Online Appendix
Equilibrium Technology Diffusion, Trade, and Growth

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A. Environment and Optimization Problems

To demonstrate the extensibility of the model, in this appendix we derive conditions for a version of the model in which there is CRRA power utility, a weakly positive probability of death δ, costs of adoption and entry can be a convex combination of labor and goods, and exogenous shocks to TFP that follow a geometric Brownian motion with drift. For expositional clarity, the body of the paper studies the special case of log utility, no exogenous productivity shocks, costs of adoption and entry in labor only, and studies the limiting economy as the death rate, and thus the BGP equilibrium entry rate, is zero.

A.1. Consumer Preferences and Labor Supply

All countries are symmetric. In each country there exists a representative consumer of measure $\bar{L}$. The utility of the consumer is given by a constant relative risk aversion (CRRA) function in final goods consumption (C), given an inelastic supply of labor ($\bar{L}$). The coefficient of relative risk aversion is $\gamma \geq 0$, and the time discount rate is ρ.

Final goods are produced through CES aggregation of an endogenous number of intermediate varieties, including those produced domestically and those imported from abroad. There is an endogenous measure $\Omega(t)$ of firms operating in each country. The flow of intermediate firms adopting a new technology is $\Omega(t)S(t)$ and the flow of firms entering the market and creating a new variety is $\Omega(t)E(t)$. The consumer purchases the final consumption good, invests in technology adoption with a real cost of $X(t)$ per upgrading intermediate firm, and invests in firm entry with a real cost of $X(t)/\chi$.[28] Consumers income consists of labor earnings paid at wage $W(t)$ and profits from their ownership of the domestic firms. Aggregate profits from selling domestically are $\bar{\Pi}_d(t)$ and aggregate profits from exporting to the $N-1$ foreign countries is $(N-1)\bar{\Pi}_x(t)$). Thus, welfare at time $\tilde{t}$ is

$$\bar{U}(\tilde{t}) = \int_{\tilde{t}}^{\infty} U(C(t))e^{-\rho(t-\tilde{t})}\mathrm{d}t$$
$$\text{s.t. } C(t) + \Omega(t)X(t)(S(t) + E(t)/\chi) = \frac{W(t)}{P(t)}\bar{L} + \bar{\Pi}_d(t) + (N-1)\bar{\Pi}_x(t). \tag{A.1}$$

With this, the period real profits as $\bar{\Pi}_i(t) = \bar{\Pi}_d(t) + (N-1)\bar{\Pi}_x(t)$ and real investment as $\bar{I}_i(t) = \Omega(t)X(t)(S(t) + E(t)/\chi)$.

[28] Firms are maximizing real profits, discounting using the interest rate determined by the consumer's marginal rate of substitution plus the death rate. Hence, the investment choice of the consumer and firm is aligned, and consumers will finance upgrades to their existing firms through equity financing. As consumers own a perfectly diversified portfolio of domestic firms, they are only diluting their own equity with this financing method.

A.2. The Static Firm Problem

Intermediate Goods Demand. There is a measure $\Omega(t)$ of intermediate firms in each country that are monopolistically competitive, and the final goods sector is perfectly competitive. The final goods sector takes prices as given and aggregates intermediate goods with a CES production function, with $\sigma > 1$ the elasticity of substitution between all available products.

The CDF of the productivity distribution at time t is $\Phi(Z, t)$, normalized such that $\Phi(\infty, t) = 1$ for all t. Therefore, the total measure of firms with productivity below Z at time t is $\Omega(t)\Phi(Z, t)$.

Drop the t subscript for clarity. The standard solutions follow from maximizing the following final goods production problem,

$$\max_{Q_d, Q_x} \left[\Omega \int_M^\infty Q_d(Z)^{(\sigma-1)/\sigma} \mathrm{d}\Phi(Z) + (N-1)\Omega \int_{\hat{Z}}^\infty Q_x(Z)^{(\sigma-1)/\sigma} \mathrm{d}\Phi(Z)\right]^{\sigma/(\sigma-1)} - \Omega \int_M^\infty p_d(Z) Q_d(Z) \mathrm{d}\Phi(Z) + (N-1)\Omega \int_{\hat{Z}}^\infty p_x(Z) Q_x(Z) \mathrm{d}\Phi(Z) \tag{A.2}$$

Since the final good market is competitive, nominal aggregate income is

$$Y = \Omega \int_M^\infty p_d(Z) Q_d(Z) \mathrm{d}\Phi(Z) + (N-1)\Omega \int_{\hat{Z}}^\infty p_x(Z) Q_x(Z) \mathrm{d}\Phi(Z). \tag{A.3}$$

Defining a price index P, the demand for each intermediate product is,

$$Q_d(Z) = \left(\frac{p_d(Z)}{P}\right)^{-\sigma} \frac{Y}{P}, Q_x(Z) = \left(\frac{p_x(Z)}{P}\right)^{-\sigma} \frac{Y}{P} \tag{A.4}$$

$$P^{1-\sigma} = \Omega \left(\int_M^\infty p_d(Z)^{1-\sigma} \mathrm{d}\Phi(Z) + (N-1) \int_{\hat{Z}}^\infty p_x(Z)^{1-\sigma} \mathrm{d}\Phi(Z)\right). \tag{A.5}$$

Static Profits. A monopolist operating domestically chooses each instant prices and labor demand to maximize profits, subject to the demand function given in equation (A.4),

$$P\Pi_d(Z) := \max_{p_d, \ell_d} \{\, (p_d Z \ell_d - W \ell_d)\} \text{ s.t. equation (A.4).} \tag{A.6}$$

Where $\Pi_d(Z)$ is the real profits from domestic production.

Firms face a fixed cost of exporting, $\kappa \geq 0$. To export, a firm must hire labor in the foreign country to gain access to foreign consumers. This fixed cost is paid in market wages, and is proportional to the number of consumers accessed. Additionally, exports are subject to a variable iceberg trade cost, $d \geq 1$, so that firm profits from exporting to a single country (i.e., export profits per market) are

$$P\Pi_x(Z) := \max_{p_x, \ell_x} \left\{\, \left(p_x \frac{Z}{d} \ell_x - W \ell_x - \bar{L}\kappa W\right)\right\} \text{ s.t. equation (A.4).} \tag{A.7}$$

Optimal firm policies consist of $p_d(Z)$, $p_x(Z)$, $\ell_d(Z)$, and $\ell_x(Z)$ and determine $\Pi_d(Z)$ and $\Pi_x(Z)$. As is standard, it is optimal for firms to charge a constant markup over marginal cost, $\bar{\sigma} := (\sigma/\sigma - 1)$.

$$p_d(Z) = \bar{\sigma}\tfrac{W}{Z}, \tag{A.8}$$

$$p_x(Z) = \bar{\sigma} d\tfrac{W}{Z}, \tag{A.9}$$

$$\ell_d(Z) = \tfrac{Q_d(Z)}{Z}, \quad \ell_x(Z) = d\tfrac{Q_x(Z)}{Z}. \tag{A.10}$$

To derive firm profits, take equation (A.6) and divide by P to get $\Pi_d(Z) = \frac{p_d(Z)}{P}Q_d(Z) - \frac{Q_d(Z)}{Z}\frac{W}{P}$. Substitute from equation (A.8), as $W = \frac{Zp_d(Z)}{\bar{\sigma}}$, to yield $\Pi_d(Z) = \frac{1}{\sigma}\frac{p_d(Z)}{P}Q_d(Z)$. Finally, use $Q_d(Z)$ from equation (A.4) to show

$$\Pi_d(Z) = \tfrac{1}{\sigma}\left(\tfrac{p_d(Z)}{P}\right)^{1-\sigma}\tfrac{Y}{P}. \tag{A.11}$$

Using similar techniques, export profits per market are

$$\Pi_x(Z) = \max\left\{0, \tfrac{1}{\sigma}\left(\tfrac{p_x(Z)}{P}\right)^{1-\sigma}\tfrac{Y}{P} - \bar{L}\kappa\tfrac{W}{P}\right\}. \tag{A.12}$$

Since there is a fixed cost to export, only firms with sufficiently high productivity will find it profitable to export. Solving equation (A.12) for the productivity that earns zero profits gives the export productivity threshold. That is, a firm will export iff $Z \geq \hat{Z}$, where $\hat{Z}$ satisfies

$$\left(\tfrac{p_x(\hat{Z})}{P}\right)^{1-\sigma} = \sigma\bar{L}\kappa\tfrac{W}{Y}, \tag{A.13}$$

$$\hat{Z} = \bar{\sigma} d(\sigma\bar{L}\kappa)^{\frac{1}{\sigma-1}}\left(\frac{W}{P}\right)\left(\frac{W}{Y}\right)^{\frac{1}{\sigma-1}}. \tag{A.14}$$

Let aggregate profits from domestic production be $\bar{\Pi}_d$ and aggregate export profits per market be $\bar{\Pi}_x$.

$$\bar{\Pi}_d := \Omega\int_M^\infty \Pi_d(Z)\mathrm{d}\Phi(Z). \tag{A.15}$$

$$\bar{\Pi}_x := \Omega\int_{\hat{Z}}^\infty \Pi_x(Z)\mathrm{d}\Phi(Z). \tag{A.16}$$

The trade share for a particular market, λ, is

$$\lambda = \Omega\int_{\hat{Z}}^\infty \left(\frac{p_x(Z)}{P}\right)^{1-\sigma}\mathrm{d}\Phi(Z). \tag{A.17}$$

A.3. Firms Dynamic Problem

Stochastic Process for Productivity Assume that operating firms have (potentially) stochastic productivity following the stochastic differential equation for geometric Brownian motion (GBM),

$$\mathrm{d}Z_t/Z_t = (\mu + \upsilon^2/2)\mathrm{d}t + \upsilon \mathrm{d}W_t, \tag{A.18}$$

where $\mu \geq 0$ is related to the drift of the productivity process, $\upsilon \geq 0$ is the volatility, and W_t is standard Brownian motion.

At any instant in time, a firm will exit if hit by a death shock, which follows a Poisson process with arrival rate $\delta \geq 0$. Thus, all firms have the same probability of exiting and the probability of exiting is independent of time.

The case of $\mu = \upsilon = \delta = 0$, is the baseline model studied in the body of the paper.

Firm's Problem. Define a firm's total real profits as

$$\Pi(Z,t) := \Pi_d(Z,t) + (N-1)\Pi_x(Z,t), \tag{A.19}$$

where from equation (A.12), $\Pi_x(Z,t) = 0$ for firms who do not export. Let $V(Z,t)$ be the value of a firm with productivity Z at time t. The firm's effective discount rate, $r(t)$, is the sum of the consumer's intertemporal marginal rate of substitution and the firm death rate δ.

Given the standard Bellman equation for the GBM of equation (A.18) and optimal static policies,

$$r(t)V(Z,t) = \Pi(Z,t) + \left(\mu + \frac{\upsilon^2}{2}\right) Z \frac{\partial V(Z,t)}{\partial Z} + \frac{\upsilon^2}{2} Z^2 \frac{\partial^2 V(Z,t)}{\partial Z^2} + \frac{\partial V(Z,t)}{\partial t}, \tag{A.20}$$

$$V(M(t),t) = \int_{M(t)}^{\infty} V(Z,t)\mathrm{d}\Phi(Z,t) - X(t), \tag{A.21}$$

$$\frac{\partial V(M(t),t)}{\partial Z} = 0. \tag{A.22}$$

Equation (A.20) is the bellman equation for a firm continuing to produce with its existing technology. It receives instantaneous profits and the value of a firm where productivity may change over time. Equation (A.21) is the value matching condition, which states that the marginal adopter must be indifferent between adopting and not adopting. $M(t)$ is the endogenous productivity threshold that defines the marginal firm. Equation (A.22) is the smooth-pasting condition.[29]

A.4. Adoption Costs

In order to upgrade its technology a firm must buy some goods and hire some labor. These costs are in proportion to the population size, reflecting market access costs as in Arkolakis (2010) or Arkolakis (2015). Whether costs are in terms of goods or labor is a common issue in growth models, with many

[29]Using the standard relationship between free boundary and optimal stopping time problems, the firm's problem could equivalently be written as the firm choosing a stopping time at which it would upgrade. If $\upsilon > 0$ this stopping time is a random variable, otherwise it is deterministic.

papers specifying goods costs and many specifying labor costs.[30] The growth literature often uses labor costs, since new ideas cannot simply be purchased, and instead must be the result of innovators doing R&D. In the technology diffusion context, in which low productivity firms are adopting already existing technologies, an adoption cost that has some nontrivial component denominated in goods is more reasonable than in the innovation case. Since there is a paucity of empirical evidence to guide our decision in the adoption context, we model the adoption cost in a way that nests the costs being in labor exclusively, in goods exclusively, or as a mix of labor and goods. Although we solve the model for this general case, our baseline is that costs are purely labor denominated.

