 + \frac{2}{E^2 \zeta N}\left(\frac{N-1}{N}\right)^2 X_{it}B_t - \frac{N-1}{N^2E^2}\frac{2N-1}{N^2\zeta}B_t^2 \\
&\qquad - \frac{\sigma_Q^2}{E\zeta N^2} - \frac{(N-1)\sigma_\epsilon^2}{E^2\zeta N^2} - \frac{\chi}{\zeta}\frac{N-1}{NE}X_{it} + \frac{\chi}{\zeta}\frac{N-1}{N^2E}B_t.
\end{aligned}$$

Substituting these moments into (E4) we find that

$$
\begin{aligned}
V(X_{it}, B_t) = X_{it}^2 & \left[\frac{-b}{N^2(1-e^{-r\Delta})} - \frac{N-1}{\zeta N^2 E}\right] \\
& + \left[\mu_\pi \Delta \frac{1}{N(1-e^{-r\Delta})} + \frac{\chi}{\zeta}\frac{N-1}{N}\right] X_{it} \\
& + \left[\mu_\pi \Delta \frac{N-1}{N^2(1-e^{-r\Delta})} - \frac{\chi}{\zeta}\frac{N-1}{N}\right] B_t \\
& - \left[-2b\frac{N-1}{N^3(1-e^{-r\Delta})} - \frac{2}{E\zeta N}(\frac{N-1}{N})^2\right] B_t X_{it} \\
& + [-b(\frac{N-1}{N})^2 \frac{1}{N^2(1-e^{-r\Delta})} + \frac{N-1}{N^2 E}\frac{2N-1}{N^2\zeta}] B_t^2 \\
& + \sigma_Q^2 \left[\frac{1}{\zeta N^2(1-e^{-r\Delta})} - \frac{b}{N^2(1-e^{-r\Delta})^2}\right] \\
& + \sigma_\epsilon^2 \left[-\frac{be^{-r\Delta}}{N(1-e^{-r\Delta})^2} - \frac{b(N-1)}{N^2(1-e^{-r\Delta})} + \frac{N-1}{E\zeta N^2}\right].
\end{aligned}
$$

Recall that an admissible strategy must lead to an inventory process that satisfies the no-Ponzi scheme condition $e^{-r\Delta t}\mathbb{E}_{i0}[X_{it}^2] \to 0$. Thus to show that $e^{-r\Delta t}\mathbb{E}_{i0}[V(X_{it}, B_t)] \to 0$ it suffices to show that $e^{-r\Delta t}\mathbb{E}_{i0}[B_t X_{it}] \to 0$ and $e^{-r\Delta t}\mathbb{E}_{i0}[B_t^2] \to 0$.

We have

$$
\begin{aligned}
e^{-r\Delta t}\mathbb{E}_{i0}\left[B_t X_{it}\right] &= e^{-r\Delta t}\mathbb{E}_{i0}\left[\frac{N}{N-1}\sum_{j\neq i}\left(X_{j,t-1} + \sum_{e\in E} q_{je,t-1}\right) X_{it}\right] \\
&= e^{-r\Delta t}\mathbb{E}_{i0}\left[\frac{N}{N-1}\left(\sum_{j\in N} X_{j,t-1} + \sum_{e\in E} Q_{e,t-1} - X_{i,t} + \epsilon_{it}\right) X_{it}\right] \\
&= \frac{N}{N-1} e^{-r\Delta t}\mathbb{E}_{i0}\left[\left(\sum_{j\in N} X_{j,t-1} + \sum_{e\in E} Q_{e,t-1}\right) X_{it} - X_{it}^2\right] \\
&\quad + e^{-r\Delta t}\frac{N}{N-1}\sigma_\epsilon^2
\end{aligned}
$$

where, for the first equality, we have used

$$
\frac{N-1}{N} B_t = \sum_{j\neq i}\left(X_{j,t-1} + \sum_{e\in E} q_{je,t-1}\right),
$$

and for the second equality we have used $X_{it} = X_{i,t-1} + \sum_{e\in E} q_{ie,t-1} + \epsilon_{it}$. Since, by Cauchy-Schwarz,

$$e^{-r\Delta t}\mathbb{E}_{i0}\left[\left(\sum_{j\in N} X_{j,t-1}+\sum_{e\in E} Q_{e,t-1}\right) X_{it}\right] \\ \leq \sqrt{\mathbb{E}_{i0}\left[\left(\sum_{j\in N} X_{j,t-1}+\sum_{e\in E} Q_{e,t-1}\right)^2\right] e^{-2r\Delta t}\mathbb{E}\left[X_{it}^2\right]},$$

it follows that

$$\lim_{t\to\infty} e^{-r\Delta t}\mathbb{E}_{i0}\left[\left(\sum_{j\in N} X_{j,t-1}+\sum_{e\in E} Q_{e,t-1}\right) X_{it}\right] = 0.$$

Thus, $\lim_{t\to\infty} e^{-r\Delta t}\mathbb{E}_{i0}[B_t X_{it}] = 0$, as desired. That $\lim_{t\to\infty} e^{-r\Delta t}\mathbb{E}_{i0}[B_t^2] = 0$ can be shown using the same method.

Step 6. We now verify that the optimal strategy of trader i is the conjectured equilibrium strategy, coinciding with (E1). For an arbitrary admissible strategy, which we denote l, let q_{iet}^l, p_{et}^l, X_{it}^l, and B_t^l denote, respectively, the induced quantity purchased on exchange e, the price on exchange e, the inventory, and the belief at date t. By recursive substitution, using the Bellman equation, for each $t \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}_{i0}\left[V(X_{i0},B_0)\right] \geq\ & \mathbb{E}_{i0}\left[\sum_{s=0}^{t} e^{-r\Delta s}\mu_\pi \Delta\left(X_{is}^l+\sum_{e\in E} q_{ies}^l\right)\right] \\ & - \mathbb{E}_{i0}\left[\sum_{s=0}^{t} e^{-r\Delta s}\left(b\left(X_{is}^l+\sum_{e\in E} q_{ies}^l\right)^2+\sum_{e\in E} p_{es}q_{ies}^l\right)\right] \\ & + \mathbb{E}_{i0}\left[e^{-r\Delta t}V(X_{i,t+1}^l, B_{t+1}^l)\right].\end{aligned}$$

The above holds with equality under the conjectured equilibrium strategy. By taking limits as $t \to \infty$, applying the transversality condition and Fatou's Lemma,

$$\begin{aligned}V(X_{i0},B_0) \geq\ & \mathbb{E}_{i0}\left[\sum_{s=0}^{\infty} e^{-r\Delta s}\left(\mu_\pi\Delta\left(X_{is}^l+\sum_{e\in E} q_{ies}^l\right)-b\left(X_{is}^l+\sum_{e\in E} q_{ies}^l\right)^2\right)\right] \\ & - \mathbb{E}_{i0}\left[\sum_{s=0}^{\infty} e^{-r\Delta s}\sum_{e\in E} p_{es}q_{ies}^l\right].\end{aligned}$$

The right-hand side is the utility of the arbitrary strategy l, whereas the left-hand side is the utility of the conjectured equilibrium strategy. This completes the proof of the Theorem.

Recall that for the proof, we used the notation σ_ϵ^2 in place of σ_X^2. Summarizing the solution for coefficients of the model, we have found that

$$E = 2 + \frac{2}{N-2} + \frac{N(N-1)}{N-2}\frac{\sigma_X^2}{\sigma_Q^2} \tag{E22}$$

$$\rho = -\frac{1}{NE} \tag{E23}$$

$$\zeta = \frac{N-2}{2b(N-1)(1+\Gamma(E-1))} - \frac{e^{-r\Delta}}{bE}, \tag{E24}$$

$$\chi = \frac{\mu_\pi \Delta}{1-e^{-r\Delta}}\zeta, \tag{E25}$$

where

$$\Gamma = \frac{(N-1)\sigma_X^2}{(N-1)\sigma_X^2 + E\sigma_Q^2}.$$

E2. Equivalence to Model with Brownian Inventory Shocks

Suppose that instead of receiving an inventory shock which is Gaussian with mean zero and variance $\sigma_\epsilon^2\Delta$ at each trading date, trader i's inventory is continually shocked by a Brownian Motion, Z_i, with volatility σ_ϵ^2. That is, $Z_{i,t\Delta}$ is trader i's cumulative inventory shock up to time $t\Delta$. We assume that the Brownian Motions $(Z_i)_{i\in N}$ are independent across traders and of all other primitive stochastic processes. Then in this setting,

$$X_{it} = X_{i,t-1} + \sum_{e\in E} q_{ie,t-1} + Z_{i,t\Delta} - Z_{i,(t-1)\Delta}$$

where $Z_{i,t\Delta} - Z_{i,(t-1)\Delta} \sim N(0, \sigma_\epsilon^2 \Delta)$. Moreover, trader i's flow utility $F_{it}(q_{it})$ is

$$\begin{aligned}
\pi_t \left(X_{it} + \sum_{e \in E} q_{iet} \right) - \sum_{e \in E} p_{et} q_{iet} \\
&- \int_t^{t+\Delta} e^{-r(s-t)} \mathbb{E}_{it} \left[\tilde{b} \left(X_{it} + \sum_{e \in E} q_{iet} + Z_{i,s\Delta} - Z_{i,t\Delta} \right)^2 \right] ds \\
&= \pi_t \left(X_{it} + \sum_{e \in E} q_{iet} \right) - \sum_{e \in E} p_{et} q_{iet} - b \left(X_{it} + \sum_{e \in E} q_{iet} \right)^2 \\
&\quad - \tilde{b} \sigma_\epsilon^2 \frac{1 - e^{-r\Delta}(r\Delta + 1)}{r^2},
\end{aligned}$$

where $q_{it} = (q_{i1t}, ..., q_{iEt})$ and $b = \tilde{b} \left(1 - e^{-r\Delta}\right)/r$.

Except for the last term which is a constant (and therefore has no effects on incentives), the flow utility is the same as in the model of the main text. Thus, the efficient PBE constructed in the main text is also an efficient PBE of this model.

Appendix F: Extension—Endogeneous Liquidity Trade

This Appendix offers an extension in which liquidity traders, who are local to each exchange and conduct no cross-exchange trade, choose the sizes of their trades.

