The Distributional Consequences of Public School Choice

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Online Appendix

B Proofs of Other Results from Main Text

B.1 Proof of Lemma 1

Proof. Under Assumption 2, a non-partisan student of type x faces maximization problem:

$$\max_y u(x,y) = v(x,y) - p(y).$$

The first-order condition is

$$\frac{\partial v}{\partial y}(x,y) - p'(y) = 0.$$

For the first-order condition to hold at $x = y$, we have $p'(y) = \frac{\partial v}{\partial y}(y,y)$, which requires

$$p(y) = \int_{z=0}^{z=y} \frac{\partial v}{\partial y}(z,z)dz,$$

while the second-order condition for maximization follows from the fact that v satisfies increasing differences in x and y.[2] Given this pricing rule,

$$\frac{\partial u(x,y)}{\partial y} = \frac{\partial v}{\partial y}(x,y) - \frac{\partial v}{\partial y}(y,y),$$

which is strictly positive for $x > y$ and strictly negative for $x < y$ by the property of increasing differences for v. Thus, $u(x,y) = v(x,y) - p(y)$ is strictly increasing in y for $y < x$ and strictly decreasing in y for $y > x$, which verifies that $u(x,y)$ is maximized at $y = x$ given this pricing rule. ■

B.2 Proof of Proposition 1

Proof. We first show that a school choice equilibrium consist of an interval of types residing in the town. Suppose that school quality in the town in a school choice equilibrium is y_{SC} with price $p_{SC} = p(y_{SC})$. A partisan of type x enrolls in the town if $v(x, y_{SC}) - p(y_{SC}) + \theta \geq v(x,x) - p(x)$ or

$$\theta \geq [v(x,x) - v(x,y_{SC})] - [p(x) - p(y_{SC})].$$

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[2]Technically, this first-order condition would yield the result that $p(y) = \int_{z=0}^{z=y} \frac{\partial v}{\partial y}(z,z)dz + C$, where C is a constant, but since we assume that $v(0,0) = 0$, we set this constant to 0.

Since $p(y) = \int_0^y \frac{\partial v}{\partial y}(z,z)dz$, the incentive condition for enrolling in the town in integral form is

$$\theta \geq \int_{y_{SC}}^{x} \left[\frac{\partial v}{\partial y}(x,z) - \frac{\partial v}{\partial y}(z,z) \right] dz.$$

If $x \geq y_{SC}$, the right-hand side of this equation is increasing in x by Assumption 1, so the condition holds for types in some range of types given by $[x, x_{SC}^H]$. Similarly, if $x \leq y_{SC}$, the right-hand side of the equation is decreasing in x, so the condition also holds for some range of types given by $[x_{SC}^L, x]$. Putting these ranges together, a range of types around y_{SC} will enroll in the town in a School Choice equilibrium.

Suppose that the school qualities in a D-district neighborhood equilibrium are $y_1, y_2, ..., y_D$ where $y_j > y_{j-1}$ and that the price for district j is $p_j = p(y_j) + \Delta_j$ where $\Delta_j \geq 0$. (At least one of the prices is the competitive price on the outside market, so $\Delta_j = 0$ for at least one j.) Following the analysis of the School Choice equilibrium above, a range of partisan types around y_j prefers district j to the outside option. Partisan type x prefers district k to district j with $k > j$ if $v(x, y_k) - p_k + \theta \geq v(x, y_j) - p_j + \theta$ which is equivalent to

$$v(x, y_k) - v(x, y_j) - [p(y_k) - p(y_j)] \geq \Delta_k - \Delta_j.$$

In integral form, the incentive condition for enrolling in district k rather than district j is then

$$\int_{y_j}^{y_k} \left[\frac{\partial v}{\partial y}(x,z) - \frac{\partial v}{\partial y}(z,z) \right] dz \geq \Delta_k - \Delta_j.$$

The integrand on the left-hand side of this equation is increasing in x, so there is some cutoff x_{jk} such that the condition holds iff $x \geq x_{jk}$. Thus, these pairwise comparisons between districts j and k with $j < k$ may adjust the lower bound of the range of types enrolling in district j and the upper bound of the range of types enrolling in district k. The end result remains the same as in the School Choice equilibrium: a range of partisan types enrolls in each district d in a Neighborhood School equilibrium. ■

B.3 Proof of Proposition 2

Proof. We initially assume a Uniform distribution of types to provide intuition and then generalize the counterexample so that it holds for other distributions of types. In a school choice equilibrium with a neighborhood assignment rule, two districts and a uniform distribution of types, types from 0 to 0.5 enroll in district 1 while types from 0.5 to 1 enroll in district 2, so $y_1 = 0.25$ and $y_2 = 0.75$. We proceed to construct a counterexample based upon the following key elements in the definition of $v(x, y)$ that are designed to counteract the proof of Theorem 1. For expositional purposes, we divide the $[0, 1]$ x $[0, 1]$ region into nine rectangular regions numbered from left to right then bottom to top.

1. Region 1 (bottom left): $[0, y_1]$ x $[0, y_1]$;
2. Region 2 (bottom center): $[0, y_1]$ x $[y_1, y_2]$;
3. Region 3 (bottom right): $[0, y_1]$ x $[y_2, 1]$;
4. Region 4 (middle left): $[y_1, y_2]$ x $[0, y_1]$;
5. Region 5 (middle center): $[y_1, y_2]$ x $[y_1, y_2]$;
6. Region 6 (middle right): $[y_1, y_2]$ x $[y_2, 1]$;
7. Region 7 (top left): $[y_2, 1]$ x $[0, y_1]$;
8. Region 8 (top center): $[y_2, 1]$ x $[y_1, y_2]$;
9. Region 9 (top right): $[y_2, 1]$ x $[y_2, 1]$;

Regions 1, 5, and 9 include the 45-degree line where the values of $v(x, y)$ determine the competitive pricing function $p(x)$. As we highlight in the proof of Theorem 1 below, θ_{SC} and θ_N only increase on ranges of the function $v(x, y)$ where $\frac{\partial^2 v}{\partial x \partial y} > 0$. So for the purpose of the counterexample, we define v to be piecewise linear in Regions 1 and 9 and to have positive mixed partial derivative in Region 5. To further simplify analysis, we set $v(x, y) = 0$ on the left and lower boundaries of Region 5 and in Regions 1, 2, and 4. We leave $v(x, y)$ momentarily undefined in Regions 3 and 7 as its value in these regions do not affect the calculations of θ_{SC} or θ_N.
Define $v(x, y)$ on Regions 1, 2, 4, 5, 6, 8 and 9 as follows:

1. Regions 1 and 2: If $y \leq 0.25, 0 \leq x \leq 0.75, v(x, y) = 0$
2. Regions 1 and 4: If $x \leq 0.25, 0 \leq y \leq 0.75, v(x, y) = 0$
3. Region 5: If $0.25 \leq x, y \leq 0.75$, $v(x, y) = 192(x - 0.25)(y - 0.25)^2$
4. Region 6: If $x \geq 0.75$ and $0.25 \leq y \leq 0.75$, $v(x, y) = 96(y - 0.25)^2 + 48(x - 0.75)$
5. Region 8: If $y \geq 0.75$ and $0.25 \leq x \leq 0.75$, $v(x, y) = 48(x - 0.25) + 96(y - 0.75)$

6. Region 9: If $x \geq 0.75$ and $y \geq 0.75$, $v(x, y) = 24 + 48(x - 0.75) + 96(y - 0.75)$

The linear coefficients of $v(x, y)$ in Region 6 (where v is linear in x), Region 8 (where v is linear in y) and Region 9 (where v is linear in both x and y are set to the values of the partial derivatives $\frac{\partial v}{\partial x} = 48$ and $\frac{\partial v}{\partial y} = 96$ at the upper right corner of Region 5. Given this information, the competitive price function is defined in three pieces:

- First, for $x, y < 0.25, v(x, y) = 0$ so clearly $p(y) = 0$.
- Next, for $0.25 \leq x, y \leq 0.75, v(y, y) = 192(x - 0.25)(y - 0.25)^2$, $\frac{\partial v}{\partial y} = 384(x - 0.25)(y - 0.25)$ and so for $0.25 \leq y \leq 0.75, p(y) = \int_{0.25}^{y} 384(z - 0.25)^2 dz = 128(y - 0.25)^3$, so $p(0.75) = 16$.
- Finally, for $x, y \geq 0.75, v(x, y) = 48x + 96y - 84$, $\frac{\partial v}{\partial y} = 96$, and so $p(y) = p(0.75) + \int_{0.75}^{y} 96 dz = 16 + (y - 0.75) = 96y - 56$.

The incentive condition for type $x = 1$ in a school choice equilibrium where all partisan types enroll in the town is

$$v(1, 0.5) - p(0.5) + \theta \geq v(1, 1) - p(1),$$

or $18 - 2 + \theta \geq 60 - 40$ or

$$\theta \geq 4.$$

The incentive condition for type $x = 0$ in a school choice equilibrium where all partisan types enroll in the town is

$$v(0, 0.5) - p(0.5) + \theta \geq v(0, 0) - p(0),$$

or $0 - 2 + \theta \geq 0 - 0$ or

$$\theta \geq 2.$$

Therefore, all partisan types will enroll in the town under school choice if

$$\theta \geq \theta_{SC} = \max(2, 4) = 4.$$

The minimum value of θ for all partisan types to enroll under neighborhood equilibrium with competitive pricing is determined by the incentive conditions for the middle type $x = 1/2$. If partisan types from 0 to 1/2 enroll in district 1 while partisan types from 1/2 to 1 enroll in district 2, then $y_1 = 1/4$ and $y_2 = 3/4$ given the uniform distribution of types. Type 1/2 prefers district 1 to the outside option if

$$v(1/2, 1/4) - p(1/4) + \theta \geq v(1/2, 1/2) - p(1/2)$$

or $0 - 0 + \theta \geq 3 - 2$ or

$$\theta \geq 1.$$

Type 1/2 prefers district 2 to the outside option if

$$v(1/2, 3/4) - p(3/4) + \theta \geq v(1/2, 1/2) - p(1/2)$$

or $12 - 16 + \theta \geq 3 - 2$ or

$$\theta \geq 5.$$

Then partisan type 1/2 requires a bonus $\theta \geq \theta_N = 5$ to enroll in either district 1 or district 2. Therefore, we have $\theta_N = 5$ and $\theta_{SC} = 4$ and thus

$$\theta_N > \theta_{SC}.$$

This incomplete definition of v provides intuition for why it is possible to have $\theta_N > \theta_{SC}$ even under the assumption of (weakly) increasing differences in $v(x, y)$. The value θ_{SC} represents the cost for an extreme type (usually $x = 1$) to select a school with $y = 1/2$ in the town rather than the outside option. With competitive pricing in a school choice equilibrium, type $x = 1$ must forego gains equal to $\pi(1) - (v(1, 0.5) - p(0.5)) = [v(1, 1) - v(1, 0.5)] - [p(1) - p(0.5)]$. In integral form (assuming that this cost is greater than the cost for type $x = 0$ to enroll in the town):

$$\theta_{SC} = \int_{1/2}^{1} [\frac{\partial v}{\partial y}(1, z) - \frac{\partial v}{\partial y}(z, z)]dz = \int_{1/2}^{1} \int_{z}^{1} \frac{\partial^2 v}{\partial x \partial z}(a, z) da dz.$$

Similarly θ_N is the maximum cost for one of four boundary types to enroll in the town rather to take the outside option: (1, 2) type $x = 0$ or type $x = 1/2$ enrolling in district 1 in the town where $y = y_1 = \mathbb{E}(x|x \leq 1/2)$; (3, 4) type $x = 1/2$ or type $x = 1$ enrolling in district 2 in the town where $y = y_2 = \mathbb{E}(x|x \geq 1/2)$. Possibilities (1) and (4) would yield $\theta_N \leq \theta_{SC}$ since types $0, 1$ are only offered $y = 1/2$ in the town in a school choice equilibrium – a school that is a worse match for their types than the best choice in a neighborhood equilibrium.[3] Thus, it is not surprising that the preliminary version of the counterexample is based on (3), the cost for type $x = 1/2$ to enroll at a school in the town with $y = y_2$.

It is natural to anticipate $\theta_{SC} > \theta_N$ since θ_N corresponds to a generally better match (type $x = 1$ choosing a school with $y = y_2$ - a distance of $1 - y_2$ from the ideal match) than the one

[3] In this discussion of the intuition for the counterexample, we leave out the price adjustment required by the boundary indifference condition for pricing in a neighborhood equilibrium, as this complicates the discussion and only makes it easier to produce a counterexample.

required for θ_{SC} (type $x = 1$ choosing a school with $y = 1/2$, a distance of $1 - 1/2 = 1/2 > 1 - y_2$ from the ideal match). These distances are represented by the areas of the regions for the double integrals above: θ_N and θ_{SC} are both double integrals of $\frac{\partial^2 v}{\partial x \partial z}$, but the double integral for θ_{SC} is over a wider region of values. But with $\frac{\partial^2 v}{\partial x \partial z} = 0$ except in Region 5 in the preliminary counterexample above, θ_N and θ_{SC} reduce to double integrals of $\frac{\partial^2 v}{\partial x \partial z}$ over Region 5. Specifically θ_N is computed for the region with x and y between 1/2 and 3/4 with $x < y$, while θ_N is computed for the region with x and y between 1/2 and 3/4 with $x > y$ – the upper and lower triangles of the same square. Since these two regions of integration have the same area, the comparison can go either way.

If function v is symmetric in x, y through Region 5 - for example, if $v(x, y) = (x-0.25)(y-0.25)$ - then the computations of θ_N and θ_{SC} are essentially identical in Region 5. In constructing the preliminary version of the counterexample, we set $v(x, y)$ to take the form $(x - 0.25)(y - 0.25)^2$ to break this tie so that $\theta_N > \theta_{SC}$. With $v(x, y) = 192(x - 0.25)(y - 0.25)^2$, the mixed partial derivative $\frac{\partial^2 v}{\partial x \partial z} = 384(y-0.25)$, which takes larger values in the square from $(0.5, 0.5)$ to $(0.75, 0.75)$ for the upper triangle with $y > x$ than for the lower triangle with $y < x$. Since the values of $\frac{\partial^2 v}{\partial x \partial z}$ in the upper triangle determines θ_N whereas the values of $\frac{\partial^2 v}{\partial x \partial z}$ in the lower triangle determines θ_{SC}, this choice of v produces $\theta_N > \theta_{SC}$.[4]

Generalizing the Counterexample

This preliminary version of the counterexample is not differentiable at the boundaries between regions. We address this point in completing the counterexample by choosing a smooth function with the same properties. We assume a general distribution of types that is symmetric about 1/2, denote $L = \mathbb{E}(x|x \leq 1/2), H = \mathbb{E}(x|x \geq 1/2)$ and choose positive constants $A, B > 0$, and use them to define v as follows:

1. Regions 1 and 2: If $y \leq L, 0 \leq x \leq$H, $v(x, y) = Ax + By$;

2. Regions 1 and 4: If $x \leq L, 0 \leq y \leq$H, $v(x, y) = Ax + By$;

3. Region 5: If $L \leq x, y \leq H$,

   $$v(x, y) = Ax+By+[-2x^3+3(H+L)x^2-6LHx+L^2(3H-L)][-2y^3+3(H+L)y^2-6LHy+L^2(3H-L)]^2;$$

4. Region 6: If $x \geq H$ and $L \leq y \leq H$, $v(x, y) = B(y - H) + v(H, y)$;

[4]By similar logic, if $v(x, y) = (x - 0.25)^2(y - 0.25)$, a value function that emphasizes the value of x rather than the value of y, the result would be reversed so that $\theta_{SC} > \theta_N$.

