Online Appendix to "Dynamic Persuasion with Outside Information" by Jacopo Bizzotto, Jesper Rüdiger and Adrien Vigier

Online Appendix A: Proof of Proposition 1

In this appendix we prove equilibrium existence and uniqueness (Proposition 1). We start with a very general result that will be used repeatedly in this and the next appendices.

Proposition A.1. *Let $t < T$. If $\phi : [0,1] \to \mathbb{R}$ is convex (respectively concave) then, irrespective of the signal-generating process, $\mathbb{E}_{s_t}[\phi(p_{t+1})|q_t]$ is convex (resp. concave) in q_t.*

Proof of Proposition A.1: Consider an arbitrary signal-generating process, with realizations in $S = \{s^i\}_{i=1}^K$. Let $\gamma_{Gi} := \mathbb{P}(s_t = s^i | \omega = G)$, $\gamma_{Bi} := \mathbb{P}(s_t = s^i | \omega = B)$, and $p^i(q) := \frac{q\gamma_{Gi}}{q\gamma_{Gi} + (1-q)\gamma_{Bi}}$. To shorten notation, in what follows we use p^i to refer to $p^i(q_t)$.

We have, with this notation,

$$\mathbb{E}_{s_t}[\phi(p_{t+1})|q_t] = \sum_{i=1}^K \mathbb{P}(s_t = s^i | q_t)\phi(p^i).$$

Thus

$$\frac{d\mathbb{E}_{s_t}[\phi(p_{t+1})|q_t]}{dq_t} = \sum_{i=1}^K \left(\frac{d\mathbb{P}(s_t = s^i | q_t)}{dq_t}\phi(p^i) + \mathbb{P}(s_t = s^i | q_t)\frac{d\phi(p^i)}{dp^i}\frac{dp^i}{dq_t} \right),$$

while

$$\begin{aligned}
\frac{d^2\mathbb{E}_{s_t}[\phi(p_{t+1})|q_t]}{dq_t^2} = \sum_{i=1}^K \Bigg(&\frac{d^2\mathbb{P}(s_t = s^i | q_t)}{dq_t^2}\phi(p^i) \\
&+ \left(2\frac{d\mathbb{P}(s_t = s^i | q_t)}{dq_t}\frac{dp^i}{dq_t} + \mathbb{P}(s_t = s^i | q_t)\frac{d^2p^i}{dq_t^2} \right)\frac{d\phi(p^i)}{dp^i} \\
&+ \mathbb{P}(s_t = s^i | q_t)\frac{d^2\phi(p^i)}{d(p^i)^2}\left(\frac{dp^i}{dq_t} \right)^2 \Bigg).
\end{aligned} \tag{A.1}$$

Notice that: (i) $\mathbb{P}(s_t = s^i | q_t) = q_t\gamma_{Gi} + (1-q_t)\gamma_{Bi}$, thus $\frac{d^2\mathbb{P}(s_t=s^i|q_t)}{dq_t^2} = 0$; moreover, (ii) $\frac{dp^i}{dq_t} = \frac{\gamma_{Gi}\gamma_{Bi}}{(\mathbb{P}(s_t=s^i|q_t))^2}$ and $\frac{d^2p^i}{dq_t^2} = -2\frac{\gamma_{Gi}\gamma_{Bi}}{(\mathbb{P}(s_t=s^i|q_t))^3}\frac{d\mathbb{P}(s_t=s^i|q_t)}{dq_t}$, implying

$$2\frac{d\mathbb{P}(s_t = s^i | q_t)}{dq_t}\frac{dp^i}{dq_t} + \mathbb{P}(s_t = s^i | q_t)\frac{d^2p^i}{dq_t^2} = 0.$$

In light of (i) and (ii), (A.1) reduces to

$$\frac{d^2 \mathbb{E}_{s_t}\left[\phi(p_{t+1})|q_t\right]}{dq_t^2} = \sum_{i=1}^{K}\left(\mathbb{P}(s_t = s^i|q_t)\frac{d^2\phi(p^i)}{d(p^i)^2}\left(\frac{dp^i}{dq_t}\right)^2\right).$$

As $\mathbb{P}(s_t = s^i|q_t)\left(\frac{dp_t^i}{dq_t}\right)^2 \geq 0$, if ϕ is convex (respectively concave) then $\mathbb{E}_{s_t}\left[\phi(p_{t+1})|q_t\right]$ is convex (resp. concave) as well. ∎

The continuation game starting in period T with beginning-of-period-T belief p_T is identical to the static Bayesian persuasion game of Kamenica and Gentzkow (2011). We summarize some of their main results in the following lemma.

Lemma A.1. *In equilibrium, at $t = T$, the agent accepts if $q_T \geq \underline{b}$ and rejects otherwise. The principal designs the experiment*

$$M_T = \begin{cases} \{0, \underline{b}\} & \text{if } p_T \in (0, \underline{b}); \\ \{p_T\} & \text{otherwise.} \end{cases}$$

The agent does not benefit from the period-T experiment, hence his equilibrium continuation payoff is convex in p_T.[1]

Lemma A.2. *Let $t < T$. Suppose that functions $\hat{g}_{t+1}(p_{t+1})$ and $\hat{f}_{t+1}(p_{t+1})$ uniquely determine the agent's (resp. the principal's) equilibrium continuation payoffs in period $t+1$. If $\hat{g}_{t+1}$ is convex, then:*

1. *in equilibrium, the principal's period-t experiment and the agent's period-t decision are both uniquely determined; the former is a function of p_t only and the latter is a function of q_t only;*

2. *functions $\hat{g}_t(p_t)$ and $\hat{f}_t(p_t)$ uniquely determine the equilibrium continuation payoffs in period t, and $\hat{g}_t$ is convex.*

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[1]The equilibrium period-T experiment generates information that has no value for the agent, since rejecting is an optimal choice for $q_T = 0$ as well as for $q_T = \underline{b}$. The agent's equilibrium continuation payoffs at the beginning of period T are thus given by the convex function $\hat{g}_T(p_T) = \max\{V_R, V_B + p_T(V_G - V_B)\}$.

Proof: Define

$$\tilde{g}_t(q_t) := \mathbb{E}_{s_t}\left[\hat{g}_{t+1}(p_{t+1})|q_t\right], \tag{A.2}$$

and

$$\tilde{f}_t(q_t) := \mathbb{E}_{s_t}[\hat{f}_{t+1}(p_{t+1}) \mid q_t]. \tag{A.3}$$

Then the agent's equilibrium continuation payoff given q_t can be written as

$$g_t(q_t) = \max\left\{V_R,\ \delta\tilde{g}_t(q_t),\ V_B + q_t(V_G - V_B)\right\}. \tag{A.4}$$

As $\hat{g}_{t+1}$ is convex by assumption, Proposition A.1 shows that $\tilde{g}_t$ is convex as well. Moreover,

$$\begin{cases} \delta\tilde{g}_t(0) = \delta V_R < V_R; \\ \delta\tilde{g}_t(1) = \delta V_G < V_G. \end{cases} \tag{A.5}$$

Then (A.4), (A.5) and convexity of $\tilde{g}_t$ give unique a_t and b_t, with $a_t \le \underline{b} \le b_t$, such that

$$\begin{cases} g_t(q_t) = V_R > \max\left\{\delta\tilde{g}_t(q_t),\ V_B + q_t(V_G - V_B)\right\} & \text{if } q_t < a_t; \\ g_t(q_t) = \delta\tilde{g}_t(q_t) > \max\left\{V_R,\ V_B + q_t(V_G - V_B)\right\} & \text{if } q_t \in (a_t, b_t); \\ g_t(q_t) = V_B + q_t(V_G - V_B) > \max\left\{V_R,\ \delta\tilde{g}_t(q_t)\right\} & \text{if } q_t > b_t. \end{cases}$$

Hence, in equilibrium, the agent rejects if $q_t < a_t$, waits if $q_t \in (a_t, b_t)$, and accepts if $q_t > b_t$. Moreover, since in equilibrium whenever indifferent the agent makes the decision preferred by the principal, the agent waits if $q_t = a_t < b_t$ and accepts if $q_t = b_t$. Hence:

$$f_t(q_t) = \begin{cases} 0 & \text{if } q_t < a_t; \\ \delta\tilde{f}_t(q_t) & \text{if } q_t \in [a_t, b_t); \\ 1 & \text{if } q_t \ge b_t. \end{cases} \tag{A.6}$$

Standard arguments yield

$$\hat{f}_t = \text{cav } f_t. \tag{A.7}$$

Furthermore, since in equilibrium whenever indifferent the principal picks the least informative experiment, the principal's experiment in period t is uniquely determined by the belief p_t at the beginning of period t. Lastly, letting $\tau_t(p_t)$ denote the principal's equilibrium experiment given p_t yields $\hat{g}_t(p_t) = \mathbb{E}_{\tau_t(p_t)}[g_t(q_t)|p_t]$.

