Online Appendix to "Voter Turnout with Peer Punishment" by David K. Levine and Andrea Mattozzi

Equilibrium

Party strategies are probability distributions over mechanisms. Given that the objective functions can be written in terms of cost functions that depend only on bids, we may as well take them to be probability distributions over bids. Formally, we take a strategy for party k to be a probability measure represented by a cumulative distribution function F_k over bids, that is, on $[\underline{b}_k, \eta_k]$. The objective function of each party is $\Pi_k(b_k, F_{-k})V - \eta_k C(b_k/\eta_k)$ where $\Pi_k(b_k, F_{-k})$ is the probability that a bid b_k wins. With this formulation we have an all-pay auction model. We do not assume common prize, we allow arbitrary $V_k \geq 0$ and common cost $C(\varphi)$ which is continuous and strictly increasing for $\varphi \geq \underline{y}$. As in the text we assume $\overline{b}_L \neq \overline{b}_S$.

Theorem 1. *There is a unique equilibrium. In this equilibrium neither party uses a pure strategy, the utility of the disadvantaged party is* 0 *and the utility of the advantaged party is* $V_{-k} - \eta_{-d}C(\overline{b}_d/\eta_{-d})$.

We also record additional facts not reported in the text but used subsequently in this Online Appendix which are the equilibrium strategies. Let $F_k^0(b)$ *record the size of the atom at* b *(if any). In* $(\eta_L \underline{y}, \overline{b}_d)$*:*

$$F_d(b_d) = 1 - \frac{\eta_{-d}C(\overline{b}_d/\eta_{-d}) - \eta_{-d}C(b_d/\eta_{-d})}{V_{-k}}$$

$$F_{-d}(\eta_{-d}\varphi_{-d}) = \frac{\eta_d C(b_{-d}/\eta_d)}{V_k}$$

The disadvantaged party has a single atom at $F_d^0(\eta_d \underline{y}) = 1 - \eta_{-d}C(\overline{b}_d/\eta_{-d})/V_{-k} + \eta_{-d}C(\eta_L \underline{y}/\eta_{-d})/V_{-d}$. *The advantaged party if it is large has an atom at* $F_L^0(\overline{b}_S) = 1 - \eta_S C(\overline{b}_S/\eta_S)/V_L$, *and whichever party is advantaged has an atom at*

$$F_{-d}^0(\underline{y}\eta_L/\eta_d) = \eta_d C(\underline{y}\eta_L/\eta_d)/V_d.$$

Proof. S will never submit a bid b_k for which $\eta_S \underline{y} < b_k < \eta_L \underline{y}$ since such a bid will be costly but losing, and neither party will submit a bid for which $b_k > \overline{b}_k$ since to do so would cost more than the value of the prize. It follows that k must either bid $\eta_k \underline{y}$ or in the range $[\eta_L \underline{y}, \overline{b}_d]$. If $V_S \leq \eta_S C\left(\underline{y}\eta_L/\eta_S\right)$, it follows that $\overline{b}_S \leq \eta_L \underline{y}$. In this case S will only turn out committed voters, that is will bid $\eta_S \underline{y}$, and L wins with probability 1 by bidding $\eta_L \underline{y}$. This case is ruled out in the text.

Consider now the case $V_S > \eta_S C\left(\underline{y}\eta_L/\eta_S\right)$. In the range $(\eta_L \underline{y}, \overline{b}_d)$ there can be no atoms by the usual argument for all-pay auctions: if there was an atom at b_k then party $-k$ would prefer to bid a bit more than b_k rather than a bit less, and since consequently there are no bids immediately below b_k party k would prefer to choose the atom at a lower bid. This also implies that S cannot have an atom at $\eta_L \underline{y}$: if L has an atom there, then S should increase its atom slightly to break the tie. If L does not have an atom there, then S should shift its atom to $\eta_S \underline{y}$ since it does not win either way.