F1. Setup

In this section we extend the baseline model by allowing liquidity traders to endogenously choose the quantity of market orders that they supply. There are M liquidity traders who are each restricted to trade on a single exchange. We assume that M is divisible by E and that a fraction $1/E$ of them trade on any given exchange. Liquidity trader j has endowment

$$H_j \sim N(0, \frac{1}{M} \sigma_H^2),$$

where the $\{H_j\}$ are mutually independent. Suppose further that each liquidity trader j has preferences of the same form that we have assumed for the strategic traders. If liquidity trader j is restricted to trade on exchange e, his or her ex-ante expected utility of purchasing h_j units via a market order is

$$\mathbb{E}[\pi h_j - c(H_j + h_j)^2 - h_j p_e \,|\, H_j, h_j].$$

Above, $c \in \mathbb{R}_+$ is the holding cost parameter of the liquidity traders. It is useful to think of c being high relative to b, the holding cost parameter of strategic agents. Finally, for simplicity, for this section, we assume that $\mu_X = 0$ and $\mu_\pi = 0$.

F2. Analysis

THEOREM 3: *There exists a symmetric affine equilibrium. In any symmetric affine equilibrium the following are true.*

1) *The quantity of market orders submitted by agent j is*

$$h_j = \frac{-cH_j}{c + \Lambda_E \frac{N-1}{N}}.$$

2) *For each $e, e' \in E$ distinct, the correlation between prices in the two exchanges from the perspective of a strategic trader is*

$$\gamma_E = \frac{(E\alpha_E)^2 \sigma_X^2 (N-1)}{(E\alpha_E)^2 \sigma_X^2 (N-1) + (\frac{c}{c+\Lambda_E \frac{N-1}{N}})^2 \sigma_H^2 E}. \tag{F1}$$

3) *A strategic trader's price impact satisfies*

$$\Lambda_E = \frac{2b((E-1)\gamma_E + 1)}{N-2}, \tag{F2}$$

while the price impact of a liquidity trader is

$$\frac{N-1}{N}\Lambda_E. \tag{F3}$$

4) *$E\alpha_E$ satisfies*

$$E\alpha_E = \frac{1}{\gamma_E(\frac{1}{N} + \frac{1}{N-2}) + (1-\gamma_E)\frac{1}{E}(\frac{1}{N} + \frac{1}{N-2}) + \frac{N-1}{N}}. \tag{F4}$$

PROOF:

We conjecture that there exists a symmetric affine equilibrium in which each strategic trader $i \in N$ submits a demand schedule of the form $-\alpha_E X_i - \zeta_E p$ and each liquidity trader j submits a market order of the form $-\tilde{\alpha}_E H_j$. We study the best response problem of trader $j \in M$. Via market clearing, we can compute the market clearing price in exchange e is

$$p_e = \frac{\sum_{i \in N} -\alpha_E X_i - \sum_{\{k \in M \,|\, k \neq j\}} \tilde{\alpha}_E H_k + h_j}{N\zeta_E}$$

if all agents $i \in N$ and $k \in M$ such that $k \neq j$ behave as conjectured and agent j purchases h_j units on the exchange. Retaining the notation that $\Lambda_E = \frac{1}{(N-1)\zeta_E}$, the price impact of liquidity trader j is $\Lambda_E \frac{N-1}{N}$. He seeks to maximize

$$\mathbb{E}[\pi h_j - c(H_j + h_j)^2 - h_j p_e \mid H_j, h_j] = -c(H_j + h_j)^2 - \Lambda_E \frac{N-1}{N} h_j^2$$

by choosing $h_j \in \mathbb{R}$. Taking a first order condition with respect to h_j we have

$$-2c(H_j + h_j) - 2h_j \Lambda_E \frac{N-1}{N} = 0,$$

which implies that

$$h_j = \frac{-cH_j}{c + \Lambda_E \frac{N-1}{N}}.$$

Thus

$$\tilde{\alpha}_E = \frac{c}{c + \Lambda_E \frac{N-1}{N}}.$$

If strategic traders take the variance of aggregate liquidity trade to be

$$\sigma_Q^2 = \left(\frac{c}{c + \Lambda_E \frac{N-1}{N}} \right)^2 \sigma_H^2,$$

we see that the analysis of the baseline model applies. That is, strategic traders maximize by submitting affine demand schedules such that equations (F1), (F2) and (F4) are satisfied. Then the analysis of the baseline model therefore ensures that provided there exists α_E and γ_E which satisfies (F1), (F2), and (F4), there exists a symmetric affine equilibrium with the four properties given in the statement of the theorem. To show existence it suffices to recognize that substituting expressions (F2) and (F4) into (F1) and re-arranging yields a cubic equation in γ_E. Since the equation is cubic there always exists at least one real root. Thus there always exists a solution to the system of equations.

The above theorem has characterized a symmetric affine equilibrium of the model with endogenous liquidity traders. The following proposition states some results relevant for assessing the allocative efficiency of the symmetric affine equilibrium.

PROPOSITION 2: *The following are true of any symmetric affine equilibrium.*

1) *$E\alpha_E \in [\frac{N-2}{N-1}, \frac{N}{N-1}]$ is always higher in fragmented markets than in centralized markets.*

2) *Fixing arbitrary E, in the limit as c tends to infinity, the expected sum of liquidity traders' holding costs tends to zero.*

3) *Fixing arbitrary $E > 1$, for all c sufficiently large, a market with E exchanges is more efficient than a market with a single exchange in the sense that the expected sum of all traders' holding costs is lower.*

4) *For any $\overline{E} > 1$, there exists an $\bar{c}$ such that if $c > \bar{c}$ then a market with $1 < E \leq \overline{E}$ exchanges is more efficient than a market with a single exchange in the sense that the expected sum of all traders' holding costs is lower.*

PROOF:

Centralized markets correspond to the case when E is 1. To prove Part 1, it is clear by inspecting (F4) that $E\alpha_E \in [\frac{N-2}{N-1}, \frac{N}{N-1}]$. Next recognize that in fragmented markets $E > 1$ and $\gamma_E < 1$ so that again by inspection, $E\alpha_E$ is always higher in fragmented markets.

To prove part 2 recognize that, using part 1 of Theorem 3, the expected sum of liquidity agents' holding costs is

$$c\left(\frac{\Lambda_E \frac{N-1}{N}}{c + \Lambda_E \frac{N-1}{N}}\right)^2 \sigma_H^2,$$

which decays to 0 as c diverges.

To prove part 3, fix $E > 1$ and inspect equation (F1). Since $E\alpha_E \in [\frac{N-2}{N-1}, \frac{N}{N-1}]$ there exists $a, b \in \mathbb{R}$ such that $1 > b > a > 0$ and $\gamma_E \in [a, b]$ for all c sufficiently large. This implies that $|1 - E\alpha_E|$ is bounded above by a constant strictly less than $\frac{1}{N-1}$ whenever c is sufficiently large. In the limit as $c \to \infty$ the aggregate quantity of liquidity trader supply absorbed by strategic traders when there is a single exchange as well as when there are E exchanges becomes arbitrarily close to $\sum_{j \in M} H_j$. Therefore, by Lemma 2, in the limit as $c \to \infty$, the expected sum of holding costs is strictly lower when there are E exchanges than when there is a single exchange since $|1 - E\alpha_E| < |1 - \alpha_1| = \frac{1}{N-1}$. However, the sum of liquidity traders' holding costs converges to 0 as $c \to \infty$. This implies the claim asserted in part 3 of the theorem.

Part 4 is an immediate implication of part 3.

We now prove the following proposition which implies that $E\alpha_E$ must be strictly monotone increasing in E at least until a certain cutoff point. As c increases the range that we can prove that $E\alpha_E$ is strictly monotone increasing in is larger.

PROPOSITION 3: *Fix $E^* \in \mathbb{N}$. If c is sufficiently large such that*

$$\left(\frac{c}{c + \frac{2bE^*}{N-2}\frac{N-1}{N}}\right)^2 > \frac{E^*}{E^* + 1},$$

then $E\alpha_E$ is strictly monotone increasing for all $E < E^*$.

PROOF:

We begin by proving that

$$\left(\frac{c}{c+\Lambda_E\frac{N-1}{N}}\right)^2 E$$

is strictly monotone increasing in E for all $E < E^*$. Since Λ_E is bounded above by $\frac{2bE^*}{N-2}$ we have that

$$\left(\frac{c}{c+\Lambda_E\frac{N-1}{N}}\right)^2 E > \frac{E^*}{E^*+1}E$$

for each $E < E^*$. Thus we have

$$\left(\frac{c}{c+\Lambda_{E+1}\frac{N-1}{N}}\right)^2 (E+1) - \left(\frac{c}{c+\Lambda_E\frac{N-1}{N}}\right)^2 E > \frac{E^*}{E^*+1}(E+1) - E$$

for each $E < E^*$. But the right hand side is equal to

$$\left(\frac{E^*}{E^*+1} - 1\right)E + \frac{E^*}{E^*+1} > \left(\frac{E^*}{E^*+1} - 1\right)E^* + \frac{E^*}{E^*+1} = 0.$$

Now we prove that $E\alpha_E$ is strictly monotone increasing at each $E < E^*$. Inspect the equation (F4). Suppose $E\alpha_E$ is decreasing in E then it must be that γ_E is increasing. Consider now (F1). Since $(\frac{c}{c+\Lambda_E\frac{N-1}{N}})^2 E$ is strictly monotone increasing and $E\alpha_E$ is decreasing it must be that γ_E is decreasing, a contradiction.

Appendix G: Extension—Private Information about Asset Payoff

This Appendix addresses an extension of the model in which strategic traders are asymmetrically informed about the asset payoff.

G1. Setup

We alter the baseline model so that each agent has private information about the asset's final payoff, $\pi \sim N(\mu_\pi, \sigma_\pi^2)$. We assume the aggregate endowment of strategic traders, $Z \equiv \sum_i X_i$, is public information. As before, liquidity traders supply a quantity $Q_e \sim N(0, \frac{\sigma_Q^2}{E})$ to each exchange, independent across exchanges. Strategic traders receive private signals of π:

$$S_i = \pi + \epsilon_i$$

where $\epsilon_i \sim N(0, \sigma_\epsilon^2)$ is i.i.d across individuals and independent of all other primitive random variables.