Finally, since $\tilde{g}_t$ is convex, (A.4) shows that g_t is as well which, in turn, implies

$$\hat{g}_t(p_t) = \mathbb{E}_{\tau_t(p_t)}[g_t(q_t)|p_t]. \qquad\qquad (A.8)$$

The properties of $\tau_t(\cdot)$ implied by (A.7) finish to establish that $\hat{g}_t$ is convex, since $\hat{g}_t$ is given by (A.8) and g_t is convex. ∎

Proof of Proposition 1: The proposition follows from Lemmata A.1 and A.2. ∎

Online Appendix B: Proof of Theorem 1

In this appendix we prove the steps leading to Theorem 1, including all lemmas of Section II except for Lemma 4, whose proof was kept in the text. The order in which we prove the results is as follows: Lemma 1, 5, 6, 7, 2, 3 and 8. The proof of Proposition B.1 concludes the appendix.

Proof of Lemma 1: For $t = T$, the result follows from Lemma A.1. For $t < T$, the result was shown within the proof of Lemma A.2. ∎

Lemma B.1. *At all $t \leq T-1$, the threshold of acceptance is at least as large as the threshold of acceptance at $t = T-1$, which itself is at least as large as the static threshold of acceptance: $b_t \geq b_{T-1} \geq b_T = \underline{b}$.*

Proof: The result follows from the arguments in the text above the statement of Lemma 2. ∎

Lemma B.2. *Each period, in equilibrium $M_t = \{p_t\}$ for all $p_t \in \{0\} \cup [b_t, 1]$. Moreover:*

(i) *either $M_t = \{0, b_t\}$ for all $p_t \in (0, b_t)$,*

(ii) *or $a_t < b_t$ and: $M_t = \{0, a_t\}$ for $p_t \in (0, a_t)$ and there exists $c_t \in [a_t, b_t)$ such that $M_t = \{p_t\}$ for $p_t \in [a_t, c_t]$, while $M_t = \{c_t, b_t\}$ for $p_t \in (c_t, b_t)$.*

Proof: Recall (A.7). If $a_t = b_t$ (so that the set of beliefs for which in equilibrium the agent waits in period t is empty) then in equilibrium $M_t = \{0, b_t\}$ for all $p_t \in (0, b_t)$. So assume $a_t < b_t$. Observe that:

(A) $\tilde{f}_t(\cdot)$ (defined by (A.3)) is concave,

(B) $f_t(q_t) = \delta \tilde{f}_t(q_t)$ for all $q_t \in \{0\} \cup [a_t, b_t)$.

(A) follows from Proposition A.1 while (B) is obtained from (A.6). In view of (A)-(B), either (i) in the statement of the lemma holds, or (ii) does. ∎

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Proof of Lemma 5: If $a_{T-1} = b_{T-1} = b_T$, the claim of the lemma is straightforward.[2] Assume now $b_{T-1} > b_T$. At $q_{T-1} = b_{T-1}$, in equilibrium the agent is indifferent between waiting and accepting. The agent's expected payoff from accepting is $b_{T-1}V_G + (1 - b_{T-1})V_B$. On the other hand, using Lemmata 1 and B.1, the agent's expected payoff from waiting can be written as $\delta[b_{T-1}V_G + (1 - b_{T-1})(\gamma V_R + (1 - \gamma)V_B)]$. So b_{T-1} is the unique solution of

$$xV_G + (1 - x)V_B = \delta[xV_G + (1 - x)(\gamma V_R + (1 - \gamma)V_B)]. \tag{B.1}$$

Next, consider $t < T - 1$ such that $b_{t+1} = b_{T-1}$. Suppose $q_t = b_t$, so that, by definition, in equilibrium the agent is indifferent between waiting and accepting. The agent's expected payoff from accepting is $b_t V_G + (1 - b_t)V_B$. On the other hand, using Lemma B.1, $q_t = b_t \geq b_{T-1} = b_{t+1}$. Hence, conditional on $s_t = g$, the agent optimally accepts in the next period. It ensues that b_t solves (B.1) and, therefore, that $b_t = b_{T-1}$. A recursive argument then yields $b_t = b_{T-1}$ for all $t < T$. ∎

Proof of Lemma 6: Suppose that in equilibrium the principal is aggressive in period $1 < t + 1 < T$. If $a_t = b_t$ the statement of the lemma is straightforward. Assume therefore that $a_t < b_t$. By virtue of Lemma B.2, in order to establish that the principal is also aggressive in period t it is enough to show that, at $p_t = a_t$, the principal strictly prefers the experiment $M_t = \{0, b_t\}$ over the uninformative experiment. On one hand, the principal's expected payoff from designing $M_t = \{0, b_t\}$ is $\frac{a_t}{b_t}$. On the other hand, her expected payoff from designing the uninformative experiment is given by $\delta \mathbb{E}_{s_t}[\hat{f}_{t+1}(p_{t+1}) \mid q_t = a_t]$. The next sequence of inequalities therefore concludes the proof:

$$\delta \mathbb{E}_{s_t}[\hat{f}_{t+1}(p_{t+1}) \mid q_t = a_t] \leq \delta \hat{f}_{t+1}(a_t) = \delta \frac{a_t}{b_{t+1}} < \frac{a_t}{b_t}.$$

The first inequality follows from noting that $\hat{f}_{t+1}$ is concave; the equality follows from the assumption that the principal is aggressive in period $t + 1$, and the second inequality is due to Lemma 5. ∎

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[2] $a_{T-1} = b_{T-1} = b_T = \underline{b}$ implies $a_{t-1} = b_{t-1} = b_t$ whenever $a_t = b_t = \underline{b}$. Hence, a recursive argument yields the result in this case.

Proof of Lemma 7: Note that, in view of Lemma 6, it is enough to show that in equilibrium the principal is aggressive at $t=1$ when T is sufficiently large. Next, Lemma 5 shows that any benefit to the principal from being conservative at $t=1$ must come from persuading the agent to accept at $t=T$ when $\omega=B$. So these benefits are bounded from above by δ^{T-1}, which tends to 0 as $T\to\infty$. On the other hand, as $b_1<1$, the corresponding loss to the principal is bounded away from zero since by being aggressive at $t=1$ the principal obtains acceptance with strictly positive probability conditional on $\omega=B$. We conclude that, for T sufficiently large, in equilibrium the principal is aggressive at $t=1$. $\blacksquare$

Proof of Lemma 2: In the perfect bad news case, the result follows from Lemma 5. In the perfect good news case, the result is easily obtained by induction using Lemma 4. $\blacksquare$

Lemma B.3. *Consider a period $t<T$ and let a_t^+ denote the beginning-of-period-$t+1$ belief given $q_t=a_t$ and $s_t=g$. Then $a_t^+>a_{t+1}$.*

Proof: The result is trivial if $a_t=b_t$, so suppose $a_t<b_t$. Assume by way of contradiction that $a_t^+\le a_{t+1}$. By definition of a_t, in equilibrium, at $q_t=a_t$ the agent is indifferent between waiting and rejecting; the corresponding remark applies to period $t+1$. Therefore $a_t^+\le a_{t+1}$ implies that, by waiting at $q_t=a_t$, the agent's expected continuation payoff is as if the agent rejected with probability 1 in period $t+1$. As $V_R>0$, rejecting in period t thus yields the agent a strictly higher expected continuation payoff than waiting. This remark contradicts the definition of a_t. $\blacksquare$

Lemma B.4. *Consider a period $t<T-1$ such that, in equilibrium, in period t the principal is not aggressive. Let c_t^+ denote the beginning-of-period-$t+1$ belief given $q_t=c_t$ and $s_t=g$. With perfect bad news, $c_t^+<b_{t+1}$.*

Proof: Suppose by way of contradiction that $c_t^+\ge b_{t+1}$. Then, using Lemma 5, given $p_t=c_t$, in equilibrium the experiment $M_t=\{0,b_t\}$ gives the principal a strictly larger expected continuation payoff than the uninformative experiment. This cannot be, since by definition of c_t (Lemma B.2) the uninformative experiment has to be optimal at $p_t=c_t$. $\blacksquare$

Lemma B.5. *Consider a period $t < T-1$ such that, in equilibrium, in period t the principal is not aggressive. With perfect bad news, $c_{t+1} = a_{t+1}$ implies $c_t = a_t$.*

Proof: Assume that the conditions in the statement of the lemma hold. Recall to begin that $c_t \geq a_t$, by definition. Suppose by way of contradiction that $c_t > a_t$. Our goal will be to show that, given $p_t = c_t$, in equilibrium the experiment $M_t = \{a_t, b_t\}$ yields the principal strictly larger expected continuation payoff than the uninformative experiment, contradicting the definition of c_t.