The amount of labor needed is parameterized by ζ, which is constant. The labor component of the adoption cost, however, increases in equilibrium in proportion to the real wage, ensuring the cost does not become increasingly small as the economy grows. The amount of goods that needs to be purchased to adopt a technology increases with the scale of the economy—otherwise the relative costs of goods would become infinitesimal in the long-run. Θ parameterizes the amount of goods required to adopt a technology, with the cost, $M(t)\,\Theta$, growing as the economy grows. Essentially, ζ controls the overall cost of technology adoption, while $\eta \in [0, 1]$ controls how much of the costs are to hire labor versus buy goods. We model the mix of labor and goods as additive in order to permit a balanced growth path equilibrium.

The real cost of adopting a technology is

$$X(t) := \bar{L}\zeta \left[(1-\eta)\frac{W(t)}{P(t)} + \eta M(t)\Theta\right]. \tag{A.23}$$

A.5. Entry and Exit

There is a large pool of non-active firms that may enter the economy by paying an entry cost—equity financed by the representative consumer—to gain a draw of an initial productivity from the same distribution from which adopters draw. Since entry and adoption deliver similar gains, we model the cost of entry as a multiple of the adoption cost for incumbents, $X(t)/\chi$, where $0 < \chi < 1$. Hence, χ is the ratio of adoption to entry costs and $\chi \in (0, 1)$ reflects that incumbents have a lower cost of upgrading to a better technology than entrants have to start producing a new variety from scratch.

Thus, the free entry condition that equates the cost of entry to the value of entry is

$$X(t)/\chi = \int_{M(t)}^{\infty} V(Z,t)\mathrm{d}\Phi(Z,t). \tag{A.24}$$

If a flow $E(t)$ of firms enter, and a flow δ exit, then the differential equation for the number of firms is $\Omega'(t) = (E(t) - \delta)\,\Omega(t)$. Since we study a stationary equilibrium, on a BGP Ω will be constant and determined by free entry, and $E(t) = \delta$ for all t.

[30] As discussed in Bollard, Klenow, and Li (2014), for examples see Hopenhayn (1992), Romer (1994), Foster, Haltiwanger, and Syverson (2008) and David (2017) for papers using output costs and Lucas (1978), Grossman and Helpman (1991), Melitz (2003), Klette and Kortum (2004), Luttmer (2007), and Acemoglu et al. (2018) for papers that use labor costs.

The costs of entry will determine the number of varieties in equilibrium, and for $\delta > 0$, there is gross entry on a balanced growth path. This model of entry and exit is very different from those in Luttmer (2007) and Sampson (2016). Here, exit is exogenous, whereas a key model mechanism studied in those papers is the endogenous selection of exit induced by fixed costs of operations. We have not modeled a fixed cost to domestic production in order to isolate our distinct mechanism. We have introduced entry and exit in our model to generate an endogenous number of varieties so that we can analyze the effect of our mechanism on welfare, taking into account changes in incumbent technology adoption behavior and changes on the extensive margin in the number of varieties produced. Given the exogenous death shock, the effect of $\delta > 0$ is only to change the firm's discount rate. For the most part, the economics are qualitatively identical to the $\delta = 0$ case and there is no discontinuity in the limit as $\delta \to 0$.[31]

A.6. Market Clearing and the Resource Constraint

Total labor demand is the sum of labor used for domestic production, export production, the fixed cost of exporting, technology adoption, and entry. Equating labor supply and demand yields

$$\bar{L} = \Omega \int_{M}^{\infty} \ell_d(Z)\mathrm{d}\Phi(Z) + (N-1)\Omega \int_{\hat{Z}}^{\infty} \ell_x(Z)\mathrm{d}\Phi(Z) + (N-1)\Omega(1-\Phi(\hat{Z}))\kappa\bar{L} + \bar{L}(1-\eta)\zeta\Omega\left(S + \delta/\chi\right). \tag{A.25}$$

The quantity of final goods must equal the sum of consumption and investment in technology adoption. Thus, the resource constraint is

$$\frac{Y}{P} = C + \Omega\bar{L}\eta\zeta M\Theta(S + \delta/\chi). \tag{A.26}$$

B. Deriving the Productivity Distribution Law of Motion and Flow of Adopters

This section describes the details of deriving the law of motion for the productivity distribution.

The Productivity Distribution Law of Motion. At points of continuity of $M(t)$, there exists a flow of adopters during each infinitesimal time period.[32] The Kolmogorov Forward Equation (KFE) for $Z > M(t)$, describes the evolution of the CDF. The KFE contains standard components accounting for the drift and Brownian motion of the exogenous GBM process detailed in equation (A.18). Furthermore, it includes the flow of adopters (source) times the density they draw from (redistribution CDF). Determining the flow of adopters is the fact that the adoption boundary $M(t)$ sweeps across the density from below at rate $M'(t)$. As adoption boundary acts as an absorbing barrier, and as it sweeps from below

[31]For special cases where $\delta = 0$ and the initial Ω is large relative to that which would be achieved on a BGP from a relatively small initial Ω, the free entry condition could hold as an inequality (i.e. $X(t)/\chi > \int_{M(t)}^{\infty} V(Z,t)d\Phi(Z)$). In that case, the lack of exit would prevent Ω from decreasing so that the free entry condition held with equality. In our baseline with $\delta = 0$, we ignore this special one-sided case, as it is economically uninteresting; it is unreasonable for the number of varieties to require major decreases in a growing economy. Also, as there is no discontinuity in the solution when taking $\delta \to 0$, we will consider our baseline economy a small δ approximation.

[32]The evolution of $M(t)$, and hence the distribution itself, can only be discontinuous at time 0 or in response to unanticipated shocks. Since this paper analyzes balanced growth path equilibria, we omit derivation of the law of motion for the distribution with discontinuous $M(t)$.

it collects $\phi(M(t),t)$ amount firms. The cdf that the flow of adopters is redistributed into is determined by two features of the environment. In the stationary equilibrium, $M(t)$ is the minimum of support of $\Phi(Z,t)$, so the adopters are redistributed across the entire support of $\phi(Z,t)$.[33] Since adopters draw directly from the productivity density, they are redistributed throughout the distribution in proportion to the density. Thus, the flow of adopters $S(t)$ multiplies the cdf, $\Phi(Z,t)$.

Since there is a constant death rate, $\delta \geq 0$, a normalized measure of $\delta\Phi(Z,t)$ exit with productivity below Z, but as new entrants of normalized flow $E(t)$ adopt a productivity through the same process as incumbents, they are added to the flow entering with measure below Z.[34]

$$\begin{aligned}\frac{\partial \Phi(Z,t)}{\partial t} &= \underbrace{\Phi(Z,t)\left(\underbrace{S(t)+E(t)}_{\text{Adopt or Enter}}\right)}_{\text{Distributed below } Z} - \underbrace{S(t)}_{\text{Adopt at } M(t)} - \underbrace{\delta\Phi(Z,t)}_{\text{Death}} \\ &\quad - \underbrace{\left(\mu - \frac{\upsilon^2}{2}\right) Z \frac{\partial \Phi(Z,t)}{\partial Z}}_{\text{Deterministic Drift}} + \underbrace{\frac{\upsilon^2}{2} Z^2 \frac{\partial^2 \Phi(Z,t)}{\partial Z^2}}_{\text{Brownian Motion}}. \qquad \text{(B.3)}\end{aligned}$$

In the baseline $\delta = \mu = \upsilon = 0$ case, this simplifies to equation (17).[35]

Normalized Productivity Distribution. Define the change of variables $z := Z/M(t)$, $g(t) := M'(t)/M(t)$, and

$$\Phi(Z,t) =: F(Z/M(t),t). \qquad \text{(B.4)}$$

Differentiating,

$$\phi(Z,t) = \frac{1}{M(t)} f(Z/M(t),t). \qquad \text{(B.5)}$$

[33]Unlike in discrete time, the distinction between drawing from the unconditional distribution or the conditional distribution of non-adopting incumbents is irrelevant. The number of adopting firms is a flow, and hence measure 0, which leads to identical conditional vs. unconditional distributions.

[34]To derive from the more common KFE written in PDFs: use the standard KFE for the pdf $\phi(Z,t)$, integrate this with respect to Z to convert into cdf $\Phi(Z,t)$, use the fundamental theorem of calculus on all terms, then the chain rule on the last term, and rearrange,

$$\frac{\partial \phi(Z,t)}{\partial t} = -\frac{\partial}{\partial Z}\left[(\mu + \frac{\upsilon^2}{2}) Z\phi(Z,t)\right] + \frac{\partial}{\partial Z^2}\left[\frac{\upsilon^2}{2} Z^2 \phi(Z,t)\right] + \ldots \qquad \text{(B.1)}$$

$$\frac{\partial \Phi(Z,t)}{\partial t} = \left(\frac{\upsilon^2}{2} - \mu\right) Z \frac{\partial \Phi(Z,t)}{\partial Z} + \frac{\upsilon^2}{2} Z^2 \frac{\partial^2 \Phi(Z,t)}{\partial Z^2} + \ldots \qquad \text{(B.2)}$$

[35]While *conditional on an optimal policy* the law of motion in equation (17) and the more general equation (16) are identical to that in Luttmer (2007), this is a mechanical result of any distribution evolution with resetting of agents through direct sampling of the distribution. Economically, the forces which determine the endogenous policy are completely different, as we concentrate on the choices of incumbents rather than entrants/exit. Some of the key differences are evident in the connection to search models as discussed in Section 5.

This normalization generates an adoption threshold that is stationary at $z = M(t)/M(t) = 1$ for all t.

Law of Motion for the Normalized Distribution. To characterize the normalized KFE, first differentiate the cdf with respect to t, yielding

$$\frac{\partial \Phi(Z,t)}{\partial t} = \frac{\partial F(Z/M(t),t)}{\partial t} - \frac{Z}{M(t)}\frac{M'(t)}{M(t)}\frac{\partial F(Z/M(t),t)}{\partial z}. \tag{B.6}$$

Differentiating the cdf with respect to Z yields

$$\frac{\partial \Phi(Z,t)}{\partial Z} = \frac{1}{M(t)}\frac{\partial F(Z/M(t),t)}{\partial z}, \tag{B.7}$$

$$\frac{\partial^2 \Phi(Z,t)}{\partial Z^2} = \frac{1}{M(t)^2}\frac{\partial^2 F(Z/M(t),t)}{\partial z^2}. \tag{B.8}$$

Given that $z := \frac{Z}{M(t)}$ and $g(t) := M'(t)/M(t)$, combining equations (B.3), (B.6), (B.7), and (B.8) provides the KFE in cdfs of the normalized distribution:

$$\frac{\partial F(z,t)}{\partial t} = (S(t) + E(t) - \delta)\, F(z,t) + \left(g(t) - \mu + \upsilon^2/2\right) z \frac{\partial F(z,t)}{\partial z} + \frac{\upsilon^2}{2} z^2 \frac{\partial^2 F(z,t)}{\partial z^2} - S(t). \tag{B.9}$$

The interpretation of this KFE is that while non-adopting incumbent firms are on average not moving in absolute terms, they are moving backwards at rate $g(t)$ relative to $M(t)$ (adjusted for the growth rate of the stochastic process). As the minimum of support is $z = M(t)/M(t) = 1$ for all t, a necessary condition is that $F(1,t) = 0$ for all t, and therefore $\frac{\partial F(1,t)}{\partial t} = 0$. Thus, evaluating equation (B.9) at $z = 1$ gives an expression for $S(t)$,[36]

$$S(t) = \left(g(t) - \mu + \frac{\upsilon^2}{2}\right)\frac{\partial F(1,t)}{\partial z} + \frac{\upsilon^2}{2}\frac{\partial^2 F(1,t)}{\partial z^2}. \tag{B.10}$$

This expression includes adopters caught by the boundary moving relative to their drift, as well as the flux from the GBM pushing some of them over the endogenously determined threshold. If $\mu = \upsilon = \delta = 0$, then a truncation at $M(t)$ solves equation (B.9) for any t and for any initial condition, as in Perla and Tonetti (2014).

$$\phi(Z,t) = \frac{\phi(Z,0)}{1 - \Phi(M(t),0)}, \quad Z \geq M(t). \tag{B.11}$$

The only non-degenerate stationary $F(z)$ consistent with equation (B.11) is given by equation (B.13)—the same form as that with $\upsilon > 0$.