G2. Analysis

THEOREM 4: *In any symmetric affine equilibrium with demand schedules which are each monotone decreasing in price,*

1) *Each strategic trader i submits a demand schedule to each exchange e of the form*
$$f_{ie}(X_i, S_i, p) = \Delta - \alpha X_i - \zeta p + w S_i.$$
where α, ζ, w, and Δ are defined by the system of equations (G2)—(G9).

2) *Price impact is*
$$\Lambda_E = \frac{(2b[(E-1)\tilde{\gamma}_1 + 1] + N\frac{\tilde{\gamma}_3}{w})}{N-2},$$
where $\tilde{\gamma}_1$ and $\tilde{\gamma}_3$ are defined by equations (G2) *and* (G4).

3) *The final inventory of strategic trader i is*
$$X_i + \sum_{e\in E} f_{ie}(X_i, S_i, p_e^f) = (1 - E\alpha)X_i + E\alpha\frac{1}{N}\sum_{j\in N} X_j + Ew\left(S_i - \frac{1}{N}\sum_{j\in N} S_j\right) + \frac{\sum_{e\in E} Q_e}{N}.$$

PROOF:

Conjecture a symmetric affine equilibrium in which agent i submits demand schedule
$$f_{ie}(X_i, S_i, p) = \Delta - \alpha X_i - \zeta p + w S_i$$
to exchange $e \in E$ for each $i \in N$ and $e \in E$. By market clearing the residual supply curve trader i faces in exchange e is
$$p_e(q) = \frac{1}{(N-1)\zeta}\left[\sum_{j\neq i}(-\alpha X_j + w S_j + \Delta) - Q_e + q\right].$$
Thus price impact is $\Lambda = \frac{1}{(N-1)\zeta}$. Also by market clearing, the equilibrium price is
$$p_e^f = \frac{1}{N\zeta}\left[\sum_{j\in N}(-\alpha X_j + w S_j + \Delta) - Q_e\right].$$

Going forward, let us define $q_{ie}^f := f_{ie}(X_i, S_i, p_e^f)$ for each $e \in E$ for ease of notation. In any equilibrium, trader i must equate marginal utility with marginal cost for every realization of the price:

$$\text{(G1)}\quad -2b\left(X_i + q^f_{i1} + (E-1)\mathbb{E}\left[q^f_{i2} \,|\, p^f_1 - \frac{1}{(N-1)\zeta}q^f_{i1}, X_i, S_i\right]\right) = p^f_1 - \mathbb{E}\left[\pi \,|\, p^f_1 - \frac{1}{(N-1)\zeta}q^f_{i1}, X_i, S_i\right] + \frac{1}{(N-1)\zeta}q^f_{i1}.$$

Above we have used symmetry. We now compute the two conditional moments $\mathbb{E}[q^f_{i2} \,|\, p^f_1 - \frac{1}{(N-1)\zeta}q^f_{i1}, S_i, X_i]$ and $\mathbb{E}[\pi \,|\, p^f_1 - \frac{1}{(N-1)\zeta}q^f_{i1}, S_i, X_i]$ by using the projection theorem. We begin with the former. We can, using the projection theorem, express

$$\mathbb{E}\left[\sum_{j\neq i} S_j \,|\, p^f_1 - \frac{q^f_{i1}}{(N-1)\zeta}, S_i\right] = \mu_\pi(N-1) + \gamma_1\Big(p^f_1 - \frac{q^f_{i1}}{(N-1)\zeta} - \frac{w\mu_\pi}{\zeta} + \alpha\frac{Z - X_i}{(N-1)\zeta} - \frac{\Delta}{\zeta}\Big) + \gamma_2(S_i - \mu_\pi).$$

Here, γ_1 and γ_2 are derived as follows. The variables, $\sum_{j\neq i} S_j$,S_i, $p^f_1 - \frac{1}{(N-1)\zeta}q^f_{i1}$ are jointly Gaussian with variance matrix

$$\Sigma = \begin{bmatrix} (N-1)^2\sigma^2_\pi + \sigma^2_\epsilon(N-1) & (N-1)\sigma^2_\pi & \frac{w}{\zeta}(\sigma^2_\pi(N-1) + \sigma^2_\epsilon) \\ (N-1)\sigma^2_\pi & \sigma^2_\pi + \sigma^2_\epsilon & \frac{w}{\zeta}\sigma^2_\pi \\ \frac{w}{\zeta}(\sigma^2_\pi(N-1) + \sigma^2_\epsilon) & \frac{w}{\zeta}\sigma^2_\pi & \frac{1}{\zeta^2}[w^2(\sigma^2_\pi + \frac{\sigma^2_\epsilon}{(N-1)}) + \frac{\sigma^2_Q}{E(N-1)^2}] \end{bmatrix}.$$

Define

$$\overline{\Sigma} \equiv \begin{bmatrix} \sigma^2_\pi + \sigma^2_\epsilon & \frac{w}{\zeta}\sigma^2_\pi \\ \frac{w}{\zeta}\sigma^2_\pi & \frac{1}{\zeta^2}[w^2(\sigma^2_\pi + \frac{\sigma^2_\epsilon}{(N-1)}) + \frac{\sigma^2_Q}{E(N-1)^2}], \end{bmatrix}$$

with

$$\overline{\Sigma}^{-1} = \left[(\sigma^2_\pi + \sigma^2_\epsilon)\frac{1}{\zeta^2}[w^2(\sigma^2_\pi + \frac{\sigma^2_\epsilon}{(N-1)}) + \frac{\sigma^2_Q}{E(N-1)^2}] - \frac{w^2}{\zeta^2}\sigma^4_\pi\right]^{-1} \times \begin{bmatrix} \frac{1}{\zeta^2}[w^2(\sigma^2_\pi + \frac{\sigma^2_\epsilon}{N-1}) + \frac{\sigma^2_Q}{E(N-1)^2}] & -\frac{w}{\zeta}\sigma^2_\pi \\ -\frac{w}{\zeta}\sigma^2_\pi & \sigma^2_\pi + \sigma^2_\epsilon \end{bmatrix}.$$

Define

$$\Sigma_{12} \equiv \begin{bmatrix} (N-1)\sigma^2_\pi & \frac{w}{\zeta}(\sigma^2_\pi(N-1) + \sigma^2_\epsilon) \end{bmatrix}.$$

By the rules of conditional normals

$$\begin{bmatrix}\gamma_2 & \gamma_1\end{bmatrix} = \Sigma_{12}\overline{\Sigma}^{-1}.$$

This yields,

$$\gamma_2 = \frac{\frac{\sigma_\pi^2\sigma_Q^2}{E(N-1)}}{(\sigma_\pi^2+\sigma_\epsilon^2)\left[w^2(\sigma_\pi^2+\frac{\sigma_\epsilon^2}{(N-1)})+\frac{\sigma_Q^2}{E(N-1)^2}\right]-w^2\sigma_\pi^4}.$$

Note that $\frac{1}{N-1}\gamma_2 \in [0,1]$. Next, we have

$$\gamma_1 = \zeta\frac{w\sigma_\pi^2\sigma_\epsilon^2(N-1)+w\sigma_\epsilon^2(\sigma_\pi^2+\sigma_\epsilon^2)}{(\sigma_\pi^2+\sigma_\epsilon^2)\left[w^2(\sigma_\pi^2+\frac{\sigma_\epsilon^2}{(N-1)})+\frac{\sigma_Q^2}{E(N-1)^2}\right]-w^2\sigma_\pi^4}.$$

Note that $\frac{w}{\zeta(N-1)}\gamma_1 \in [0,1]$. We have

$$\begin{aligned}\mathbb{E}\left[q_{i2}^f \,|\, p_1^f - \frac{1}{(N-1)\zeta}q_{i1}^f, S_i\right] &= -\alpha X_i + wS_i + \Delta + \frac{\alpha Z}{N} - \frac{wS_i}{N} - \Delta\\ &- \frac{w}{N}\left[\mu_\pi(N-1) + \gamma_1\left(p_1^f - \frac{1}{(N-1)\zeta}q_{i1}^f - \frac{w\mu_\pi}{\zeta} + \alpha\frac{Z-X_i}{(N-1)\zeta} - \frac{\Delta}{\zeta}\right)\right]\\ &- \frac{w}{N}\left[\gamma_2(S_i-\mu_\pi)\right].\end{aligned}$$

Next, we move on to compute, $\mathbb{E}[\pi \,|\, p_1^f - \frac{1}{(N-1)\zeta}q_{i1}^f, S_i, X_i]$. We can, using the rules of conditional normals, express

$$\begin{aligned}\mathbb{E}\left[\pi \,|\, p_1^f - \frac{q_{i1}^f}{(N-1)\zeta}, S_i\right] &= \mu_\pi + \gamma_4(S_i-\mu_\pi)\\ &+ \gamma_3\left(p_1 - \frac{q_{i1}^f}{(N-1)\zeta} - \frac{w\mu_\pi}{\zeta} + \alpha\frac{Z-X_i}{(N-1)\zeta} - \frac{\Delta}{\zeta}\right).\end{aligned}$$

The variables, π,S_i, $p_1^f - \frac{q_{i1}^f}{(N-1)\zeta}$ are jointly Gaussian with variance matrix

$$\Sigma = \begin{bmatrix}\sigma_\pi^2 & \sigma_\pi^2 & \frac{w}{\zeta}\sigma_\pi^2\\ \sigma_\pi^2 & \sigma_\pi^2+\sigma_\epsilon^2 & \frac{w}{\zeta}\sigma_\pi^2\\ \frac{w}{\zeta}\sigma_\pi^2 & \frac{w}{\zeta}\sigma_\pi^2 & \frac{1}{\zeta^2}\left[w^2(\sigma_\pi^2+\frac{\sigma_\epsilon^2}{(N-1)})+\frac{\sigma_Q^2}{E(N-1)^2}\right]\end{bmatrix}.$$