Next we observe that in $(\eta_L \underline{y}, \overline{b}_d)$ there can be no open interval with zero probability. If party k has such an interval, then party $-k$ will not submit bids in that interval since the cost of the bid is strictly increasing it would do strictly better to bid at the bottom of the interval. Hence there would have to be an interval in which neither party submits bids. But then, for the same reason, it would be strictly better to lower the bid for bids slightly above the interval.

Let U_k be the equilibrium expected utility of party k. In equilibrium the disadvantaged party must earn zero since it must make bids with positive probability arbitrarily close to $\overline{b}_d$, while the advantaged party gets at least $U_{-d} \geq V_{-d} - \eta_{-d} C(\overline{b}_d/\eta_{-d}) > 0$ since by bidding slightly more than $\overline{b}_d$ it can win for sure, but gets no more than that since it must make bids with positive probability arbitrarily close to $\overline{b}_d$. We conclude that the equilibrium payoff of the advantaged party must be exactly $U_{-d} \geq V_d - \eta_{-d} C(\overline{b}_d/\eta_{-d})$.

From the absence of zero probability open intervals in $(\eta_L \underline{y}, \overline{b}_d)$ it follows that the indifference condition for the advantaged party

$$F_d(b_d)V_d - \eta_{-d} C(b_d/\eta_{-d}) = V_d - \eta_{-d} C(\overline{b}_d/\eta_{-d})$$

must hold for at least a dense subset. Similarly for the disadvantaged party

$$F_{-d}(b_{-d})V_{-d} - \eta_d C(b_{-d}/\eta_d) = 0$$

for at least a dense subset. This uniquely defines the cdf for each party in $(\eta_L \underline{y}, \overline{b}_d)$:

$$F_d(b_d) = 1 - \frac{\eta_{-d}C(\overline{b}_d/\eta_{-d}) - \eta_{-d}C(b_d/\eta_{-d})}{V_d}$$

$$F_{-d}(b_{-d}) = \frac{\eta_d C(b_{-d}/\eta_d)}{V_{-d}}.$$

As these are differentiable they can be represented by continuous density functions which are found by taking the derivative.

Evaluating $F_d(b_d)$ at $\eta_d \underline{y}$ gives

$$F_d^0(\eta_d \underline{y}) = 1 - \eta_{-d}C(\overline{b}_d/\eta_{-d}/V_{-d}) + \eta_{-d}C(\eta_L \underline{y}/\eta_{-d})/V_{-d}.$$

Since $F_d(\overline{b}_d) = 1$ and we already proved that S has no atom at $\eta_L \underline{y}$ this is in fact the only atom for the disadvantaged party.

As for the advantaged party, if -$d = S$ then $\eta_L > \eta_S \geq \overline{b}_S > \overline{b}_L$ implies that $F_S(\overline{b}_L) = \eta_L C(\overline{b}_L/\eta_L)/V_L = 1$. If instead $-d = L$ then $F_L(\overline{b}_S) = \eta_S C(\overline{b}_S/\eta_S)/V_L$. If $\overline{b}_S < \eta_S$ then this is 1 and there is no atom, otherwise there must be an atom of size $F_L^0(\overline{b}_S) = 1 - \eta_S C(\overline{b}_S/\eta_S)/V_L$. Turning to $\eta_L \underline{y}$ we see that the atom there is given by

$$F_{-d}^0 = \frac{\eta_d C(\underline{y}\eta_L/\eta_d)}{V}$$

since the advantaged group never bids less. □

Which party is advantaged?

Theorem 2. *For* any *individual cost function $c(y)$ with corresponding committed voters $\underline{y}$ there exist $\underline{\theta}^S < 1, \underline{\eta}_S < 1/2$ and $\overline{V} > \underline{V}^S > 0$ such that if all the conditions $\theta > \underline{\theta}^S, \eta_S \geq \underline{\eta}_S$ and $\underline{V}^S < V < \overline{V}$ hold the small party is advantaged. Conversely if $\underline{y} > 0$ for any values of the other parameters there* *exist* $\underline{\theta}^L > 0, \overline{\eta}_S > 0$ *and* $\underline{V}^L > 0$ *such that if any of the conditions* $\theta < \underline{\theta}^L$, $\eta_S < \overline{\eta}_S$, $V < \underline{V}^L$ *or* $V > \overline{V}$ *are satisfied then the large group is advantaged.*