[36] Equivalently, the flow of adopters can be derived as the net flow of the probability current through the adoption threshold.

The Stationary Normalized Productivity Distribution. From equation (B.9), the stationary KFE is

$$0 = SF(z) + \left(g - \mu + \frac{\upsilon^2}{2}\right) zF'(z) + \frac{\upsilon^2}{2} z^2 F''(z) - S. \tag{B.12}$$

subject to $F(1) = 0$ and $F(\infty) = 1$.

Moreover, for any strictly positive $\upsilon > 0$, the KFE will asymptotically generate a stationary Pareto distribution for some tail parameter from any initial condition. While many $\theta > 1$ could solve this differential equation, the particular θ tail parameter is determined by initial conditions and the evolution of $M(t)$. As in Luttmer (2007) and Gabaix (2009), the geometric random shocks leads to an endogenously determined power-law distribution.

$$F(z) = 1 - z^{-\theta}, \quad z \geq 1. \tag{B.13}$$

From equations B.12 and B.13, evaluating at $z = 1$ for a given g and θ,

$$S = \theta\left(g - \mu - \theta\frac{\upsilon^2}{2}\right) \tag{B.14}$$

Therefore, given an equilibrium g and θ, the CDF in equation (B.13) and S from equation (B.14) characterize the stationary distribution. The relationship between g and θ is determined by the firms' decisions given S and $F(z)$. It is independent of δ on a BGP since the exit rate is constant and uniform across firms (in contrast to Luttmer (2007), where selection into exit is generated by fixed costs of operations and is not independent of firm productivity).

The Stationary Distribution with no GBM. For $\upsilon = 0$, the lack of random shocks means that the stationary distribution will not necessarily become a Pareto distribution from arbitrary initial conditions. However, if the initial distribution is Pareto, the normalized distribution will be constant. If the initial distribution is a power-law, then the stationary distribution is asymptotically Pareto.

Since a Pareto distribution is attained for any $\upsilon > 0$, we consider our baseline ($\upsilon = 0$, $\mu = 0$) case with an initial Pareto distribution as a small noise limit of the full model with GBM from some arbitrary initial condition. Beyond endogenously determining the tail index and changing the expected time to execute the adoption option, exogenous productivity volatility of incumbent firms has qualitatively little impact on the model. For the baseline case, from equation (B.14), the flow of adopters is $S = \theta g$.

C. Normalized Static Equilibrium Conditions

To aid in computing a balanced growth path equilibrium, in this section we transform the problem and derive normalized static equilibrium conditions.

Expectations using the Normalized Distribution. If an integral of the following form exists, for some unary function $\Psi(\cdot)$, substitute for $f(z)$ from B.5, then do a change of variables of $z = \frac{Z}{M}$ to obtain a useful transformation of the integral.

$$\int_{M}^{\infty} \Psi\left(\frac{Z}{M}\right) \phi(Z) \mathrm{d}Z = \int_{M}^{\infty} \Psi\left(\frac{Z}{M}\right) f\left(\frac{Z}{M}\right) \frac{1}{M} \mathrm{d}Z = \int_{M/M}^{\infty} \Psi(z) f(z) \mathrm{d}z. \tag{C.1}$$

The key to this transformation is that moving from ϕ to f introduces a $1/M$ term. Thus, abusing notation by using an expectation of the normalized variable,

$$\int_{M}^{\infty} \Psi\left(\frac{Z}{M}\right) \phi(Z) \mathrm{d}Z = \mathbb{E}\left[\Psi(z)\right]. \tag{C.2}$$

C.1. Normalizing the Static Equilibrium

Define the following normalized, real, per-capita values:

$$\hat{z} := \frac{\hat{Z}}{M} \tag{C.3}$$

$$y := \frac{Y}{\bar{L}MP} \tag{C.4}$$

$$c := \frac{C}{\bar{L}M} \tag{C.5}$$

$$q_d(Z) := \frac{Q_d(Z)}{\bar{L}M} \tag{C.6}$$

$$x := \frac{X}{\bar{L}Mw} \tag{C.7}$$

$$w := \frac{W}{MP} \tag{C.8}$$

$$\pi_d(Z) := \frac{\Pi_d(Z)}{\bar{L}Mw}. \tag{C.9}$$

In order to simplify algebra and notation, we normalize profits and adoption costs relative to real, normalized wages. This, for example, means the normalized cost of adoption $x = \zeta$.

Combining the normalized variables with equations (A.8) and (A.9) provides the real prices in terms of real, normalized wages.

$$\frac{p_d(Z)}{P} = \bar{\sigma} \frac{w}{Z/M}, \tag{C.10}$$

$$\frac{p_x(Z)}{P} = \bar{\sigma} d \frac{w}{Z/M}. \tag{C.11}$$

Substituting equations (C.10) and (C.11) into equation (A.4) and dividing by $\bar{L}M$ yields normalized quantities,

$$q_d(Z) = \bar{\sigma}^{-\sigma} w^{-\sigma} y \left(\frac{Z}{M}\right)^{\sigma}, \tag{C.12}$$

$$q_x(Z) = \bar{\sigma}^{-\sigma} w^{-\sigma} d^{-\sigma} y \left(\frac{Z}{M}\right)^{\sigma}. \tag{C.13}$$

Divide equation (A.10) by $\bar{L}$, then substitute from equations (A.4) and (C.10). Finally, divide the top and bottom by M to obtain normalized demand for production labor

$$\ell_d(Z)/\bar{L} = \bar{\sigma}^{-\sigma} w^{-\sigma} y \left(\tfrac{Z}{M}\right)^{\sigma-1}, \tag{C.14}$$
$$\ell_x(Z)/\bar{L} = \bar{\sigma}^{-\sigma} w^{-\sigma} y d^{1-\sigma} \left(\tfrac{Z}{M}\right)^{\sigma-1}. \tag{C.15}$$

Divide equation (A.5) by $P^{1-\sigma}$ and then substitute from equation (C.10) for $p_d(Z)/P$ to obtain

$$1 = \Omega \bar{\sigma}^{1-\sigma} w^{1-\sigma} \left[\int_M^\infty \left(\tfrac{Z}{M}\right)^{\sigma-1} \mathrm{d}\Phi(Z) + (N-1) \int_{\hat{Z}}^\infty d^{1-\sigma} \left(\tfrac{Z}{M}\right)^{\sigma-1} \mathrm{d}\Phi(Z) \right]. \tag{C.16}$$

Simplify equation (C.16) by defining $\bar{z}$, a measure of effective aggregate productivity. Then use equation (C.2) to give normalized real wages in terms of parameters, $\hat{z}$, and the productivity distribution

$$\bar{z} := \left[\Omega \left(\mathbb{E}\left[z^{\sigma-1}\right] + (N-1)(1-F(\hat{z})) d^{1-\sigma} \mathbb{E}\left[z^{\sigma-1} \mid z > \hat{z}\right] \right) \right]^{\frac{1}{\sigma-1}}, \tag{C.17}$$
$$w^{\sigma-1} = \bar{\sigma}^{1-\sigma} \bar{z}^{\sigma-1}, \tag{C.18}$$
$$w = \tfrac{1}{\bar{\sigma}} \bar{z}. \tag{C.19}$$

Note that if $d = 1$ and $\hat{z} = 1$, then $w = \frac{1}{\bar{\sigma}} \left(\Omega N \mathbb{E}\left[z^{1-\sigma}\right]\right)^{1/(\sigma-1)}$. Divide equations (A.11) and (A.12) by $\bar{L} M w$ and substitute with equation (C.18) to obtain normalized profits,

$$\pi_d(Z) = \tfrac{1}{\sigma} \left(\tfrac{p(Z)}{P}\right)^{1-\sigma} \tfrac{y}{w} = \tfrac{1}{\sigma \bar{z}^{\sigma-1}} \tfrac{y}{w} \left(\tfrac{Z}{M}\right)^{\sigma-1}, \tag{C.20}$$
$$\pi_x(Z) = \tfrac{1}{\sigma \bar{z}^{\sigma-1}} \tfrac{y}{w} d^{1-\sigma} \left(\tfrac{Z}{M}\right)^{\sigma-1} - \kappa. \tag{C.21}$$

Divide equations (A.15) and (A.16) by $\bar{L} M w$ and use equations (C.20) and (C.21) to find aggregate profits from domestic production and from exporting to one country,

$$\bar{\pi}_d = \Omega \tfrac{1}{\sigma \bar{z}^{\sigma-1}} \tfrac{y}{w} \mathbb{E}\left[z^{\sigma-1}\right], \tag{C.22}$$
$$\bar{\pi}_x = \Omega \left(\tfrac{1}{\sigma \bar{z}^{\sigma-1}} \tfrac{y}{w} d^{1-\sigma} (1-F(\hat{z})) \mathbb{E}\left[z^{\sigma-1} \mid z > \hat{z}\right] - (1-F(\hat{z}))\kappa \right). \tag{C.23}$$

Divide equation (A.25) by $\bar{L}$, and aggregate the total labor demand from equations (C.14) and (C.15) to obtain normalized aggregate labor demand

$$\begin{aligned} 1 &= \Omega \bar{\sigma}^{-\sigma} w^{-\sigma} y \left(\mathbb{E}\left[z^{\sigma-1}\right] + (N-1)(1-F(\hat{z})) d^{1-\sigma} \mathbb{E}\left[z^{\sigma-1} \mid z > \hat{z}\right] \right) \\ &\quad + \Omega (N-1)(1-F(\hat{z}))\kappa + \Omega(1-\eta)\zeta S && \text{(C.24)} \\ &= \bar{\sigma}^{-\sigma} w^{-\sigma} y \bar{z}^{\sigma-1} + \Omega \left((N-1)(1-F(\hat{z}))\kappa + (1-\eta)\zeta \left(S + \delta/\chi\right) \right). && \text{(C.25)} \end{aligned}$$

Define $\tilde{L}$ as a normalized quantity of labor used outside of variable production. Multiply equation (C.25) by w, and use equation (C.18) to show that

$$\tilde{L} := \Omega\left[(N-1)(1-F(\hat{z}))\kappa + (1-\eta)\zeta\left(S+\delta/\chi\right)\right], \tag{C.26}$$

$$w = \tfrac{1}{\bar{\sigma}}y + \tilde{L}w, \tag{C.27}$$

$$1 = \tfrac{1}{\bar{\sigma}}\tfrac{y}{w} + \tilde{L}. \tag{C.28}$$

Reorganize to find real output as a function of the productivity distribution and labor supply (net of labor used for the fixed costs of exporting and adopting technology)

$$\tfrac{y}{w} = \bar{\sigma}\left(1-\tilde{L}\right), \tag{C.29}$$

$$y = \left(1-\tilde{L}\right)\bar{z}. \tag{C.30}$$

This equation lends interpretation to $\bar{z}$ as being related to the aggregate productivity. Substituting equation (C.29) into equations (C.20) and (C.21) to obtain a useful formulation of firm profits

$$\pi_d(Z) = \tfrac{1-\tilde{L}}{(\sigma-1)\bar{z}^{\sigma-1}}\left(\tfrac{Z}{M}\right)^{\sigma-1}, \tag{C.31}$$

$$\pi_x(Z) = \tfrac{1-\tilde{L}}{(\sigma-1)\bar{z}^{\sigma-1}}d^{1-\sigma}\left(\tfrac{Z}{M}\right)^{\sigma-1} - \kappa. \tag{C.32}$$

Define the common profit multiplier $\bar{\pi}_{\min}$ as

$$\bar{\pi}_{\min} := \tfrac{1-\tilde{L}}{(\sigma-1)\bar{z}^{\sigma-1}} = \tfrac{1-\tilde{L}}{(\sigma-1)(\bar{\sigma}w)^{\sigma-1}} = \frac{(\bar{\sigma}w)^{1-\sigma}y}{\sigma w}, \tag{C.33}$$

$$\pi_d(Z) = \bar{\pi}_{\min}\left(\tfrac{Z}{M}\right)^{\sigma-1}, \tag{C.34}$$

$$\pi_x(Z) = \bar{\pi}_{\min}d^{1-\sigma}\left(\tfrac{Z}{M}\right)^{\sigma-1} - \kappa. \tag{C.35}$$