Define

$$\overline{\Sigma} \equiv \begin{bmatrix} \sigma_\pi^2 + \sigma_\epsilon^2 & \frac{w}{\zeta}\sigma_\pi^2 \\ \frac{w}{\zeta}\sigma_\pi^2 & \frac{1}{\zeta^2}\left[w^2(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{(N-1)}) + \frac{\sigma_Q^2}{E(N-1)^2}\right] \end{bmatrix}$$

and

$$\Sigma_{12} \equiv \begin{bmatrix} \sigma_\pi^2 & \frac{w}{\zeta}\sigma_\pi^2 \end{bmatrix}.$$

Then

$$\begin{bmatrix} \gamma_4 & \gamma_3 \end{bmatrix} = \Sigma_{12}\overline{\Sigma}^{-1}.$$

We obtain,

$$\gamma_4 = \frac{\sigma_\pi^2 \frac{1}{\zeta^2}\left[w^2(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{(N-1)}) + \frac{\sigma_Q^2}{E(N-1)^2}\right] - \frac{w^2}{\zeta^2}\sigma_\pi^4}{(\sigma_\pi^2 + \sigma_\epsilon^2)\frac{1}{\zeta^2}\left[w^2(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{(N-1)}) + \frac{\sigma_Q^2}{E(N-1)^2}\right] - \frac{w^2}{\zeta^2}\sigma_\pi^4},$$

and

$$\gamma_3 = \frac{\frac{w}{\zeta}\sigma_\pi^2\sigma_\epsilon^2}{(\sigma_\pi^2 + \sigma_\epsilon^2)\frac{1}{\zeta^2}\left[w^2(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{(N-1)}) + \frac{\sigma_Q^2}{E(N-1)^2}\right] - \frac{w^2}{\zeta^2}\sigma_\pi^4}.$$

Note that $\gamma_3\frac{w}{\zeta(N-1)} \in [0,1]$ and $\gamma_4 \in [0,1]$. It is useful, for the analysis to follow, to redefine the inference coefficients so that they all lie in the interval $[0,1]$. Specifically, define $\tilde{\gamma}_1 = \frac{w}{\zeta(N-1)}\gamma_1$, $\tilde{\gamma}_2 = \frac{1}{N-1}\gamma_2$, $\tilde{\gamma}_3 \equiv \frac{w}{\zeta(N-1)}\gamma_3$, and $\tilde{\gamma}_4 = \gamma_4$. Then

$$\text{(G2)} \qquad \tilde{\gamma}_1 = \frac{w^2\sigma_\pi^2\sigma_\epsilon^2 + \frac{w^2\sigma_\epsilon^2(\sigma_\pi^2+\sigma_\epsilon^2)}{N-1}}{w^2\left[(\sigma_\pi^2 + \sigma_\epsilon^2)(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{N-1}) - \sigma_\pi^4\right] + \frac{\sigma_Q^2}{E(N-1)^2}(\sigma_\pi^2 + \sigma_\epsilon^2)}$$

$$\text{(G3)} \qquad \tilde{\gamma}_2 = \frac{\frac{\sigma_\pi^2\sigma_Q^2}{E(N-1)^2}}{w^2\left[(\sigma_\pi^2 + \sigma_\epsilon^2)(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{N-1}) - \sigma_\pi^4\right] + \frac{\sigma_Q^2}{E(N-1)^2}(\sigma_\pi^2 + \sigma_\epsilon^2)}.$$

$$\text{(G4)} \qquad \tilde{\gamma}_3 = \frac{w^2\sigma_\pi^2\frac{\sigma_\epsilon^2}{N-1}}{w^2\left[(\sigma_\pi^2 + \sigma_\epsilon^2)(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{N-1}) - \sigma_\pi^4\right] + \frac{\sigma_Q^2}{E(N-1)^2}(\sigma_\pi^2 + \sigma_\epsilon^2)}.$$

$$\text{(G5)} \qquad \tilde{\gamma}_4 = \frac{\sigma_\pi^2 \left[w^2(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{(N-1)}) + \frac{\sigma_Q^2}{(E(N-1)^2)} \right] - w^2\sigma_\pi^4}{w^2 \left[(\sigma_\pi^2 + \sigma_\epsilon^2)(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{N-1}) - \sigma_\pi^4 \right] + \frac{\sigma_Q^2}{E(N-1)^2}(\sigma_\pi^2 + \sigma_\epsilon^2)}.$$

We can now use the equation (G1) together with the conditional moments we just computed, to match coefficients and pin down α, ζ, w, and Δ. The coefficient of q_{i1} gathered on to the LHS is

$$-2b - \frac{1}{(N-1)\zeta} - 2b(E-1)\frac{1}{N}\tilde{\gamma}_1 - \frac{\tilde{\gamma}_3}{w}.$$

The coefficient of p_1 gathered on to the RHS is

$$1 - 2b(E-1)\frac{1}{N}\tilde{\gamma}_1(N-1)\zeta - \zeta\frac{(N-1)\tilde{\gamma}_3}{w}.$$

The coefficient of S_i gathered on to the RHS is

$$2b(E-1)w\left(\frac{N-1}{N})(1-\tilde{\gamma}_2\right) - \gamma_4.$$

The coefficient of X_i gathered on to the RHS is

$$2b + 2b(E-1)\left[-\alpha + \alpha\frac{\tilde{\gamma}_1}{N}\right] + \frac{\tilde{\gamma}_3}{w}\alpha.$$

The constant coefficient gathered on to the RHS is

$$2b(E-1)\left[\frac{\alpha Z}{N} - \frac{w}{N}\left(\mu_\pi(N-1)\left(1-\tilde{\gamma}_2-\tilde{\gamma}_1\right) + \frac{\tilde{\gamma}_1\alpha Z}{w} - \frac{\Delta\tilde{\gamma}_1(N-1)}{w}\right)\right]$$

$$-\mu_\pi + \tilde{\gamma}_3\mu_\pi(N-1) - \frac{\tilde{\gamma}_3\alpha Z}{w} + \tilde{\gamma}_3\frac{(N-1)\Delta}{w} + \tilde{\gamma}_4\mu_\pi.$$

We now match coefficients to compute ζ as a function of $\tilde{\gamma}_1$ and $\tilde{\gamma}_3$:

$$\zeta = \frac{N-2}{N-1}\frac{1}{(2b[(E-1)\tilde{\gamma}_1 + 1] + N\frac{\tilde{\gamma}_3}{w})}.$$

Price impact is therefore

$$\text{(G6)} \qquad \frac{1}{(N-1)\zeta} = \frac{(2b[(E-1)\tilde{\gamma}_1 + 1] + N\frac{\tilde{\gamma}_3}{w})}{N-2}.$$

Notice that compared with the model without private information about asset payoffs, there is now a $\frac{N\tilde{\gamma}_3}{w}$ term which is a result of using the price in an exchange to do inference on the asset's payoff, π. We now match coefficients to derive an equation which characterizes w:

$$\text{(G7)}\quad -2b(E-1)w\frac{N-1}{N}(1-\tilde{\gamma}_2)+\gamma_4 = w[2b+\frac{(2b[(E-1)\tilde{\gamma}_1+1]+\frac{N}{w}\tilde{\gamma}_3)}{N-2}+2b(E-1)\frac{1}{N}\tilde{\gamma}_1+\frac{\tilde{\gamma}_3}{w}].$$

Notice that after substituting in the expressions for the inference coefficients and rearranging, we would obtain a cubic equation in w. We now match coefficients to compute α as a function of the inference coefficients:

$$\text{(G8)}\qquad \alpha = \frac{2b}{2b+\frac{(2b[(E-1)\tilde{\gamma}_1+1]+N\frac{\tilde{\gamma}_3}{w})}{N-2}+2b(E-1)\frac{1}{N}\tilde{\gamma}_1+2b(E-1)(1-\frac{\tilde{\gamma}_1}{N})}.$$

We now match coefficients to compute Δ as a function of the inference coefficients:

(G9)

$$\Delta = -\frac{2b(E-1)\left[\frac{\alpha Z}{N}-\frac{w}{N}(\mu_\pi(N-1)+\frac{\tilde{\gamma}_1\alpha Z}{w}-\tilde{\gamma}_2(N-1)\mu_\pi-\tilde{\gamma}_1(N-1)\mu_\pi)\right]}{2b+\frac{1}{(N-1)\zeta}+\frac{2b(E-1)\tilde{\gamma}_1}{N}+\frac{\tilde{\gamma}_3}{w}+\frac{2b(E-1)\tilde{\gamma}_1(N-1)}{N}+\frac{\tilde{\gamma}_3(N-1)}{w}} - \frac{-\mu_\pi+\tilde{\gamma}_3\mu_\pi(N-1)-\frac{\tilde{\gamma}_3\alpha Z}{w}+\tilde{\gamma}_4\mu_\pi}{2b+\frac{1}{(N-1)\zeta}+\frac{2b(E-1)\tilde{\gamma}_1}{N}+\frac{\tilde{\gamma}_3}{w}+\frac{2b(E-1)\tilde{\gamma}_1(N-1)}{N}+\frac{\tilde{\gamma}_3(N-1)}{w}}.$$

Thus equations (G8), (G6), (G7), (G9), (G2), (G3), (G4), and (G5) are necessary conditions that any symmetric affine equilibrium must satisfy. An argument analogous to that of Theorem 1 can be used to show that a solution to these equations constitute a symmetric affine equilibrium provided that ζ is positive. Part 2 follows from equation (G6). This completes the proof of parts 1 and 2. We omit the proof of part 3 since it is a straightforward computation.

PROPOSITION 4: *For any value of E, if there exists a symmetric affine equilibrium with $\zeta > 0$ then $w > 0$.*

PROOF:

The equation characterizing w is

$$\tilde{\gamma}_4-\tilde{\gamma}_3 = w[2b+\frac{1}{\zeta(N-1)}+2b(E-1)\frac{1}{N}\tilde{\gamma}_1+2b(E-1)(\frac{N-1}{N})(1-\tilde{\gamma}_2)].$$

The left hand side is positive as seen by inspecting the equations defining the inference coefficients. The bracketed term on the right hand side is also always positive if the demand schedules are downward sloping since the inference coefficients are in the unit interval. Thus the only way for the cubic equation to be satisfied is if w is positive.