As a preliminary step, notice that, since in equilibrium the principal is not aggressive in period $t+1$ (Lemma 6), at $p_{t+1} = a_{t+1}$ the principal must weakly prefer the uninformative experiment over $M_{t+1} = \{0, b_{t+1}\}$ (Lemma B.2), that is,

$$f_{t+1}(a_{t+1}) \geq \frac{a_{t+1}}{b_{t+1}} f_{t+1}(b_{t+1}) + \left(1 - \frac{a_{t+1}}{b_{t+1}}\right) f_{t+1}(0). \tag{B.2}$$

Next, Lemmata B.3 and B.4 together imply $c_t^+ \in (a_{t+1}, b_{t+1})$. Thus, using Lemma B.2, at $p_t = c_t$, in equilibrium the principal's expected continuation payoff from the uninformative experiment can be expressed as $\delta \mathbb{E}[f_{t+1}(X)]$, where X is a random variable with mean c_t and support $\{0, a_{t+1}, b_{t+1}\}$, and $\mathbb{P}(X=0) = (1-c_t)\gamma$. Call this remark A.

Now, at $p_t = c_t$, reasoning similarly as above and using $b_{t+1} = b_t$ (Lemma 5) establishes that the principal's expected continuation payoff from designing the experiment $M_t = \{a_t, b_t\}$ can be bounded from below (strictly) by $\delta \mathbb{E}[f_{t+1}(Y)]$, where Y is a random variable with mean c_t and support $\{0, a_{t+1}, b_{t+1}\}$, and $\mathbb{P}(Y=0) = \frac{b_t - c_t}{b_t - a_t}(1-a_t)\gamma$. Call this remark B.

The last step of the proof is as follows. First, straightforward algebra shows $1 - c_t > \frac{b_t - c_t}{b_t - a_t}(1-a_t)$. Hence, $\mathbb{P}(X=0) > \mathbb{P}(Y=0)$. As X and Y have the same mean and are both supported on $\{0, a_{t+1}, b_{t+1}\}$, we conclude that X is a mean-preserving spread of Y. Inequality (B.2) then implies $\delta \mathbb{E}[f_{t+1}(X)] \leq \delta \mathbb{E}[f_{t+1}(Y)]$. Hence, combining remarks A and B, given $p_t = c_t$, in equilibrium the experiment $M_t = \{a_t, b_t\}$ yields the principal strictly larger expected continuation payoff than the uninformative experiment, contradicting the definition of c_t. ∎

Lemma B.6. *With perfect bad news, in any period such that in equilibrium the principal is not aggressive, $c_t = a_t$.*

Proof: Consider t such that, in equilibrium, the principal is not aggressive in period t. Then, by Lemma 6, the principal is not aggressive in period $T-1$. Moreover, using Lemma B.3, for any $q_{T-1} \in [a_{T-1}, b_{T-1})$, the agent waits and: (i) accepts following $s_{T-1} = g$, (ii) rejects following $s_{T-1} = b$. Thus f_{T-1} is linear in q_{T-1} over the belief interval $[a_{T-1}, b_{T-1})$. It ensues that $c_{T-1} = a_{T-1}$. Reasoning by induction using Lemma B.5 then establishes $c_t = a_t$. ∎

Proof of Lemma 3: By the definitions of aggressive and conservative, Lemma 3 follows from Lemma B.2 if we can show that, in equilibrium, each period either (a) the principal is aggressive or (b) $c_t = a_t$. Lemma 4 shows that (a) always holds in the perfect good news case. In the perfect bad news case, by Lemma B.6, either (a) holds or (b) does. ∎

Lemma B.7. *In equilibrium, conditional on $\omega = G$ the agent accepts with probability 1.*

Proof: Lemma 1 ensures that in equilibrium the agent rejects in period t if and only if $q_t \in [0, a_t)$. Lemma 3 ensures that in equilibrium $q_t \notin (0, a_t)$. Thus if the agent rejects in period t, it must be the case that $q_t = 0$. ∎

Lemma B.8. *Let $\tilde{\gamma}(\delta) := \left(\frac{1-\delta}{\delta}\right) \frac{V_R(V_G - V_B)}{(V_G - V_R)(V_R - V_B)}$. With perfect bad news, $\gamma > \tilde{\gamma}(\delta)$ if and only if $b_{T-1} > \underline{b}$; in this case,*

$$b_{T-1} = \frac{\delta(\gamma V_R + (1-\gamma)V_B) - V_B}{(V_G - V_B)(1-\delta) + \delta\gamma(V_R - V_B)}. \tag{B.3}$$

Proof: The arguments in the proof of Lemma A.2 show that $b_{T-1} > \underline{b}$ if and only if given $q_{T-1} = \underline{b}$ the agent strictly prefers waiting over rejecting, that is, if and only if

$$\delta \left[\underline{b} V_G + (1 - \underline{b})(\gamma V_R + (1-\gamma)V_B)\right] > V_R,$$

which, upon rearrangement, yields $\gamma > \tilde{\gamma}(\delta)$. Solving (B.1) gives (B.3). ∎

Proof of Lemma 8: By Lemmata 5 and B.8, if $\gamma \le \tilde{\gamma}(\delta)$ then in equilibrium $a_t = b_t = \underline{b}$ in every period, and so the principal is aggressive in period $T-1$. Since $\tilde{\gamma}(\delta) > 1$ for δ sufficiently

small, we conclude that, in equilibrium, for δ small enough the principal is aggressive in period $T-1$ regardless of γ.

Suppose next that $\gamma > \tilde{\gamma}(\delta)$. Then at $q_{T-1} = a_{T-1}$, in equilibrium the agent is indifferent between rejecting and waiting. Hence, $V_R = \delta[a_{T-1}V_G + (1-a_{T-1})(\gamma V_R + (1-\gamma)V_B)]$, giving

$$a_{T-1} = \frac{V_R - \delta(\gamma V_R + (1-\gamma)V_B)}{\delta(V_G - \gamma V_R - (1-\gamma)V_B)} \tag{B.4}$$

after rearrangement. Now, using Lemma B.2, the necessary and sufficient condition for the principal to not be aggressive in period $T-1$ in equilibrium is $f_{T-1}(a_{T-1}) \geq \frac{a_{T-1}}{b_{T-1}}$. Noting that $f_{T-1}(a_{T-1}) = \delta[a_{T-1} + (1-\gamma)(1-a_{T-1})]$ and substituting for a_{T-1} and b_{T-1} using (B.3) and (B.4), the former inequality becomes

$$V_B\delta(\gamma-1) + \delta V_G(1-\gamma) + V_R\gamma(1-\delta) \tag{B.5}$$
$$\geq \frac{[V_B\delta(\gamma-1) + V_R(1-\delta\gamma)][\delta\gamma(V_B - V_R) + (V_B - V_G)(1-\delta)]}{\delta[(V_B - V_R)\delta\gamma + V_B(1-\delta)]}.$$

One checks that for $\delta = 1$ the quadratic equation in γ obtained from (B.5) has roots $\gamma = 0$ and $\gamma = 1$. On the other hand, for $\delta < 1$, (B.5) is violated when either $\gamma = 1$, or $\gamma = \tilde{\gamma}(\delta)$. So (B.5) holds for all values of γ in between the roots of the corresponding quadratic equation. Letting $\underline{\gamma}(\delta)$ and $\overline{\gamma}(\delta)$ denote the real roots, the previous remarks yield $\tilde{\gamma}(\delta) < \underline{\gamma}(\delta) \leq \overline{\gamma}(\delta) < 1$ and show that these roots only exist for $\delta > \underline{\delta}$, where $\underline{\delta} > 0$ is defined implicitly by $\underline{\gamma}(\underline{\delta}) = \overline{\gamma}(\underline{\delta})$. Noting that, by Lemma 3, whenever the principal is not aggressive she is conservative concludes the proof. ∎

Proposition B.1. *Let b_t^G (respectively b_t^B) denote the period-t threshold of acceptance under perfect good news (resp. perfect bad news). Then $b_t^G \leq b_t^B$ for all t, with $b_T^G = b_T^B = \underline{b}$.*

Proof: First, notice that, under perfect bad news, b_{T-1}^B satisfies the following fixed-point property: at $q_{T-1} = b_{T-1}^B$ the agent is indifferent between (a) accepting and (b) making his final decision next period given that next period the principal designs $M_T = \{0, b_{T-1}^B\}$ if $p_T \in (0, b_{T-1}^B)$ and $M_T = \{b_{T-1}^B, 1\}$ if $p_T \in (b_{T-1}^B, 1)$. Next, let X denote the random variable representing the belief at which, under perfect bad news, the agent makes his final decision following scenario (b). Then $\text{supp}(X) = \{0, b_{T-1}^B, 1\}$, $\mathbb{E}[X] = b_{T-1}^B$ and $\mathbb{P}(X=0) = (1 - b_{T-1}^B)\gamma$. Straightforward algebra therefore yields $\mathbb{P}(X=1) = b_{T-1}^B\gamma$ and $\mathbb{P}(X = b_{T-1}^B) = 1 - \gamma$.