Use equation (C.35) set to zero to solve for $\hat{z}$. This is an implicit equation as $\bar{\pi}_{\min}$ is a function of $\hat{z}$ through $\bar{z}$

$$\hat{z} = d\left(\tfrac{\kappa}{\bar{\pi}_{\min}}\right)^{\frac{1}{\sigma-1}}. \tag{C.36}$$

Substitute equations (C.34) and (C.35) into equations (C.22) and (C.23) to obtain a useful formulation for aggregate profits

$$\bar{\pi}_d = \Omega\bar{\pi}_{\min}\mathbb{E}\left[z^{\sigma-1}\right], \tag{C.37}$$

$$\bar{\pi}_x = \Omega\left((1-F(\hat{z}))\left(\bar{\pi}_{\min}d^{1-\sigma}\mathbb{E}\left[z^{\sigma-1} \;\middle|\; z>\hat{z}\right] - \kappa\right)\right). \tag{C.38}$$

Combine to calculate aggregate total profits

$$\bar{\pi}_d + (N-1)\bar{\pi}_x = \Omega\bar{\pi}_{\min}\left[\mathbb{E}\left[z^{\sigma-1}\right] + (N-1)(1-F(\hat{z}))d^{1-\sigma}\mathbb{E}\left[z^{\sigma-1} \;\middle|\; z > \hat{z}\right]\right] \\ - \Omega(N-1)(1-F(\hat{z}))\kappa. \tag{C.39}$$

Rewriting aggregate total profits using the definition of $\bar{z}$ yields

$$\bar{\pi}_{\text{agg}} := \bar{\pi}_{\min}\bar{z}^{\sigma-1} - \Omega(N-1)(1-F(\hat{z}))\kappa. \tag{C.40}$$

Note that in a closed economy, $\bar{z} = \left(\Omega\mathbb{E}\left[z^{\sigma-1}\right]\right)^{1/(\sigma-1)}$ and therefore aggregate profits relative to wage are a markup dependent fraction of normalized output relative to wage $\bar{\pi}_d = \frac{1-\tilde{L}}{\sigma-1}$. Take the resource constraint in equation (A.26) and divide by $M\bar{L}w$ and then use equation (C.29) to get an equation for normalized, per-capita consumption

$$\tfrac{c}{w} = \tfrac{y}{w} - \Omega\eta\zeta\Theta\left(S+\delta/\chi\right)/w = \bar{\sigma}\left(1-\tilde{L}\right) - \Omega\eta\zeta\tfrac{\Theta}{w}\left(S+\delta/\chi\right), \tag{C.41}$$

$$c = \left(1-\tilde{L}\right)\bar{z} - \eta\zeta\Omega\Theta\left(S+\delta/\chi\right). \tag{C.42}$$

Normalize the cost in equation (A.23) by dividing by $\bar{L}PMw$. This is implicitly a function of $\hat{z}$ through w

$$x = \zeta\left(1-\eta+\eta\Theta/w\right). \tag{C.43}$$

Normalize the trade share in equation (A.17) by substituting from equations (C.11) and (C.18)

$$\lambda = (1-F(\hat{z}))d^{1-\sigma}\frac{\Omega\mathbb{E}\left[z^{\sigma-1} \;\middle|\; z > \hat{z}\right]}{\bar{z}^{\sigma-1}}. \tag{C.44}$$

Starting from C.40, use C.33, and C.26, to derive that

$$\bar{\pi}_{\text{agg}} = \frac{1}{\sigma-1}\left(1-\Omega\left[(1-\eta)\zeta\left(S+\delta/\chi\right) - \sigma(N-1)(1-F(\hat{z}))\kappa\right]\right). \tag{C.45}$$

Stationary Trade Shares and Profits. Using the stationary distribution in equation (B.13), calculate average profits from C.40,

$$\frac{\bar{\pi}_{agg}}{\Omega} = \frac{\bar{\pi}_{\min}\theta}{1+\theta-\sigma} + \frac{(\sigma-1)(N-1)\kappa\hat{z}^{-\theta}}{1+\theta-\sigma}. \tag{C.46}$$

Note that the minimum profits are at $z = 1$ and equal to $\bar{\pi}_{\min}$ as long as $\hat{z} > 1$. Using this to define the profit spread between the average and worst firm in the economy,

$$\frac{\bar{\pi}_{agg}}{\Omega} - \bar{\pi}_{\min} = \underbrace{\frac{\bar{\pi}_{\min}\theta}{1+\theta-\sigma} + \frac{(\sigma-1)(N-1)\kappa\hat{z}^{-\theta}}{1+\theta-\sigma} - \bar{\pi}_{\min}}_{\text{ProfitSpread}} = \frac{(\sigma-1)\bar{\pi}_{\min}}{1+\theta-\sigma} + \frac{(\sigma-1)(N-1)\kappa\hat{z}^{-\theta}}{1+\theta-\sigma}. \quad \text{(C.47)}$$

Define the ratio of mean to minimum profits as $\bar{\pi}_{\text{rat}} := \frac{\bar{\pi}_{agg}/\Omega}{\bar{\pi}_{\min}}$. From C.47 and C.36 find that

$$\bar{\pi}_{\text{rat}} = \frac{\theta}{1+\theta-\sigma} + (N-1)d^{1-\sigma}\frac{(\sigma-1)\hat{z}^{\sigma-1-\theta}}{1+\theta-\sigma}. \quad \text{(C.48)}$$

Take equations (C.44) and (C.17) to find

$$\bar{z}^{\sigma-1} = \Omega\mathbb{E}\left[z^{\sigma-1}\right] + (N-1)\lambda\bar{z}^{\sigma-1}. \quad \text{(C.49)}$$

Solving gives an expression for a function of aggregate productivity in terms of underlying productivity and trade shares. Defining the home trade share as $\lambda_{ii} := 1-(N-1)\lambda$,

$$\bar{z}^{\sigma-1} = \Omega\frac{\mathbb{E}\left[z^{\sigma-1}\right]}{1-(N-1)\lambda} = \Omega\frac{\mathbb{E}\left[z^{\sigma-1}\right]}{\lambda_{ii}}. \quad \text{(C.50)}$$

From equations (C.19) and (C.50),

$$w = \frac{1}{\bar{\sigma}}\Omega^{\frac{1}{\sigma-1}}\mathbb{E}\left[z^{\sigma-1}\right]^{\frac{1}{\sigma-1}}\lambda_{ii}^{\frac{1}{1-\sigma}}. \quad \text{(C.51)}$$

This relates the real normalized wage to the aggregate productivity, the home trade share, and the number of varieties. Given that $\sigma > 1$, this expression implies that the larger the share of goods purchased at home, the lower the real wage is.

From C.44 and C.50 ,

$$\lambda = (1-F(\hat{z}))d^{1-\sigma}\frac{\mathbb{E}\left[z^{\sigma-1} \mid z > \hat{z}\right]\lambda_{ii}}{\mathbb{E}\left[z^{\sigma-1}\right]}. \quad \text{(C.52)}$$

Using the stationary distribution,

$$\lambda = \hat{z}^{-\theta}d^{1-\sigma}\frac{\hat{z}^{\sigma-1}\frac{\theta}{\theta-(\sigma-1)}\lambda_{ii}}{\frac{\theta}{\theta-(\sigma-1)}} = \hat{z}^{-\theta}d^{1-\sigma}\hat{z}^{\sigma-1}\lambda_{ii}. \quad \text{(C.53)}$$

Using the definition of the home trade share,

$$\lambda_{ii} = \frac{1}{1+(N-1)\hat{z}^{\sigma-1-\theta}d^{1-\sigma}}. \quad \text{(C.54)}$$

Furthermore, multiplying the numerator and denominator by $\bar{\pi}_{\min}$ and using equation (C.36),

$$\lambda_{ii} = \frac{\bar{\pi}_{\min}}{\bar{\pi}_{\min} + (N-1)\hat{z}^{-\theta}\kappa}. \tag{C.55}$$

Which gives an alternative expression for $\bar{\pi}_{\min}$ when $\kappa > 0$,

$$\bar{\pi}_{\min} = (N-1)\hat{z}^{-\theta}\kappa \frac{\lambda_{ii}}{1-\lambda_{ii}}. \tag{C.56}$$

D. Normalized and Stationary Dynamic Equilibrium Conditions

This section derives normalized stationary dynamic balanced growth path equilibrium conditions.

D.1. Utility and Welfare on a BGP

Using the substitution $C(t) = c\bar{L}M(t)$ shows time 0 welfare $\bar{U}$ as a function of c and g is

$$\bar{U}(c,g) = \tfrac{1}{1-\gamma}\tfrac{(c\bar{L}M(0))^{1-\gamma}}{\rho+(\gamma-1)g}. \tag{D.1}$$

With $\log$ utility

$$\bar{U}(c,g) = \frac{\rho\log(c\bar{L}M(0)) + g}{\rho^2}. \tag{D.2}$$

Using the standard IES of the consumer, adjusted for stochastic death of the firm, the firms' effective discount rate on a BGP is,

$$r = \rho + \gamma g + \delta. \tag{D.3}$$

With $\log$ utility

$$r = \rho + g + \delta. \tag{D.4}$$

We restrict parameters such that $g(1-\gamma) < \rho$ in equilibrium to ensure finite utility. In the $\log$ utility case, this is simply $\rho > 0$.

D.2. Normalization of the Firm's Dynamic Problem

We proceed to derive the normalized continuation value function, value matching condition, and smooth pasting condition originally specified in equations (A.20)–(A.22). Define the normalized real value of the firm relative to normalized wages as

$$v(z,t) := \tfrac{V(Z,t)}{\bar{L}M(t)w(t)}. \tag{D.5}$$

Rearranging

$$V(Z,t) = \bar{L}w(t)M(t)v(Z/M(t),t). \tag{D.6}$$

First differentiate the continuation value $V(Z,t)$ with respect to t in equation (D.6) and divide by $w(t)M(t)\bar{L}$, using the chain and product rule. This gives,

$$\frac{1}{w(t)M(t)\bar{L}}\frac{\partial V(Z,t)}{\partial t} = \frac{M'(t)}{M(t)}v(z,t) - \frac{M'(t)}{M(t)}\frac{Z}{M(t)}\frac{\partial v(z,t)}{\partial z} + \frac{M(t)}{M(t)}\frac{\partial v(z,t)}{\partial t} + \frac{w'(t)}{w(t)}v(z,t). \tag{D.7}$$

Defining the growth rate of $g(t) := M'(t)/M(t)$ and $g_w(t) := w'(t)/w(t)$. Substitute these into equation (D.7), cancel out $M(t)$, and group $z = Z/M(t)$ to give

$$\frac{1}{w(t)M(t)\bar{L}}\frac{\partial V(Z,t)}{\partial t} = (g(t) + g_w(t))v(z,t) - g(t)z\frac{\partial v(z,t)}{\partial z} + \frac{\partial v(z,t)}{\partial t}. \tag{D.8}$$

Differentiating equation (D.6) with respect to Z yields

$$\frac{\partial V(Z,t)}{\partial Z} = \frac{\bar{L}M(t)w(t)}{M(t)}\frac{\partial v(Z/M(t),t)}{\partial z} = \bar{L}w(t)\frac{\partial v(z,t)}{\partial z}. \tag{D.9}$$

Similarly,

$$\frac{\partial^2 V(Z,t)}{\partial Z^2} = \frac{\bar{L}w(t)}{M(t)}\frac{\partial^2 v(z,t)}{\partial z^2}. \tag{D.10}$$

Define the normalized profits from equation (A.19) as

$$\pi(z,t) := \frac{\Pi(zM(t),t)}{w(t)M(t)\bar{L}}. \tag{D.11}$$

Divide equation (A.20) by $M(t)w(t)\bar{L}$, then substitute for $\frac{\partial V(Z,t)}{\partial t}$, $\frac{\partial V(Z,t)}{\partial Z}$, and $\frac{\partial^2 V(Z,t)}{\partial Z^2}$ from eqs. (D.8), (D.9), and (D.10) in to (A.20). Finally, group the normalized profits using equation (D.11):

$$(r(t) - g(t) - g_w(t))v(z,t) = \pi(z,t) + \left(\mu + \frac{\upsilon^2}{2} - g(t)\right)z\frac{\partial v(z,t)}{\partial z} + \frac{\upsilon^2}{2}z^2\frac{\partial^2 v(z,t)}{\partial z^2} + \frac{\partial v(z,t)}{\partial t}. \tag{D.12}$$

Equation (D.12) is the normalized version of the value function of the firm in the continuation region. The stationary version of this equation is,

$$(r-g)v(z) = \pi(z) + \left(\mu + \frac{\upsilon^2}{2} - g\right)zv'(z) + \frac{\upsilon^2}{2}z^2v''(z). \tag{D.13}$$