5. Region 8: If $y \geq H$ and $L \leq x \leq H$, $v(x,y) = A(x-H) + v(x,H)$:

6. Region 9: If $x \geq H$ and $y \geq H$, $v(x,y) = A(x-H) + B(y-H) + v(H,H)$.

The function $g(x) = [-2x^3 + 3(H+L)x^2 - 6LHx + L^2(3H-L)]$ is central to this construction. It has the following desirable properties:

1. $g(L) = 0$;

2. $g'(x) = 6(-x^2 + x - 6LH) = 6(H-x)(x-L)$, so $g'(L) = g'(H) = 0$;

3. Both $g(x)$ and $g'(x)$ are strictly positive for $L < x < H$.

Because of these properties, it is possible to add $g(x)[g(y)]^2$ to the baseline value $Ax + By$ in Region 5 while maintaining both differentiability and continuity of v on all boundaries between other regions and Region 5. In addition, $\frac{\partial^2 v}{\partial x \partial y} = 2g'(x)g'(y)g(y)$ on Region 5, which is strictly positive on the interior of Region 5. As in the preliminary version of the counterexample, θ_N is determined by the double integral of $\frac{\partial^2 v}{\partial x \partial z}$ over the x, y between 1/2 and 3/4 with $x < y$, while θ_N is computed for the region with x, y between 1/2 and 3/4 with $x > y$ – the upper and lower triangles of the same square. Since these regions are reflections of each other, we can combine two separate double integrals into one:

$$\theta_N - \theta_{SC} = \int_{0.5}^{H} \int_{0.5}^{z} \frac{\partial^2 v}{\partial a \partial z}(a,z) da dz - \int_{0.5}^{H} \int_{z}^{H} \frac{\partial^2 v}{\partial a \partial z}(a,z) da dz$$

or

$$\theta_N - \theta_{SC} = \int_{0.5}^{H} \int_{0.5}^{z} [\frac{\partial^2 v}{\partial a \partial z}(a,z) - \frac{\partial^2 v}{\partial a \partial z}(z,a)] da dz.$$

Since v takes form $Ax + By + g(x)[g(y)]^2$ on Region 5, $\frac{\partial^2 v}{\partial a \partial z}(a,z) = 2g'(a)g'(z)g(z) > 2g'(a)g'(z)g(a)$ if $z > a$. This shows that the integrand in the calculation of $\theta_N - \theta_{SC}$ is positive, so $\theta_N > \theta_{SC}$.

One last point is that the generalized counterexample exhibits weakly rather than strictly increasing differences in x, y. To address this point, we can add a term of infinitesimal magnitude, such as $\epsilon x^2 y^2$ (where ϵ is a very small positive constant) to the definition of $v(x,y)$ in all regions. The adjusted definition of v provides a continuous and differentiable function with strictly increasing differences in x, y, and for ϵ sufficiently small, it remains the case that $\theta_N > \theta_{SC}$. This completes the definition of the generalized counterexample. ■

B.4 Proof of Proposition 3

Proof. We define the left-hand loss for a district with partisan types on the interval $[a, b]$ as $C(a, y(a, b))$, where $y(a, b) = \mathbb{E}(x|a \leq x \leq b)$. Similarly, we define the right-hand loss for that district as $C(b, y(a, b))$. Since $y(a, b)$ is increasing in each of a and b, the left-hand loss function for interval $[a, b]$ is increasing in b while the right-hand loss function is decreasing in a.[5]

Claim 1: *There exists a School Choice equilibrium.*

Define $h(y)$ to be the expected value of partisan types enrolling in the town when everyone anticipates school quality y and (competitive) price $p(y)$ in the town. This function h is continuous in y since v is continuous in x and y. If $\theta > 0$, then some types $x > 0$ enroll in the town if y is anticipated to be equal to 0, so $h(0) > 0$ and similarly $h(1) < 1$. Therefore, there is a fixed point of h between 0 and 1 and this fixed point corresponds to a School Choice equilibrium. □

Claim 2: *If there is a unique School Choice equilibrium, then there is a two-district Neighborhood School equilibrium where a superset of the partisan types who enroll in the town under school choice enroll under neighborhood assignment.*

Define $L^{2R}(x, \theta)$ as the right-side loss function in district 2 when

(a) district 1 consists of types $(x, x_M(x, \theta))$ where $x_M(x, \theta)$ is chosen so that the left-hand loss function is equal to θ, and

(b) district 2 consists of types $(x_M(x, \theta), x_H(x, \theta))$ where $x_H(x, \theta)$ is chosen so that the left-hand side loss function in district 2 is equal to right-hand side loss function in district 1.

Here $x_M(x, \theta)$ and $x_H(x, \theta)$ are well defined since $C(x, b)$ is increasing in b for $b \geq x$.[6]

First, suppose that there is a boundary equilibrium where partisan types $[0, x_H^{SC} < 1]$ enroll in the town under school choice, and that this is the unique school choice equilibrium.[7] Then the left-hand loss on $[0, x_H^{SC}]$ is less than θ and the right-hand loss on $[0, x_H^{SC}]$ equals θ. We expect to find a

[5]Since a change in a increases both the lower limit of the interval and $y(a, b)$, the effect of a change in a on the left-hand loss function is ambiguous in sign and similarly the effect of a change in b on the right-hand loss function is ambiguous in sign.

[6]If the left-hand loss on $[x, 1]$ is less than θ, then x is greater than the lowest partisan type enrolling under school choice and is not relevant to the analysis. If the left-hand loss on $[x_M(x, \theta), 1]$ is less than the right-hand loss on $[x, x_M(x, \theta)$, then set $L^{2R}(x, \theta)$ as the right-hand loss on $[x_M(x, \theta), 1]$.

[7]Given the assumptions of symmetric single-peaked distribution and positive third-order mixed partial derivatives of v, the right side loss for any interval $[x, 1]$ is greater than the left-side loss for that interval, so there cannot be a boundary equilibrium where types from x to 1 enroll in town t under school choice.

similar boundary equilibrium under neighborhood equilibrium. To look for this, find the minimum value of z such that $L^{2R}(0,z) = \theta$, with intervals $[0, x_M(z,\theta)]$ in district 1 and $[x_M(z,\theta), x_H(z,\theta)]$ in district 2.[8] Denote x^M and x^H as the values of $x^M(z,\theta)$ and $x^H(z,\theta)$ corresponding to this minimum value of z.

If the left-hand loss in district 2 (which equals the right-hand loss in district 1 by construction) is greater than θ, then (1) the boundary school choice equilibrium with interval $[0, x_{SC}]$ has right-hand loss equal to θ and left-hand loss less than θ, and (2) the left-hand loss on $[x_M, x_H]$ is greater than θ and the right-hand loss on this interval is equal to θ. By the Intermediate Value Theorem, there must be an interval $[x_1, x_2]$ where $x_{SC} < x_2 < x_H$ with left-hand and right-hand losses each equal to θ. Then there is a second school choice equilibrium where types $[x_1, x_2]$ enroll in the town, which contradicts the assumption of a unique school choice equilibrium.

If the left-hand loss in district 1 is greater than θ (i.e. $z > \theta$), then there is one interval $[0, x_{SC}]$ with left-hand loss less than θ and right-hand loss equal to θ and another interval $[0, x_H]$ with left-hand loss greater than θ and right-hand loss less than θ, so $x_{SC} < x_H$. If we keep extending the right-hand limit of the interval, then there must be some point $x_{HH} > x_H$ so that the interval $[0, x_{HH}]$ has left-hand loss greater than θ and right-hand loss equal to θ. Therefore, there must be a value m between x_H and x_{HH} with left-hand and right-hand losses each equal to θ on $[0, m]$. But then this would be a second school choice equilibrium, which is a contradiction.

Therefore, if there is a unique school choice equilibrium and it is a boundary equilibrium where partisan types on $[0, x^{SC} < 1]$ enroll in the town under school choice, there is also a boundary equilibrium under neighborhood assignment with $[0, x_M]$ enrolling in district 1 and $[x_M, x_H]$ enrolling in district 2. The right hand loss in the interval $[x_M, x_H] = \theta$, so the right-hand loss for the interval

[8] If there is no such value of z, then there must be intervals $[0,a], [a,1]$ where the right-hand loss on $[0,a]$ equals the left-hand loss on $[a,1]$ and the right-hand loss on $[a,1]$ is less than θ. Since the right-hand loss is greater than the left-hand loss on all intervals of form $[a,1]$, the left-hand loss on $[a,1]$, which equals the left-hand loss on $[0,a]$, is less than θ. If the left-hand loss on $[0,a]$ is also less than or equal to θ, then we have identified a neighborhood equilibrium where all partisan types enroll in the town. Alternately, if the left-hand loss on $[0,a]$ is greater than θ, then $a > x_{SC}$ since by assumption the left-hand loss on $[0, x_{SC}]$ is less than θ. Then the left-hand loss on $[0,1]$ is larger than the left-hand loss on $[0,a]$ so must be greater than θ and the right-hand loss on $[0,1]$ is even larger, so also is greater than θ. We can find a value b between a and 1 so that the right-hand loss on $[0,b]$ is equal to θ. Since right-hand loss on $[0, x^{SC}]$ and right-hand loss on $[0,b]$ are each equal to θ, while left-hand loss on $[0, x^{SC}]$ is less than θ and left-hand loss on $[0,b]$ is greater than θ, there must be some type d between a and b such that left-hand loss equals right-hand loss equals θ on $[0,d]$. This would indicate a second school choice equilibrium with enrollment of partisan types on $[0,d]$, but this contradicts the assumption of a unique school choice equilibrium.

$[0, x_H]$ must be greater than θ. If $x_H < x^{SC}$, then there is a value $x_2^{SC} < x_H < x^{SC}$ such that the right-hand loss on the interval $[0, x_2^{SC}] = \theta$ and there is a second boundary equilibrium under school choice where types from 0 to x_2^{SC} enroll in the town. This is a contradiction.

Next suppose that there is an interior equilibrium under school choice where partisan types on the interval $[x_L^{SC}, x_H^{SC}]$ enroll in the town with $0 < x_L^{SC} < x_H^{SC} < 1$ and that this is the unique equilibrium under school choice. Then since there is no boundary equilibrium under school choice, the right-side loss for interval $[0, x_\theta(0)]$ (where $x_\theta(0)$ is defined to make the left-side loss on this interval equal to θ) must be less than θ. Then since the school choice equilibrium is assumed to be unique, the right-side loss on the interval $(x, x_\theta(x))$ must be less than θ for each $x < x_L^{SC}$ and greater than or equal to θ for $x \geq x_L^{SC}$.

If $L^{2R}(0, \theta) \geq \theta$, then there is a value $\theta^0 \leq \theta$ such that $L^{2R}(x, \theta_0) = \theta$, which produces a candidate neighborhood equilibrium with intervals $[x_L = 0, x_M = x_\theta(0)]$ of types enrolling in district 1 and $[x_M, x_H]$ enrolling in district 2. If instead, $L^{2R}(0, \theta) < \theta$, then either there is a value $x_L > 0$ such that $L^{2R}(x_L, \theta) = \theta$, or $L^{2R}(x, \theta) < \theta$ for each x. In the first case, where $L^{2R}(x_L, \theta) = \theta$, there is a candidate equilibrium with intervals $[x_L, x_M]$ and $[x_M, x_H]$ enrolling in the two districts where the left-hand loss in district 1 and the right-hand loss in district 2 are equal to θ while the left-hand loss in district 2 equals the right-hand loss in district 1 by construction. In the second case, there is a candidate equilibrium with intervals $[x_L, x_M]$ and $[x_M, 1]$ enrolling in the two districts where the left-hand loss in district 1 equals θ, the right-hand loss in district 2 is less than θ and the left-hand loss in district 2 equals the right-hand loss in district 1 by construction.

Each of these candidate neighborhood equilibria (one for the case where $L^{2R}(0, \theta) \geq \theta$, one for the case where $L^{2R}(x, \theta) \leq \theta$ for each x and a third one for the case where $L^{2R}(0, \theta) \leq \theta$ and there exists x_L such that $L^{2R}(x_L, \theta) = \theta$) is an actual neighborhood equilibrium (with competitive pricing) if the right-hand loss in district 1, which equals the left-hand loss in district 2, is less than θ. If instead, the right-hand loss in district 1 is greater than θ in any of these candidate equilibria, then there must be some value of Δ such that the left-hand loss equals θ and the right-hand loss is less than θ on the interval $[x_M + \Delta, x_H]$. By the Intermediate Value Theorem, there must be an interval $[a_2, b_2]$ with $x_L < a < x_M + \Delta$ such that left-hand loss and right-hand loss each equal θ, so that there is a school choice equilibrium with types from a_2 to b_2 enrolling in the town. But since the right-hand loss on the interval $[x, x_\theta(0)]$ is less than θ there must be another interval $[a_1, b_1]$ where $0 < a_1 < x_L$ where left-hand loss and right-hand loss are each equal to θ, producing another school choice equilibrium.[9] This is a contradiction, so given the assumption of a unique school choice equilibrium, the candidate equilibria in each of these three cases is an actual neighborhood equilibrium.

In the case of a unique interior school choice equilibrium, we have identified a neighborhood equilibrium where partisan types on $[x_L, x_M]$ enroll in district 1, partisan types on the interval $[x_M, x_H]$ enroll in district 2, the left-hand loss in district 1 and the right-hand loss in district 2 equal θ and right-hand loss in district 1 and left-hand loss in district 2 are less than θ. We now want to show that all types who enroll in the unique interior school choice equilibrium also enroll in the associated neighborhood equilibrium that we have identified.