We claim that, under perfect good news, b_{T-1}^G satisfies the same fixed-point property as above. To see this, consider $q_{T-1} = b_{T-1}^G$ and let Y denote the random variable representing the belief at which, under perfect good news, the agent makes his final decision following scenario (b). Then $\text{supp}(Y) = \{0, b_{T-1}^G, 1\}$, $\mathbb{E}[Y] = b_{T-1}^G$ and $\mathbb{P}(Y=1) = b_{T-1}^G \gamma$. Straightforward algebra yields $Y \sim X$. The claim ensues.

We now show by induction that $b_t^G \leq b_t^B$ for all $t < T$ (the case $t = T$ is obvious). For $t = T-1$ the result immediately follows from the claim above. Next, suppose $b_t^G \leq b_t^B$ for $1 < t < T$. By Lemma 5: $b_t^B = b_{T-1}^B$. Hence $b_t^G \leq b_{T-1}^B$. But then the claim above implies that, under perfect good news, at $q_{t-1} = b_{T-1}^B$ in equilibrium the agent must weakly prefer accepting over waiting. So $b_{t-1}^G \leq b_{T-1}^B = b_{t-1}^B$. ∎

Online Appendix C: Technical Appendix for Subsection II.B

Pareto Efficiency

Define an outcome Z by two probability distributions over $\{\text{accept}, \text{reject}\} \times \{1, \ldots, T\}$, specifying, respectively, the probability $Z(x, t \,|\, G)$ that the agent makes the final decision x in period t when $\omega = G$, and the corresponding probability $Z(x, t \,|\, B)$ when $\omega = B$.

Proposition C.1. *An outcome Z is Pareto efficient if and only if (i) $Z(accept, 1 \,|\, G) = 1$, and (ii) $Z(accept, 1 \,|\, B) + Z(reject, 1 \,|\, B) = 1$.*

Proof: Let $\mathcal{Z}^*$ denote the set of Pareto efficient outcomes and $\mathcal{Z}^\dagger$ the set of outcomes satisfying conditions (i) and (ii) in the statement of the lemma.

We start by showing that $\mathcal{Z}^* \subseteq \mathcal{Z}^\dagger$. First, as (a) $V_R > 0$ and (b) the principal gets 0 from rejection:

$$Z \in \mathcal{Z}^* \Rightarrow Z(\text{reject}, t | \omega) = 0, \ \forall t > 1. \tag{C.1}$$

Second, as (a) the principal prefers acceptance over rejection and (b) $V_G > V_R$:

$$Z \in \mathcal{Z}^* \Rightarrow Z(\text{reject}, t | G) = 0, \ \forall t. \tag{C.2}$$

Third, since (a) $V_G > 0$ and (b) $\delta < 1$:

$$Z \in \mathcal{Z}^* \Rightarrow Z(\text{accept}, 1 | G) = 1. \tag{C.3}$$

Fourth, we claim that

$$Z \in \mathcal{Z}^* \Rightarrow Z(\text{accept}, t | B) = 0, \ \forall t > 1. \tag{C.4}$$

Suppose for a contradiction that this is not the case and that we can find $Z \in \mathcal{Z}^*$ and $\hat{t} > 1$ such that $Z(\text{accept}, \hat{t} | B) > 0$. Define $\tilde{Z}$ as follows:

(i) $\tilde{Z}(\cdot, \cdot | G) = Z(\cdot, \cdot | G)$,

(ii) $\tilde{Z}(\cdot, t | B) = Z(\cdot, t | B), \ \ \forall t \notin \{1, \hat{t}\}$,

(iii) $\tilde{Z}(\text{accept}, \hat{t} | B) = 0$,

(iv) $\tilde{Z}(\text{accept}, 1 | B) = Z(\text{accept}, 1 | B) + \delta^{\hat{t}-1} Z(\text{accept}, \hat{t} | B)$,

(v) $\tilde{Z}(\text{reject}, 1|B) = Z(\text{reject}, 1|B) + (1 - \delta^{\hat{t}-1})Z(\text{accept}, \hat{t}|B)$.

Applying (C.1)-(C.3), the principal's expected payoff under $\tilde{Z}$ can be written as

$$p_1 \tilde{Z}(\text{accept}, 1|G) + (1 - p_1) \sum_{t=1}^{T} \delta^{t-1} \tilde{Z}(\text{accept}, t|B)$$

$$= p_1 Z(\text{accept}, 1|G) + (1 - p_1) \sum_{t \notin \{1, \hat{t}\}} \delta^{t-1} Z(\text{accept}, t|B)$$

$$+ (1 - p_1) \left[Z(\text{accept}, 1|B) + \delta^{\hat{t}-1} Z(\text{accept}, \hat{t}|B) \right]$$

$$= p_1 + (1 - p_1) \sum_{t=1}^{T} \delta^{t-1} Z(\text{accept}, t|B).$$

Thus $\tilde{Z}$ and Z give the same expected payoff to the principal. On the other hand, the agent's expected payoff under $\tilde{Z}$ can be written as

$$p_1 V_G \tilde{Z}(\text{accept}, 1|G) + (1 - p_1) \left[V_R \tilde{Z}(\text{reject}, 1|B) + V_B \sum_{t=1}^{T} \delta^{t-1} \tilde{Z}(\text{accept}, t|B) \right]$$

$$= p_1 V_G Z(\text{accept}, 1|G) + (1 - p_1) V_B \sum_{t \notin \{1, \hat{t}\}} \delta^{t-1} Z(\text{accept}, t|B)$$

$$+ (1 - p_1) \left[V_R \left(Z(\text{reject}, 1|B) + (1 - \delta^{\hat{t}-1}) Z(\text{accept}, \hat{t}|B) \right) \right.$$

$$\left. + V_B \left(Z(\text{accept}, 1|B) + \delta^{\hat{t}-1} Z(\text{accept}, \hat{t}|B) \right) \right]$$

$$= p_1 V_G + (1 - p_1) \left[V_R Z(\text{reject}, 1|B) + V_B \sum_{t=1}^{T} \delta^{t-1} Z(\text{accept}, t|B) \right]$$

$$+ (1 - p_1) V_R (1 - \delta^{\hat{t}-1}) Z(\text{accept}, \hat{t}|B),$$

where the first two terms in the final sum represent the agent's expected payoff under Z. As $(1 - p_1) V_R (1 - \delta^{\hat{t}-1}) Z(\text{accept}, \hat{t}|B) > 0$, we find that the agent's expected payoff under $\tilde{Z}$ is greater than it is under Z. Thus, $\tilde{Z}$ and Z give the same expected payoff to the principal, but the agent's expected payoff is strictly greater under $\tilde{Z}$ than it is under Z, contradicting the initial assumption that $Z \in \mathcal{Z}^*$. We conclude that (C.4) holds. Combining (C.1)-(C.4) shows that $\mathcal{Z}^* \subseteq \mathcal{Z}^\dagger$.

We next show that $\mathcal{Z}^\dagger \subseteq \mathcal{Z}^*$. Let $Z \in \mathcal{Z}^\dagger$. If $Z \notin \mathcal{Z}^*$, we can find Z' which Pareto dominates Z. Either $Z' \in \mathcal{Z}^\dagger$ or, by the first part of the proof, we can find $Z'' \in \mathcal{Z}^\dagger$ which Pareto dominates

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Z', in which case Z'' Pareto dominates Z, by transitivity. Hence, assume without loss of generality that $Z' \in \mathcal{Z}^\dagger$. Since both Z and Z' belong to $\mathcal{Z}^\dagger$ we have $Z'(x, t|\omega) = Z(x, t|\omega)$ unless $t = 1$ and $\omega = B$. But then either $Z'(\text{accept}, 1|B) < Z(\text{accept}, 1|B)$ and then the principal strictly prefers Z over Z', or $Z'(\text{accept}, 1|B) > Z(\text{accept}, 1|B)$ and then the agent strictly prefers Z over Z'. Therefore, Z' does not Pareto dominate Z, contradicting the definition of Z'. This shows that $Z \in \mathcal{Z}^*$. ∎

Comparison with the Single-Player Setting

We show here that increasing γ may increase the probability of type II errors made and lower the agent's expected payoff. We start with the following useful lemma.