To derive the normalized smooth pasting condition, use equation (A.22) to show that equation (D.9) evaluated at $Z = M(t)$ equals 0, delivering

$$\frac{\partial v(1,t)}{\partial z} = 0. \tag{D.14}$$

To arrive at the normalized value matching condition, divide equation (A.21) by $M(t)w(t)\bar{L}$ to obtain

$$\frac{V(M(t),t)}{M(t)w(t)\bar{L}} = \int_{M(t)}^{\infty} \frac{V(Z,t)}{M(t)w(t)\bar{L}} \phi(Z,t) \mathrm{d}Z - \frac{X(t)}{M(t)w(t)\bar{L}} \tag{D.15}$$

Substituting using equation (D.5) and the definition of $x(t)$ yields

$$v(M(t)/M(t),t) = \int_{M}^{\infty} v(Z/M,t)\phi(Z,t)\mathrm{d}Z - x(t). \tag{D.16}$$

Finally, normalize the integral, realizing it is of the form discussed in equation (C.2), to obtain the normalized value matching condition:

$$v(1,t) = \int_{1}^{\infty} v(z,t) f(z,t) \mathrm{d}z - x(t). \tag{D.17}$$

D.3. Normalization of the Free Entry Condition

Normalizing the free entry condition given in equation (A.24), following similar steps that delivered the normalized value matching condition in equation (D.17), gives

$$x(t)/\chi = \int_{1}^{\infty} v(z,t) f(z,t) \mathrm{d}z. \tag{D.18}$$

Relating this to the value-matching condition of the adopting firm given in equation (D.17) provides a simple formulation of the stationary free entry condition that is useful in determining Ω and g:

$$v(1) = x\frac{1-\chi}{\chi}. \tag{D.19}$$

E. Solving for the Continuation Value Function

Although our baseline model does not feature exogenous productivity shocks, in this section we solve for the value function of a more general model that has GBM with $\upsilon \geq 0$. Our baseline case of $\upsilon = 0$ is nested in this formulation. The differential equation for the value function is solved using the method of undetermined coefficients. The goal of this section is to solve for the value function as a function of parameters, g, Ω, and $\hat{z}$ (sometimes implicitly through $\bar{\pi}_{min}$).

Selection into Exporting. If $\kappa > 0$, generically some firms will choose to be exporters and some firms will only sell domestically. The value function will have a region of productivities representing the value of firms that only sell domestically and a region representing firms that also export. That is,

$$v(z) = \begin{cases} v_d(z) & \text{if } z \leq \hat{z} \\ v_x(z) & \text{if } z \geq \hat{z}. \end{cases}$$

We guess the value function is of the following form, with undetermined constants a, ν, and b,[37]

$$v_d(z) = a\bar{\pi}_{\min}\left(z^{\sigma-1} + \frac{\sigma-1}{\nu}z^{-\nu}\right), \tag{E.1}$$

$$v_x(z) = a\bar{\pi}_{\min}\left((1+(N-1)d^{1-\sigma})z^{\sigma-1} + \frac{\sigma-1}{\nu}z^{-\nu} + (N-1)\frac{1}{a(r-g)}\left(bz^{-\nu} - \frac{\kappa}{\bar{\pi}_{\min}}\right)\right). \tag{E.2}$$

The value of a firm can be decomposed into the value of operating with its current productivity forever and the option value of adopting a better technology. The constant a is a discounting term on the value of earning the profits from producing with productivity z in perpetuity. The constant ν reflects the rate at which the option value of technology adoption goes to zero as productivity increases. The constant b is an adjustment to the perpetuity profits that reflects a firm with productivity z will eventually switch from exporting to being a domestic producer if z is constant in a growing economy.

By construction, the form of these guesses ensures that value matching and smooth pasting are satisfied, both at the adoption threshold ($z = 1$) and the exporter threshold ($z = \hat{z}$). To solve for a and ν, substitute equation (E.1) into the continuation value function in equation (D.13) using $\pi_d(z)$ from equation (C.34). This generates an ODE and, grouping terms, the method of undetermined coefficients provides a system of 2 equations in the 2 unknowns.[38]

Solving the system gives,

$$\nu = \frac{\mu - g}{\upsilon^2} + \sqrt{\left(\frac{g-\mu}{\upsilon^2}\right)^2 + \frac{r-g}{\upsilon^2/2}}, \tag{E.3}$$

$$a = \frac{1}{r - g - (\sigma-1)(\mu - g + (\sigma-1)\upsilon^2/2)}. \tag{E.4}$$

To solve for b, plug equation (E.2) into the continuation value function in equation (D.13) using $\pi_x(z)$ from equation (C.35). This generates an ODE and, grouping terms, the method of undetermined coefficients provides a system of 3 equations in the 3 unknowns. By construction, the a and ν terms match those previously found, giving a consistent solution for b:

$$b = (1 - a(r-g))\, d^{1-\sigma}\, \hat{z}^{\nu+\sigma-1}. \tag{E.5}$$

Note, the main effect of the GBM is to modify the ν constant to reflect changes in the expected execution

[37] This guess is also applying a standard transversality condition to eliminate an explosive root.

[38] Instead of the method of undetermined coefficients, a direct solution approach would be to solve the continuation value function ODEs in the domestic sales and exporter regions, using the smooth pasting condition as the boundary value.

time of the option value of technology diffusion, and hence the exponent for discounting.

As will be useful in solving for g, evaluating at the adoption threshold yields,

$$v(1) = a\bar{\pi}_{\min}\left(1 + \frac{\sigma - 1}{\nu}\right). \tag{E.6}$$

For the baseline case of $\mu = \upsilon = 0$,

$$a = \frac{1}{r + (\sigma - 2)g}, \tag{E.7}$$

$$\nu = \frac{r}{g} - 1, \tag{E.8}$$

$$b = \frac{\sigma - 1}{\nu + \sigma - 1} d^{1-\sigma} \hat{z}^{\nu+\sigma-1}. \tag{E.9}$$

F. Computing the BGP Equilibrium when All Firms Export

In the case of $\kappa = 0$ all firms export (given d s.t. the economy is not in autarky), and the value function has only one region. We guess that the value function will take the following form,

$$v(z) = a\bar{\pi}_{\min}\left(1 + (N-1)d^{1-\sigma}\right)\left(z^{\sigma-1} + \frac{\sigma - 1}{\nu} z^{-\nu}\right). \tag{F.1}$$

Substituting equation (F.1) into the continuation value function in equation (D.13) with profits from equation (C.35), the ν and a are identical to those from equations (E.3) and (E.4). Evaluating at the threshold,

$$v(1) = a\left(1 + (N-1)d^{1-\sigma}\right)\bar{\pi}_{\min}\left(1 + \frac{\sigma - 1}{\nu}\right). \tag{F.2}$$

Solving for the Growth Rate and measure of Varieties when All Firms Export. Using the free entry condition from equation (D.19) with equation (F.2) to find,

$$\frac{x}{\bar{\pi}_{\min}} = a\left(1 + (N-1)d^{1-\sigma}\right)\frac{\chi}{1-\chi}\frac{\sigma + \nu - 1}{\nu}. \tag{F.3}$$

Substitute equations (F.1) and (F.2) into the value matching condition of equation (D.17), and divide by $a\bar{\pi}_{\min}(1 + (N-1)d^{1-\sigma})$

$$1 + \frac{\sigma - 1}{\nu} = \frac{\theta(\nu + \sigma - 1)(\theta + \nu - \sigma + 1)}{\nu(\theta + \nu)(\theta - \sigma + 1)} - \frac{x}{\bar{\pi}_{\min} a\left(1 + (N-1)d^{1-\sigma}\right)}. \tag{F.4}$$

Combine equations (F.3) and (F.4), and solve for ν. For any cost function x and minimum profits $\bar{\pi}_{\min}$,

$$\nu = \frac{\chi\theta(\theta + 1 - \sigma)}{\sigma - 1 - \theta\chi}. \tag{F.5}$$

The aggregate growth rate is found by equating equations (E.3) and (F.5) to find

$$g = \underbrace{\mu}_{\text{Drift}} + \underbrace{\frac{(r-\mu)((\sigma-1)/\chi-\theta)}{\theta^2-\theta\sigma+(\sigma-1)/\chi}}_{\text{Deterministic}} + \frac{\upsilon^2}{2}\underbrace{\frac{\theta^2(\theta+1-\sigma)^2}{(\theta^2-\theta\sigma+(\sigma-1)/\chi)(\theta-(\sigma-1)/\chi)}}_{\text{Stochastic}}. \tag{F.6}$$

In the baseline case of $\upsilon = \mu = 0$

$$g = \frac{(\rho+\delta)(\sigma-1-\chi\theta)}{\theta\chi(\gamma+\theta-\sigma)-(\gamma-1)(\sigma-1)}, \tag{F.7}$$

and,

$$\frac{x}{\bar{\pi}_{\min}} = \left(1+(N-1)d^{1-\sigma}\right)\frac{\chi}{1-\chi}\frac{1}{r-g}. \tag{F.8}$$

The growth rate is independent of the trade costs, the population, and the number of countries. The intuition–as discussed in the body of the paper–is that the growth rate is driven by the ratio of the minimum to the mean profits, which are proportional and independent of the scale or integration of economies in the absence of any export selection. The constant death rate only enters to increase the discount rate.

Note in the baseline case of $\upsilon = \mu = \delta = 0$ and $\log$ utility

$$g = \frac{\rho(\sigma-1-\chi\theta)}{\theta^2\chi}\bar{\pi}^k_{\text{rat}}, \tag{F.9}$$

where (from eqs. C.40 and C.17 with $\kappa = 0$ and $\hat{z} = 1$) the ratio of average profits to minimum profits is

$$\bar{\pi}^k_{\text{rat}} = \frac{\bar{\pi}^k_{\text{agg}}/\Omega}{\bar{\pi}^k_{\min}} = \frac{(1+(N-1)d^{1-\sigma})\bar{\pi}_{\min}E[z^{\sigma-1}]}{(1+(N-1)d^{1-\sigma})\bar{\pi}_{\min}} = E[z^{\sigma-1}] = \frac{\theta}{1+\theta-\sigma}. \tag{F.10}$$

The Measure of Varieties Ω. Here we solve for the measure of varieties Ω in the baseline case of $\upsilon = \mu = \delta = 0$.

Substitute into the free entry condition equation (F.8) using the definition of $\bar{\pi}_{\min}$ in terms of $\bar{z}$ and $\tilde{L}$ from equation (C.33), the definition of $\bar{z}$ from equation (C.17), and the definition of $\tilde{L}$ from equation (C.26), to obtain the implicit equation. In the baseline case where $\eta = 0$, an explicit solution is[39]

[39]The solution for the simplest case of $\gamma = 1$ and $\delta = 0$ is,

$$g = \frac{\rho(\sigma-1-\theta\chi)}{\theta\chi(1+\theta-\sigma)} \tag{F.11}$$

$$\Omega = \frac{\chi(1+\theta-\sigma)}{\zeta\rho\left((1+\theta)(\sigma-1)-\sigma\theta\chi\right)} \tag{F.12}$$

$$\Omega = \frac{\chi((\gamma-1)(\sigma-1)-\theta\chi(\gamma+\theta-\sigma))}{\zeta\left(\theta\chi(-\gamma\delta-\sigma(\theta(\delta+\rho)+\rho)+\delta+\theta\rho+\rho)+(\gamma-1)\delta(\sigma-1)+\theta^2\sigma\chi^2(\delta+\rho)\right)}. \tag{F.13}$$

Note that the only place that the adoption cost, ζ, has come into the system of Ω and g is in the denominator of F.13. For this reason, the ζ parameter (along with $\bar{L}$) determines the scale of the economy.

It can be shown that in the Krugman model for all cases with $\eta = 0$, the number of domestic varieties is independent of trade costs d. Thus, both Ω and g are independent of d if $\eta = 0$. This implies through equations (C.26) and (B.14) that the amount of labor dedicated to technology adoption, $\tilde{L}$, is also independent of d if $\eta = 0$.

Through C.42, since $\tilde{L}$, Ω, and g are independent of d when $\eta = 0$, in response to a decrease in trade costs d, c increases only due to the $\left(1+(N-1)d^{1-\sigma}\right)^{1/(\sigma-1)}$ term in $\bar{z}$.