Proof. For completeness we allow in this theorem the possibility that $\overline{b}_S < \eta_L \underline{y}$, that is the small party may or may not be willing to turn out at least the number of committed voters of the large party. To prove the first half of the theorem, observe that marginal cost is $C'(\varphi) = (1-\theta)c(\varphi) + \theta(1-\varphi)c'(\varphi)$ so if $\theta = 1$ then $C'(1) = 0$. Since average cost at $\varphi = 1$ is $C(1) > 0$, average cost is strictly larger than marginal cost at $\theta = 1, \varphi = 1$. Therefore from continuity it must be so for θ, φ both sufficiently close to 1. That is for $1 \geq \theta > \underline{\theta}^S$ and $1 > \overline{\varphi} > \varphi > \underline{\varphi}$ average cost is declining. Having fixed $\overline{\varphi}$ we may choose $\underline{\eta}_S < 1$ large enough that for $\eta_S \geq \underline{\eta}_S$ the small party is large enough to outbid $\overline{\varphi}\eta_L$, that is $1/2 > \underline{\eta}_S \geq \overline{\varphi}(1 - \underline{\eta}_s)$. Hence if we choose the prize V so that the large party's maximal willingness to turn out lies in this range, that is, $\overline{\varphi} > \overline{b}_L/\eta_L > \underline{\varphi}$ then the small party must be advantaged as is able to outbid the large party and has a lower average cost of matching the large party bid. For the second half of the Theorem, the large party is advantaged for $\theta = 0$ hence by continuity for small θ. For $\eta_S < \eta_L \underline{y}$ the small party is unable to overcome the committed voters of the large party. If $V < C(\eta_L \underline{y}/\eta_S)$ then the small party is unwilling to bid. If $V > \eta_L C(\eta_S/\eta_L) = \overline{V}$ then $\overline{b}_L > \eta_S$ so the large party is surely advantaged. □

Who Wins?

Theorem 3. *The equilibrium bidding function of a strongly advantaged party FOSD that of the disadvantaged party.*

Proof. At $\overline{b}_d$ we have $F_d(\overline{b}_d) = \hat{F}_{-d}(\overline{b}_d) = 1$ so this is irrelevant for FOSD. For $\eta_S \underline{y} \leq b < \eta_L \underline{y}$ we have $F_L(b) = 0$ while $F_S(b) > 0$ if and only if S is disadvantaged. Hence when S is disadvantaged its bidding schedule cannot FOSD that of L, while if it is advantaged this range is irrelevant for FOSD.

It remains to examine the range $\eta_L \underline{y} \leq b < \overline{b}_d$. In this range the equilibrium bid distributions are given by

$$F_d(b) = 1 - \frac{\eta_{-d} C(\overline{b}/\eta_{-d})}{V} + \frac{\eta_{-d} C(b/\eta_{-d})}{V}$$

$$F_{-d}(b) = \frac{\eta_d C(b/\eta_d)}{V}.$$

Hence for FOSD of the advantaged party, we must have

$$1 - \frac{\eta_{-d} C(\hat{b}_d/\eta_{-d})}{V} + \frac{\eta_{-d} C(b/\eta_{-d})}{V} - \frac{\eta_d C(b/\eta_d)}{V} > 0.$$

Moreover since $\eta_d v_d \geq \eta_d C(\overline{b}_d/\eta_d)$ this is true if

$$1 - \frac{\eta_{-d} C(\overline{b}_d/\eta_{-d})}{\eta_d C(\overline{b}_d/\eta_d)} + \frac{\eta_{-d} C(b/\eta_{-d})}{\eta_d C(\overline{b}_d/\eta_d)} - \frac{\eta_d C(b/\eta_d)}{\eta_d C(\overline{b}_d/\eta_d)} > 0$$

and if and only if the disadvantaged party is not constrained in bidding. This is equivalent to

$$(\eta_{-d} C(b/\eta_{-d}) - \eta_d C(b/\eta_d)) - \left(\eta_{-d} C(\overline{b}_d/\eta_{-d}) - \eta_d C(\overline{b}_d/\eta_d)\right) > 0.$$