Lemma C.1. *Either $a_t = b_t = \underline{b}$ in all periods or the interval of beliefs at which the agent waits is constant for the first $x \leq T-1$ periods, and then strictly decreasing over the remaining periods.*

Proof: In the perfect good news case, the result follows from Lemma 4. Below, we focus on the perfect bad news case. If in equilibrium the principal is aggressive at $t = T-1$ then the result is a consequence of Lemmata 5, 6 and A.1. Therefore, suppose henceforth that the principal is conservative at $t = T-1$.

Notice to begin with that by Lemma 5 all we need to show is that the sequence a_t increases with t. Reasoning as in Lemma B.4 establishes that $a_{T-1}^+ \in (a_{T-1}, b_{T-1})$. Since $\hat{g}_{T-1} > \hat{g}_T$ over the belief interval (a_{T-1}, b_{T-1}), we obtain, using definition (A.2),

$$
\begin{aligned}
\tilde{g}_{T-2}(a_{T-1}) &= \mathbb{P}(s_{T-2} = b|q_{T-2} = a_{T-1})V_R + \mathbb{P}(s_{T-2} = g|q_{T-2} = a_{T-1})\hat{g}_{T-1}(a_{T-1}^+) \\
&> \mathbb{P}(s_{T-1} = b|q_{T-1} = a_{T-1})V_R + \mathbb{P}(s_{T-1} = g|q_{T-1} = a_{T-1})\hat{g}_T(a_{T-1}^+) \\
&= \tilde{g}_{T-1}(a_{T-1}).
\end{aligned}
$$

We conclude from the arguments in the proof of Lemma A.2 that $a_{T-2} < a_{T-1}$.

Now, if the principal is aggressive in period $T-2$ then Lemmata 5 and 6, immediately give $a_1 = \cdots = a_{T-3} < a_{T-2} < a_{T-1}$. So suppose that the principal is conservative in period $T-2$. Since $a_{T-2} < a_{T-1}$ and $b_{T-2} = b_{T-1}$, Lemmata B.2 and B.6 establish that $\hat{g}_{T-2} > \hat{g}_{T-1}$ over the

belief interval (a_{T-2}, b_{T-2}). Moreover, $a_{T-2}^+ \in (a_{T-2}, b_{T-2})$. Hence:

$$\tilde{g}_{T-3}(a_{T-2}) = \mathbb{P}(s_{T-3} = b | q_{T-3} = a_{T-2})V_R + \mathbb{P}(s_{T-3} = g | q_{T-3} = a_{T-2})\hat{g}_{T-2}(a_{T-2}^+)$$
$$> \mathbb{P}(s_{T-2} = b | q_{T-2} = a_{T-2})V_R + \mathbb{P}(s_{T-2} = g | q_{T-2} = a_{T-2})\hat{g}_{T-1}(a_{T-2}^+)$$
$$= \tilde{g}_{T-2}(a_{T-2}).$$

We conclude from the arguments in the proof of Lemma A.2 that $a_{T-3} < a_{T-2}$. Pursuing the recursion completes the proof. ∎

Section II revealed the existence of two possible equilibrium regimes. In one regime, the principal is aggressive at $t = 1$, and triggers the agent's final decision in the first period. In the other regime, the principal is conservative, and seeks to sustain uncertainty until $t = T$. We next establish that, as long as no regime switch occurs, increasing the amount of exogenous outside information weakly increases the welfare of the agent.

Lemma C.2. *Let $(T'', \gamma'') \geq (T', \gamma')$. Assume that, in equilibrium, at $t = 1$, either the principal is aggressive given $(T, \gamma) = (T', \gamma')$ as well as given $(T, \gamma) = (T'', \gamma'')$, or the principal is conservative in both cases. Then the agent's equilibrium expected payoff is greater given $(T, \gamma) = (T'', \gamma'')$ than given $(T, \gamma) = (T', \gamma')$.*

Proof: We show the proof for the perfect bad news case (the proof for the perfect good news case is similar but easier). Throughout the proof primes will be used for all objects corresponding to the situation in which $(T, \gamma) = (T', \gamma')$. Similarly, double primes will be used for all objects corresponding to the situation in which $(T, \gamma) = (T'', \gamma'')$.

In the case in which at $t = 1$ the principal is aggressive given $(T, \gamma) = (T', \gamma')$ as well as given $(T, \gamma) = (T'', \gamma'')$, the result immediately follows from Lemma 5 and noting that $b''_{T''-1} \geq b'_{T'-1}$ (which, in turn, follows from Lemma B.8). Below we deal with the other case.

Since $\gamma'' \geq \gamma'$ notice first that, by Lemma A.1, $a''_{T''-1} \leq a'_{T'-1}$ and $b''_{T''-1} \geq b'_{T'-1}$. Moreover, since at $t = 1$ the principal is conservative, the same must be true at all $t < T$ (Lemma 6). Hence, $\hat{g}''_{T''-1}(\cdot)$ is piecewise linear with kinks at $a''_{T''-1}$ and $b''_{T''-1}$, $\hat{g}''_{T''-1}(a''_{T''-1}) = V_R$ and $\hat{g}''_{T''-1}(b''_{T''-1}) = V_B + b''_{T''-1}(V_G - V_B)$. A similar remark applies to $\hat{g}'_{T'-1}(\cdot)$. We conclude that

$\hat{g}''_{T''-1}(\cdot) \geq \hat{g}'_{T'-1}(\cdot)$. If $T'=2$ then Lemma C.1 finishes the proof. Otherwise,

$$\begin{aligned}
\tilde{g}''_{T''-2}(a'_{T'-2}) &= \mathbb{E}_{s''_{T''-2}}[\hat{g}''_{T''-1}(p_{T''-1})|q_{T''-2}=a'_{T'-2}] \\
&\geq \mathbb{E}_{s'_{T''-2}}[\hat{g}''_{T''-1}(p_{T''-1})|q_{T''-2}=a'_{T'-2}] \\
&\geq \mathbb{E}_{s'_{T''-2}}[\hat{g}'_{T''-1}(p_{T''-1})|q_{T''-2}=a'_{T'-2}] \\
&= \tilde{g}'_{T''-2}(a'_{T'-2}).
\end{aligned}$$

The first inequality follows from convexity of $\hat{g}''_{T''-1}(\cdot)$ and the fact that, since $\gamma'' \geq \gamma'$, $s''_{T''-2}$ is Blackwell-more-informative than $s'_{T''-2}$. The second inequality follows from the previously established inequality $\hat{g}''_{T''-1}(\cdot) \geq \hat{g}'_{T'-1}(\cdot)$. Hence, $\delta\tilde{g}''_{T''-2}(a'_{T'-2}) \geq \delta\tilde{g}'_{T''-2}(a'_{T'-2}) = V_R$ which, in turn, implies $a''_{T''-2} \leq a'_{T'-2}$ and, reasoning as above, $\hat{g}''_{T''-2}(\cdot) \geq \hat{g}'_{T'-2}(\cdot)$. If $T'=3$ then Lemma C.1 finishes the proof. Otherwise, we can repeat the last step. ∎

Lemma C.3. *Fix* $\gamma \in (0,1)$. *There exists* $\underline{\delta}(\gamma,T) < 1$ *such that, in equilibrium, whenever* $\delta > \underline{\delta}(\gamma,T)$, *at* $t=1$ *the principal is conservative.*

Proof: Fix $\gamma \in (0,1)$. First, notice that

$$\lim_{\delta \to 1} b_t = 1 \tag{C.5}$$

for all $t<T$. Next, let each element of the sequence $\{x_t\}_{t=1}^{T-1}$ be defined implicitly as follows:

$$\mathbb{P}(\omega = G|p_t = x_t, s_t = g, \ldots, s_{T-1} = g) = \underline{b}.$$

Thus, $x_1 < x_2 < \cdots < x_{T-1} < \underline{b}$. Moreover notice that, for all $t<T$:

$$\limsup_{\delta \to 1} a_t \leq x_t. \tag{C.6}$$

Otherwise, we could find a δ sufficiently close to 1 such that given $q_t = a_t$ the agent would strictly prefer waiting until the deadline over rejection (contradicting the definition of a_t). Let $1 - x_1 > \epsilon > 0$. Applying (C.6), we can find $\underline{\delta} < 1$ such that $\delta > \underline{\delta}$ implies

$$\hat{f}_1(x_1 + \epsilon) \geq \delta^{T-1}\left[1 - \left(1 - (x_1 + \epsilon)\right)\left(1 - (1-\gamma)^{T-1}\right)\right]. \tag{C.7}$$