The key relationship can be summarized by the following elasticities.

$$\frac{\mathrm{d}\log\bar{\pi}^k_{\text{rat}}(d)}{\mathrm{d}\log(d)} = 0. \tag{F.14}$$

Using equation (C.54) with $\hat{z} = 1$ shows

$$\frac{\mathrm{d}\log\lambda_{ii}(d)}{\mathrm{d}\log(d)} = (\sigma-1)\left(1+\frac{d^{\sigma-1}}{N-1}\right)^{-1} = (\sigma-1)(1-\lambda_{ii}) > 0. \tag{F.15}$$

Furthermore, when $\eta = 0$,

$$\frac{\mathrm{d}\log(1-\tilde{L}(d))}{\mathrm{d}\log(d)} = \frac{\mathrm{d}\log\Omega(d)}{\mathrm{d}\log(d)} = \frac{\mathrm{d}\log g(d)}{\mathrm{d}\log(d)} = 0. \tag{F.16}$$

For the case with $\eta = 0$, the ratio of c for different trade costs d_1 and d_2 that both feature positive trade is

$$\frac{\bar{c}_{d_1}}{\bar{c}_{d_2}} = \frac{\bar{z}_{z_{d_1}}}{\bar{z}_{d_2}} = \left(\frac{1+(N-1)d_1^{1-\sigma}}{1+(N-1)d_2^{1-\sigma}}\right)^{\frac{1}{\sigma-1}}. \tag{F.17}$$

Using the normalized welfare function in equation (D.1), since g is independent of d in the $\kappa = 0$ case, the ratio of welfare for different trade costs d_1 and d_2 that both feature positive trade (for $\gamma > 0$) is,

$$\frac{\bar{U}_{d_1}}{\bar{U}_{d_2}} = \left(\frac{1+(N-1)d_1^{1-\sigma}}{1+(N-1)d_2^{1-\sigma}}\right)^{\frac{1-\gamma}{\sigma-1}}. \tag{F.18}$$

Comparing welfare between free trade ($d = 1$) and autarky (sufficiently high d s.t. there are no exporters) gives (for $\gamma \neq 1$)

$$\frac{\bar{U}_{free}}{\bar{U}_{autarky}} = N^{\frac{1-\gamma}{\sigma-1}}. \tag{F.19}$$

G. Computing the BGP Equilibrium with Selection into Exporting ($\kappa > 0$)

We solve for g and Ω by reducing the equilibrium conditions to a system of two equations in these two unknowns. First, combining the free entry condition from equation (D.19) with $v(1)$ from equation (E.6) yields

$$\frac{x}{\bar{\pi}_{\min}} = a\frac{\chi}{1-\chi}\frac{\sigma+\nu-1}{\nu}. \tag{G.1}$$

The second equation is found by evaluating the value matching condition of equation (D.17) by substituting in the domestic and exporter value functions in equations E.1 and E.2, the export threshold $\hat{z}$ in equation (C.36), and the value at the adoption threshold $v(1)$ in equation (E.6) and dividing by $a\bar{\pi}_{\min}$. That is, evaluate

$$\frac{v(1)}{a\bar{\pi}_{\min}} = \frac{\int_1^\infty v(z,t)f(z,t)\mathrm{d}z}{a\bar{\pi}_{\min}} - \frac{x}{a\bar{\pi}_{\min}} \tag{G.2}$$

to obtain

$$1 + \frac{\sigma-1}{\nu} = \frac{\frac{\nu(N-1)(\theta-\sigma+1)\left(d^{1-\sigma}(\theta+\nu)\hat{z}^{-\theta+\sigma-1} - b\theta\hat{z}^{-\theta-\nu}\right)}{a(g-r)} + \theta\left(\nu(N-1)d^{1-\sigma}(\theta+\nu)\hat{z}^{-\theta+\sigma-1} + (\nu+\sigma-1)(\theta+\nu-\sigma+1)\right)}{\nu(\theta+\nu)(\theta-\sigma+1)} - \frac{\chi}{1-\chi}\frac{\sigma+\nu-1}{\nu}. \tag{G.3}$$

As detailed in Section H, use the definition of $\bar{\pi}_{\min}$ and substitute for $x, \nu, a, b, \hat{z}$ and r into equations (G.1) and (G.3) to find a system of 2 equations in Ω and g. Note that the adoption cost x does not appear in equation (G.3), which is why g is independent of the specification of the cost of adoption. Equation (G.1) does explicitly depend on x, which is why the number of varieties is a function of the adoption cost, and ultimately why welfare is also a function of x.

G.1. Case with $\upsilon = \mu = \delta = 0$

As the GBM does not qualitatively impact the solution, we concentrate our analytical theory on the simple baseline case. The cost of adoption is a function of minimum profits, parameters, and $r - g$:

$$x = \bar{\pi}_{\min}\frac{\chi}{1-\chi}\frac{1}{r-g}. \tag{G.4}$$

Evaluating the general equation (G.3) with the substitutions for $x, \nu, a, b, \hat{z}$ and r that correspond to the baseline case of $\upsilon = \mu = \delta = 0$ yields a unique g that satisfies the value function and the value matching equations, given by the implicit equation,

$$g = \frac{\left(\sigma - 1 + (N-1)\theta d^{1-\sigma}\hat{z}^{-\theta+\sigma-1}\right)\bar{\pi}_{\min} + (N-1)(-\theta+\sigma-1)\hat{z}^{-\theta}\kappa}{x(\gamma+\theta-1)(\theta-\sigma+1)} - \frac{\rho}{\gamma+\theta-1}. \tag{G.5}$$

Using $\kappa/\bar{\pi}_{\min} = \hat{z}^{\sigma-1}d^{1-\sigma}$ from equation (C.36), simplify to

$$g = \frac{(\sigma-1)\left[\bar{\pi}_{\min} + (N-1)\kappa\hat{z}^{-\theta}\right]}{x(\gamma+\theta-1)(\theta-\sigma+1)} - \frac{\rho}{\gamma+\theta-1}. \tag{G.6}$$

Using equation (C.47), realize this relates the growth rate to the difference between average and minimum profits

$$g = \left(\frac{\bar{\pi}_{\text{agg}}/\Omega - \bar{\pi}_{\min}}{x(\gamma+\theta-1)}\right) - \frac{\rho}{\gamma+\theta-1}. \tag{G.7}$$

Substitute for x using the free entry condition in equation (G.4), use the definition of the average to minimum profit ratio $\left(\bar{\pi}_{\text{rat}} := \frac{\bar{\pi}/\Omega}{\bar{\pi}_{\min}}\right)$, and substitute for r using equation (D.3) to obtain

$$g = (\rho + (\gamma-1)g)\frac{1-\chi}{\chi(\gamma+\theta-1)}\left(\bar{\pi}_{\text{rat}} - 1\right) - \frac{\rho}{\gamma+\theta-1}. \tag{G.8}$$

Solving for g gives an equation for g as a function exclusively of parameters and the ratio of average to minimum profits

$$g = \frac{\rho}{\chi\theta\left((1-\chi)\bar{\pi}_{\text{rat}} - 1\right)^{-1} + 1 - \gamma}. \tag{G.9}$$

Furthermore, with log utility $\gamma = 1$ and

$$g = \frac{\rho(1-\chi)}{\chi\theta}\bar{\pi}_{\text{rat}} - \frac{\rho}{\chi\theta}. \tag{G.10}$$

The relationship between growth and trade costs. First, note that the growth rate is increasing in the profit ratio, (since $\chi \in (0,1)$):

$$\frac{\mathrm{d}g(\bar{\pi}_{\text{rat}})}{\mathrm{d}\bar{\pi}_{\text{rat}}} = \frac{\rho\theta\chi(1-\chi)}{\left((\gamma-1)(1-(1-\chi)\bar{\pi}_{\text{rat}} + \theta\chi\right)^2} > 0. \tag{G.11}$$

To determine if whether growth is increasing in d, use the chain rule

$$\frac{\mathrm{d}g(d)}{\mathrm{d}d} = \frac{\mathrm{d}g(\bar{\pi}_{\text{rat}})}{\mathrm{d}\bar{\pi}_{\text{rat}}}\frac{\mathrm{d}\bar{\pi}_{\text{rat}}(d)}{\mathrm{d}d}. \tag{G.12}$$

Given equations (G.11) and (G.12), a sufficient condition to conclude that $\frac{\mathrm{d}g(d)}{\mathrm{d}d} < 0$ is $\frac{\mathrm{d}\bar{\pi}_{\mathrm{rat}}(d)}{\mathrm{d}d} < 0$. To show this differentiate C.48 w.r.t. d to find,

$$\frac{\mathrm{d}\bar{\pi}_{\mathrm{rat}}(d)}{\mathrm{d}d} \propto -\left((\sigma-1)\hat{z}(d) + d(1+\theta-\sigma)\frac{\mathrm{d}\hat{z}(d)}{\mathrm{d}d}\right). \tag{G.13}$$

Since $d > 0, \hat{z}(d) > 0$, $\sigma > 1$, and $1+\theta-\sigma > 0$, a sufficient condition for $\frac{\mathrm{d}\bar{\pi}_{\mathrm{rat}}(d)}{\mathrm{d}d} < 0$ is if $\frac{\mathrm{d}\hat{z}(d)}{\mathrm{d}d} > 0$. Differentiate equation (C.36) to find

$$\frac{\mathrm{d}\hat{z}(d)}{\mathrm{d}d} \propto (\sigma-1) - \frac{d}{\bar{\pi}_{\min}(d)}\frac{\mathrm{d}\bar{\pi}_{\min}(d)}{\mathrm{d}d}. \tag{G.14}$$

Therefore, a sufficient condition to conclude that $\frac{\mathrm{d}\hat{z}(d)}{\mathrm{d}d} > 0$ is

$$\frac{\mathrm{d}\log\bar{\pi}_{\min}(d)}{\mathrm{d}d} < \frac{\sigma-1}{d}. \tag{G.15}$$

Summarizing,

$$\mathrm{sign}\frac{\mathrm{d}g(d)}{\mathrm{d}d} = \mathrm{sign}\frac{\mathrm{d}\bar{\pi}_{\mathrm{rat}}}{\mathrm{d}d} = -\mathrm{sign}\frac{\mathrm{d}\hat{z}(d)}{\mathrm{d}d}. \tag{G.16}$$

G.2. Baseline Case with $\upsilon = \mu = \delta = \eta = 0$ and $\log$ utility.

Adding the restriction that $\eta = 0$ and $\gamma = 1$ to the adoption cost equation (C.43) and the firms' discount rate equation (D.4) delivers the key simplifications that permit solving for the BGP equilibrium in closed form:

Calculating the Growth Rate and the Measure of Varieties.

$$x = \zeta, \tag{G.17}$$

$$r - g = \rho. \tag{G.18}$$

Using equation (G.4) gives an expression for $\bar{\pi}_{\min}$ in terms of model parameters

$$\bar{\pi}_{\min} = \frac{(1-\chi)\zeta\rho}{\chi}. \tag{G.19}$$

Substitute equation (G.19) into equation (C.36) to find the export threshold in terms of parameters,

$$\hat{z} = d\left(\frac{1}{\zeta}\frac{\kappa\chi}{\rho(1-\chi)}\right)^{\frac{1}{\sigma-1}}. \tag{G.20}$$

Substitute $\gamma = 1$ and equations (G.17), (G.19), and (G.20) into equation (G.6) and simplify to obtain g in closed form:

$$g = \frac{\rho(1-\chi)}{\chi\theta}\frac{\sigma-1}{(\theta-\sigma+1)}\left(1+(N-1)d^{-\theta}\left(\frac{\kappa}{\zeta}\frac{\chi}{\rho(1-\chi)}\right)^{1-\frac{\theta}{\sigma-1}}\right)-\frac{\rho}{\theta}, \tag{G.21}$$

$$= \frac{\rho(1-\chi)}{\chi\theta}\frac{\left(\theta+(N-1)(\sigma-1)d^{-\theta}\left(\frac{1}{\zeta}\frac{\kappa\chi}{\rho(1-\chi)}\right)^{1-\frac{\theta}{\sigma-1}}\right)}{(\theta-\sigma+1)}-\frac{\rho}{\chi\theta}. \tag{G.22}$$

For ease of comparison to g as a function of $\bar{\pi}_{\text{rat}}$ use equation (G.8) evaluated at $\gamma = 1$ with equation (G.21) to realize

$$\bar{\pi}_{\text{rat}} - 1 = \frac{\left(1+(N-1)d^{-\theta}\left(\frac{1}{\zeta}\frac{\kappa\chi}{\rho(1-\chi)}\right)^{1-\frac{\theta}{\sigma-1}}\right)}{\left(\frac{\theta}{\sigma-1}-1\right)}, \tag{G.23}$$

or equation (G.10) with equation (G.22) to see

$$\bar{\pi}_{\text{rat}} = \frac{\left(\theta+(N-1)(\sigma-1)d^{-\theta}\left(\frac{1}{\zeta}\frac{\kappa\chi}{\rho(1-\chi)}\right)^{1-\frac{\theta}{\sigma-1}}\right)}{(\theta-\sigma+1)}. \tag{G.24}$$

Note, g is decreasing in d and κ (since $\theta > 0$, $\sigma > 1$, and $1+\theta-\sigma > 0$). Thus, from equation (G.16) or direct differentiation of equation (G.21),

$$\frac{\mathrm{d}g}{\mathrm{d}d} < 0; \quad \frac{\mathrm{d}\bar{\pi}_{\text{rat}}}{\mathrm{d}d} < 0; \quad \frac{\mathrm{d}\hat{z}}{\mathrm{d}d} > 0; \quad \frac{\mathrm{d}g}{\mathrm{d}\kappa} < 0. \tag{G.25}$$

See by comparing to equation (F.7) that the limit of g as $d \to \infty$ in equation (G.22) equals the autarky and all-export economy growth rates, so there is no discontinuity in the economy in this direction.