We now focus on characterizing how Ew and $E\alpha$ change as E varies. In this model, the efficient allocation is the same as that of the baseline model. Thus by part 3 of Theorem 4, perfect allocative efficiency is acheived if $Ew = 0$ and $E\alpha = 1$.

PROPOSITION 5: *The following are true.*

1) *There exists a unique symmetric affine equilibrium when* $E = 1$.
2) *When there is just a single exchange,*

$$0 < w_1 < \frac{1}{2b}\frac{\sigma_\pi^2}{\sigma_\pi^2 + \sigma_\epsilon^2},$$

where w_1 *corresponds to the unique symmetric affine equilibrium.*

3) *There exist at least one and at most three symmetric affine equilibria for all* E *sufficiently large.*
4) *For any sequence* $\{Ew_E\}$ *corresponding to symmetric affine equilibria,*

$$Ew_E \to \frac{1}{2b}\frac{N}{N-1}\frac{\sigma_\pi^2}{\sigma_\epsilon^2} > \frac{1}{2b}\frac{\sigma_\pi^2}{\sigma_\pi^2 + \sigma_\epsilon^2}$$

as $E \to \infty$.

5) *For any sequence,* $\{E\alpha_E\}m$ *corresponding to symmetric affine equilibria,* $E\alpha_E \to 1$*, which is strictly greater than* α_1.

PROOF:

Part 1. When there is a single exchange,

$$w_1 = \frac{\tilde{\gamma}_4 - (1 + \frac{N}{N-2})\tilde{\gamma}_3}{2b(1 + \frac{1}{N-2})}. \tag{G10}$$

Rearranging (G10), we derive

$$2b(1 + \frac{1}{N-2})w^3\left[(\sigma_\pi^2 + \sigma_\epsilon^2)\left(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{N-1}\right) - \sigma_\pi^4\right] + w\left(1 + \frac{1}{N-2}\right)\frac{2b\sigma_Q^2}{E(N-1)^2}(\sigma_\pi^2 + \sigma_\epsilon^2)$$
$$= \sigma_\pi^2\left[w^2\left(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{(N-1)}\right) + \frac{\sigma_Q^2}{E(N-1)^2}\right] - w^2\sigma_\pi^4 - \left(1 + \frac{N}{N-2}\right)w^2\sigma_\pi^2\frac{\sigma_\epsilon^2}{N-1}.$$

Thus, when E is 1, w_1 satisfies a cubic equation with coefficients:

$$[w_1^3] : 2b\left(1 + \frac{1}{N-2}\right)\left[(\sigma_\pi^2 + \sigma_\epsilon^2)\left(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{N-1}\right) - \sigma_\pi^4\right]$$

$$[w_1^2] : \frac{N}{N-2}\sigma_\pi^2 \frac{\sigma_\epsilon^2}{N-1}$$

$$[w_1] : 2b\left(1 + \frac{1}{N-2}\right)\frac{\sigma_Q^2}{E(N-1)^2}(\sigma_\pi^2 + \sigma_\epsilon^2)$$

$$[1] : -\frac{\sigma_\pi^2\sigma_Q^2}{E(N-1)^2}.$$

Since the coefficient of w_1^2 is positive, the coefficient of w_1^3 is positive, and the constant is negative, there always exists exactly one positive real root. Let p, q, and r denote the roots of the cubic equation. Then $pqr = -\frac{constant\ coefficient}{coefficient\ of\ w_1^3} > 0$. Thus if there is one real root and 2 complex roots, the real root must be positive. If there are are three real roots, at least one must be positive. Next, $p + q + r = -\frac{coefficient\ of\ w_1^2}{coefficient\ of\ w_1^3} < 0$ so if there are three real roots, two must be negative and one must be positive. There always exists a unique positive real root. Take this positive real root. For this value of w_1, by (G6), ζ_1 is positive. An approach analogous to that of Theorem 1 (which we omit) can then be used to verify that there is a symmetric affine equilibrium corresponding to this value of w_1. To prove uniqueness, it can be shown that $\tilde{\gamma}_4 - (1 + \frac{N}{N-2})\tilde{\gamma}_3$ is monotone decreasing in w_1 as seen by using (G4) and (G5). Since

$$w_1 - \frac{\tilde{\gamma}_4 - (1 + \frac{N}{N-2})\tilde{\gamma}_3}{2b(1 + \frac{1}{N-2})}$$

is monotone increasing in w_1 when viewing $\tilde{\gamma}_3$ and $\tilde{\gamma}_4$ as functions of w_1, the equilibrium is unique since (G10) is a necessary condition which must be satisfied in any symmetric affine equilibrium.

Part 2. We rearrange (G5) to derive

$$\tilde{\gamma}_4 = \frac{\sigma_\pi^2 \left(w^2 \left(\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{N-1} \right) + \frac{\sigma_Q^2}{(N-1)^2} \right) - w^2 \sigma_\pi^4}{(\sigma_\pi^2 + \sigma_\epsilon^2) \left(w^2 (\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{N-1}) + \frac{\sigma_Q^2}{(N-1)^2} \right) - w^2 \sigma_\pi^4}$$

$$< \frac{\frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2} \left[(\sigma_\pi^2 + \sigma_\epsilon^2) \left(w^2 (\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{N-1}) + \frac{\sigma_Q^2}{(N-1)^2} \right) - w^2 \sigma_\pi^4 \right]}{(\sigma_\pi^2 + \sigma_\epsilon^2) \left(w^2 (\sigma_\pi^2 + \frac{\sigma_\epsilon^2}{N-1}) + \frac{\sigma_Q^2}{(N-1)^2} \right) - w^2 \sigma_\pi^4}$$

$$= \frac{\sigma_\pi^2}{\sigma_\pi^2 + \sigma_\epsilon^2}.$$

Inspecting (G10) together with the above inequality gives the result.

Parts 3 and 4. Rearranging equation (G7), we derive

$$w_E = \frac{\tilde{\gamma}_4 - (1 + \frac{N}{N-2})\tilde{\gamma}_3}{2b + \frac{2b}{N-2} + 2b(E-1)(\frac{1}{N} + \frac{1}{N-2})\tilde{\gamma}_1 + 2b(E-1)(\frac{N-1}{N})(1 - \tilde{\gamma}_2)}.$$

we observe that $|\, w_E \,|$ is less than $\frac{C}{E}$ for large E for some constant C since $\tilde{\gamma}_2$ is by inspection bounded away from 1 (we can derive a bound which holds for all E) and the numerator is bounded above by $2 + \frac{N}{N-2}$. Thus, it must be the case that $\tilde{\gamma}_4 \to \frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}$ in the limit as $E \to \infty$. By inspection $\tilde{\gamma}_1$ and $\tilde{\gamma}_3$ converges to 0 while $\tilde{\gamma}_2 \to \frac{\sigma_\pi^2}{\sigma_\pi^2+\sigma_\epsilon^2}$. We can express

$$E w_E = \frac{\tilde{\gamma}_4 - (1 + \frac{N}{N-2})\tilde{\gamma}_3}{\frac{2b+\frac{2b}{N-2}}{E} + 2b\frac{(E-1)}{E}(\frac{1}{N} + \frac{1}{N-2})\tilde{\gamma}_1 + 2b\frac{(E-1)}{E}(\frac{N-1}{N})(1 - \tilde{\gamma}_2)}.$$

Thus in the limit as $E \to \infty$,

$$E w_E \to \frac{1}{2b} \frac{1}{\frac{N-1}{N} \frac{\sigma_\epsilon^2}{\sigma_\pi^2+\sigma_\epsilon^2}} \frac{\sigma_\pi^2}{\sigma_\epsilon^2 + \sigma_\pi^2} = \frac{1}{2b} \frac{N}{N-1} \frac{\sigma_\pi^2}{\sigma_\epsilon^2}.$$

Note that this implies that for large enough E, any real root of the cubic equation for w_E must be positive, which by (G6) implies that ζ_E is positive for any real root. An argument analogous to Theorem 1 can then be used to verify that there is a symmetric affine equilibrium corresponding to any positive root of the cubic equation for w_E. Since a cubic equation always has at least one real root and at most three, there always exists at least one and at most three symmetric affine equilibrium for E sufficiently large.

Part 5. Using earlier results we can write

$$E\alpha_E = \frac{2bE}{2b + \frac{(2b[(E-1)\tilde{\gamma}_1+1]+\frac{N}{w_E}\tilde{\gamma}_3)}{N-2} + 2b(E-1)\frac{1}{N}\tilde{\gamma}_1 + 2b(E-1)(1-\frac{\tilde{\gamma}_1}{N})}.$$

Thus, if $\sigma_Q^2 > 0$, as $E \to \infty$,

$$E\alpha_E \to 1.$$

When $E = 1$,

$$\alpha_1 = \frac{2b}{2b + \frac{2b+\frac{N\tilde{\gamma}_3}{w_1}}{N-2}} < 1.$$

Thus, an increase in fragmentation means a more efficient redistribution of endowments, at least in the limit.

Next, we give a coarse analysis of welfare which compares the expected holding costs of strategic agents as E tends infinity with the case of centralized exchange when $E = 1$.

PROPOSITION 6: *If $\frac{\sigma_\pi^4}{\sigma_\epsilon^2}$ is sufficiently small, then for all E sufficiently large the allocation of any symmetric affine equilibrium is more efficient than the allocation of the unique symmetric affine equilibrium when E is 1.*

PROOF:

By symmetry it suffices to study the expected holding cost of an individual agent. Recall, in what follows, that we have assumed for simplicity that the mean of the liquidity trader supply is zero. The expected holding cost of an agent is

$$\mathbb{E}\left[b\left((1-E\alpha_E)X_i + E\alpha_E\frac{Z}{N} + Ew_E\left(S_i - \frac{1}{N}\sum_{j\in N} S_j\right) + \frac{\sum_{e\in E} Q_e}{N}\right)^2\right] =$$
$$b\left[\left((1-E\alpha_E)X_i + E\alpha_E\frac{Z}{N}\right)^2 + (Ew_E)^2\left(\left(\frac{N-1}{N}\right)^2 + \frac{N-1}{N^2}\right)\sigma_\epsilon^2 + \frac{\sigma_Q^2}{N^2}\right].$$

Consider taking a limit as $E \to \infty$ of the above expression. Then we obtain

$$b\left[\frac{Z^2}{N^2} + \frac{\sigma_Q^2}{N^2} + \left(\frac{1}{2b}\frac{N}{N-1}\right)^2\frac{\sigma_\pi^4}{\sigma_\epsilon^2}\left(\left(\frac{N-1}{N}\right)^2 + \frac{N-1}{N^2}\right)\right].$$

The only difference between this expected holding cost and the expected holding cost at the efficient allocation is the last term. Thus when $\frac{\sigma_\pi^4}{\sigma_\epsilon^2}$ is small, a large level of fragmentation is preferred to centralized exchange.