Let $t(\eta, b) \equiv \eta C(b/\eta)$. The derivative with respect to η is $t_\eta(\eta, b) = C(b/\eta) - (b/\eta)C'(b/\eta)$ so the cross partial is $t_{\eta b}(\eta, b) = -(b/\eta^2)C''(b/\eta)$. Observe that the sufficient condition may be written as

$$0 < (t(\eta_{-d}, b) - t(\eta_d, b)) - \left(t(\eta_{-d}, \overline{b}_d) - t(\eta_d, \overline{b}_d)\right)$$

$$= \int_{\eta_d}^{\eta_{-d}} \left(t_\eta(\eta, b) - t_\eta(\eta, \overline{b}_d)\right) d\eta$$

$$= -\int_{\eta_d}^{\eta_{-d}} \int_b^{\overline{b}_d} t_{\eta b}(\eta, b') d\eta db' = \int_{\eta_d}^{\eta_{-d}} \int_b^{\overline{b}_d} (b'/\eta^2)C''(b'/\eta) d\eta db'$$

This is positive if $\eta_{-d} > \eta_d$ and C is convex or if $\eta_d > \eta_{-d}$ and C is concave, which gives the primary result. On the other hand, in the case of a common prize, L advantaged, and S unconstrained, it is negative and gives the exact sign of $F_d(b) - F_{-d}(b)$ (it is necessary and sufficient). Hence, since the difference between F_d and F_{-d} is positive for $\eta_S \underline{y} \leq b < \eta_L \underline{y}$ and negative for $\eta_L \underline{y} \leq b < \hat{b}_d$ neither bidding schedule FOSD the other. □

The next proposition studies the case where costs are incrementally concave and yet the large party is advantaged as noted in footnote 12 in the text. It shows how the *FOSD* result can fail in the strong sense that the disadvantaged small party turns out more members in expectation and has a higher probability of winning than the large advantaged party.

Proposition. *Suppose that cost is quadratic so that for $\theta > 1/2$ it is incrementally concave. For any η_S there exists a $\underline{\varphi} > 0$ such that for any $\underline{y} < \underline{\varphi}$ there is an open set of V's and for any such V there are bounds $1/2 < \underline{\theta} < \theta^* < \overline{\theta} \leq 1$ such that*

a. for $\overline{\theta} > \theta > \theta^*$ the small party is advantaged

b. for $\theta^* > \theta > \underline{\theta}$ the large party is advantaged yet the small party turns out more expected voters and has a higher probability of winning the election.

Proof. Recall the quadratic case $C(\varphi_k) = (1-2\theta)(\varphi_k - \underline{y})^2 + 2\theta(1-\underline{y})(\varphi_k - \underline{y})$. Hence $C'(\underline{y}) = 2\theta(1-\underline{y})$ and $C''(\varphi_k) = 2(1-2\theta)$. We fix $\theta > 1/2$ so that cost is incrementally concave.

We first establish that for sufficiently small $\underline{y}$ there is a range of $V's$ such $\eta_L \underline{y} < \overline{b}_S < \eta_S$ for $1/2 \leq \theta \leq 1$ and such that S is advantaged at $\theta = 1$.

Since the derivative of C with respect to θ is $2(\varphi_k - \underline{y})(1-\varphi_k) > 0$ the greatest willingness to bid is at $\theta = 1/2$ and the least is at $\theta = 1$. At $\theta = 1/2$ the utility of S is $V - (1-\underline{y})(\varphi_k - \underline{y})$ and so $\overline{b}_k < 1$ for $V < (1-\underline{y})^2 = \overline{V}_S$. At $\theta = 1$ the utility of S is $V + (\varphi_k - \underline{y})^2 - 2(1-\underline{y})(\varphi_k - \underline{y})$ so $\overline{b}_k > \eta_L \underline{y}$ for $V > 2(1-\underline{y})(\eta_L \underline{y}/\eta_S - \underline{y}) - (\eta_L \underline{y}/\eta_S - \underline{y})^2 = \underline{V}_S$.