Note that the parameter ζ (previously interpreted as the scale in equation F.13) and κ only enter the growth rate multiplicatively. This is because the fixed costs of adoption, entry, and export in levels are proportional to the scale of the economy. Since the calibration strategy targets relative moments (i.e., proportion of exporters, relative size of exporters to domestic firms, growth rates, trade shares), κ is not separately identifiable from ζ without some moment that targets the level of the economy.

To find the number of varieties, maintain $\gamma = 1, \eta = 0$. To solve for Ω, start with the definition of $\bar{\pi}_{\min}$ from equation (C.33):

$$\bar{\pi}_{\min} = \tfrac{1-\tilde{L}}{(\sigma-1)\bar{z}^{\sigma-1}}. \tag{G.26}$$

For $\bar{\pi}_{\min}$, substitute from equation (G.19). For the right hand side, substitute for $\tilde{L}$, $\bar{z}$ and S with equations (C.26), (C.17), and (B.14). Then, use g and $\hat{z}$ from equations (G.21), (G.20), and solve for Ω in terms of model parameters:

$$\Omega = \frac{1}{\zeta}\frac{\chi(1+\theta-\sigma)}{(1-\chi)\theta\sigma\rho}\left(1+(N-1)d^{-\theta}\left(\frac{\kappa\chi}{\zeta\rho(1-\chi)}\right)^{1-\frac{\theta}{\sigma-1}} - \frac{1+\theta-\sigma}{\theta\sigma(1-\chi)}\right)^{-1}. \tag{G.27}$$

Note, from equations (C.54) and (G.20), the home trade share is,

$$\lambda_{ii} = \frac{1}{1+(N-1)d^{-\theta}\left(\frac{\kappa\chi}{\zeta\rho(1-\chi)}\right)^{1-\frac{\theta}{\sigma-1}}}. \tag{G.28}$$

Note, by substituting eq. G.28 into eqs. G.21 and G.27, the growth rate and Ω can be written as a function of the home trade share:

$$g = \frac{\rho(1-\chi)}{\chi\theta}\frac{\sigma-1}{(\theta-\sigma+1)}\lambda_{ii}^{-1} - \frac{\rho}{\theta}. \tag{G.29}$$

$$\Omega = \frac{\chi}{\zeta\rho}\left(\frac{(1-\chi)\theta\sigma}{1+\theta-\sigma}\lambda_{ii}^{-1} - 1\right)^{-1}. \tag{G.30}$$

Calculating Consumption. By the resource constraint, $c = y$ when $\eta = 0$ (equation C.41). Thus, using equation (C.30), consumption is given by

$$c = y = (1-\tilde{L})\bar{z}. \tag{G.31}$$

Equations (C.26), (B.14), (C.36), and (G.29) combine to yield the amount of labor dedicated to variable goods production in terms of the home trade share:

$$1-\tilde{L} = (\sigma-1)\left(\sigma - \frac{1+\theta-\sigma}{\theta(1-\chi)}\lambda_{ii}\right)^{-1}. \tag{G.32}$$

Equation (C.50) gives

$$\bar{z} = \Omega^{\frac{1}{\sigma-1}}\lambda_{ii}^{\frac{1}{1-\sigma}}\left(\mathbb{E}\left[z^{\sigma-1}\right]\right)^{\frac{1}{\sigma-1}}. \tag{G.33}$$

Substituting equations (G.32), (G.33), and (G.30) into equation (G.31) yields consumption as a function of parameters and the home trade share:

$$c = \frac{(\sigma-1)\theta\sigma(1-\chi)}{\sigma(1+\theta-\sigma)}\left(\frac{\chi}{\rho\zeta}\right)^{\frac{1}{\sigma-1}}\left(\frac{(1-\chi)\theta\sigma}{1+\theta-\sigma} - \lambda_{ii}\right)^{\frac{\sigma}{1-\sigma}}\left(\mathbb{E}\left[z^{\sigma-1}\right]\right)^{\frac{1}{\sigma-1}}. \tag{G.34}$$

Trade Cost Elasticities. Comparative statics are analyzed by calculating elasticities with respect to trade costs using equations (G.20), (G.21), (G.24), (G.27), and (G.28):

$$\frac{\mathrm{d}\log \hat{z}(d)}{\mathrm{d}\log(d)} = 1, \tag{G.35}$$

$$\frac{\mathrm{d}\log g(d)}{\mathrm{d}\log(d)} = -\theta\left(1 + \frac{\rho d^{\theta}(-\theta\chi + \sigma - 1)\left(\frac{\kappa\chi}{\zeta\rho(1-\chi)}\right)^{\frac{\theta}{\sigma-1}}}{\kappa(N-1)(\sigma-1)\chi}\right)^{-1} < 0, \tag{G.36}$$

$$\frac{\mathrm{d}\log \bar{\pi}_{\text{rat}}(d)}{\mathrm{d}\log(d)} = -\theta\left(1 + \frac{\theta d^{\theta}\left(\frac{\kappa\chi}{\zeta\rho(1-\chi)}\right)^{\frac{\theta}{\sigma-1}-1}}{(N-1)(\sigma-1)}\right)^{-1} < 0, \tag{G.37}$$

$$\frac{\mathrm{d}\log \Omega(d)}{\mathrm{d}\log(d)} = \theta\left(1 + \frac{\rho d^{\theta}((\theta+1)(\sigma-1) - \theta\sigma\chi)\left(\frac{\kappa\chi}{\zeta\rho(1-\chi)}\right)^{\frac{\theta}{\sigma-1}}}{\theta\kappa(N-1)\sigma\chi}\right)^{-1} > 0, \tag{G.38}$$

$$\frac{\mathrm{d}\log \lambda_{ii}(d)}{\mathrm{d}\log(d)} = \theta\left(1 + \frac{d^{\theta}}{N-1}\left(\frac{\kappa\chi}{\zeta\rho(1-\chi)}\right)^{\frac{\theta}{\sigma-1}-1}\right)^{-1} > 0. \tag{G.39}$$

Let $\varepsilon_{f,x}$ be the elasticity of any $f(x)$ w.r.t. x. These elasticities can be rearranged to highlight their relationship to trade volume. To see this, first define the ratio of the home trade share to the share of goods purchased away from home:

$$\frac{\lambda_{ii}}{1-\lambda_{ii}} = \frac{d^{\theta}\left(\frac{\kappa\chi}{\zeta\rho(1-\chi)}\right)^{\frac{\theta}{\sigma-1}-1}}{N-1}. \tag{G.40}$$

Substituting this into the result above, yields

$$\frac{\mathrm{d}\log \lambda_{ii}(d)}{\mathrm{d}\log(d)} = \theta\left(1 + \frac{\lambda_{ii}}{1-\lambda_{ii}}\right)^{-1} = \theta(1-\lambda_{ii}), \tag{G.41}$$

$$\frac{\mathrm{d}\log g(d)}{\mathrm{d}\log(d)} = -\theta\left[1 + \left(\frac{-\theta\chi + \sigma - 1}{(\sigma-1)(1-\chi)}\right)\frac{\lambda_{ii}}{1-\lambda_{ii}}\right]^{-1} \tag{G.42}$$

$$= \left(\frac{\chi(1+\theta-\sigma)}{(\sigma-1)(1-\chi)}\lambda_{ii} - 1\right)^{-1}\varepsilon_{\lambda_{ii},d}, \tag{G.43}$$

$$\frac{\mathrm{d}\log \Omega_{ii}(d)}{\mathrm{d}\log(d)} = \left(1 - \frac{1+\theta-\sigma}{\theta\sigma(1-\chi)}\lambda_{ii}\right)^{-1}\varepsilon_{\lambda_{ii},d}. \tag{G.44}$$

The elasticity of $(1-\tilde{L})$ w.r.t. d is

$$\frac{\mathrm{d}\log(1-\tilde{L}(d))}{\mathrm{d}\log(d)} = \left(\frac{\theta\sigma(1-\chi)}{1+\theta-\sigma}\lambda_{ii}^{-1} - 1\right)^{-1}\varepsilon_{\lambda_{ii},d} > 0. \tag{G.45}$$

The elasticity of $\bar{z}$ w.r.t. d is

$$\frac{\mathrm{d}\log(\bar{z}(d))}{\mathrm{d}\log(d)} = \frac{\varepsilon_{\Omega,d} - \varepsilon_{\lambda_{ii},d}}{\sigma - 1} > 0. \tag{G.46}$$

From equation (G.31)

$$\varepsilon_{c,d} = \varepsilon_{1-\tilde{L},d} + \varepsilon_{\bar{z},d}. \tag{G.47}$$

Using equations (G.34) and (G.41) yields

$$\varepsilon_{c,d} = -\frac{\sigma}{\sigma - 1}\left(1 - \frac{\theta\sigma(1-\chi)}{(1+\theta-\sigma)}\lambda_{ii}^{-1}\right)^{-1} \varepsilon_{\lambda_{ii},d}. \tag{G.48}$$

Finally, from D.2

$$\varepsilon_{\bar{U},d} = \frac{\rho\,\varepsilon_{c,d} + g\,\varepsilon_{g,d}}{g + \rho\log(c\bar{L}M(0))} \tag{G.49}$$

$$= \frac{\rho^2}{\bar{U}}\,(\rho\,\varepsilon_{c,d} + g\,\varepsilon_{g,d}). \tag{G.50}$$

This can be further organized by substitution for $\varepsilon_{c,d}$ and $\varepsilon_{g,d}$ from equations (G.48) and (G.43) into equation (G.50).[40]

$$\varepsilon_{\bar{U},d} = -\varepsilon_{\lambda_{ii},d}\frac{\rho^3}{\bar{U}}\left(\frac{\sigma}{(\sigma-1)\left(1 - \frac{\theta\sigma(1-\chi)}{(\theta-\sigma+1)\lambda_{ii}}\right)} + \frac{(\sigma-1)(1-\chi)}{\theta\chi(\theta-\sigma+1)\lambda_{ii}}\right). \tag{G.51}$$

Therefore, if the term in brackets is positive, then the elasticity of utility is of the opposite sign of the home trade share, and hence always decreasing in trade costs. The first term in the brackets is always negative and the second term is always positive. There exist parameter values such that sum is negative, such that economies with lower trade costs have lower welfare in a comparison of steady states. This occurs when $\sigma - 1$ is close to its lower bound of $\theta\chi$. This unintuitive result is possible because this analysis ignores transition dynamics and because this is an inefficient economy, so economics of the second best applies.

Firm Adoption Timing. A firm adopts when normalized productivity equals 1 by definition. On the BGP, firms drift backwards towards $z = 1$ at constant rate g. Thus, the time until adoption $\tau(z)$ is given

[40] For determining the direction of the change, as an elasticity is $d\bar{U}'(d)/\bar{U}(d)$, and since $d > 1$, if $\bar{U}(d) > 0$ then the sign of this elasticity calculation matches the sign of the derivative. Otherwise, the sign of the derivative is negative of the elasticity. As equation (G.50) divided by the utility in the calculation, this sign cancels, and ensures that negative utility does not affect the direction of the changes (as expected with a monotone function with the possibility of arbitrarily small initial conditions).

by

$$
\begin{aligned}
e^{-g\tau} z &= 1 \\
\tau(z) &= \frac{\log(z)}{g} \qquad \text{(G.52)}
\end{aligned}
$$

The expected time until adoption for a firm that is about to draw a new productivity, $\bar{\tau}$, is just the expected adoption time integrated over the distribution of the new z:

$$\bar{\tau} = \int_1^\infty \frac{\log(z)}{g} dF(z) = \frac{1}{g}\int_1^\infty \log(z)\theta z^{-\theta-1} = \frac{1}{\theta g} \qquad \text{(G.53)}$$

Since firms draw a new z from the unconditional distribution, the expected time to adoption for a newly adopting firm is the same as the average time to adoption.