Since the left-hand loss on the interval $[x_L, x_M]$ is equal to θ, the left-hand loss on $[x_L, x_H]$ must be greater than θ. Since the left-hand loss on the interval $[x_L, x_H]$ is greater than θ and the left-hand loss on the interval $[x_M, x_H]$ is less than θ, there must be a type x_{LM} with $x_L < x_{LM} < x_M$ such that the left-hand loss on the interval $[x_{LM}, x_H]$ is equal to θ. Since the right-hand loss on the interval $[x_M, x_H]$ equals θ and the right-hand loss and $x_{LM} < x_M$, the right-hand loss on the interval $[x_{LM}, x_H]$ must be greater than θ. In sum, we have identified intervals $[x_L, x_M]$ and $[x_{LM}, x_H]$ with left-hand loss equal to θ in each interval, right-hand loss less than θ for the first interval and right-hand loss greater than θ in the second interval. Since $x_L < x_{LM}$, there must be an interval with starting point between x_L and x_{LM} where left-hand loss and right-hand loss are each equal to θ – this interval must be exactly the range of types enrolling in the school choice equilibrium, so $x_L < x_{SC} < x_{LM} < x_M$, so every type enrolling in the (interior) school choice equilibrium also enrolls in the neighborhood equilibrium. ■

[9] In fact, there at least three school choice equilibria in this case because the left-hand loss is less than the right-hand loss for any interval $[x, 1]$.

C Additional Analysis and Examples (Online Appendix)

Example 2 discusses a variant of Example 1 where both partisanship and type are continuously distributed. Example 3 demonstrates that it is possible to have a unique boundary equilibrium under school choice. Example 4 shows that it is possible to have a unique interior School Choice equilibrium (neither of which is true for all values of θ in Example 1). Example 5 shows that our assumption of increasing differences does not always imply that average utility is higher under neighborhood assignment.

C.1 Example 2: Continuous Partisanship

Suppose that $v(x, y) = xy$ and that both types and the partisan bonus are drawn from two independent continuous distributions. Furthermore, suppose that type x is distributed $U(0, 1)$ as in Example 1.

School Choice Equilibrium: There is a school choice equilibrium where a partisan with bonus θ_j enrolls in the town if $\frac{1}{2} - \sqrt{2\theta_j} \le x_j \le \frac{1}{2} + \sqrt{2\theta_j}$. Each decision rule is symmetric about $\frac{1}{2}$, so with x distributed uniformly on $(0, 1)$, the school quality is 1/2 and the equilibrium price is 1/8, just as in Example 1.

Neighborhood School Equilibrium: There is also a neighborhood equilibrium with two districts and equilibrium school qualities $y_1 = \frac{1}{4}$ and $y_2 = \frac{3}{4}$. Here a partisan with bonus θ_j enrolls in the town if $\frac{1}{2} - \sqrt{2\theta_j} \le x_j \le \frac{1}{2} + \sqrt{2\theta_j}$. Each decision rule is symmetric about $\frac{1}{2}$, so with x distributed uniformly on (0, 1), the school quality is still 1/2 and the equilibrium price is 1/8, just as in Example 1.

In this extended example, as in Example 1, for each partisan bonus, a wider range of type x's choose the town under neighborhood assignment than under school choice. One difference here, however, is that for the smallest values of θ_j, the set of type x's enrolling under neighborhood assignment does not necessarily subsume the set of types x's enrolling under school choice, as the neighborhood ranges may be near $\frac{1}{4}$ and $\frac{3}{4}$ while the school choice range is near to $\frac{1}{2}$.

A special feature of this case is that partial derivatives of the value function are the same everywhere. So it is also possible to construct equilibrium, which moves the center of the school choice interval. For instance, if range is $[0.4, 0.6]$ for SC, there will be any SC equilibrium with length 0.5. However, given a SC interval, we can construct a Neighborhood equilibrium with middle of SC interval and have that more people stay in the town under the Neighborhood equilibrium than under a School Choice equilibrium, consistent with Theorem 1.

C.2 Example 3: Unique Boundary Equilibrium under School Choice

Suppose that the distribution of types is $U(0,1)$ as in Example 1 and that $v(x,y) = xy^2$. Now $\frac{\partial^2 v}{\partial x \partial y} = 2y$, which is constant in x and strictly increasing in y. With a Uniform distribution of types, the school quality for any interval of types $[a,b]$ is exactly in the middle of the range of types at $\frac{a+b}{2}$. Further, since $\frac{\partial^3 v}{\partial x^2 \partial y} = 0$ and $\frac{\partial^3 v}{\partial x \partial y^2} > 0$, if types in a range $[a,b]$ enroll in a school, the right-hand loss function at type b is greater than the left-hand loss function at type a. Thus, any school choice equilibrium is a boundary equilibrium where types from $x = 0$ to $x = x_{SC}$ enroll in the town, with school quality $y_{SC} = x_{SC}/2$ and price $p_{SC} = p(y_{SC}) = (2/3)y_{SC}^3 = x_{SC}^3/16$.

In equilibrium, type x_{SC} must be exactly indifferent between enrolling in the town and taking the outside option. That is $x_{SC}(x_{SC}/2))^2 - p(y_{SC}) + \theta = \pi(x_{SC})$, or $x_{SC}^3/4 - (2/3)(x_{SC}^3)(1/8) + \theta = (1/3)(x_{SC}^3)$ or $\theta = x_{SC}^3(1/3 + 1/12 - 1/4) = x_{SC}^3/6$. Solving for x_{SC} as a function of θ, $x_{SC} = (6\theta)^{1/3}$ identifies a unique school choice equilibrium for this case where types on $[0, x_{SC}]$ enroll in the town. For example, with $\theta = \frac{1}{48}, x_{SC} = \frac{1}{2}$, so types $[0, 1/2]$ enroll in the town with $y_{SC} = 1/4$ and

$$p_{SC} = p(y_{SC}) = (2/3)(1/4)^3 = 1/96.$$

For similar reasons, we anticipate a boundary equilibrium with neighborhood assignment where types $[x_L = 0, x_M]$ enroll in district 1 and types $[x_M, x_H]$ enroll in district 2 in the town. Since the left-hand loss is greater than the right-hand loss in each district (and the price in each district should equal the competitive price), the equilibrium conditions are that (1) the right-hand loss in district 2 equals θ; (2) the left-hand loss in district 2 equals the right-hand loss in district 1; (3) the left endpoint in district 1 is equal to 0. These equilibrium conditions yield polynomial equations that are not easily solvable in closed form, so instead we used numerical methods to approximate the equilibrium with $\theta = 1/48$: $x_M \approx 0.471$ and $x_H \approx 0.8158$. Thus with $\theta = 1/2$, types from 0 to 1/2 enroll in the town under school choice, while types from 0 to about 0.8158 enroll in equilibrium with a neighborhood assignment rule. Consistent with Proposition 3, any type enrolling under school choice will also enroll in equilibrium under a neighborhood assignment rule.

C.3 Example 4: Unique Interior Equilibrium under School Choice

Suppose that $v(x) = xy$ as in Example 1, but that the distribution of types is triangular on $(0,1) : f(x) = 4x$ if $x \leq 1/2$ and $f(x) = 4(1-x)$ if $x \geq 1/2$. With this distribution function, the expected value of types on an interval $[a, b]$ is greater than $\frac{a+b}{2}$ if $b < 1 - a$. This rules out the possibility of a boundary equilibrium (which would require a lower loss at the boundary rather than the other end of the interval of types enrolling in the town) under school choice. An interior equilibrium requires the school quality to be exactly equal to $\frac{a+b}{2}$, which is only possible if the interval of types is symmetric about $1/2 : [1/2 - \Delta, 1/2 + \Delta]$, with school quality $y_{SC} = 1/2$ and $p_{SC} = p(1/2) = 1/8$. In equilibrium, type $1/2 - \Delta$ must be indifferent between enrolling in the town and taking the outside option: $(1/2 - \Delta)(1/2) - 1/8 + \theta = (1/2 - \Delta)^2/2$, with solution $\Delta = (2\theta)^{1/2}$. For example, with $\theta = 0.02$, $2\theta^{1/2} = 0.2$, so partisan types in the range $[0.3, 0.7]$ enroll in the unique school choice equilibrium with $y_{SC} = 1/2$ and $p_{SC} = p(1/2) = 1/8$.

By the same logic, there is also an interior equilibrium with neighborhood assignment whenever $\theta < \theta_N$ in this example, again with $x = 1/2$ as the midpoint of enrollment in the town. With two districts and neighborhood assignment, a symmetric equilibrium has types in the range $[1/2-\Delta, 1/2]$ enroll in district 1 and types in the range $[1/2, 1/2 + \Delta]$ enroll in district 2. With a triangular distribution, the expected value of types on the range $[a, b]$ with $a < b \leq 1/2$ is $\frac{2(b^3-a^3)}{3(b^2-a^2)}$. Using this formula with $a = 1/2 - \Delta$ and $b = 1/2$, we find $y_1 = \frac{3-6\Delta+4\Delta^2}{6-6\Delta}$. Type $x = 1/2 - \Delta$ is indifferent between enrolling in district 1 in the town and taking the outside option if $(1/2-\Delta)y_1 - 2y_1^2/2 + \theta = (1/2 - \Delta)^2/2$. Substituting $y_1 = \frac{3-6\Delta+4\Delta^2}{6-6\Delta}$ and $\theta = 0.02$, this indifference condition produces a quartic equation in $\Delta : 8\Delta^4 - 24\Delta^3 + 15.12\Delta^2 + 5.76\Delta - 2.88 = 0$. Only one of the four roots of this equation, partisan types from $[0.158872, 1/2]$ enroll in district 1 while partisan types in the range $[0.5, 0.841128]$ enroll in district 2 in the town with $y_1 \approx 0.358872$ and $y_2 \approx 0.641128$. Once again, any type enrolling in the school choice equilibrium also enrolls in this neighborhood assignment equilibrium.

C.4 Example 5: Aggregate Welfare Can be Higher Under School Choice

Suppose that there are three types $x = 0$, $\frac{1}{2}$, and 1 with associated probabilities $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{1}{4}$ that $v(x, y) = xy$, and that $\theta = 1/32$.

In a Neighborhood School equilibrium with two equal sized districts, low types enroll in district 1, high types enroll in district 2 and middle types divide equally between the two districts. Then, as in Example 1, $y_1 = \frac{1}{4}$, $y_2 = \frac{3}{4}$, $p_1 = 1/32$ and $p_2 = 9/32$. Further, the value of θ is exactly high enough so that each type is indifferent between enrolling in the town and choosing the outside option. (We assume that all types break ties by remaining in the town.)

With a school choice rule, the value of θ is not high enough for all types to enroll in the town. But there is an equilibrium where high and low types choose the outside option, while middle types enroll in the town with $y_{SC} = \frac{1}{2}$ and $p_{SC} = 1/8$. Thus, high and low types are indifferent between the two assignment rules, as they get utility equal to the value of the outside option in each case. However, middle types strictly prefer the School Choice equilibrium.

C.5 Unequal Size Districts

C.5.1 Computing θ_N when $m < 1$

Proposition 4 *When $m < 1$, $D = 2$, $M_{d_1} = M_{d_2} = \frac{1}{2}$, then if θ is sufficiently large, there is an equilibrium where partisan types $[0, x^*]$ enroll in district 1 and partisan types $[x^*, 1]$ enroll in district 2 for some x^* between 0 and 1.*

Proof. Consider a possible assignment of partisan types from 0 to a to district 1 and of types from a to 1 to district 2 with associated school qualities $y_1(a) = E(x|0 \leq x \leq a)$ and $y_2(a) = E(x|a \leq x \leq 1)$, ignoring for the moment whether this assignment is feasible given the measure of houses in each district. Define $\gamma(a) = [v(a, y_1(a)) - p(y_1(a))] - [v(a, y_2(a)) - p(y_2(a))]$ to be the relative gain for the boundary type to choose district 1 rather than district 2 at competitive prices in each district. In terms of the cost function, $\gamma(a) = C(a, y_2(a)) - C(a, y_1(a))$, so $\gamma(0) = C(0, \frac{1}{2}) - C(0, 0) > 0$ and $\gamma(1) = C(1, 1) - C(1, \frac{1}{2}) < 0$. Since C is continuous in each argument, γ is continuous and so there exists some value a^* such that $\gamma(a^*) = 0$.

By construction, if $m \leq \frac{1}{2}$ and θ is sufficiently large, there is an equilibrium where partisan types $[0, a^*]$ enroll in district 1, partisan types $[a^*, 1]$ enroll in district 2 and the prices are competitive: $p(y_1(a^*))$ and $p(y_2(a^*))$ respectively. Since $\gamma(a^*) = 0$, the boundary type at a^* is indifferent between enrolling in districts 1 and 2 under these conditions. Further, since $m \leq \frac{1}{2}$, the measure of partisan types enrolling in each district is less than the assumed capacity (measure $\frac{1}{2}$) of houses in each district. Thus, the allocation is feasible and the remaining houses in each district will be occupied by non-partisans since the equilibrium price is competitive in each district.

If $m > \frac{1}{2}$, define $x_{B1} = F^{-1}(\frac{1}{2m})$ and $x_{B2} = F^{-1}(1 - \frac{1}{2m})$. For $\frac{1}{2} < m < 1$, $\frac{1}{2} < \frac{1}{2m} < 1$, so $\frac{1}{2m} > 1 - \frac{1}{2m}$ and thus $x_{B1} > x_{B2}$. These values are chosen so that when $M_{d_1} = M_{d_2} = \frac{1}{2}$, a division of partisan types $\{[0, x_{B1}], [x_{B1}, 1]\}$ to the two districts fills district 1 with partisan types while a division of partisan types $\{[0, x_{B2}], [x_{B2}, 1]\}$ fill district 2 with partisan types. That is, the measures of partisan types in $[0, x_{B1}]$ and in $[x_{B2}, 1]$ are each equal to $\frac{1}{2}$.

Case 1: If $\gamma(x_{B2}) \leq 0$, then if θ is sufficiently large there is an equilibrium with partisan types in $[0, x_{B2}]$ choosing district 1 and partisan types in $[x_{B2}, 1]$ choosing district 2. In this equilibrium, district 2 includes only partisan types and the equilibrium price in that district may be greater than the competitive price given school quality $y_2(x_{B2})$ while by contrast, district 1 enrolls both partisan and non-partisan types at equilibrium price equal to competitive price given school quality $y_1(x_{B1})$. (Note that since $\gamma(x_{B2} \leq 0)$, partisan boundary types prefer district 2 to district 1 at competitive prices, so the equilibrium price is competitive in district 1 while the equilibrium price in district 2 may be higher than the competitive price to sustain the boundary indifference condition.)

Case 2: If $\gamma(x_{B1}) \geq 0$, then if θ is sufficiently large there is an equilibrium with partisan types in $[0, x_{B1}]$ choosing district 1 and partisan types in $[x_{B1}, 1]$ choosing district 2. In this equilibrium, district 1 includes only partisan types and the equilibrium price in that district may be greater than the competitive price given school quality $y_1(x_{B1})$, while district 2 enrolls both partisan and non-partisan types at equilibrium price equal to competitive price given school quality $y_2(x_{B2})$. (Note that since $\gamma(x_{B1} \geq 0)$, partisan boundary types prefer district 1 to district 2 at competitive prices, so the equilibrium price is competitive in district 2 while the equilibrium price in district 1 may be higher than the competitive price to sustain the boundary indifference condition.)