Appendix H: Extension—Arbitrary Covariance Matrix

In this Appendix, we extend the baseline model to allow for correlation among the primitive asset quantities $\{X_1, \ldots, X_N, Q_1, \ldots, Q_E\}$ setting the sizes of trading interests. This model variant nests the baseline model. Consequently, many of the proofs are quite similar.

H1. Setup

We retain the same model setup as in the baseline but alter the assumptions about the joint distribution of $(X_1, \ldots, X_N, Q_1, \ldots, Q_E)$. We assume that $Q = C + \sum_{e \in E} \xi_e$ and $Q_e = \frac{C}{E} + \xi_e$ for each $e \in E$, where C and $\{\xi_e\}_{e \in E}$ are random variables in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. Here, C is the component of liquidity trader supply which is common across exchanges and ξ_e is the component idiosyncratic to exchange e. We assume that the distribution of C does not depend on E and that $\{\xi_e\}_{e \in E}$ is a collection of i.i.d, Gaussian distributed random variables with a mean of 0 and variance of $\frac{\sigma_\xi^2}{E}$ that are independent of $X_1, \ldots, X_N$, and C. Under these assumptions, the distribution of Q does not depend on E. Next, we assume that $X_1, \ldots, X_N, C$ are jointly Gaussian with $\mathbb{E}[C] = \mu_Q$, $\text{var}[C] = \rho$, $\text{cov}(X_i, X_j) = \Sigma$ for all $i, j \in N$ such that $i \neq j$, and $\text{cov}(X_i, C) = \eta$, $\mathbb{E}[X_i] = \mu_X$, and $\text{var}[X_i] = \sigma_X^2$ for all $i \in N$. For the distribution to be well defined, ρ, Σ, η, and σ_X^2 are such that the covariance matrix of $X_1, \ldots, X_N, C$ is positive definite.

H2. Analysis

LEMMA 5: *The condition, $\sigma_X^2 + (N-1)\Sigma > 0$, holds.*

PROOF:
The covariance matrix of $(X_1, \ldots, X_N)$ is positive definite. Denote the covariance matrix V_X. Each element of the diagonal of V_X is σ_X^2 while all other elements are Σ. This implies that $\mathbf{1}^T V_X \mathbf{1} = N[\sigma_X^2 + (N-1)\Sigma] > 0$ where $\mathbf{1}$ is an $N \times 1$ vector of ones.

THEOREM 6: *For each $E \in \mathbb{N}$, there exists at least one and up to three symmetric affine equilibria. If either $\eta \geq 0$ or $\sigma_\xi^2 = 0$, there is a unique symmetric affine equilibrium. Given an arbitrary $E \in \mathbb{N}$ let $(\Delta_E, \alpha_E, \zeta_E)$ be an arbitrary corresponding symmetric affine equilibrium. Then Δ_E, α_E, and ζ_E satisfy equations* (H16), (H17), *and* (H18). *Moreover:*

1) For each $e \in E$,

$$\Lambda_E = \frac{2b(1 + \gamma_E(E-1))}{N-2}$$

where

$$\gamma_E \equiv \text{corr}_{X_i}(p_e^*, p_k^*),$$

for $k \neq e$ such that $k \in E$.

2) Price in exchange $e \in E$ is

$$p_e^* = \frac{N-1}{N}\Lambda_E[\sum_{i\in N} -\alpha_E X_i - Q_e + N\Delta_E].$$

3) The final asset position of trader $i \in N$ is

$$(1-E\alpha_E)X_i + E\alpha_E \frac{\sum_{i\in N} X_j}{N} + \frac{Q}{N}.$$

4) If $\sigma_\xi^2 = 0$ or $E = 1$, for each $E \in \mathbb{N}$, the equilibrium allocation corresponds with that of the centralized benchmark.

5) If $\sigma_\xi^2 > 0$, given an arbitrary sequence of symmetric affine equilibria, $\{(\Delta_E, \alpha_E, \zeta_E)\}_{E\in\mathbb{N}}$, we have

$$E\alpha_E \to \frac{N}{N-1}\frac{1+\frac{\eta}{N\sigma_X^2}}{1-\frac{\Sigma}{\sigma_X^2}}.$$

PROOF:

The proof proceeds in 3 steps. In the step 1 we compute some relevant moments corresponding to a symmetric affine equilibrium, $(\Delta_E, \alpha_E, \zeta_E)$. In step 2, we substitute the derived moments from step 1 into the optimality condition for a traders' demand submission problem and match coefficients to derive a system of three equations for Δ_E, α_E, and ζ_E. In step 3 we prove existence of a symmetric affine equilibrium and uniqueness when $\eta \geq 0$. We then prove parts 1 through 5.

Step 1: To begin we conjecture an arbitrary symmetric affine equilibrium $(\Delta_E, \alpha_E, \zeta_E)$ in which each trader submits a demand schedule of the form in (2) to each exchange e. For ease of notation define

$$q_{ie}^f := f_{ie}(X_i, p_e^f).$$

We compute the following unconditional moments.

(H1) $$\mathbb{E}\left[\frac{-\alpha_E(\sum_i X_i) + \Delta_E N - Q_{e'}}{\zeta_E N}\right] = \frac{-\alpha_E\mu_X + \Delta_E}{\zeta_E} - \frac{\mu_Q}{E\zeta_E N}$$

(H2) $$\mathbb{E}\left[\frac{\sum_{j\neq i} -\alpha_E X_j}{\zeta_E(N-1)} - \frac{Q_e}{\zeta_E(N-1)} + \frac{\Delta_E}{\zeta_E}\right] = \frac{-\alpha_E\mu_X + \Delta_E}{\zeta_E} - \frac{\mu_Q}{E\zeta_E(N-1)}$$

$$\text{(H3)}\quad \text{var}\left[\sum_i X_i\right] = N\sigma_X^2 + 2\Sigma\sum_{i=1}^{N}(i-1) = N\sigma_X^2 + \Sigma(N-1)N.$$

Using the above moments we can then compute the following moments, conditional on X_i, using the projection theorem.

$$\text{(H4)}\quad \mathbb{E}\left[\frac{-\alpha_E(\sum_i X_i) + \Delta_E N - Q_{e'}}{\zeta_E N}\,\middle|\, X_i\right] = \frac{-\alpha_E\mu_X + \Delta_E}{\zeta_E} - \frac{\mu_Q}{E\zeta_E N} + \frac{\frac{1}{\zeta_E N}(-\alpha_E(N-1)\Sigma - \alpha_E\sigma_X^2 - \frac{\eta}{E})}{\sigma_X^2}(X_i - \mu_X)$$

$$\text{(H5)}\quad \mathbb{E}\left[\frac{(\sum_{j\neq i} -\alpha_E X_j) - Q_e + \Delta_E(N-1)}{\zeta_E(N-1)}\,\middle|\, X_i\right] = \frac{-\alpha_E\mu_X + \Delta_E}{\zeta_E} - \frac{\mu_Q}{E\zeta_E(N-1)} + \frac{\frac{1}{\zeta_E(N-1)}(-\alpha_E\Sigma(N-1) - \frac{\eta}{E})}{\sigma_X^2}(X_i - \mu_X)$$

$$\text{(H6)}\quad \text{var}\left[-\alpha_E(\sum_{j\neq i} X_j) + \Delta_E(N-1) - Q_{e'}\,\middle|\, X_i\right] = \alpha_E^2(N-1)\sigma_X^2 + \alpha_E^2\Sigma(N-2)(N-1) + \frac{\rho}{E^2} + \frac{\sigma_\xi^2}{E} + \frac{2\eta\alpha_E(N-1)}{E} - \frac{[(-\alpha_E\Sigma(N-1) - \frac{\eta}{E})]^2}{\sigma_X^2}$$

$$\text{(H7)}\quad \text{cov}_{X_i}\left(\sum_j -\alpha_E X_j - Q_{e'}, \sum_{j\neq i} -\alpha_E X_j - Q_e\right) = \text{var}\left[\sum_{j\neq i} -\alpha_E X_j \mid X_i\right] - 2\text{cov}_{X_i}(Q_{e'}, \sum_{j\neq i} -\alpha_E X_j) + \text{cov}_{X_i}(Q_{e'}, Q_e).$$

Using the above moments, we compute the following moments, conditional on X_i and $p_e^f - \Lambda q_{ie}^f$, (the portion of price in exchange e which is unknown to agent

i—see equation (H14)) by using the projection theorem. We have,

$$\text{(H8)}\quad \mathbb{E}\left[p^f_{e'} \,|\, p^f_e - \frac{q^f_{ie}}{\zeta_E(N-1)}, X_i\right] =$$
$$(1 - \frac{N-1}{N}\gamma_E)\frac{-\alpha_E\mu_X + \Delta_E}{\zeta_E} - (1-\gamma_E)\frac{\mu_Q}{E\zeta_E N} + \frac{N-1}{N}\gamma_E p^f_e - \gamma_E \frac{q^f_{ie}}{\zeta_E N}$$
$$+ (1-\gamma_E)\frac{\frac{1}{\zeta_E N}(-\alpha_E(N-1)\Sigma - \frac{\eta}{E})}{\sigma^2_X}(X_i - \mu_X) + \frac{-\alpha_E}{\zeta_E N}(X_i - \mu_X)$$

$$\text{(H9)}\quad \mathbb{E}\left[q^f_{ie'} \,|\, p^f_e - \frac{q^f_{ie}}{\zeta_E(N-1)}, X_i\right] =$$
$$-\alpha_E X_i \frac{N-1}{N} - (1 - \frac{N-1}{N}\gamma_E)(-\alpha_E\mu_X + \Delta_E) + (1-\gamma_E)\frac{\mu_Q}{EN} - \frac{\alpha_E}{N}\mu_X$$
$$-(1-\gamma_E)\frac{\frac{1}{N}(-\alpha_E(N-1)\Sigma - \frac{\eta}{E})}{\sigma^2_X}(X_i - \mu_X) - \frac{N-1}{N}\gamma_E\zeta_E p^f_e + \gamma_E\frac{q^f_{ie}}{N} + \Delta_E.$$