H. Computing the BGP Equilibrium in General

In the general case of the $\kappa = 0$, the equilibrium g can be calculated through an explicit equation, and the Ω found separately. If $\kappa > 0$, then a system of 2 non-linear algebraic equations in g and Ω are solved. Summarizing equations for easy reference against the code:

General Substitutions. The following substitutions are used in reducing the equilibrium conditions into a simple system of equations that can be solved for g and Ω. Given g and Ω all other equilibrium values are determined. We use equations (B.13), (B.14), (E.3), (E.4), (E.5), (D.3), (C.26), (C.17), (C.36), (C.19), and (C.43):

$$
\begin{aligned}
F(z) &= 1 - z^{-\theta} && \text{(H.1)} \\
S &= \theta\left(g - \mu - \theta\frac{\upsilon^2}{2}\right) && \text{(H.2)} \\
\nu &= \frac{\mu - g}{\upsilon^2} + \sqrt{\left(\frac{g-\mu}{\upsilon^2}\right)^2 + \frac{r-g}{\upsilon^2/2}} && \text{(H.3)} \\
a &= \frac{1}{r - g - (\sigma-1)(\mu - g + (\sigma-1)\upsilon^2/2)} && \text{(H.4)} \\
b &= (1 - a(r-g))\, d^{1-\sigma}\hat{z}^{\nu+\sigma-1} && \text{(H.5)} \\
r &= \rho + \gamma g + \delta && \text{(H.6)} \\
\tilde{L} &= \Omega\left[(N-1)(1-F(\hat{z}))\kappa + (1-\eta)\zeta\left(S + \delta/\chi\right)\right] && \text{(H.7)} \\
\bar{z} &= \left[\Omega\left(\mathbb{E}\left[z^{\sigma-1}\right] + (N-1)(1-F(\hat{z}))d^{1-\sigma}\mathbb{E}\left[z^{\sigma-1} \mid z > \hat{z}\right]\right)\right]^{1/(\sigma-1)} && \text{(H.8)} \\
\hat{z} &= d\left(\frac{\kappa}{\bar{\pi}_{\min}}\right)^{\frac{1}{\sigma-1}} && \text{(H.9)} \\
w &= \tfrac{1}{\sigma}\bar{z} && \text{(H.10)} \\
x &= \zeta\left(1 - \eta + \eta\Theta/w\right) && \text{(H.11)}
\end{aligned}
$$

(Note, since $\bar{\pi}_{\min}$ is an implicit function through the $\hat{z}$ in $\bar{z}$, it is easiest to add it to the system of equations instead of substituting it out).

All Firms Export Case. For any $\upsilon \geq 0$, the growth rate is given by equation (F.6), substituting for r from D.3.

$$g = \mu + \frac{(r-\mu)((\sigma-1)/\chi - \theta)}{\theta^2 - \theta\sigma + (\sigma-1)/\chi} + \frac{\upsilon^2}{2} \frac{\theta^2(\theta+1-\sigma)^2}{(\theta^2 - \theta\sigma + (\sigma-1)/\chi)(\theta - (\sigma-1)/\chi)}. \tag{H.12}$$

Using the equilibrium g above, Ω can be found by solving the following system of equations in Ω and $\bar{\pi}_{\min}$ from equations (F.3) and (C.33) (where Ω is implicitly in the $\bar{z}$ and $\tilde{L}$ terms):

$$\frac{x}{\bar{\pi}_{\min}} = a\left(1 + (N-1)d^{1-\sigma}\right)\frac{\chi}{1-\chi}\frac{\sigma+\nu-1}{\nu}. \tag{H.13}$$

$$\bar{\pi}_{\min} = \tfrac{1-\tilde{L}}{(\sigma-1)\bar{z}^{\sigma-1}}. \tag{H.14}$$

Selection into Exporting Case. Equations (G.1), (G.3), and (C.33) provide a system of 3 equations in $g, \Omega,$ and $\bar{\pi}_{\min}$. To solve this non-linear system, substitute for $\bar{\pi}_{\min}, \nu, a, b, x, r, S, \tilde{L}, \bar{z}$ and $\hat{z}$ using the general substitutions listed above to eliminate dependence on all other endogenous variables.

$$\frac{x}{\bar{\pi}_{\min}} = a\frac{\chi}{1-\chi}\frac{\sigma+\nu-1}{\nu}, \tag{H.15}$$

$$1 + \frac{\sigma-1}{\nu} = \frac{\frac{\nu(N-1)(\theta-\sigma+1)\left(d^{1-\sigma}(\theta+\nu)\hat{z}^{-\theta+\sigma-1} - b\theta\hat{z}^{-\theta-\nu}\right)}{a(g-r)} + \theta\left(\nu(N-1)d^{1-\sigma}(\theta+\nu)\hat{z}^{-\theta+\sigma-1} + (\nu+\sigma-1)(\theta+\nu-\sigma+1)\right)}{\nu(\theta+\nu)(\theta-\sigma+1)} - \frac{\chi}{1-\chi}\frac{\sigma+\nu-1}{\nu}, \tag{H.16}$$

$$\bar{\pi}_{\min} = \tfrac{1-\tilde{L}}{(\sigma-1)\bar{z}^{\sigma-1}}. \tag{H.17}$$

This system of equations holds for both the general case and the baseline case of $\upsilon = \mu = 0$ (if using Mathematica, make sure to substitute ν to reorganize the formulas to avoid any singularity). Alternatively, for the baseline case, together with the same definition of $\bar{\pi}_{\min}$ use equations (G.4) and G.9 as the system of equations, using the same substitutions as in the more general case.

Post Solution Calculations. In either case, given the equilibrium g and Ω, the following equilibrium values can be calculated through equations (C.40), (C.30), (D.1), (D.2), (C.54), and (C.42).

$$\bar{\pi}_{\text{agg}} = \bar{\pi}_{\min}\bar{z}^{\sigma-1} - \Omega(N-1)(1-F(\hat{z}))\kappa, \tag{H.18}$$

$$y = \left(1-\tilde{L}\right)\bar{z}, \tag{H.19}$$

$$\bar{U} = \begin{cases} \frac{1}{1-\gamma}\frac{(c\bar{L}M(0))^{1-\gamma}}{\rho+(\gamma-1)g} & \gamma \neq 1 \\ \frac{\rho\log(c\bar{L}M(0))+g}{\rho^2} & \gamma = 1 \end{cases}, \tag{H.20}$$

$$\lambda_{ii} = \frac{1}{1+(N-1)\hat{z}^{\sigma-1-\theta}d^{1-\sigma}}, \tag{H.21}$$

$$c = \left(1-\tilde{L}\right)\bar{z} - \eta\zeta\Omega\Theta\left(S+\delta/\chi\right). \tag{H.22}$$

Consumption Equivalents. Here we focus on the log utility case. Let subscript t represent variables associated with the initial equilibrium and subscript T denote variables in the new equilibrium. From equation (46), the present discounted value of utility in steady state is

$$\bar{U}(c,g) = \frac{\rho\log(c)+g}{\rho^2}. \tag{H.23}$$

Consumption equivalent welfare is the permanent change in the level of consumption c needed to make the representative consumer in the initial equilibrium indifferent to living in the new equilibrium. That is, the consumption equivalent α is such that

$$\bar{U}((1+\alpha)c_t, g_t) = \bar{U}(c_T, g_T) \tag{H.24}$$

$$\frac{\rho\log((1+\alpha)c_t)+g_t}{\rho^2} = \frac{\rho\log(c_T)+g_T}{\rho^2} \tag{H.25}$$

$$1+\alpha = \frac{c_T}{c_t}\exp\left(\frac{g_T-g_t}{\rho}\right) = \exp\left(\rho\left(\bar{U}_T-\bar{U}_t\right)\right) \tag{H.26}$$

I. Notation

General notation principle for normalization: move to lowercase after normalizing to the scale of the economy, from nominal to real, per-capita, and relative wages (all where appropriate). For symmetric countries, denote variables related to the trade sector with an x subscript. An overbar denotes an aggregation of the underlying variable. Drop the t subscript where possible for clarity in the static equilibrium conditions.

Notation Summary

Parameters	
Multiplicative Adoption Cost	$\zeta > 0$
Labor/Goods Proportion of Adoption Cost	$0 \leq \eta \leq 1$
Goods Adoption Cost	$\Theta \geq 0$
Iceberg Trade Cost	$d \geq 1$
Fixed Export Cost	$\kappa \geq 0$
Consumer CRRA	$\gamma \geq 0$
Elasticity of Substitution between Products	$\sigma > 1$
Markup	$\bar{\sigma} := \sigma/(\sigma - 1)$
Tail Parameter of the Pareto Distribution	$\theta > 1$
Consumer's Discount Rate	$\rho > 0$
Population per Country	$\bar{L} > 0$
Adoption to Entry Cost Ratio	$\chi < 1$
Firm Death Rate	$\delta \geq 0$
Drift of Productivity Process	$\mu \geq 0$
Standard Deviation of Productivity Process	$\upsilon \geq 0$

Notation Summary

Equilibrium Variables	
Productivity	Z
CDF of the Productivity Distribution	$\Phi(Z,t)$
PDF of the Productivity Distribution	$\phi(Z,t)$
Representative Consumers Flow Utility	$U(t)$
Representative Consumers Welfare	$\bar{U}(t)$
Real Firm Value	$V(Z,t)$
Optimal Search Threshold	$M(t)$
Optimal Export Threshold	$\hat{Z}(t)$
Aggregate Nominal Expenditures on Final Goods	$Y(t)$
Aggregate Real Consumption	$C(t)$
Domestic Labor demand	$\ell_d(Z,t)$
Export Labor demand	$\ell_x(Z,t)$
Domestic Quantity	$Q_d(Z,t)$
Export Quantity	$Q_x(Z,t)$
Real Search Cost	$X(t)$
Domestic idiosyncratic prices	$p_d(Z,t)$
Export idiosyncratic prices	$p_x(Z,t)$
Nominal Wages	$W(t)$
Real Domestic Profits	$\Pi_d(Z,t)$
Real Per-market Export Profits	$\Pi_x(Z,t)$
Firm Effective Discount Rate	$r(t)$
Trade Share	$\lambda(t)$
Price level	$P(t)$
Number of Varieties	$\Omega(t)$

Normalization Notation Summary (implicit t where appropriate)

Real, Normalized, and Per-Capita Variables	
Per-capita Labor Demand/Supply	$L := \bar{L}/\bar{L}$
Normalized Productivity	$z := Z/M$
Normalized Optimal Export Threshold	$\hat{z} := \hat{Z}/M$
Normalized CDF of the Productivity Distribution	$F(z,t) := \Phi(zM(t),t)$
Normalized PDF of the Productivity Distribution	$f(z,t) := M(t)\phi(zM(t),t)$
Expectation of the Normalized Productivity Distribution	$\mathbb{E}\left[\Psi(z)\right] := \int_1^\infty \Psi(z)f(z)\mathrm{d}z$
Conditional Expectation of the Normalized Productivity Distribution	$\mathbb{E}\left[\Psi(z) \mid z > \hat{z}\right] := \int_{\hat{z}}^\infty \Psi(z)\frac{f(z)}{1-F(\hat{z})}\mathrm{d}z$
Normalized, Per-capita Real Firm Value Normalized by Real Wages	$v(z,t) := \frac{1}{LMw}V(Z,t)$
Normalized, Per-capita, Real Expenditures on Final Goods (i.e., Output)	$y := \frac{1}{LMP}Y$
Normalized, Per-capita Real consumption	$c := \frac{1}{LM}C$
Normalized, Per-capita, Real Adoption Cost Relative to Real Wages	$x := \frac{1}{LMw}X$
Normalized Real Wages	$w := \frac{1}{PM}W$
Normalized, Per-capita, Real, Aggregate Domestic Profits	$\bar{\pi}_d$
Normalized, Per-capita, Real, Aggregate Per-market Export Profits	$\bar{\pi}_x$