Case 3: If $\gamma(x_{B2}) > 0$ and $\gamma(x_{B1} < 0)$, there is a value a^* between x_{B2} and x_{B1} where $\gamma(a^*) = 0$. In this case, if θ is sufficiently large there is an equilibrium where partisan types $[0, a^*]$ enroll in district 1, partisan types $[a^*, 1]$ enroll in district 2 and the prices are competitive: $p(y_1(a^*))$ and $p(y_2(a^*))$ respectively, analogous to the case where $m \leq \frac{1}{2}$ as discussed above. By construction, since a^* is between x_{B2} and x_{B1}, the measure of partisans enrolling in each district is less than $\frac{1}{2}$ with non-partisans occupying houses in each district since the equilibrium prices are competitive in each district. ■

Example 2 *Suppose that the distribution of types is uniform $f(x) = 1$ for $0 \leq x \leq 1$ and that $v(x, y) = x^2y^2$.*

With a uniform distribution of types, $y_1(a) = \frac{a}{2}$ and $y_2(a) = \frac{a+1}{2}$. The equilibrium price function is given by $p(y) = y^4/2$. (The general form for the equilibrium price function if $v(x, y) = x^\alpha y^\beta$ is $p(y) = \frac{\beta}{\alpha+\beta}y^{\alpha+\beta}$). The boundary indifference condition for type a at competitive prices is $v(a, y_1(a)) - p(y_1(a)) = v(a, y_2(a)) - p(y_2(a))$, or $a^2(\frac{a}{2})^2 - a^4/2 = a^2(\frac{1+a}{2})^2 - (\frac{1+a}{2})^4/2$. The left-hand side of this equation simplifies to $\frac{7a^4}{32}$, while the right-hand side is $\frac{7a^4+12a^3+2a^2-4a-1}{32}$. Setting these equal gives the cubic equation $12a^3+2a^2-4a-1 = 0$, which factors to $(2a+1)(6a^2-2a-1) = 0$.

Solving the quadratic equation $6a^2 - 2a - 1 = 0$ gives the relevant solution $a^* = \frac{1+\sqrt{7}}{6}$, approximately equal to $a^* = 0.608$. Since there is a single value of a in $[0, 1]$ that satisfies the boundary indifference condition, $\gamma(a) > 0$ for $a < a^*$ and $\gamma(a) < 0$ for $a > a^*$.

There is an equilibrium with partisan types $[0, a^*]$ in district 1 and $[a^*, 1]$ in district 2 and competitive prices in each district so long as each district has a sufficient number of houses to accommodate these partisans (and θ is sufficiently large, as discussed in more detail below). If each district includes measure $\frac{1}{2}$ of houses, then there are enough houses in each district for these partisans if $m\frac{1+\sqrt{7}}{6} \leq 12$ or $m \leq \frac{3}{1+\sqrt{7}}$, which is equivalent to $m \leq \frac{\sqrt{7}-1}{2}$. So, if $0 \leq m \leq \frac{\sqrt{7}-1}{2}$ (approximately 0.823), there is an equilibrium with boundary indifference at competitive prices with partisan types $[0, a^*]$ choosing district 1, partisan types $[a^*, 1]$ choosing district 2, and non-partisan types occupying the remaining houses in each district. To sustain this equilibrium, θ must be sufficiently large so that partisan types $0, a^*, 1$ prefer to locate in the town than to take the outside option, or $\theta \geq \max[C(0, y_1(a^*)), C(a^*, y_1(a^*)), C(1, y_2(a^*))]$.

If $\frac{\sqrt{7}-1}{2} \leq m \leq 1$, there are not enough houses in district 1 to accommodate all partisan types in the range $0 \leq x \leq a^*$. In this case (once again, assuming that θ is sufficiently large), there is an equilibrium with no non-partisan types in district 1. In this equilibrium, partisan types $[0, \frac{1}{2m}]$ choose district 1 and remaining partisan types $[\frac{1}{2m}, 1]$ choose district 2. By construction, $\frac{1}{2m} < a^*$ for $m \geq \frac{\sqrt{7}-1}{2}$, so $\gamma(\frac{1}{2m}) > 0$. At competitive prices in the two districts, a boundary type prefers district 1 to district 2, so the boundary indifference condition requires a competitive price in district 2 and a larger than competitive price in district 1. Thus, in equilibrium, non-partisans fill the remaining houses in district 2; non-partisans are not willing to locate in district 1 under these conditions, which is consistent with equilibrium because partisan types occupy all the houses in district 1. To sustain this equilibrium, θ must be sufficiently large so that partisan types $0, a^*, 1$ prefer to locate in the town than to take the outside option.

Equilibrium with Different Measures of Houses in Each District

Our analysis of the example illustrates a generalized method for finding equilibria with any distribution of houses across the two districts. First, find a^*.[10] If there are enough houses for district 1 to accommodate the partisan types in $[0, a^*]$ and for district 2 to accommodate the partisan types in $[a^*, 1]$, then there is an equilibrium with precisely this assignment of houses and

[10] The discussion in the text assumes a unique solution a^* to $\gamma(a) = 0$. If there are multiple solutions to $\gamma(a) = 0$, the analysis may be slightly more complicated, but the general approach described in the text should still identify an equilibrium.

competitive prices in each district. Otherwise, if there are not enough houses in district 1 to accommodate all partisan types in $[0, a^*]$, then there should be an equilibrium where partisan types in $[0, x_{B1}]$ enroll in district 1 and non-partisans only enroll in district 2. Similarly, if there are not enough houses in district 2 to accommodate all partisan types in $[a^*, 1]$, then there should be an equilibrium where partisan types in $[x_{B2}, 1]$ enroll in district 2 and non-partisans only enroll in district 1.

C.5.2 Extending the Main Result

We now compare θ_N and θ_{SC} in cases where there are not necessarily an equal measure of partisans in the districts, once again focusing on the case of $D = 2$, which requires greatest difference on average between actual and ideal school quality at competitive prices for partisans. We assume Strong Assortative Matching with positive third-order mixed partials of v. We define θ_a as the value of θ (as a function of a) required to sustain a neighborhood equilibrium where partisan types $[0, a]$ enroll in district 1 and partisan types $[a, 1]$ enroll in district 2.

Proposition 5 *With two districts, $\theta_a < \theta_{SC}$ if $a \leq \frac{1}{3}$ or $a \geq \frac{2}{3}$*

The proof relies on two lemmas.

Lemma 4 *The school qualities satisfy $y_1 \geq \frac{a}{2}$ and $y_2 \leq \frac{1+a}{2}$.*

We show the result for y_2, as the mirror-image argument demonstrates the result for y_1. If $a \geq \frac{1}{2}$, then since f is symmetric and single-peaked, f is declining on $[a, 1]$ with the immediate implication that $E(x|a \leq x \leq 1) \leq \frac{1+a}{2}$.

If $a < \frac{1}{2}$, school quality y_2 is a weighted average of $E(x|a \leq x \leq 1-a)$ and $E(x|1-a \leq x \leq 1)$, where $E(x|a \leq x \leq 1-a) = \frac{1}{2}$ since f is symmetric about $x = \frac{1}{2}$ and $E(x|1-a \leq x \leq 1) \leq 1 - \frac{a}{2}$ since f is single-peaked. Specifically, $y_2 = \frac{[F(1-a)-F(a)]E(x|a \leq x \leq 1-a)+[1-F(1-a)]E(x|1-a \leq x \leq 1)}{1-F(a)}$, or $y_2 = \frac{[F(1-a)-F(a)]/2+[1-F(1-a)]E(x|1-a \leq x \leq 1)}{1-F(a)}$ since $E(x|a \leq x \leq 1-a) = \frac{1}{2}$. Further, since f is single-peaked and symmetric about $x = \frac{1}{2}$, $F(1-a) - F(a) \geq 2a - 1$ and $F(1-a) \geq 1 - a$ for $a \leq \frac{1}{2}$, so $1 - F(a) \leq a$. Thus $y_2 \leq [(1-2a)\frac{1}{2} + a(1 - \frac{a}{2})]/[1 - 2a + a] = \frac{1-a^2}{2}/(1-a) = \frac{1+a}{2}$.

Lemma 5 *The incentive conditions for types $x = 0$ and $x = 1$ to enroll in the town in a two-district neighborhood equilibrium are sufficient to ensure the incentive condition for type $x = a$.*

By Lemma 4, $y_2 \leq \frac{1+a}{2}$. Thus $1 - y_2 \geq y_2 - m$ and so given Strong Assortative Matching, $C(1, y_2) \geq C(a, y_2)$. That is, the incentive condition for type $x = 1$ to choose district 2 at a competitive price rather than the outside option guarantees the incentive condition for type $x = a$ to do so. If the price in district 2 is above the competitive price with $p_2 = p(y_2) + \Delta$, then the costs for types $x = 1$ and $x = m$ to choose district 2 rather than the outside option are each increased by Δ to $C(1, y_2) + \Delta$ and $C(a, y_2) + \Delta$. In this case, once again, the incentive condition for type $x = 1$ ensures the incentive condition for type $x = a$ to choose district 2 rather than the outside option. In addition, the boundary indifference condition requires type $x = a$ to be indifferent between choosing district 1 and district 2 in the town in equilibrium. By transitivity, the incentive condition for type $x = 1$ to choose district 2 also ensures the incentive condition for type $x = m$ to choose district 1 rather than the outside option.

Corollary 4 *$\theta_N > \theta_{SC}$ is only possible if partisan type $x = 1$ pays a price above the competitive price for school quality y_2 in district 2 or if partisan type $x = 0$ pays a price above the competitive price for school quality y_1 in district 1.*

Lemma 5 indicates that with $D = 2$, $\theta_N = \max[C(0, y_1) + p_1 - p(y_1), C(1, y_2) + p_2 - p(y_2)]$, whereas $\theta_{SC} = \max\{C(0, \frac{1}{2}), C(1, \frac{1}{2})\} = C(1, \frac{1}{2})$ given Strong Assortative Matching with positive third-order mixed partials of v. With a positive measure of partisans in each district, $y_1 < \frac{1}{2}$ and $y_2 < \frac{1}{2}$, so $\theta_N > \theta_{SC}$ is only possible if θ_N is determined by the incentive conditions in a district with price strictly greater than the competitive price for its school quality.

Proof of Proposition 5:

As suggested by Corollary 4, we consider the two cases $p_2 > p(y_2)$ and $p_1 > p(y_1)$, where one of the two prices is given by the competitive price for the school quality associated with the given district and the other is above that competitive price because of the boundary indifference condition. That is, $p_2 > p(y_2)$ if $v(a, y_2) - p(y_2) > v(a, y_1) - p(y_1)$, so $p_2 = \max\{p(y_2), p(y_1) + v(a, y_2) - v(a, y_1)\}$ and similarly $p_1 = \max\{p(y_1), p(y_2) - v(a, y_2) + v(a, y_1)\}$.

<u>Case 1: $p_2 = p(y_1) + v(a, y_2) - v(a, y_2)$</u>

We first consider the possibility that $\theta_N > \theta_{SC}$ because partisan type 1 prefers the school choice equilibrium to enrolling in district 2 with $D = 2$. Since this is only possible if $p_2 > p(y_2)$, we assume that $p_2 = p(y_1) + v(a, y_2) - v(a, y_2)$. Our analysis below includes instances where $p(y_1) + v(a, y_2) - v(a, y_2) < p(y_2)$ and so our computations sometimes involve a *sub-competitive price* in district 2, but in such cases $p_2 = p(y_2)$ and so partisan types choosing district 2 prefer the proposed neighborhood equilibrium to the school choice equilibrium with $y_{SC} = \frac{1}{2}$. Similarly, our computations for this case assume that θ_N is determined by the incentives for partisan type $x = 1$ to enroll in district 2 when in fact it is determined by the incentives for partisan type $x = 0$ to enroll in district 1 as covered in Case 2 below.

When $p_2 = p(y_1) + v(a, y_2) - v(a, y_1)$, $\theta_N = v(1, 1) - p(1) + p_2 - v(1, y_2)$. So when $p_H = p(y_1) + v(a, y_2) - v(a, y_1)$, $\theta_2 = v(1, 1) - v(1, y_2) - p(1) + p(y_1) + v(a, y_2) - v(a, y_1)$.

In integral form,

$$\theta_N = \int_{y_2}^{1} \frac{\partial v}{\partial y}(1, z), dz - \int_{y_1}^{1} \frac{\partial v}{\partial y}(z, z) dz - \int_{y_1}^{y_2} \frac{\partial v}{\partial y}(m, z) dz.$$

Combining these integrals gives

$$\theta_N = \int_{y_2}^{1} [\frac{\partial v}{\partial y}(1, z) - \frac{\partial v}{\partial y}(z, z)] dz - \int_{a}^{y_2} [\frac{\partial v}{\partial y}(z, z) - \frac{\partial v}{\partial y}(a, z)] dz + \int_{y_1}^{a} [\frac{\partial v}{\partial y}(a, z) - \frac{\partial v}{\partial y}(z, z)] dz.$$

In double-integral form,

$$\theta_N = \int_{y_2}^{1} \int_{z}^{1} \frac{\partial^2 v}{\partial x \partial y}(w, z)] dw dz - \int_{a}^{y_2} \int_{a}^{z} \frac{\partial^2 v}{\partial x \partial y}(w, z) dw dz + \int_{y_1}^{a} \int_{z}^{a} \frac{\partial^2 v}{\partial x \partial y}(w, z) dw dz.$$

By contrast,

$$\theta_{SC} = \int_{1/2}^{1} \int_{z}^{1} \frac{\partial^2 v}{\partial x \partial y}(w, z)] dw dz.$$

Comparing these double integrals for θ_{SC} and θ_N, the positive integrands that are unique to θ_{SC} occur at lower values of (x, y) than those for θ_N. Therefore, θ_N can only be larger than θ_{SC} if the double integral covers a larger proportion of the plane with $0 \le x \le 1$ and $0 \le y \le 1$, so it suffices to consider the case $v(x, y) = xy$ and $\frac{\partial^2 v}{\partial x \partial y} = 1$.

With $v(x, y) = xy$, $p(y) = \frac{y^2}{2}$, as in Example 1. If the boundary indifference condition determines the price in district 2, $p_2 = p(y_1) + v(a, y_2) - v(a, y_1) = \frac{y_1^2}{2} + a(y_2 - y_1)$. Then θ_N is given by $v(1, y_2) - p_2 + \theta_N = v(1, 1) - p(1)$ or $\theta_N = v(1, 1) - v(1, y_2) - [p(1) - p_2]$. With $v(x, y) = xy$, then $\theta_N = (1 - y_2) - [\frac{1}{2} - \frac{y_1^2}{2} - a(y_2 - y_1)] = \frac{1}{2} - y_2 + \frac{y_1^2}{2} + a(y_2 - y_1)$, which is decreasing in both y_1 and y_2 since $y_1 \le a$.