Above, γ_E denotes

$$\text{(H10)}\qquad \frac{\text{cov}_{X_i}(\sum_i -\alpha_E X_i - Q_e, \sum_{j\neq i} -\alpha_E X_j - Q_{e'})}{\text{var}[\sum_{j\neq i} -\alpha_E X_j - Q_e \,|\, X_i]}.$$

Of course, $\mathbb{E}[q^f_{ie'} \,|\, p^f_e - \frac{q^f_{ie}}{\zeta(N-1)}, X_i]$ could have been computed in one step by just a single application of the projection theorem, but we found it less algebraicly taxing to apply the projection theorem twice. To finish deriving $\mathbb{E}[q^f_{ie'} \,|\, p^f_e - \frac{q^f_{ie}}{\zeta(N-1)}, X_i]$, we must compute an expression for γ_E. The denominator was computed earlier in equation (6). To compute the numerator, we make use of the decomposition in equation (H7). The terms $\sum_{j\neq i} X_j$, Q'_e, Q_e, and X_i are jointly normally distributed with covariance matrix

$$\overline{\Sigma} = \begin{bmatrix} (N-1)\sigma^2_X + \Sigma(N-2)(N-1) & \frac{\eta(N-1)}{E} & \frac{\eta(N-1)}{E} & \Sigma(N-1) \\ \frac{\eta(N-1)}{E} & \frac{\rho}{E^2} + \frac{\sigma^2_\xi}{E} & \frac{\rho}{E^2} & \frac{\eta}{E} \\ \frac{\eta(N-1)}{E} & \frac{\rho}{E^2} & \frac{\rho}{E^2} + \frac{\sigma^2_\xi}{E} & \frac{\eta}{E} \\ \Sigma(N-1) & \frac{\eta}{E} & \frac{\eta}{E} & \sigma^2_X \end{bmatrix}.$$

The goal is to derive the covariance matrix of $\sum_{j\neq i} X_j$, $Q_{e'}$, Q_e conditional on

X_i, which we denote $\tilde{\Sigma}$. To do this we can apply the projection theorem. Then

$$\tilde{\Sigma} = \begin{bmatrix} (N-1)\sigma_X^2 + \Sigma(N-2)(N-1) & \frac{\eta(N-1)}{E} & \frac{\eta(N-1)}{E} \\ \frac{\eta(N-1)}{E} & \frac{\rho}{E^2} + \frac{\sigma_\xi^2}{E} & \frac{\rho}{E^2} \\ \frac{\eta(N-1)}{E} & \frac{\rho}{E^2} & \frac{\rho}{E^2} + \frac{\sigma_\xi^2}{E} \end{bmatrix} - \frac{1}{\sigma_X^2} \begin{bmatrix} \Sigma^2(N-1)^2 & \frac{\Sigma\eta(N-1)}{E} & \frac{\Sigma\eta(N-1)}{E} \\ \frac{\Sigma\eta(N-1)}{E} & \frac{\eta^2}{E^2} & \frac{\eta^2}{E^2} \\ \frac{\Sigma\eta(N-1)}{E} & \frac{\eta^2}{E^2} & \frac{\eta^2}{E^2} \end{bmatrix}.$$

From above, we have

$$\mathrm{cov}_{X_i}\left(-\alpha_E X_i + \sum_{j\neq i} -\alpha_E X_j - Q_{e'}, \sum_{j\neq i} -\alpha_E X_j - Q_e\right) =$$
$$\alpha_E^2\left((N-1)\sigma_X^2 + \Sigma(N-2)(N-1) - \frac{\Sigma^2(N-1)^2}{\sigma_X^2}\right) + \frac{2\alpha_E\eta(N-1)}{E}\left(1 - \frac{\Sigma}{\sigma_X^2}\right) + \frac{\rho}{E^2} - \frac{\eta^2}{E^2\sigma_X^2}.$$

We finally derive that

$$\gamma_E = \frac{\Xi}{\Xi + \frac{\sigma_\zeta^2}{E}}, \tag{H11}$$

where

$$\Xi = \alpha_E^2((N-1)\sigma_X^2 + \Sigma(N-2)(N-1)) + \frac{\rho}{E^2} + 2\frac{\eta}{E}\alpha_E(N-1) - \frac{\big(-\alpha_E\Sigma(N-1) - \frac{\eta}{E}\big)^2}{\sigma_X^2}. \tag{H12}$$

This concludes step 1.

Step 2. By market clearing, we have

$$p_e^f = \frac{-\alpha_E(\sum_i X_i) + \Delta_E N - Q_e}{\zeta_E N}. \tag{H13}$$

Also by market clearing, the residual supply curve trader i faces in exchange e

is

$$p_e(q) = \frac{-\alpha_E(\sum_{j\neq i} X_i) + q + \Delta_E(N-1) - Q_e}{\zeta_E(N-1)}. \tag{H14}$$

This implies that the price impact agent i faces in exchange e is $\Lambda := \frac{1}{\zeta_E(N-1)}$, which by symmetry, is the price impact each agent i faces in all exchanges. In equilibrium trader i equates his expected marginal utility conditional on $p_e^f - \frac{q_{ie}^f}{\zeta_E(N-1)}$ and X_i, with his marginal cost. That is

$$\mu_\pi - 2b\left(X_i + q_{ie}^f + (E-1)\mathbb{E}\left[q_{i2}^f \,|\, p_e^f - \frac{q_{ie}^f}{\zeta_E(N-1)}, X_i\right]\right) = p_e^f + \Lambda q_{ie}^f. \tag{H15}$$

Substiting equation (H9) into (H15) and matching coefficients we obtain a system of three equations which characterize the three unknowns, Δ_E, α_E, and ζ_E. We do not explicitly list the algebraic steps here. Matching the coefficients on price, we obtain

$$\zeta_E = \frac{1}{2b((E-1)\gamma_E + 1)} \frac{N-2}{N-1}. \tag{H16}$$

Matching the coefficients on X_i we obtain

$$\alpha_E = \frac{1 + \frac{E-1}{E}\frac{(1-\gamma_E)\eta}{N\sigma_X^2}}{E\gamma_E(\frac{1}{N} + \frac{1}{N-2} + \frac{N-1}{N}\frac{\Sigma}{\sigma_X^2}) + (1-\gamma_E)(\frac{1}{N} + \frac{1}{N-2} + \frac{N-1}{N}\frac{\Sigma}{\sigma_X^2}) + E\frac{N-1}{N}(1 - \frac{\Sigma}{\sigma_X^2})}. \tag{H17}$$

Matching the constant coefficients, we obtain

$$\Delta_E = \frac{\mu_\pi - 2b(E-1)\mu_X\left(\frac{(1-\gamma_E)\mu_Q}{EN\mu_X} - \frac{(1-\gamma_E)\frac{1}{N}(\alpha_E(N-1)\Sigma + \frac{\eta}{E})}{\sigma_X^2} + \alpha_E\frac{N-1}{N}(1-\gamma_E)\right)}{2b\frac{N-1}{N-2}(1 + \gamma_E(E-1))}. \tag{H18}$$

Above, γ_E, as we saw in equation (H11) is dependent on α_E. By inspecting (H17) and (H11) we see that α_E satisfies a cubic equation. This cubic equation can be derived by multiplying both sides of (H17) by the denominator on the right hand side of (H17), and then multiplying both sides by the denominator in the expression for γ_E. Note that this does not add "solutions" since the denominator in (H17) is strictly positive since $\gamma_E > -\frac{1}{E-1}$ (which can be seen by a proof analogous to that of Lemma 5) and since the denominator in the expression for γ_E is also strictly positive since it is a variance.

Step 3. By Theorem 1, equations (H17), (H16), and (H18) are necessary and sufficient conditions for $(\Delta_E, \alpha_E, \zeta_E)$ to be a symmetric affine equilibrium. To prove existence of at least one and up to three such symmetric affine equilibria, we observe that equilibrium existence is equivalent to the existence of a real root of the cubic equation that characterizes α_E. But since the equation is cubic it must have at least one real root and up to three real roots. We now prove uniqueness of the equilibrium when $\eta \geq 0$. Fix $E \geq 1$, denote $y \equiv \alpha_E$, and define

$$g(y) \equiv$$
$$y - \frac{1 + \frac{E-1}{E}\frac{(1-\gamma_E)\eta}{N\sigma_X^2}}{E\gamma_E(\frac{2N-2}{N(N-2)} + \frac{N-1}{N}\frac{\Sigma}{\sigma_X^2}) + (1-\gamma_E)(\frac{2N-2}{N(N-2)} + \frac{N-1}{N}\frac{\Sigma}{\sigma_X^2}) + E\frac{N-1}{N}(1 - \frac{\Sigma}{\sigma_X^2})}.$$

Above we view each γ_E also as a function of y as defined by (H11) with y in place of α_E. There exists a symmetric affine equilibrium for each y positive such that $g(y) = 0$. Using the assumption that $\eta \geq 0$, the second term in the above expression is strictly monotone decreasing in γ_E when $\gamma_E \in (0, 1]$. By inspecting equation (H11) (and using Lemma 5) we can show that $\gamma_E \in (0, 1]$. Moreover it is strictly monotone increasing in y. Thus $g(y)$ is strictly monotone increasing in y. Hence there can exist at most one value of $y \in \mathbb{R}$ such that $g(y) = 0$.