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Noting that

$$1 - \left(1 - (x_1 + \epsilon)\right)\left(1 - (1 - \gamma)^{T-1}\right) = (1 - \gamma)^{T-1} + (x_1 + \epsilon)\left(1 - (1 - \gamma)^{T-1}\right) > x_1 + \epsilon,$$

combining (C.5)-(C.7) yields, for δ sufficiently large:

$$\hat{f}_1(x_1 + \epsilon) > \frac{x_1 + \epsilon}{b_1}.$$

If in equilibrium the principal were aggressive in period 1 we would have $\hat{f}_1(x_1 + \epsilon) = \frac{x_1 + \epsilon}{b_1}$. ∎

Proposition C.2. *With perfect good news, the agent's equilibrium expected payoff is monotonically increasing in T and γ. With perfect bad news, the agent's equilibrium expected payoff is monotonically increasing in T and, if players are sufficiently impatient, also monotonically increasing in γ; however, if players are patient enough, the agent's equilibrium expected payoff is non-monotonic in γ.*

Proof: For the perfect good news case, the result follows from Lemma C.2. We show the proof of the result for the perfect bad news case. We start with three observations:

- *Observation 1:* if in equilibrium at $t=1$ the principal is aggressive then $\hat{g}_1$ is piecewise linear with a kink at b_1, $\hat{g}_1(0) = V_R$, $\hat{g}_1(b_1) = V_B + b_1(V_G - V_B)$ and $\hat{g}_1(1) = V_G$.

- *Observation 2:* if in equilibrium at $t=1$ the principal is conservative then $\hat{g}_1$ is piecewise linear with kinks at a_1 and b_1, $\hat{g}_1(0) = \hat{g}_1(a_1) = V_R$, $\hat{g}_1(b_1) = V_B + b_1(V_G - V_B)$ and $\hat{g}_1(1) = V_G$.

- *Observation 3:* b_1 is both non-decreasing, and continuous in γ.

Observations 1 and 2 immediately follow from the definitions of a_1, b_1, and the experiments designed by the principal when she is aggressive and conservative. Observation 3 follows from Lemmata 5 and B.8.

Now, let $T'' > T'$. We want to show that the agent's equilibrium expected payoff is at least as large in the game of length $T = T''$ as in the game of length $T = T'$. If in equilibrium the principal is aggressive at $t=1$ given $T = T''$ and given $T = T'$, the result then follows from Lemma C.2, and similarly if in equilibrium the principal is conservative at $t=1$ given both game lengths. Hence, by Lemma 6, the only case left to consider is when in equilibrium the

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principal is aggressive at $t=1$ given $T=T''$, but conservative at $t=1$ given $T=T'$. In the latter case, the result follows from Observations 1-2 combined with Lemma 5.

Next, let $\gamma''>\gamma'$. We first want to show that, if players are sufficiently impatient, then the agent's equilibrium expected payoff is at least as large under $\gamma=\gamma''$ than under $\gamma=\gamma'$. We know from Lemmata 6 and 8 that, for $\delta<\underline{\delta}$, in equilibrium the principal is aggressive at $t=1$ regardless of γ. So the result follows from Observations 1 and 3.

Finally, we want to show that, if players are patient enough, then the agent's equilibrium expected payoff is non-monotone in γ. This result follows from Lemmata B.8 and C.3, combined with Observations 1-2 showing that an equilibrium switch from aggressive at $t=1$ to conservative at $t=1$ triggers a drop of the agent's equilibrium expected payoff. ∎

The rest of this appendix considers a hypothetical planner with payoffs W_{aG} from acceptance in state G, $W_{rG}<W_{aG}$ from rejection in state G, W_{rB} from rejection in state B, and $W_{aB}<W_{rB}$ from acceptance in state B. We are interested in this planner's equilibrium expected payoff, Q. For concreteness, we henceforth refer to Q as the (equilibrium) quality of the agent's final decision.[3]

The planner's welfare differs from the agent's in two ways: first, while the planner cares about errors made by the agent, the planner is indifferent about the timing of said errors; second the planner and the agent may weigh type I and type II errors differently. Notwithstanding these differences, the effect of exogenous outside information on Q mirrors its effect on the welfare of the agent (Proposition C.2).

Proposition C.3. *With perfect good news, the quality of the agent's final decision is monotonically increasing in T and γ. With perfect bad news, Q is monotonically increasing in T and, if players are sufficiently impatient, also monotonically increasing in γ. However, if players are patient enough, then Q is non-monotonic in γ.*

Proof: We focus as usual on the perfect bad news case (the perfect good news case being similar and easier). Let X denote the random variable representing the belief at which in equilibrium the agent makes his final decision. Let $\phi:[0,1]\to\mathbb{R}$ denote the piecewise linear function with a kink at $\underline{b}$ such that $\phi(0)=W_{rB}$, $\phi(\underline{b})=W_{aB}+\underline{b}(W_{aG}-W_{aB})$ and $\phi(1)=W_{aG}$. Then:

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[3]If $W_{aG}=W_{rB}=1$ and $W_{rG}=W_{aB}=0$, then $1-Q$ represents the equilibrium probability that the agent makes a type I or type II error.

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(a) $\phi(\cdot)$ is convex

(b) $\mathbb{E}[X] = p_1$;

(c) if in equilibrium the principal is aggressive at $t = 1$ then $\mathrm{supp}(X) \subseteq \{0, b_1\}$;

(d) if in equilibrium the principal is conservative at $t = 1$ then $\mathrm{supp}(X) = \{0, a_{T-1}^+, b_1\}$, where $a_{T-1}^+ > \underline{b}$ denotes the beginning of period-T belief given $q_{T-1} = a_{T-1}$ and $s_{T-1} = g$;

(e) $Q = \mathbb{E}[\phi(X)]$.

We are now ready to prove the various parts of the proposition. First, we know from Lemmata 6 and 8 that, for $\delta < \underline{\delta}$, in equilibrium the principal is aggressive at $t = 1$ regardless of γ. Hence, suppose $\delta < \underline{\delta}$. Let $\gamma'' > \gamma'$. Then, by Lemmata 5 and B.8, $b_1'' \geq b_1'$. That $Q'' \geq Q'$ now follows from remarks (a), (b), (c) and (e) above.

Next, if players are patient enough, Lemmata B.8 and C.3 establish that, starting from $\gamma = 0$ and increasing γ, in equilibrium, at $t = 1$ the principal is aggressive at first but then switches to being conservative. Since b_1 is continuous in γ, remarks (a)-(e) establish that this equilibrium switch induces a drop in Q.

Lastly, let $T'' > T'$. If in equilibrium the principal is aggressive at $t = 1$ given $T = T''$ and given $T = T'$ then, by Lemma 5, $Q'' = Q'$. If in equilibrium the principal is aggressive at $t = 1$ given $T = T''$ but conservative at $t = 1$ given $T = T'$ then, by Lemma 5 and remarks (a)-(e), $Q'' > Q'$. By Lemma 6, the last case remaining is when in equilibrium the principal is conservative at $t = 1$ given $T = T''$ and given $T = T'$. A simple recursive argument based on Proposition A.1 then establishes $Q'' > Q'$. ∎

Online Appendix D: Frequent signals

Proof of Proposition 2: We will prove part (ii) of the proposition (the proof of part (i) uses arguments similar to those used in the proof of Lemma 7 and is therefore omitted). Let $b_{T_n-1,n}$ denote the agent's period-(T_n-1) standard of approval given signal frequency n. We now show that $b_{T_n-1,n}$ is monotonically increasing in n. Fix n. By Lemma A.1 of Online Appendix A, in equilibrium the agent does not benefit from the period-T_n experiment. So $b_{T_n-1,n}$ is independent of information generated by the principal. However, with signal frequency $n+1$ the amount of exogenous outside information that the agent obtains by waiting two periods is the same as what he obtains with signal frequency n by waiting one period. These remarks imply

$$b_{T_{n+1}-2,n+1} \geq b_{T_n-1,n}.$$

By Lemma 5,

$$b_{T_{n+1}-2,n+1} = b_{T_{n+1}-1,n+1}.$$

Hence,

$$b_{T_{n+1}-1,n+1} \geq b_{T_n-1,n}.$$

In what follows, let $\bar{b} := \lim_{n\to\infty} b_{T_n-1,n}$. The condition $\lambda > \varphi(r)$ is equivalent to $b_1 > \underline{b}$. So whenever this condition holds, $\bar{b} > \underline{b}$. But then, for L sufficiently small, for all sufficiently large n being aggressive in the first period cannot be optimal for the principal: keeping uncertainty high until the last period enables the principal to benefit (at a very small cost) from a strictly lower standard of approval. ∎

Online Appendix E: Different Discount Factors

In this appendix we prove Proposition 3. All the results in this appendix refer to the setting in which the players' discount rates are δ_P and δ_A.