<u>Case 1a: $a \ge \frac{2}{3}$</u>

When $a \ge \frac{2}{3}$, the constraint $y_1 \ge \frac{a}{2}$ from Lemma 4 is sufficient to ensure $\theta_N \le \theta_{SC}$. In this case, since the projected value for θ_N is decreasing in y_1 and y_2, we know that θ_N is greatest at the extreme values $y_2 = \frac{2}{3}$ and $y_1 = \frac{a}{2} = \frac{1}{3}$. With these values, $\theta_N = \frac{1}{2} - \frac{2}{3} + \frac{1}{18} + \frac{2}{3}\frac{1}{3} = \frac{1}{9} < \theta_{SC} = \frac{1}{8}$.

<u>Case 1b: $0 \le a \le \frac{1}{3}$</u>

In this case, we begin with the initial restrictions $y_2 \geq \frac{1}{2}$ and $y_1 \geq \frac{a}{2}$. Substituting these values into $\theta_N = (1 - y_2) - [\frac{1}{2} - \frac{y_1^2}{2} - a(y_2 - y_1)]$ gives $\theta_N = \frac{a}{2} - \frac{3a^2}{8}$, which is increasing for $a \leq \frac{2}{3}$ and equal to $\frac{1}{8}$ at $a = \frac{1}{3}$. This result is comparable to the result from Case 1a: if $a \leq \frac{1}{3}$, then $\theta_N \leq \theta_{SC}$.

<u>Case 2: $p_1 = p(y_2) - v(a, y_2) + v(a, y_1)$</u>

We can apply the same series of steps for the alternative possibility of a larger than competitive price in district 1. In this case, $p_1 = p(y_2) - v(a, y_2) + v(a, y_1) = \int_0^{y_2} \frac{\partial v}{\partial y}(z, z)dz - \int_{y_1}^{y_2} \frac{\partial v}{\partial y}(a, z,)dz$. Then θ_N corresponds to $v(0, y_1) - p_1 + \theta_N = v(0, 0) - p(0)$ or $\theta_N = p_1 - v(0, y_1)$ given the normalizing assumptions $v(0, 0) = p(0) = 0$.

Using the definition of p_1, $\theta_N = \int_0^{y_2} \frac{\partial v}{\partial y}(z, z)dz - \int_{y_1}^{y_2} \frac{\partial v}{\partial y}(a, z)dz - \int_0^{y_1} \frac{\partial v}{\partial y}(0, z)dz$, or

$$\theta_N = \int_0^{y_1} [\frac{\partial v}{\partial y}(z, z) - \frac{\partial v}{\partial y}(0, z)]dz - \int_{y_1}^{a} [\frac{\partial v}{\partial y}(a, z) - \frac{\partial v}{\partial y}(z, z)]dz + \int_a^{y_2} [\frac{\partial v}{\partial y}(z, z) - \frac{\partial v}{\partial y}(a, z)]dz.$$

In double-integral form,

$$\theta_N = \int_0^{y_1} \int_0^{z} \frac{\partial^2 v}{\partial x \partial y}(w, z)dwdz - \int_{y_1}^{a} \int_z^{a} \frac{\partial^2 v}{\partial x \partial y}(w, z)dwdz + \int_a^{y_2} \int_a^{z} \frac{\partial^2 v}{\partial x \partial y}(w, z)dwdz.$$

Whereas θ_{SC} is given by the double integral of the mixed-partial derivative of v on the region with $y \geq a, x \geq y$, the last term in this equation for θ_N includes the double integral of the mixed partial derivative of v on the region with $a \leq y \leq y_2, x \leq y$. None of the terms in these two integrals overlap. We can still carry out a comparison of the integrals because Lemma 4 indicates that $y_2 \leq \frac{a+1}{2}$ since f is assumed to be single-peaked and symmetric about $\frac{1}{2}$. The upper-right portion of the double integral for θ_{SC} on the region $y_2 \leq y \leq 1, x \geq y$ covers a triangle with area $\frac{(1-y_2)^2}{2}$. The farther-right portion of the double integral for θ_N (given that p_1 is above the competitive price for the school quality y_1) covers a triangle with area $\frac{(y_2-a)^2/2}{2}$. Since $y_2 \leq \frac{a+1}{2}$, the upper right section of the double integral for θ_{SC} covers a greater area than the farthest-right section of the double integral for θ_N and lies strictly to the right and above the farthest-right section of the double integral for θ_N. So, as in Case 1, it suffices once again to consider $v(x, y) = xy$ where the mixed partial derivative is equal to 1.

<u>Case 2a and 2b:</u>

With $v(x, y) = xy$, Case 2 provides a mirror image to Case 1, whereby the result in Case 1a that $\theta_{SC} > \theta_N$ if $a \geq \frac{2}{3}$ (for computations based on p_2) carries over to show that $\theta_{SC} > \theta_N$ if $a \leq \frac{1}{3}$ (for computations based on p_1). Similarly the result in Case 1b that $\theta_{SC} > \theta_N$ if $a \leq \frac{1}{3}$ (for computations based on p_2) carries over to show that $\theta_{SC} > \theta_N$ if $a \geq \frac{2}{3}$ (for computations based on p_1).

Proposition 5 limits the range of possibilities for which $\theta_N > \theta_{SC}$. We now show that it is possible to identify conditions that yield $\theta_N > \theta_{SC}$ for values of a closer to $\frac{1}{2}$ but also that these conditions are highly artificial.

Proposition 6 *$\theta_N > \theta_{SC}$ is possible with partisan types $[0, a]$ enrolling in district 1 and partisan types enrolling in district 2 and price $p_2 > p(y_2)$ if $\frac{1}{3} < a < \frac{1}{2}$ and $F(m) < \frac{3a-1}{3-a}$.*

We first consider the possibility that $p_2 > p(y_2)$, following the methods from the proof of Proposition 4. For consistency with the discussion above, we label the subcases with $p_2 > p(y_2)$ and $\frac{1}{3} \leq a \leq \frac{1}{2}$ as Cases 1c ($\frac{1}{2} \leq a \leq \frac{2}{3}$) and Cases 1d ($\frac{1}{3} \leq a \leq \frac{1}{2}$). As in Cases 1a and 1b, our analysis of Cases 1c and 1d includes instances where $p(y_1) + v(a, y_2) - v(a, y_1) < p(y_2)$ and so our computations sometimes involve a *sub-competitive price* in district 2 but in these cases $p_2 = p(y_2)$ and so partisan types choosing district 2 prefer the proposed neighborhood equilibrium to the school choice equilibrium with $y_{SC} = \frac{1}{2}$.

Case 1c: $\frac{1}{2} \leq a \leq \frac{2}{3}$

In this case, the expected value formula $F(a)y_1 + (1 - F(a))y_2 = \frac{1}{2}$ is a stronger restriction on y_1 than $y_1 \geq \frac{a}{2}$. Denoting $y_2 = \frac{1}{2} + \Delta$ and solving the expected value formula for y_1 gives $y_1 = \frac{1}{2} - \Delta\frac{1-F(a)}{F(a)} \geq \frac{1}{2}(1 - \Delta)$, i.e. $y_1 \geq 1 - y_2$ if $a \geq \frac{1}{2}$ (since $a \geq \frac{1}{2}$ implies $F(a) \geq \frac{1}{2}$ and in turn $\frac{1-F(a)}{F(a)} \leq 1$).

So with $\frac{1}{2} \leq a \leq \frac{2}{3}$, we can substitute $y_1 \geq 1 - y_2$ in the expression for θ_N to conclude that $\theta_N \leq (1-y_2)-[\frac{1}{2}-\frac{1-y_2^2}{2}-a(2y_2-1)]$. The right-hand side of this inequality is (still) decreasing in y_2 since $y_2 \geq a$, so it takes its maximum value in y_2 at $y_2 = a$ for each a. At $y_2 = a, y_1 = 1-y_1 = 1-a$, we have $\theta_N = 1 - 3a + 5\frac{a^2}{2}$, which is a convex function in a so it takes a maximum value at either extreme of the range $\frac{1}{2} \leq a \leq \frac{2}{3}$. At $a = \frac{1}{2}$, this formula yields $\theta_N = 1 - \frac{3}{2} + \frac{5}{8} = \frac{1}{8}$; at $a = \frac{2}{3}$, it yields $\theta_N = 1 - 2 + 5\frac{4}{18} = \frac{1}{9}$ (the same result as for $a = \frac{2}{3}$) in Case 1a. These values are both less than or equal to $\theta_{SC} = \frac{1}{8}$.

Case 1d: $\frac{1}{3} \leq a \leq \frac{1}{2}$

Once again, we apply the expected value equation $F(a)y_1 + [1 - F(a)]y_2 = \frac{1}{2}$ or $y_2 = \frac{1-aF(a)}{2[1-F(a)]}$. Thus,

$$\theta_N = \frac{1}{2} - y_2 + \frac{y_1^2}{2} + a(y_2 - y_1) = \frac{1}{2} - \frac{1 - 2F(a)y_1}{1 - F(a)} + \frac{y_1^2}{2} + a(\frac{1 - 2y_1}{2[1 - F(a)]}).$$

(Note that $y_2 - y_1 = \frac{1-2F(a)y_1-2y_1+2F(a)y_1}{2[1-F(a)]} = \frac{1-2y_1}{(2[1-F(a))]}$.)

Since θ_N is a convex function of y_1, it takes a maximum at an extreme value of y_1. For fixed $a \leq \frac{1}{2}$, y_1 is bounded below by $\frac{a}{2}$ and above by a. Therefore, as a function of $a \leq \frac{1}{2}$, θ_N takes a maximum value either at $x_1 = \frac{a}{2}$ or at $x_1 = a$.

Case 1d1: $\frac{1}{3} \leq a \leq \frac{1}{2}$ and $y_1 = a$

In this case,

$$\theta_N = \frac{1}{2} - y_2 + \frac{y_1^2}{2} + a(y_2 - y_1) = \frac{1}{2} - \frac{1 - 2y_1 F(a)}{2[1 - F(a)]} + \frac{a^2}{2} + \frac{1 - 2y_1 F(a)}{2[1 - F(a)]} - a.$$

Simplifying this expression gives

$$\theta_N = \frac{a + a^2 F(a) - a^2 - F(a) + 2a(1-a)F(a)}{2[1 - F(a)]}.$$

We are interested in $\theta_N \geq \theta_{SC} = \frac{1}{8}$, or $a + a^2 F(a) - a^2 - F(a) + 2a(1-a)F(a) \geq \frac{1-F(a)}{4}$. Reorganizing the inequality to solve for $F(a)$ gives $F(m)(-3 + 8a - 4a^2) \geq 1 - 4a + 4a^2$. Since $-3 + 8a - 4a^2 = (3 - 2a)(2a - 1)$, the coefficient for $F(a)$ on the left-hand side of the inequality is negative for $a < \frac{1}{2}$ and equal to 0 for $a = \frac{1}{2}$. Further, $1 - 4a + 4a^2 = (2a - 1)^2$, so the right-hand side of the inequality is positive for $a < \frac{1}{2}$. Therefore, the inequality fails for $a \leq \frac{1}{2}$ and $\theta_N < \theta_{SC}$ when $a < \frac{1}{2}$ and the boundary indifference condition yields competitive price in district 1 and a larger than competitive price in district 2. The boundary case $a = \frac{1}{2}, y_1 = \frac{1}{2}$ corresponds to the degenerate case where the distribution of types consists only of a point mass at $x = \frac{1}{2}$.

Case 1d2: $\frac{1}{3} < a < \frac{1}{2}$ and $y_1 = \frac{a}{2}$

At $x_1 = \frac{a}{2}$,

$$\theta_N = \frac{1}{2} - \frac{1 - aF(a}{2[1 - F(a)]} + \frac{a^2}{8} + \frac{a}{2[1 - F(a)]} - \frac{a^2}{2[1 - F(a)]}.$$

Regrouping terms over a common denominator,

$$\theta_N = \frac{a - a^2 + aF(a) - F(a)}{2[1 - F(a)]} + \frac{a^2}{8} = \frac{4a - 3a^2 + 4aF(a) - 4F(a) - a^2 F(a)}{8[1 - F(a)]}.$$

We are interested in comparing θ_N to $\theta_{SC} = \frac{1}{8}$, where $\theta_N > \frac{1}{8}$ is equivalent to

$$4a - 3a^2 + 4aF(a) - 4F(a) - a^2 F(a) > 1 - F(a).$$

Reorganizing and solving for $F(a)$ gives $F(a) < \frac{3a^2 - 4a + 1}{4a - 3 - a^2}$, which is equivalent to $F(a) < \frac{3a-1}{3-a}$, where $\frac{3a-1}{3-a}$ is strictly increasing for $\frac{1}{3} < a < \frac{1}{2}$. The boundary cases are $F(\frac{1}{3}) < 0$ and $F(\frac{1}{2}) < 0.2$.

In sum, Case 1d2 with $\frac{1}{3} < a < \frac{1}{2}$ and $y_1 = \frac{a}{2}$ is the only one where $\theta_N > \theta_{SC}$ is possible with $p_2 > p(y_2)$. In each case where $F(a) < \frac{3a-1}{3-a}$ and $y_1 = E(x|x \leq a) = \frac{a}{2}$ (i.e. where x is uniformly distributed on $[0, a]$ with $f(a) = \frac{F(a)}{a}$ on this range), there is an equilibrium with $v(x, y) = xy$ and measures of houses, $M_{d_2} = M(1 - F(m)), M_{d_1} = 1 - M_{d_2}$ where partisan types $[0, a]$ enroll in district 1, partisan types $[a, 1]$ enrolling in district 2 and $\theta_N > \theta_{SC}$. Since $p_2 > p(y_2)$, district 2 consists solely of partisan types while district 1 includes both partisan types and non-partisans. This same constructive method also identifies additional equilibria with $\frac{1}{3} < a < \frac{1}{2}$, $y_1 > \frac{m}{2}$ and $\theta_N > \theta_{SC}$, though with tighter restrictions on $F(a)$ than $F(a) < \frac{3a-1}{3-a}$.

Corollary 5 *$\theta_N > \theta_{SC}$ is possible with partisan types $[0, a]$ enrolling in district 1 and partisan types enrolling in district 2 and price $p_1 > p(y_1)$ if $\frac{1}{2} < a < \frac{2}{3}$ and $F(a) < \frac{2-3a}{2+a}$.*

The corollary follows applying the analysis of Proposition 6 with $p_1 > p(y_1)$, which produces the mirror image result, with $\theta_N > \theta_{SC}$ under analogous conditions with $\frac{1}{2} < a < \frac{2}{3}$. The constraint $F(m) < \frac{2-3a}{2+a}$ with $\frac{1}{2} < a < \frac{2}{3}$ is the same constraint as in Proposition 6 after replacing a with $1-a$ in the constraint for $F(a)$.