We now prove the remaining parts of the theorem. Part 1 follows immediately from (H16). Part 2 follows immediately from (H14). Part 3 of the theorem is true of any symmetric affine equilibrium independent of the joint distribution of the random variables and the proof is analogous to that of Theorem 1. Part 4 follows from part 3 and (H17) when substituting in $\gamma_E = 1$ which is the value γ_E takes on when $\sigma_Q^2 = 0$. To prove part 5, observe that using Proposition 8, $\gamma_E \to 0$. By equation (H17),

$$E\alpha_E = \frac{1 + \frac{(E-1)}{E}\frac{(1-\gamma_E)\eta}{N\sigma_X^2}}{\frac{1}{E} + \frac{\gamma_E(E-1)}{EN} + \frac{(E-1)\gamma_E+1}{E(N-2)} + (E-1)\frac{N-1}{EN} - (1-\gamma_E)(E-1)\frac{N-1}{EN}\frac{\Sigma}{\sigma_X^2}}.$$

Since $\gamma_E \to 0$, $E\alpha_E \to \frac{1+\frac{\eta}{N\sigma_X^2}}{\frac{N-1}{N}(1-\frac{\Sigma}{\sigma_X^2})}$.

COROLLARY 6.1: *Let $\{E\alpha_E\}_{E\in\mathbb{N}}$ be defined as in Theorem 6. Then $-El_E$ converges to a constant that exceeds 1 if and only if $\sigma_\xi^2 > 0$ and $\eta > -[\sigma_X^2 + (N-1)\Sigma]$, where, by the positive definiteness of the covariance matrix of $X_1, \ldots, X_N$, we have $\sigma_X^2 + (N-1)\Sigma \geq 0$. Further, $E\alpha_E$ converges to a constant that exceeds $\frac{N-2}{N-1}$ if and only if $\sigma_\xi^2 > 0$ and $\eta > -[2\sigma_X^2 + (N-2)\Sigma]$.*

PROOF:

Theorem 6 supplies a closed form expression for the limiting value of $E\alpha_E$ as $E \to \infty$. The rest of the proof is a simple computation.

PROPOSITION 7: *Let*

$$E^* \equiv \frac{\hat{c} - \sigma_\xi^2(\frac{1}{N} + \frac{1}{N-2} + \frac{N-1}{N}\frac{\Sigma}{\sigma_X^2}) - \sigma_\xi^2\frac{\eta}{N\sigma_X^2}}{\sigma_\xi^2\frac{N-1}{N}(1-\frac{\Sigma}{\sigma_X^2}) - \sigma_\xi^2(1+\frac{\eta}{N\sigma_X^2})},$$

where

$$\hat{c} \equiv$$
$$-\frac{(N-1)\sigma_X^2 + \Sigma(N-2)(N-1) - \frac{\Sigma^2(N-1)^2}{\sigma_X^2} + 2\eta(N-1)(1-\frac{\Sigma}{\sigma_X^2}) + \rho - \frac{\eta^2}{\sigma_X^2}}{N-2}.$$

If E^* is in $\mathbb{N}$, there is a unique symmetric affine equilibrium when $E = E^*$ whose allocation is the efficient allocation. If $\eta \geq 0$, by Theorem 6, there is a unique symmetric affine equilibrium allocation associated with each $E \in \mathbb{N}$. The $E \in \mathbb{N}$ whose symmetric affine equilibrium is most efficient is either $\lfloor E^* \rfloor$ or $\lceil E^* \rceil$.

PROOF:

Let $(\alpha_E, \zeta_E, \Delta_E)$ denote an arbitrary symmetric affine equilibrium. Define $g_E \equiv E\alpha_E$. Substituting equation (H11) into (H17) and rearranging yields a cubic equation in g_E with coefficients

$$[g_E^3] : A(1 + \frac{1}{N-2})$$

$$[g_E^2] : B(1 + \frac{1}{N-2}) - A$$

$$[g_E] : F(1 + \frac{1}{N-2}) + \sigma_\xi^2(\frac{1}{N} + \frac{1}{N-2} + \frac{N-1}{N}\frac{\Sigma}{\sigma_X^2}) + \sigma_\xi^2 E\frac{N-1}{N}(1 - \frac{\Sigma}{\sigma_X^2}) - B$$

$$[1] : -F - E\sigma_\xi^2(1 + \frac{\eta}{N\sigma_X^2}) + \sigma_\xi^2\frac{\eta}{N\sigma_X^2},$$

where

$$A \equiv ((N-1)\sigma_X^2 + \Sigma(N-2)(N-1) - \frac{\Sigma^2(N-1)^2}{\sigma_X^2}),$$

$$B \equiv 2\eta(N-1)(1 - \frac{\Sigma}{\sigma_X^2})$$

and

$$F \equiv \rho - \frac{\eta^2}{\sigma_X^2}.$$

By definition, at E^*, $g_{E^*} = 1$. Therefore, we have

$$A(1+\frac{1}{N-2}) + B(1+\frac{1}{N-2}) - A + F(1+\frac{1}{N-2}) \\ + \sigma_\xi^2(\frac{1}{N} + \frac{1}{N-2} + \frac{N-1}{N}\frac{\Sigma}{\sigma_X^2}) + \sigma_\xi^2 E^* \frac{N-1}{N}(1-\frac{\Sigma}{\sigma_X^2}) - B - F \\ - E^*\sigma_\xi^2(1+\frac{\eta}{N\sigma_X^2}) + \sigma_\xi^2\frac{\eta}{N\sigma_X^2} = 0.$$

Solving for E^* we obtain,

$$E^* = \frac{-\frac{A+B+F}{N-2} - \sigma_\xi^2(\frac{1}{N} + \frac{1}{N-2}\frac{N-1}{N}\frac{\Sigma}{\sigma_X^2}) - \sigma_\xi^2\frac{\eta}{N\sigma_X^2}}{\sigma_\xi^2\frac{N-1}{N}(1-\frac{\Sigma}{\sigma_X^2}) - \sigma_\xi^2(1+\frac{\eta}{N\sigma_X^2})}.$$

That the $E \in \mathbb{N}$ whose symmetric affine equilibrium allocation is most efficient is either $\lfloor E^* \rfloor$ or $\lceil E^* \rceil$ when $\eta \geq 0$ follows from Proposition 11.

PROPOSITION 8: *For each $E \in \mathbb{N}$ denote an arbitrary corresponding symmetric affine equilibria, $\{(\Delta_E, \alpha_E, \zeta_E)\}_{e\in E}$. Let Λ_E be the corresponding equilibrium price impact and γ_E the equilibrium inference coefficient. Then, if $\sigma_\xi^2 = 0$, $\{\Lambda_E\}_{E\in\mathbb{N}}$ diverges to ∞ and $\{\gamma_E\}_{E\in\mathbb{N}}$ is the constant sequence of ones. If $\sigma_\xi^2 > 0$, $\{\Lambda_E\}_{E\in\mathbb{N}}$ converges to*

$$\frac{1+c^*}{\frac{1}{2b}(N-2)},$$

where

$$c^* = \frac{1}{\sigma_\xi^2}\left[(\frac{1+\frac{\eta}{N\sigma_X^2}}{\frac{N-1}{N}(1-\frac{\Sigma}{\sigma_X^2})})^2((N-1)\sigma_X^2 + \Sigma(N-2)(N-1) - \frac{\Sigma^2(N-1)^2}{\sigma_X^2}) \\ + 2N(1+\frac{\eta}{N\sigma_X^2})\eta + \rho - \frac{\eta^2}{\sigma_X^2}\right],$$

while $\{\gamma_E\}_{E\in\mathbb{N}}$ converges to 0.

PROOF:

The claims when $\sigma_\xi^2 = 0$ are obvious in light of Theorem 6. We prove the claims when $\sigma_\xi^2 > 0$. By inspecting equation (H17), and recognizing that Lemma 5 implies that $\frac{1}{N} + \frac{1}{N-2} + \frac{N-1}{N}\frac{\Sigma}{\sigma_X^2} > 0$, and that $\gamma_E > -\frac{1}{E-1}$ we see that

$$|\,\alpha_E\,| < \frac{1 + \frac{E-1}{E}\frac{(1-\gamma_E)\,|\,\eta\,|}{N\sigma_X^2}}{E\frac{N-1}{N}(1-\frac{\Sigma}{\sigma_X^2})}.$$

Thus inspecting the equation (H11), we see that for large E, the numerator of γ_E is $O(\frac{1}{E^2})$ while the denominator is $\omega(\frac{1}{E^2})$ so that $\gamma_E \to 0$. To prove that Λ_E converges and compute its limit point, we can use (H11) to derive an explicit expression for $E\gamma_E$ and then use part 5 of Theorem 6. We find that Λ_E converges to

$$\frac{1+c^*}{\frac{1}{2b}(N-2)}.$$

PROPOSITION 9: *Suppose $\eta \geq 0$. For each $E \in \mathbb{N}$, let Λ_E denote the equilibrium price impact in the unique symmetric affine equilibrium. The sequence, $\{-\Lambda_E\}_{E\in\mathbb{N}}$, is strictly monotone increasing.*

PROOF:

The proof is analogous to that of Proposition 1.

PROPOSITION 10: *The total expected payment of liquidity traders is*

$$\frac{N-1}{N}\Lambda_E(-\mu_Q N\Delta_E + \sigma_\xi^2 + \frac{\rho+\mu_Q^2}{E} - \alpha_E N(\eta + \mu_X\mu_Q)).$$

PROOF:

We compute

$$\begin{aligned} -\mathbb{E}\left[\sum_{e\in E} p_e^* Q_e\right] &= -\frac{N-1}{N}\Lambda_E \mathbb{E}\left[\sum_{e\in E}(\sum_{i\in N} -\alpha_E X_i + N\Delta_E - Q_e)Q_e\right] \\ &= \frac{N-1}{N}\Lambda_E(-\mu_Q N\Delta_E + \sigma_\xi^2 + \frac{\rho+\mu_Q^2}{E} + \alpha_E N(\eta+\mu_X\mu_Q)). \end{aligned}$$

PROPOSITION 11: *Suppose $\sigma_\xi^2 > 0$ and $\eta \geq 0$. For each, $E \in \mathbb{N}$, denote the unique symmetric affine equilibrium, $(\Delta_E, \alpha_E, \zeta_E)$. The sequence, $\{E\alpha_E\}_{E\in\mathbb{N}}$, is strictly monotone increasing.*

PROOF:

The proof is analogous to that of part 6 of Theorem 1.