Lemma E.1. *In equilibrium, at $t = T$, the agent accepts if $q_T \geq \underline{b}$ and rejects otherwise. The principal designs the experiment*

$$M_T = \begin{cases} \{0, \underline{b}\} & \text{if } p_T \in (0, \underline{b}); \\ \{p_T\} & \text{otherwise.} \end{cases}$$

The agent does not benefit from the period-T experiment, hence his equilibrium continuation payoff is convex in p_T.

Proof: See the proof of Lemma A.1 in Online Appendix A. ∎

Lemma E.2. *Let $t < T$. Suppose that functions $\hat{g}_{t+1}(p_{t+1})$ and $\hat{f}_{t+1}(p_{t+1})$ uniquely determine the agent's (resp. the principal's) equilibrium continuation payoffs in period $t+1$. If $\hat{g}_{t+1}$ is convex, then:*

1. *in equilibrium, the principal's period-t experiment and the agent's period-t decision are both uniquely determined; the former is a function of p_t only and the latter is a function of q_t only;*

2. *functions $\hat{g}_t(p_t)$ and $\hat{f}_t(p_t)$ uniquely determine the equilibrium continuation payoffs in period t, and $\hat{g}_t$ is convex.*

Proof: Let $\tilde{g}_t(q_t)$ and $\tilde{f}_t(q_t)$ be defined as in equations (A.2) and (A.3), respectively. Then the agent's equilibrium continuation payoff given q_t can be written as

$$g_t(q_t) = \max\left\{ V_R, \, \delta_A \tilde{g}_t(q_t), \, V_B + q_t(V_G - V_B) \right\}. \tag{E.1}$$

As $\hat{g}_{t+1}$ is convex by assumption, Proposition A.1 shows that $\tilde{g}_t$ is convex as well. Moreover,

$$\begin{cases} \delta_A \tilde{g}_t(0) = \delta_A V_R < V_R; \\ \delta_A \tilde{g}_t(1) = \delta_A V_G < V_G. \end{cases} \tag{E.2}$$

Then (E.1), (E.2) and convexity of $\tilde{g}_t$ give unique a_t and b_t, with $a_t \leq \underline{b} \leq b_t$, such that

$$
\begin{cases}
g_t(q_t) = V_R > \max\left\{\delta_A \tilde{g}_t(q_t),\, V_B + q_t(V_G - V_B)\right\} & \text{if } q_t < a_t; \\
g_t(q_t) = \delta_A \tilde{g}_t(q_t) > \max\left\{V_R,\, V_B + q_t(V_G - V_B)\right\} & \text{if } q_t \in (a_t, b_t); \\
g_t(q_t) = V_B + q_t(V_G - V_B) > \max\left\{V_R,\, \delta_A \tilde{g}_t(q_t)\right\} & \text{if } q_t > b_t.
\end{cases}
$$

Hence, in equilibrium, the agent rejects if $q_t < a_t$, waits if $q_t \in (a_t, b_t)$, and accepts if $q_t > b_t$. Moreover, since in equilibrium whenever indifferent the agent makes the decision preferred by the principal, it ensues that the agent waits if $q_t = a_t < b_t$ and accepts if $q_t = b_t$. This gives

$$
f_t(q_t) = \begin{cases}
0 & \text{if } q_t < a_t; \\
\delta_P \tilde{f}_t(q_t) & \text{if } q_t \in [a_t, b_t); \\
1 & \text{if } q_t \geq b_t.
\end{cases}
\tag{E.3}
$$

Standard arguments yield $\hat{f}_t = \operatorname{cav} f_t$. Since in equilibrium whenever indifferent the principal picks the least informative experiment, the principal's experiment in period t is uniquely determined by the belief p_t at the beginning of period t. Lastly, letting $\tau_t(p_t)$ denote the principal's equilibrium experiment given p_t yields $\hat{g}_t(p_t) = \mathbb{E}_{\tau_t(p_t)}[g_t(q_t)|p_t]$.

Finally, since $\tilde{g}_t$ is convex, (E.1) shows that g_t is as well. Since $\hat{g}_t(p_t) = \mathbb{E}_{\tau_t(p_t)}[g_t(q_t)|p_t]$, convexity of g_t together with the properties of $\tau_t(\cdot)$ establish that $\hat{g}_t$ is convex. ∎

Proposition E.1. *There exists a unique equilibrium.*

Proof: The proposition follows from Lemmata E.1 and E.2. ∎

Lemma E.3. *Each period, cutoffs $0 < a_t \leq b_t < 1$ exist such that, in equilibrium the agent rejects if $q_t < a_t$, waits if $q_t \in [a_t, b_t)$, and accepts if $q_t \geq b_t$.*

Proof: For $t = T$, the result follows from Lemma E.1. For $t < T$, the result was shown within the proof of Lemma E.2. ∎

Lemma E.4. *At all $t \leq T-1$, the threshold of acceptance is at least as large as the threshold of acceptance at $t = T-1$, which itself is at least as large as the static threshold of acceptance: $b_t \geq b_{T-1} \geq b_T = \underline{b}$.*

Proof: See the proof of Lemma B.1 in Online Appendix B. ∎

Lemma E.5. *Each period, in equilibrium $M_t = \{p_t\}$ for all $p_t \in \{0\} \cup [b_t, 1]$. Moreover:*

(i) *either $M_t = \{0, b_t\}$ for all $p_t \in (0, b_t)$,*

(ii) *or $a_t < b_t$ and: $M_t = \{0, a_t\}$ for $p_t \in (0, a_t)$ and there exists $c_t \in [a_t, b_t)$ such that $M_t = \{p_t\}$ for $p_t \in [a_t, c_t]$, while $M_t = \{c_t, b_t\}$ for $p_t \in (c_t, b_t)$.*

Proof: Recall $\hat{f}_t = \text{cav } f_t$. If $a_t = b_t$ (so that the set of beliefs for which in equilibrium the agent waits in period t is empty) then in equilibrium $M_t = \{0, b_t\}$ for all $p_t \in (0, b_t)$. So assume $a_t < b_t$. Observe that:

(A) $\tilde{f}_t(\cdot)$ (defined by equation (A.3) in Online Appendix A) is concave,

(B) $f_t(q_t) = \delta_P \tilde{f}_t(q_t)$ for all $q_t \in \{0\} \cup [a_t, b_t)$.

(A) follows from Proposition A.1 while (B) is obtained from (E.3). In view of (A)-(B), either (i) in the statement of the lemma holds, or (ii) does. ∎

Lemma E.6. *With perfect good news, in equilibrium the principal is aggressive at $t = 1$.*

Proof: The proof follows the same steps of the proof of Lemma 4. ∎

Lemma E.7. *With perfect bad news, either $a_t = b_t = \underline{b}$ in all periods, or for all $t < T$: $b_t = b_{T-1} > b_T = \underline{b}$.*

Proof: If $a_{T-1} = b_{T-1} = b_T$, the claim of the lemma is straightforward. Assume now $b_{T-1} > b_T$. At $q_{T-1} = b_{T-1}$, the agent is indifferent between waiting and accepting. The agent's expected payoff from accepting is $b_{T-1} V_G + (1 - b_{T-1}) V_B$. On the other hand, using Lemmata E.3 and E.4 (and noting that in equilibrium, in period T, the agent accepts for $p_T \geq b_T$), the agent's

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expected payoff from waiting can be written as $\delta_A[b_{T-1}V_G+(1-b_{T-1})(\gamma V_R+(1-\gamma)V_B)]$. So b_{T-1} is the unique solution of

$$xV_G+(1-x)V_B=\delta_A[xV_G+(1-x)(\gamma V_R+(1-\gamma)V_B)]. \tag{E.4}$$

Next, consider $t<T-1$ such that $b_{t+1}=b_{T-1}$, and $q_t=b_t$, so that, by definition, the agent is indifferent between waiting and accepting. The agent's expected payoff from accepting is $b_tV_G+(1-b_t)V_B$. On the other hand, using Lemma E.4, $q_t=b_t\geq b_{T-1}=b_{t+1}$. Hence, conditional on $s_t=g$, the agent optimally accepts in the next period. It ensues that b_t solves (E.4) and, therefore, that $b_t=b_{T-1}$. A recursive argument then yields $b_t=b_{T-1}$ for all $t<T$. ∎

Lemma E.8. *Let $\tilde{\gamma}(\delta_A):=\left(\frac{1-\delta_A}{\delta_A}\right)\frac{V_R(V_G-V_B)}{(V_G-V_R)(V_R-V_B)}$. Then $\gamma>\tilde{\gamma}(\delta_A)$ if and only if $b_{T-1}>\underline{b}$, and either condition implies*

$$b_{T-1}=\frac{\delta_A(\gamma V_R+(1-\gamma)V_B)-V_B}{(V_G-V_B)(1-\delta_A)+\delta_A\gamma(V_R-V_B)}. \tag{E.5}$$