■

Discussion:

The cases where $\theta_N > \theta_{SC}$ result from divisions of partisan types in a neighborhood equilibrium that resemble the school choice outcome, with almost all partisan types choosing to locate in a single district with school quality close to $y_{SC} = \frac{1}{2}$. The requirement $F(a) \leq \frac{3a-1}{3-a}$ yields a continuous and increasing constraint over the range $a \in [\frac{1}{3}, \frac{1}{2}]$ with boundary cases $F(\frac{1}{3}) \leq 0$ and $F(\frac{1}{2}) \leq \frac{1}{2}$.

Example: Consider the value $a = 0.45$ where $\frac{3a-1}{3-a} = \frac{7}{51} \approx 0.137$. With $F(0.45) = 0.1$ (an arbitrary value chosen to satisfy $F(0.45) < \frac{7}{51}$), $y_1 = \frac{m}{2}$ corresponds to $y_1 = 0.225$, $y_2 \approx 0.531$ and $\theta_N \approx 0.132 > \theta_{SC} = 0.125$.[11] This value of θ_N results from an allocation of houses that can accommodate 90% of partisans in district 2 (i.e. $M_{d_2} \geq 0.9m$) and requires an imbalance of houses between the two districts if m is greater than $\frac{5}{9}$.

In addition, distribution f must have two contrary features to support $\theta_N > \theta_{SC}$. First, the distribution must be relatively tight around $a = \frac{1}{2}$ to satisfy the restriction $F(a) \leq \frac{2-3a}{2+a}$. Second, the distribution must be relatively flat from $x = 0$ to $x = a$ so that $y_1 = E(x|x \leq a)$ is close to the minimum possible value $\frac{a}{2}$ identified by Lemma 4. These properties are broadly in conflict with each other because most distributions that are concentrated around a central value tend to fall off quite quickly with distance from that central point. For instance, a Normal distribution

[11] By symmetry, the mirror image case with 90% of partisan types (those with $x \in [0, 0.55]$) in district 2, $y_1 \approx 0.469$ and $y_2 = 0.775$ also produces $\theta_N \approx 0.132$.

with mean $\frac{1}{2}$ has $F(0.45) = 0.1$ if its standard deviation is approximately 0.039, in which case the proposed school quality $y_1 = 0.225$ is a full seven standard deviations below the mean. In fact, $y_1 = E(a|a \leq 0.45) \approx 0.432$ for a Normal distribution with mean 0.5 and standard deviation 0.039, but $\theta_N \approx 0.121 < \theta_{SC}$ for $a = 0.45, F(0.45) = 0.1$ and $y_1 = 0.432$. (With $a = 0.45, F(a) = 0.1$, $y_1 \leq 0.278$ is the approximate requirement for $\theta_N > \theta_{SC}$.) That is, even when the distribution of types is quite tight around $x = \frac{1}{2}$, and most of the partisan types choose to locate in one of two districts under neighborhood assignment, allowing for a distribution with reasonable shape on $[0, a]$ eliminates the possibility that $\theta_N > \theta_{SC}$.

D Two-Town Model (Online Appendix)

D.1 Setup

We now alter the analysis to consider a general equilibrium version of the model with two towns, A and B. We also assume that an equal number of partisans are attached to each town, there are no non-partisans, and that each family must choose a house in either town A or town B. In the two-town model, outside options are determined endogenously in equilibrium in contrast to Assumption 2 in the main text.

As before, we assume that the utility function for each family is given by

$$u(x_i, y_j, p_j) = \theta_{ij} + v(x_i, y_j) - p_j,$$

where $\theta_{ij} = \theta > 0$ if family i is partisan to town t and school j is in town t, and $\theta_{ij} = 0$ if family i is partisan to town t and school j is not in town t. We assume a continuum of partisan families of measure 1 for each town and that partisans of both types have identical distributions for student type $f(x)$ on [0, 1], maintaining all other properties assumed for f and v from the one-town model.

Each town has D equal-size districts, which we label as $A_1, A_2, ..., A_D$ for town A and $B_1, B_2, ..., B_D$ for town B. Districts are ordered in ascending school quality: $y_{t_1} \leq y_{t_2} \leq \ldots \leq y_{t_D}$ for each town $t \in \{A, B\}$. We denote the sets of town-A and town-B partisans choosing district d in town t as α_{t_d} and β_{t_d}, respectively, and denote an assignment of town-A partisans to districts by $\alpha = \{\alpha_{A_1}, \alpha_{A_2}, ..., \alpha_{A_D}, \alpha_{B_1}, \alpha_{B_2}, ..., \alpha_{B_D}\}$ and an assignment of town-B partisans to districts by $\beta = \{\beta_{A_1}, \beta_{A_2}, ..., \beta_{A_D}, \beta_{B_1}, \beta_{B_2}, ..., \beta_{B_D}\}$.

Definition 1 *A **two-town equilibrium** is an allocation of families to schools, α and β, associated average types in each district $\{y_{A_1}, y_{A_2}, ..., y_{A_D}, y_{B_1}, y_{B_2}, ..., y_{B_D}\}$ and prices $(p_{A_1}, p_{A_2}, ..., p_{A_D}, p_{B_1}, p_{B_2}, ..., p_{B_D})$ where*

(1) Each student maximizes utility $u(x_i, y_d, p_d)$ with the choice of school district d,

(2) Each district d enrolls $\frac{1}{D}$ students,

(3) If town $t \in \{A, B\}$ uses a school choice rule, then $y_{t_1} = y_{t_2} = ... = y_{t_D} = \mathbb{E}[x|$ enroll in town $t]$.

A focal outcome in this model is one where all partisans of a given town reside in that town. We define this as a non-integrated equilibrium. Our first result is on the existence of a symmetric non-integrated equilibrium where partisans make the same housing decision in each town.

Definition 2 *In a **non-integrated equilibrium**, all town-A partisans live in town A and all town-B partisans live in town B. In an **integrated equilibrium**, either some town-A partisans live in town B or some town-B partisans live in town A.*

Proposition 1 *If both towns use the same assignment rule, then there is a symmetric non-integrated equilibrium with cutoffs $\{x_0 = 0, x_1, x_2, ..., x_{D-1}, x_D = 1\}$, where students of type $x \in [x_{d-1}, x_d]$ enroll in district d of their partisan town.*

This result is immediate whether both towns use neighborhood assignment or school choice. In both cases, the options and prices for schooling in two towns are identical, so clearly town-A partisans will choose to live in town A and town-B partisans will choose to live in town B. With a neighborhood schooling rule in both towns, (1) the type cutoffs are determined by the capacities in each district and the implicit equation $F(x_d) = \frac{d}{D}$ for each d, (2) the school qualities equal the conditional expectation $y_{A_d} = y_{B_d} = y_d = \mathbb{E}[x|x_{d-1} < x < x_d]$ for each d, and (3) price increments between districts in towns A and B are determined by the boundary indifference conditions

$$p_d - p_{d-1} = v(x_d, y_d) - v(x_d, y_{d-1})$$

for districts $d = 2, ..., D$. Then by construction, given the property of increasing differences of v in x and y, any choice of price for district 1, $p_{A_1} = p_{B_1} = p_1$ induces the precise sorting of students to districts as stated in the proposition. The resulting symmetric non-integrated equilibrium is stable for either assignment rule if θ is strictly greater than 0, in the sense that a small change in locational choices will not induce any partisan to switch towns at the cost of θ.[12]

We use the non-integrated equilibrium with neighborhood school assignment in each town as the baseline outcome for comparisons to the results when one town adopts school choice primarily because it is the unique symmetric equilibrium when both towns use neighborhood assignment rules and all districts are the same size. Furthermore, there is perfect sorting of partisans within each town in this non-integrated neighborhood school equilibrium, so the adoption of a school choice rule necessarily reduces inequalities in school assignment if families do not move.

[12]There may also be equilibria other than the non-integrated outcome when both towns use the same school assignment rule. For example, if both towns use a school choice rule, there could be an equilibrium where one town has higher school quality than the other and town-A partisans and town-B partisans of highest types both choose the higher quality school. One complication is that if town A has the higher quality school in this case, then partisans of town B must forego θ to attend that school, while partisans of town A gain θ by choosing it, so any equilibrium other than the no mixing equilibrium involves asymmetric decision rules for partisans of town A and partisans of town B.

D.2 School Choice in Town A and Neighborhood Schools in Town B

Suppose that town A uses the school choice rule and town B uses the neighborhood assignment rule. At this point, we specialize in the analysis to the case with $D = 2$ districts in town B. To simplify notation, we denote the equilibrium school quality and price (for each district) in town A as y_A and p_A and the corresponding values in town B as y_{B_1} and p_{B_1} for district 1 and y_{B_2} and p_{B_2} for district 2, using the convention that $y_{B_1} \leq y_{B_2}$.

Much of the intuition from the one town model carries over to the two-town model. In particular, Proposition 2 indicates that when town A adopts school choice, partisan enrollment takes the form of intervals in each district. Furthermore, the range of types of town-A partisans enrolling in town A subsumes the range of types of town-B partisans who enroll in town A.

Proposition 2 *In any equilibrium where town A uses school choice and town B uses neighborhood assignment, an interval $[x_A^L, x_A^H]$ for town-A partisans and an interval $[x_B^L, x_B^H]$ of town-B partisans enroll in town A, where $x_A^L \leq x_B^L \leq x_B^H \leq x_A^H$.*[13]

We distinguish between three types of equilibria according to the ordering of school quality y_A in town A relative to school qualities in the two districts in town B, y_{B1} and y_{B2}.

1. In a **Type 1 equilibrium**, $y_A > y_{B_2} > y_{B_1}$;
2. In a **Type 2 equilibrium**, $y_{B_2} > y_A > y_{B_1}$;
3. In a **Type 3 equilibrium**, $y_{B_2} > y_{B_1} > y_A$.[14]

For relatively large values of θ, partisans of each town have a strong incentive to enroll in that town. In the limiting case, partisans of town B with types above the median enroll in district B_2, partisans of town B with types below the median enroll in district B_1 and all partisans of town A enroll in town A, producing a unique equilibrium which happens to be of Type 2. For relatively small values of θ, however, this logic need not hold and it is possible that there can be equilibria (one or more) of each type for a given value of θ.

[13] In a non-integrated equilibrium, since all town-A partisans and no town-B partisans enroll in town A, $x_A^L = 0$ and $x_A^H = 1$. In this case, we set $x_B^L = x_B^H = y_{SC}$ and the result holds. It is natural to set $x_B^L = x_B^H = y_{SC}$ because the first town-B partisans to enroll in town A will be those of types nearest to y_{SC}.

[14] We make an explicit choice to use strict rather than weak inequalities in these definitions. As long as $\theta > 0$, $y_A = y_{B_1}$ or $y_A = y_{B_2}$ is only possible in an equilibrium where only town-A partisans enroll in town A, but this in turn implies the trivial equilibrium with $y_A = y_{B_1} = y_{B_2} = 0.5$. Similarly, if $y_{B_1} = y_{B_2}$, then we have identified an equilibrium where the two districts in B are identical and so it is as if B has adopted school choice rather than a neighborhood assignment rule.

D.3 Example

Example 3 *Suppose that the distribution of types is Uniform on (0, 1) for partisans of each town and that the utility function is* $u(x, y) = xy$*.*

We consider several possibilities in turn. We leave out detailed computations for equilibria of Type 3. Given the symmetry of the example (and the fact that third-order cross partials of $u(x, y) = xy$ are equal to 0), Type 1 and Type 3 equilibria are essentially mirror images of each other.

Case 1: Non-Integrated Equilibrium

In a non-integrated equilibrium, town-B partisans are partitioned into districts with types $[0, 1/2]$ in district 1 and types $[1/2, 1]$ in district 2 so that $y_1 = 1/4$ and $y_2 = 3/4$, while all town-A partisans choose town A so that $y_A = 1/2$. We work backwards from the equilibrium conditions to identify equilibrium prices and restrictions on θ to construct a non-integrated equilibrium. A marginal town-B partisan at $x = 1/2$ must be indifferent between districts 1 and 2. Thus,

$$\frac{1}{2}y_1 - p_1 = \frac{1}{2}y_2 - p_2,$$

or equivalently $p_2 - p_1 = 1/4$.

Given $p_2 - p_1 = 1/4$, partisans of either town with $x < 1/2$ prefer district 1 to 2 in town B. The incentive condition for town-A partisans with $x < 1/2$ to choose A is $x/2 + \theta - p_A \geq x/4 - p_1$, or $\theta \geq p_A - p_1$ at $x = 0$ where the condition is most binding. Similarly, the incentive condition for partisans of town B with $x < 1/2$ to choose B_1 is $x/4 + \theta - p_1 \geq x/2 - p_A$, or $\theta \geq 1/8 - p_A + p_1$ at $x = 1/2$ where the condition is most binding. Thus, the smallest value for which both conditions hold jointly is $\theta_{NI} \equiv 1/16$, and hence $p_A - p_1 = 1/16$. (A similar argument shows that the incentive conditions for partisans with types $x > 1/2$ also hold simultaneously at $\theta = 1/16$ when $p_2 - p_A = 3/16$).

In sum, there is a Non-Integrated Equilibrium if $\theta \geq 1/16$.

Case 2: Integrated Equilibrium of Type 2

For values of $\theta < 1/16$, we simplify computations by looking for an integrated equilibrium with symmetric cutoffs x_A^L and $x_A^H = 1 - x_A^L$. Given the constraints that 1/4 of all students must enroll in each district in town B (and half of all students must enroll in town A),

$$x_B^L = \frac{1}{2} - x_A^L$$

and

$$x_B^H = 3/2 - x_A^H = \frac{1}{2} + x_A^L.$$

Thus, under the assumption that $x_A^H = 1 - x_A^L$, equilibrium assignments can be described as a function of x_A^L alone. Furthermore, by Proposition 2, $x_B^L \geq x_A^L$, which implies that x_A^L must be less than or equal to 1/4. We provide detailed computations in Appendix E.1 to show that there is a unique equilibrium of this form for each value $\theta < \theta_{NI}$, and further that x_A^L is decreasing in θ, so that fewer partisans of town A choose to live in town B as θ increases.

Case 3: Integrated Equilibrium of Type 1

The analysis for equilibria of Type 1 and Type 3 are much simpler than that of an equilibrium of Type 2 because one of the boundary indifference conditions is between districts B_1 and B_2. A town-B partisan with type x is indifferent between enrolling in these two districts if

$$v(x, y_{B_1}) - p_{B_1} + \theta = v(x, y_{B_2}) - p_{B_2} + \theta.$$

Similarly, a town-A partisan with type x is indifferent between enrolling in these two districts if

$$v(x, y_{B_1}) - p_{B_1} = v(x, y_{B_2}) - p_{B_2}.$$

That is, the equilibrium indifference conditions are the same and so the cutoff determining whether types enroll in district B_2 or in district B_1 must be the same for partisans of each town. In a Type 1 equilibrium, lowest types enroll in B_1. Since enrollment in B_1 must equal measure 1/2 in equilibrium, the cutoff between districts B_1 and B_2 must be the 25th percentile, $x_{0.25}$, for partisans of each town.

Given this observation, there are only two degrees of freedom in the enrollment pattern, specifically the values x_A and x_B which are the cutoffs distinguishing between town A and district B_2 for partisans of the two towns, respectively. Then the two equilibrium conditions are that partisans of town A with type x_A and also partisans of town B with type x_B are indifferent between living in A and in B_2. These conditions are the following:

$$v(x_A, y_A) - p_A + \theta = v(x_A, y_{B_2}) - p_{B_2}$$
$$v(x_{B_2}, y_{B_2}) - p_{B_2} + \theta = v(x_B, y_A) - p_A.$$

Combining these equations, we have a single condition

$$2\theta = v(x_{B_2}, y_{B_2}) - v(x_{B_2}, y_A) - v(x_A, y_{B_2}) - v(x_A, y_A).$$

With a Uniform distribution of types, the market clearing condition in town A requires $x_B = 1 - x_A$, and with $v(x, y) = xy$, this condition can be simplified to $(x_{B2} - x_A)(y_{B2} - y_A) = 2\theta$, or $(1-2x_A)(y_{B2} - y_A) = 2\theta$. Town-B partisans with types between 0.25 and $x_B = 1 - x_A$ enroll in B_2 . Similarly, partisans of town A with types between 0.25 and x_A enroll in district B_2. Therefore, $y_{B_2} = 2x_A^2 - 2x_A + 7/8$ and $y_A = 0.5 + x_A - x_A^2$. Substituting these values in the market clearing condition gives $\theta = 3x_A^3 - 4.5x_A^2 + 15x_A/8 - 3/16$. This cubic equation also has a unique solution in the relevant range for $\theta < 3/64$.[15]

Comparisons of Equilibria for a Single Value of θ

We compare the equilibria of each type for the particular value $\theta = 37/2000$.[16] Table C1 lists the enrollment patterns for partisans of each town in the three equilibria corresponding to Types 1, 2, and 3. There is considerable overlap in the enrollment patterns for partisans of the two towns. In each equilibrium, lowest types enroll in the district with lowest school quality, while highest types enroll in the district with highest school quality, regardless of partisanship. Furthermore, many "middle" types enroll in the district with middling school quality in each equilibrium, regardless of partisanship, though the definition of a "middle" type varies endogenously across the three types of equilibria. Of these three distinct equilibria with $\theta = 37/2000$, the Type 2 equilibrium is closest in nature to the equilibrium from Example 1 in the one town model; in the Type 2 equilibrium, the choice by town A to adopt school choice induces a change in school quality towards the middle, but induces flight of lowest and highest partisan types.

	Town-A Partisan			Town-B Partisan		
Equilibrium	District B_1	District B_2	Town A	District B_1	District B_2	Town A
Type 1	$0 \le x \le 1/4$	$1/4 \le x \le 0.55$	$x \ge 0.55$	$0 \le x \le 1/4$	$1/4 \le x \le 0.45$	$x \ge 0.45$
Type 2	$0 \le x \le 0.2$	$x \ge 0.8$	$0.2 \le x \le 0.8$	$0 \le x \le 0.3$	$x \ge 0.7$	$0.3 \le x \le 0.7$
Type 3	$0.55 \le x \le 3/4$	$x \ge 3/4$	$x \le 0.55$	$0.45 \le x \le 3/4$	$x \ge 3/4$	$x \le 0.45$

[15]The value $\theta = 3/64$ yields a knife-edge equilibrium of this sort where $x_A = 1/4$, meaning that no partisans of town A enroll in district B_2. In this equilibrium, partisans of town A with type 1/4 are exactly indifferent between all three options: enrolling in A, B_2, or B_1. For $\theta < 3/64$, there is a unique Type 1 equilibrium in this example, where x_A is strictly decreasing in θ with $x_A = 0.5$ at $\theta = 0$ and $x_A = 1/4$ at $\theta - 3/64$.

[16]The interior cutoffs with values $(0.2, 0.8)$ and $(0.3, 0.7)$ are exact for the Type 2 equilibrium. The interior cutoffs with values 0.45 and 0.55 for the Type 1 and the Type 3 equilibria are approximate to three decimal places.

Equilibrium	Quality y_{B_1}	Quality y_{B_2}	Quality y_A
Type 1	1/8	1/2	11/16
Type 2	13/100	87/100	1/2
Type 3	1/2	7/8	5/16

Table C1. Comparison of Equilibria in Example 3

D.4 General Properties of the Two-Town Model

We next ask whether we can generalize the insights of this example. Proposition 3 shows that a non-integrated equilibrium of Type 2 exists when partisanship is not too large.

Proposition 3 *There exists a value θ_{NI} such that there is a non-integrated equilibrium of the two-town model where town A adopts school choice and town B adopts a neighborhood assignment rule iff $\theta \geq \theta_{NI}$ and there is an integrated equilibrium of Type 2 for each $\theta < \theta_{NI}$.*

Our proof of Proposition 3 relies on a fixed point argument. Intuitively, if $\theta < \theta_{NI}$, then there are incentives for highest and/or lowest type town-A partisans to trade places with marginal type town-B partisans. But as trades of these sorts occur in equilibrium, then the identities of marginal type families change and specifically the marginal low-type town-A partisans increases, when the marginal low-type town-B partisans decreases. Thus, for each θ with $0 < \theta < \theta_{NI}$, there must be a critical point (with $x_A^L < x_B^L$ and associated values for x_A^H and x_B^H) where the pair of values of marginal types (x_A^L, x_B^L) yields exactly equal utility gains (excluding prices) for each of these two marginal types to choose town A rather than district 1 in town B, thereby producing an integrated equilibrium.

Corollary 6 *In an integrated equilibrium of Type 2 where town A uses school choice and town B uses neighborhood assignment, the lowest-type partisans of each town enroll in schools with lower qualities and highest-type partisans of each town enroll in schools with higher qualities than they would in a non-integrated equilibrium with neighborhood assignment in both towns.*

Corollary 6 follows from the observation that any type x student will choose the same district within town B whether that student is partisan to town A or to town B. In a Type 2 equilibrium, highest and lowest type students (regardless of partisanship) enroll in town B in an integrated equilibrium. Since partisans of each town with x close to 0 enroll in district B_1 while partisans of each town with x close to 1 enroll in district B_2, the quality of these districts must be spread farther than in the non-integrated equilibrium. Thus, if $\theta < \theta_{NI}$, town A's adoption of school choice rule only increases inequality of educational opportunities in a Type 2 equilibrium (as measured by the spread between the highest and lowest quality schools chosen by partisans of town A.)

Proposition 4 extends Proposition 3 to confirm the existence of Type 1 and Type 3 equilibria for relatively small values of θ. As suggested by our analysis of the example above, the proof of Proposition 4 is much simpler than that of Proposition 3 because we know in advance that $x_{0.25}$ is the marginal type between B_1 and B_2 for partisans of either town in a Type 1 equilibrium and that $x_{0.75}$ is the marginal type between B_1 and B_2 for partisans of either town in a Type 3 equilibrium. The proof follows by another fixed point argument.

Proposition 4 *There exists values θ_1 and θ_3 such that there an integrated equilibrium of Type j if $\theta < \theta_j$ for $j = 1, 3$.*

D.5 Welfare Analysis for the Two-Town Model

Welfare analysis in the two-town model shares features of the one-town model but is complicated both by multiplicity of equilibria and by the fact that outside options are generated endogenously rather than fixed exogenously. In the one-town model, when a student enrolls in town t in equilibrium 1 but takes the outside option in equilibrium 2, by revealed preference, that student must prefer equilibrium 1 since the same outside option is available in both cases. However, this is not the case in the two-town model, for a change from neighborhood assignment to school choice in town A, likely improves outside options in town B for some town-A partisans but degrades them for others.

Given these complexities, we focus our welfare analysis on lowest types, in particular lowest type town-A partisans, which adopts school choice, since proponents of school choice typically argue it is beneficial for lower-type families. Suppose that town A has a neighborhood rule with two districts, A_1 and A_2 in the base case. We will consider what happens when town A offers school choice.

Equilibrium	District	School Quality
Base Case	A_1	$\mathbb{E}[x \mid 0 < x < 0.5]$
Type 1	B_1	$\mathbb{E}[x \mid 0 < x < 0.25]$
Type 2	B_1	$\mathbb{E}[x \mid 0 < x < 0.25] < y_{B_1} < \mathbb{E}[x \mid 0 < x < 0.5]$
Type 3	A	$\mathbb{E}[x \mid 0 < x < 0.25] < y_A < \mathbb{E}[x \mid 0 < x < 0.5]$

Table C2. Equilibrium School Qualities for Lowest-Type Town-A Partisans

In any integrated equilibrium where town A offers school choice and town B uses a neighborhood assignment rule, the district with the lowest school quality enrolls the lowest types of partisans of each town. From a purely mechanical standpoint, in any integrated equilibrium where district B_1 has lowest school quality, that school quality is lower than in the baseline case of a non-integrated equilibrium where only partisans of town B enroll in district B_1. For instance, in Example 3, school quality is $y_{B_1} = 0.25$ in the base case, but this school quality falls to 1/8 (the minimum possible) in an integrated equilibrium of Type 1 and to 13/100 in an integrated equilibrium of Type 2. The only exception is in a Type 3 equilibrium, where town A has the lowest quality schools and those schools have higher quality than district B_1 in the base case. In Example 3, lowest types enroll in schools with quality 1/4 in the base case and with quality 5/16 in a Type 3 equilibrium where town A offers school choice.

As a general observation, the value of the lowest quality school in town A increases when town A adopts school choice, whether the equilibrium is Type 1, 2, or 3. Yet, in similar fashion to the one town model, this increase in quality of schools in district $A1$ need not directly affect lowest type partisans of town A because they move to town B in either a Type 1 or a Type 2 equilibrium. Lowest-type partisans do attend higher quality schools in a Type 3 equilibrium when town A adopts school choice than in the base case. However, this only occurs because property values throughout town A are uniformly lower than in town B in a Type 3 equilibrium.

There is a degree of freedom in the description of any equilibrium in the two town model, namely the price of houses in the district with lowest school quality. One seemingly natural rule would be to set the price in this district to the competitive price for schools of this quality in the one town model. Given this assumption, it is possible to have a clean comparison of realized utility values for lowest-type town A partisans in the base (non-integrated) case and in a Type 3 equilibrium. In the two equilibria, lowest type town A partisans pay the competitive price to live in town A. Therefore, as in the one town model, they achieve higher utility when attending a lower quality school in the base case than in an integrated equilibrium of Type 3. Also, as in the one town model, it is possible to take the paternalistic point of view that the Type 3 equilibrium is preferable to the base case for these lowest type partisans of town A.

By contrast, comparisons of the base case equilibrium to an integrated equilibrium of either Type 1 or Type 2 are not as clear from the perspective of lowest-type partisans of town A because they move to town B in an integrated equilibrium of Type 1 or 2. Since there is no consistent outside option, we cannot use revealed preference to compare realized utility values across a pair of equilibria. What we can say is that lowest-type partisans of town A attend unambiguously lower quality schools in a Type 1 or 2 equilibrium after the adoption of school choice by town A than beforehand and also lose the benefit of the partisan bonus after moving to town B.

E Additional Details and Proofs for Two-Town Model (Online Appendix)

E.1 Calculations for Example 3

Suppose that town-A partisans enroll in district B_1 if $x \in [0, x_A^L]$, enroll in town A if $x \in (x_A^L, 1-x_A^L)$, and enroll in district B_2 if x in $[1-x_A^L, 1]$. Then, given that each district enrolls an equal number of students and the types of partisans of both towns are distributed according to $U(0,1)$, this means that the cutoffs for town-B partisans are given by $x_B^L = \frac{1}{2} - x_A^L$ and $x_B^H = \frac{1}{2} + x_A^L$. That is, town-B partisans enroll in district B_1 if $x \in [0, \frac{1}{2} - x_A^L]$, enroll in town A if $x \in (\frac{1}{2} - x_A^L, \frac{1}{2} + x_A^L)$, and enroll in district B_2 if $x \in [\frac{1}{2} + x_A^L, 1]$.

Given these choices, the average type of town-A partisans is $\frac{x_A^L}{2}$ in district B_1, $\frac{1}{2}$ in town A and $(1 - x_A^L/2)$ in district B_2. Similarly, the average type of town-B partisans is $\frac{1}{4} - x_A^L/2$ in district B_1, $\frac{1}{2}$ in town A and $(\frac{3}{4} + \frac{x_A^L}{2})$ in district B_2. Taking weighted averages, we have

$$\begin{aligned}
y_{B_1} &= \left[x_A^L \frac{x_A^L}{2} + (\frac{1}{2} - x_A^L)(\frac{1}{4} - \frac{x_A^L}{2})\right] / \left[\frac{1}{2}\right] &&= 2(x_A^L)^2 - x_A^L + \frac{1}{4}, \\
y_{B_2} &= \left[x_A^L \frac{(1 - x_A^L)}{2} + (\frac{1}{2} - x_A^L)(\frac{3}{4} + x_A^L)\right] / \left[\frac{1}{2}\right] &&= \frac{3}{4} + x_A^L - 2(x_A^L)^2.
\end{aligned}$$

In equilibrium,

(1) Town-A partisans with $x = x_A^L$ obtain equal utility from A and B_1.

(2) Town-B partisans with $x = x_B^L$ obtain equal utility from A and B_1.

(3) Town-A partisans with $x = 1 - x_A^L$ obtain equal utility from A and B_2.

(4) Town-B partisans with $x = 1 - x_B^L$ obtain equal utility from A and B_2.

Given $v(x, y) = xy$, these conditions can be represented as

$$x_A^L y_A - p_A + \theta = x_A^L y_{B_1} - p_{B_1}, \tag{6}$$

$$x_B^L y_A - p_A = x_B^L y_{B_1} - p_{B_1} + \theta, \tag{7}$$

$$x_A^H y_A - p_A + \theta = x_A^H y_{B_2} - p_{B_2}, \tag{8}$$

$$x_B^H y_A - p_A = x_B^H y_{B_2} - p_{B_2} + \theta. \tag{9}$$

Solving for $p_A - p_{B_1}$ in (6) and (7) gives $(x_A^L - x_B^L)(y_A - y_{B1}) + 2\theta = 0$, or equivalently $2\theta = (\frac{1}{2} - 2x_A^L)(y_A - y_{B_1})$ after substituting $x_B^L = \frac{1}{2} - x_A^L$. Then substituting $y_A = \frac{1}{2}$ and

$y_{B_1} = \frac{1}{4} - x_A^L/2$, we have $y_A - y_{B_1} = \frac{1}{4} + x_A^L/2$, and thus $2\theta = \frac{1}{8} - 3(x_A^L)^2 + 4(x_A^L)^3)$. Solving (8) and (9) for θ as a function of x_A^L yields the identical equation.