Proof: The arguments in the proof of Lemma E.2 show that $b_{T-1}>\underline{b}$ if and only if given $q_{T-1}=\underline{b}$ the agent strictly prefers waiting over rejection, that is, if and only if

$$\delta_A\left[\underline{b}V_G+(1-\underline{b})\left(\gamma V_R+(1-\gamma)V_B\right)\right]>V_R,$$

which, upon rearrangement, yields $\gamma>\tilde{\gamma}(\delta)$. Solving (E.4) yields (E.5). ∎

Lemma E.9. *Let $t<T-1$. In equilibrium, if the principal is aggressive in period $t+1$, then the principal is also aggressive in period t.*

Proof: Suppose that in equilibrium the principal is aggressive in period $1<t+1<T$. If $a_t=b_t$ the statement of the lemma is straightforward. Assume therefore that $a_t<b_t$. By virtue of Lemma E.5, in order to establish that the principal is also aggressive in period t it is enough to show that when $p_t=a_t$ the principal strictly prefers the experiment $M_t=\{0,b_t\}$ over the uninformative experiment. On one hand, the principal's expected payoff from designing $M_t=\{0,b_t\}$ is $\frac{a_t}{b_t}$. On the other hand, her expected payoff from designing the uninformative

experiment is given by $\delta_P \mathbb{E}_{s_t}[\hat{f}_{t+1}(p_{t+1}) \mid q_t = a_t]$. The next sequence of inequalities therefore concludes the proof:

$$\delta_P \mathbb{E}_{s_t}[\hat{f}_{t+1}(p_{t+1}) \mid q_t = a_t] \leq \delta_P \hat{f}_{t+1}(a_t) = \delta_P \frac{a_t}{b_{t+1}} < \frac{a_t}{b_t}.$$

The first inequality follows from noting that $\hat{f}_{t+1}$ is concave (which we show formally in the appendix); the equality follows from the assumption that the principal is aggressive in period $t+1$, and the second inequality is due to Lemma E.7. ∎

Lemma E.10. *There exists $\tilde{T}(\gamma, \delta_A, \delta_P) < \infty$ such that, in equilibrium, the principal is aggressive in period 1 if and only if $T > \tilde{T}(\gamma, \delta_A, \delta_P)$.*

Proof: Note that in view of Lemma E.9 it is enough to show that, for T sufficiently large, in equilibrium the principal is aggressive at $t=1$. Next, part (ii) of Lemma E.7 shows that any benefit to the principal from not being aggressive at $t=1$ must come from persuading the agent to accept at $t=T$ when $\omega = B$. So these benefits are bounded from above by δ_P^{T-1}, which tends to 0 as $T \to \infty$. On the other hand, as $b_1 < 1$, the corresponding loss to the principal is bounded away from zero since by being aggressive at $t=1$ the principal obtains acceptance with strictly positive probability conditional on $\omega = B$. We conclude that, for T sufficiently large, in equilibrium the principal is aggressive at $t=1$. ∎

Lemma E.11. *In equilibrium, each period either the principal is aggressive, or the principal is conservative.*

Proof: The proof follows the same steps as the proof of Lemma 3. ∎

Lemma E.12. *There exist cutoffs $\underline{\delta}_A \in (0,1)$ and $\underline{\delta}_P(\delta_A) \in (0,1]$ and, for $\delta_A > \underline{\delta}_A$ and $\delta_P > \underline{\delta}_P(\delta_A)$, functions $0 < \underline{\gamma}(\delta_A, \delta_P) < \overline{\gamma}(\delta_A, \delta_P) < 1$ such that, in equilibrium, the principal is conservative in period $T-1$ if and only if $\delta_A > \underline{\delta}_A$, $\delta_P > \underline{\delta}_P(\delta_A)$, and $\gamma \in (\underline{\gamma}(\delta_A, \delta_P), \overline{\gamma}(\delta_A, \delta_P))$.*

Proof: We saw in the proof of Lemma E.8 that $\gamma \leq \tilde{\gamma}(\delta_A)$ implies that in equilibrium the agent never waits. So whenever $\gamma \leq \tilde{\gamma}(\delta_A)$, in equilibrium the principal has to be aggressive in

period $T-1$. In particular, since $\tilde{\gamma}(\delta_A)>1$ for δ_A sufficiently small, we find that for δ_A small enough the principal is aggressive in period $T-1$ irrespective of γ and of δ_P.

Suppose next that $\gamma>\tilde{\gamma}(\delta_A)$. Then for $q_{T-1}=a_{T-1}$ in equilibrium the agent is indifferent between waiting and rejection. The agent's expected payoff from rejection is given by V_R. His expected payoff from waiting is on the other hand given by $\delta_A[a_{T-1}V_G+(1-a_{T-1})(\gamma V_R+(1-\gamma)V_B)]$, where we deduced from Lemma E.1 that $s_{T-1}=g$ induces $p_T>b_T=\underline{b}$. We therefore obtain $V_R=\delta_A[a_{T-1}V_G+(1-a_{T-1})(\gamma V_R+(1-\gamma)V_B)]$, giving

$$a_{T-1}=\frac{V_R-\delta_A(\gamma V_R+(1-\gamma)V_B)}{\delta_A(V_G-\gamma V_R-(1-\gamma)V_B)}. \tag{E.6}$$

Now, using Lemma E.5, the necessary and sufficient condition for the principal not to be aggressive in period $T-1$ in equilibrium is $f_{T-1}(a_{T-1})\geq\frac{a_{T-1}}{b_{T-1}}$.[4] Noting that $f_{T-1}(a_{T-1})=\delta_P[a_{T-1}+(1-\gamma)(1-a_{T-1})]$ and substituting for a_{T-1} and b_{T-1} using (E.5) and (E.6), the former inequality becomes

$$V_B\delta_A(\gamma-1)+\delta_AV_G(1-\gamma)+V_R\gamma(1-\delta_A) \tag{E.7}$$
$$\geq\frac{[V_B\delta_A(\gamma-1)+V_R(1-\delta_A\gamma)]\,[\delta_A\gamma(V_B-V_R)+(V_B-V_G)(1-\delta_A)]}{\delta_P\,[(V_B-V_R)\delta_A\gamma+V_B(1-\delta_A)]}.$$

One checks that if (E.7) holds for some δ_P', it must hold for $\delta_P''>\delta_P'$: either the right-hand side is positive, and therefore decreasing in δ_P, or it is negative, but the left-hand side is always positive,[5] so in this case the inequality is always satisfied. Moreover, for $\delta_A=1$ the quadratic equation in γ obtained from (E.7) has roots $\gamma=0$ and $\gamma=1$. On the other hand, for $\delta_A<1$, (E.7) is violated whenever either $\gamma=1$, or $\gamma=\tilde{\gamma}(\delta_A)$. So (E.7) holds for all values of γ in between the roots of the corresponding quadratic equation. Letting $\underline{\gamma}(\delta_A,\delta_P)$ and $\overline{\gamma}(\delta_A,\delta_P)$ denote the real roots, the previous remarks yield $\tilde{\gamma}(\delta_A,\delta_P)<\underline{\gamma}(\delta_A,\delta_P)\leq\overline{\gamma}(\delta_A,\delta_P)<1$ and show that these roots only exist for $\delta_A>\underline{\delta}_A$ and $\delta_P>\underline{\delta}_P(\delta_A)$, where (i) $\underline{\delta}_A$ is defined implicitly by $\underline{\gamma}(\underline{\delta}_A,1)=\overline{\gamma}(\underline{\delta}_A,1)$ and (ii) $\underline{\delta}_P(\delta_A)$ is defined implicitly for $\delta_A>\underline{\delta}_A$ by $\underline{\gamma}(\underline{\delta}_P(\delta_A),\delta_A)=\overline{\gamma}(\underline{\delta}_P(\delta_A),\delta_A)$. Noting that, by Lemma E.11, whenever the principal is not aggressive she is conservative concludes the proof. ■

Lemmata E.1 to E.12 together with Proposition E.1 now prove Proposition 3 following the same steps as the proof of Theorem 1.

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[4]That is, at $p_{T-1}=a_{T-1}$ the principal must prefer the uninformative experiment over $M_{T-1}=\{0,b_{T-1}\}$.
[5]Since $V_G>V_G$ and $V_R>0$ imply $V_B\delta_A(\gamma-1)+\delta_AV_G(1-\gamma)+V_R\gamma(1-\delta_A)>\delta_A(1-\gamma)(V_G-V_B)>0$.