Based on these computations, there is an equilibrium of the given form whenever $2\theta = 1/8 - 3(x_A^L)2 + 4(x_A^L)^3$ or equivalently $\theta = 1/16 - (3/2)(x_A^L)^2 + 2(x_A^L)^3$, and $x_A^L \leq 1/4$ so that $x_A^L \geq x_B^L$. This is a cubic equation for θ as a function of x_A^L, so is not naturally conducive to an analytic solution with x_A^L as a function of θ. However, we can identify some of the properties of $x_A^L(\theta)$ by studying comparative statics of this equation with θ as a function of x_A^L.

Differentiating $\theta(x_A^L) = 1/16-(3/2)(x_A^L)^2+2(x_A^L)^3$ with respect to x_A^L gives $d\theta/dx_A^L = 6(x_A^L)^2 - 3x_A^L < 0$ for $x_A^L < \frac{1}{2}$. So θ is declining as a function of x_A^L over the relevant range of values of x_A^L from 0 to 1/4 and further $\theta(x_A^L = 0) = 1/16$, corresponding to the cutoff $\theta_{NI} = 1/16$ for a non-integrated equilibrium, and $\theta(x_A^L = 1/4) = 0$, corresponding to an integrated equilibrium where partisans of both towns follow identical decision rules. That is, there is a one-to-one relationship between θ and x_A^L for x_A^L between 0 and 1/4, and therefore a unique equilibrium of this form for each value $\theta < \theta_{NI}$. Substituting $x_A^L = 0.2$ into the equations above yields $\theta(x_A^L = 0.2) = 37/2000$, $y_{B_1} = 0.13$ and $y_{B_2} = 0.87$ – the values used in the example in the text.

E.2 Proof of Proposition 2

Proof. Suppose that a town-B partisan of type x_h enrolls in district d in town B where $y_d > y_A$. Then since this student prefers district d in town B to enrolling in town A,

$$v(x_h, y_d) + \theta - p_d \geq v(x_h, y_A) - p_A,$$

or equivalently,

$$\theta \geq p_d - p_A + v(x_h, y_d) - v(x_h, y_A).$$

By the property of increasing differences of v, the difference $v(x, y_d) - v(x, y_A)$ is strictly increasing in x given $y_d > y_A$, so any partisan of town B with $x' > x_h$ strictly prefers district d in town B to enrolling in town A and will not enroll in town A. By similar reasoning, if type x_l enrolls in a district in town B with school quality less than y_A, then town-B partisans of type $x'' < x_l$ also will not enroll in town A. Thus, the set of town-B partisans who enroll in town A must be an interval of types $[x_B^L, x_B^H]$. An essentially identical argument extends this result to show that the set of town-A partisans who enroll in town A is an interval of types $[x_A^L, x_A^H]$.

Since town-A partisans receive a bonus for enrolling in town A, while town-B partisans receive a bonus for enrolling in town B, if a town-B partisan of type x enrolls in town A, then a town-A partisan of type x will also enroll in town A in equilibrium. This shows that $x^L_A \leq x^L_B \leq y_A$, $x^H_A \geq x^H_B$. A town-B partisan of type $x <$ x^L_A enrolls in a school in town B, so $v(x, y_d) + \theta - p_d$ $\geq v(x, y_A) - p_A$ for some district d in town B. We can rewrite this inequality as

$$v(x, y_d) - v(x, y_A) \geq p_d - p_A - \theta.$$

But if $y_d \geq y_A$, then this inequality would hold for all types greater than x (by the property of increasing differences for v), and so none of them would enroll in town B.[17] Thus, town-B partisans with types below x^A_L enroll in districts in town B with qualities less than y_A. By a similar argument, town-B partisans with types above x^L_A enroll in districts in town B with qualities greater than y_A, with analogous properties holding for town-A partisans. ■

E.3 Proof of Proposition 3

Proof. Suppose that there are two districts of equal size in each town, that there is measure 1 each of town-A partisans and of town-B partisans (so that each district has capacity equal to measure 1/2), and that the distribution of types is identical for partisans of each town. By Proposition 2, when town A uses a school choice rule and town B uses a neighborhood school assignment rule, in any equilibrium, an interval of partisans of type A $[x^L_A, x^H_A]$ and an interval of town-B partisans $[x^L_B, x^H_B]$ enroll in town A, where $x^L_A \leq x^L_B \leq x^H_B \leq x^H_A$ and these cutoffs are determined endogenously in equilibrium.

Given these enrollment constraints, the choice of x^L_A implicitly determines the choice of x^L_B given the enrollment constraint $F(x^L_A) + F(x^L_B) = 1/2$. Then since $x^L_A \leq x^L_B$, x^L_A takes possible values on $[0, x_{0.25}]$, where $x_{0.25}$ is defined by $F(x_{0.25}) = 1/4$. Similarly, x^H_A takes possible values on $[x_{0.75}, 1]$ where $F(x_{0.75}) = 3/4$ and x^H_B is an implicit function of x^H_A according to the equation $(1 - F(x^H_A)) + (1 - F(x^H_B)) = 1/2$ or equivalently $F(x^H_A) + F(x^H_B) = 3/2$.

Define

$$\lambda_L(x^L_A, x^H_A) = [v(x^L_B, y_A) - v(x^L_A, y_A)] - [v(x^L_B, y_1) - v(x^L_A, y_1)] - 2\theta$$

and

$$\lambda_H(x^L_A, x^H_A) = [v(x^H_A, y_2) - v(x^H_B, y_2)] - [v(x^H_A, y_A) - v(x^H_B, y_A)] - 2\theta,$$

The arguments to λ_L and λ_H exploit the fact that x^L_B (x^H_B) can be written in terms of x^L_A (x^H_A), and the value of y_A depends on x^L_A and x^H_A.

[17] We assume that town-B partisans enroll in town B in case of a tie in utility between the most preferred district in town B and the most preferred district in town A.

There is no integration at the bottom if $x_A^L = 0$ and no integration at the top if $x_A^H = 1$. Given school qualities $y_1 < y_A < y_2$ and prices p_A, p_1, and p_2, there is no integration at the bottom if $v(x_A^L = 0, y_A) + \theta - p_A \geq v(x_A^L, y_1) - p_1$ and $v(x_B^L, y_{B_1}) + \theta - p_1 \geq v(x_B^L, y_A) - p_A$, so that marginal (boundary) types of partisans of each town each prefer not to integrate. Combining these two equations to eliminate the prices gives the condition $\lambda_L(x_A^L = 0, x_A^H) \leq 0$ as a necessary condition for an equilibrium with non-integration at the bottom. If there is integration at the bottom, then both incentive conditions must hold with equality so that $\lambda_L(x_A^L = 0, x_A^H) = 0$ is a necessary condition for an equilibrium with $x_A^L > 0$. Similarly, $\lambda_H(x_A^L, x_A^H = 1) \leq 0$ is a necessary condition for equilibrium with no integration at the top and $\lambda_H(x_A^L, x_A^H) = 0$ is necessary for an equilibrium with integration at the top.

Holding x_A^L fixed, increased integration at the top, as represented by a reduction in x_A^H, yields an increase in y_2 and a decline in y_A. That is, y_A is strictly increasing and y_2 is strictly decreasing in x_A^H, while x_A^L, x_B^L, and y_1 are constant in x_A^H. By increasing differences of v in both arguments, $\lambda_L(x_A^L, x_A^H)$ is strictly increasing in x_A^H, so it takes its maximum value at $x_A^H = 1$ for each value of x_A^L. Thus, for each x_A^L, there is at most one value of x_A^H such that $v(x_A^L, x_A^H) = 0$. Further, when $x_A^L = x_{0.25}$ (its maximum possible value), then $x_A^L = x_B^L$ and $\lambda_H(x_{0.25}, x_A^H) = -2\theta$ for each value of x_A^H. Since v is continuous and $\lambda_L(x_{0.25}, 1) < 0$, then either (1) there exists some value $\bar{x} < x_{0.25}$ such that $\lambda_L(\bar{x}, 1) = 0$ and $\lambda_L(x_A^L, 1) < 0$ for $x_A^L > \bar{x}$ or (2) $\lambda_L(x_A^L, 1) < 0$ for all $x_A^L \leq x_{0.25}$.

In case (1), by construction, there exists a uniquely defined function $\varphi(x_A^L)$ for $\underline{\mathrm{x}} \leq x_A^L \leq \bar{x}$ such that $\lambda_L(x_A^L, \varphi(x_A^L)) = 0$. From above, we know $\varphi(x) = x_{0.75}$ and $\varphi(x) = 1$ Furthermore, since $\lambda_L(\bar{x}, 1) = 0$, then $\lambda_L(\bar{x}, x_{0.75}) < 0$ since λ_L is strictly increasing in its second argument. Then since v is continuous, there either

(1A) exists a value $\underline{x} < \bar{x}$ such that $\lambda_L(x, x_{0.75}) = 0$ and $\lambda_L(x_A^L, x_{0.75}) < 0$ for each x_A^L such that $\underline{x} < x_A^L < \bar{x}$, or

(1B) $\lambda_L(x_A^L, x_{0.75}) < 0$ for each $x_A^L < \bar{x}$.

When $x_A^H = x_{0.75}$ (its minimum possible value), then $x_A^H = x_B^H$ and so $\lambda_H(x_A^L, x_{0.75}) = -2\theta$ for each value of x_A^L. So, in particular, in Case (1A), $\lambda_H(x, x_{0.75}) = \lambda_H(x, \varphi(x)) = -2\theta$. Then, since v (and therefore λ_H) is continuous in each argument, there either exists x_A^L between $\underline{x}$ and $\bar{x}$ so that $\lambda_H(x_A^L, \varphi(x_A^L)) = 0$, in which case there is an equilibrium with integration at top and bottom at $[x_A^L, x_A^H = \varphi(x_A^L)]$ or $\lambda_H(\bar{x}, \varphi(\bar{x}) = 1) \leq 0$, in which case there is an equilibrium with integration at the bottom and non-integration at the top at $[x_A^L = \bar{x}, x_A^H = 1]$.

Similarly, in Case (1B), there exists a uniquely defined function $\varphi(x_A^L)$ for each $x_A^L \leq \bar{x}$ such that $\lambda_L(x_A^L, \varphi(x_A^L)) = 0$. The distinction between Case (1A) and Case (1B) is that since the range $(0, \bar{x})$ of relevant values of x_A^L includes 0, it is now possible to find an equilibrium with non-integration at the bottom. Since $\lambda_H(0, x_{0.75}) = -2\theta < 0$, either there exists a value x_A^H between $x_{0.75}$ and $\varphi(x_A^L = 0)$ such that $\lambda_H(0, x_A^H) = 0$, in which case there is an equilibrium with integration at the top and non integration at the bottom at $(0, x_A^H)$ or $\lambda_H(0, \varphi(0)) < 0$, in which case the logic from (1A) implies that there exists an equilibrium.

In case (2), $\lambda_L(x_A^L, 1) < 0$ for all $x_A^L \leq x_{0.25}$, so in fact $\lambda_L(x_A^L, x_A^H) < 0$ in all cases. This rules out the possibility of an equilibrium with integration at the bottom, so assume that $x_A^L = 0$ and look for an equilibrium with non integration at the bottom. Since $\lambda_H(0, x_{0.75}) = -2\theta < 0$, either $\lambda_H(0, 1) \leq 0$, in which case there is an equilibrium with non-integration at top or bottom, or there exists some value x_A^H between $x_{0.75}$ and 1 such that $\lambda_H(0, x_A^H) = 0$ in which case there is an equilibrium at $(0, x_A^H)$ with integration at the top and no integration at the bottom. ■

E.4 Proof of Proposition 4

Proof. In a Type 1 integrated equilibrium, $y_A > y_{B_1} > y_{B_2} = \mathbb{E}[x|0 < x < 0.25]$. Since we know that $x_{0.25}$ is the type-cutoff between district B_1 and B_2 in this equilibrium, the only remaining parameters to identify are the cutoffs for partisans of each town between district B_2 and town A. As described in the text, denote x_A as the cutoff for town-A partisans and x_B be the cutoff for town-B partisans. The boundary indifference condition for partisans of type A is

$$v(x_A, y_A) - p_A + \theta = v(x_A, y_{B_2}) - p_{B_2}. \tag{10}$$

Similarly, the boundary indifference condition for partisans of type B is

$$v(x_B, y_A) - p_A = v(x_B, y_{B_2} - p_{B_2}) + \theta. \tag{11}$$

Subtracting equation (11) from (10) gives

$$2\theta + v(x_A, y_A) - v(x_A, y_{B_2}) = v(x_B, y_A) - v(x_B, y_{B_2}).$$

Since we know $y_A > y_{B_2}$, we can convert this to integral form:

$$2\theta + \int_{y_{B_2}}^{y_A} \frac{\partial v}{\partial y}(x_A, z)dz = \int_{y_{B_2}}^{y_A} \frac{\partial v}{\partial y}(x_B, z)dz.$$

This equation can only hold if $x_A < x_B$ given that $\theta > 0$, so we can rewrite it in double integral form:

$$2\theta = \int_{x_A}^{x_B} \int_{y_{B_2}}^{y_A} \frac{\partial^2 v}{\partial x \partial y}(a, z) da dz.^{18} \tag{12}$$

The market clearing condition here requires

$$1 - F(x_B) + 1 - F(x_A) = 1,$$

or

$$F(x_B) = 1 - F(x_A).$$

Consider two extreme possibilities: $x_A = x_{0.25}$ and $x_B = x_{0.75}$, where no town-A partisans enroll in district B_2 and $x_A = x_B = 1/2$, where the same number of partisans of each town enroll in district B_2. If $x_A = x_B = 1/2$, the right-hand side of the last equation is 0, so less than the left-hand side which is equal to 2θ. Next define

$$\theta^{*1} = 0.5 \int_{x_{0.25}}^{x_{0.75}} \int_{y_{B2}}^{y_A} \frac{\partial^2 v}{\partial x \partial y}(a, z) da dz,$$

where y_A and y_{B_2} take the appropriate values corresponding to $x_A = x_{0.25}$ and $x_B = x_{0.75}$. By construction, if $x_A = x_{0.25}$ and $x_B = x_{0.75}$, the right-hand side of the equation (12) is greater than the left-hand side if $\theta < \theta^{*1}$. Then by the Intermediate Value Theorem, there is a value of x_A between $x_{0.25}$ and $1/2$ where the combined boundary indifference condition holds with equality. This value will support an equilibrium of Type 1 where price p_{B_1} is set to the competitive price for y_{B_1} from the one-town model and the price increments between districts B_1 and B_2 as well as B_2 and A are determined by the boundary indifference conditions.

An essentially identical argument proves the existence of a Type 3 integrated equilibrium for θ below a cutoff θ^{*3}. ■

[18] We note that y_A and y_{B_2} are in fact functions of x_A and x_B and allow for this in the analysis described below.