Set $\theta = 1$ and let φ^* be defined by $A(\varphi^*) = A(\eta_L \varphi^*/\eta_S)$. Some algebra yields $\varphi^* = \sqrt{(\eta_S/\eta_L)\underline{y}(2-\underline{y})}$. This will be less than η_S/η_L provided $\underline{y}(2-\underline{y}) < \eta_S/\eta_L$. At $\theta = 1$ the utility for L is $V - \eta_L\left(-(\varphi_L - \underline{y})^2 + 2(1-\underline{y})(\varphi_L - \underline{y})\right)$. Hence L would like to bid greater than $\eta_L \varphi^*$when

$$V > \eta_L\left(-(\sqrt{(\eta_S/\eta_L)\underline{y}(2-\underline{y})} - \underline{y})^2 + 2(1-\underline{y})(\sqrt{(\eta_S/\eta_L)\underline{y}(2-\underline{y})} - \underline{y})\right) = \underline{V}_L.$$

It is smaller than η_S when $V < \eta_L\left(-(\eta_S/\eta_L - \underline{y})^2 + 2(1-\underline{y})(\eta_S/\eta_L - \underline{y})\right) = \overline{V}_L$. Hence for V in this range and $\theta = 1$ S is advantaged.

We observe that when $\underline{y} = 0$ we have $\overline{V}_S = 1, \underline{V}_S = 0, \overline{V}_L = \eta_S\,(2 - \eta_S/\eta_L) > \eta_S$ and $\underline{V}_L = 0$. This establishes that for sufficiently small $\underline{y}$ there is a range of $V's$ such that $\eta_L \underline{y} < \overline{b}_S < \eta_S$ for $1/2 \leq \theta \leq 1$ and such that tS is advantaged at $\theta = 1$. Fix such a V.

Define the desire to bid as the solution of

$$(1-2\theta)(b_k/\eta_k - \underline{y})^2 + 2\theta(1-\underline{y})(b_k/\eta_k - \underline{y}) = V/\eta_k$$

and for S at least this is also the willingness to bid, and it will be the willingness to bid of L provided the constraint $b_L < \eta_L$ is satisfied. Since the last equation is quadratic in b_k it can be solved by the quadratic formula from which it is apparent that $\overline{b}_k(\theta)$ is a continuous function. This implies as well that the strategies are continuous in θ, since the support of the continuous part of the density is continuous as is the upper bound. We can also conclude that $\overline{b}_S = \overline{b}_L = b$ if and only if

$$(1-2\theta)(b - \eta_S \underline{y})^2 + 2\theta(1-\underline{y})\eta_S(b - \eta_S \underline{y}) = \eta_S V$$

and

$$(1-2\theta)(b - \eta_S \underline{y})^2 + 2\theta(1-\underline{y})\eta_S(b - \eta_S \underline{y}) - \eta_S V =$$

$$(1-2\theta)(b - \eta_L \underline{y})^2 + 2\theta(1-\underline{y})\eta_L(b - \eta_L \underline{y}) - \eta_L V.$$

The latter equation is linear in b since the b^2 terms are the same on both sides. Hence the equation has a unique solution $b(\theta)$ which is a rational function of θ. Substituting that into the first equation we find that those values of θ for which $\overline{b}_S = \overline{b}_L$ are zeroes of a rational function. Hence, either there must be a finite number of zeroes or the function must be identically equal to zero. But it cannot be identically zero since $\overline{b}_S - \overline{b}_L$ is negative at $\theta = 1/2$ and positive at $\theta = 1$. We conclude that there is some point θ^* at which $\overline{b}_S = \overline{b}_L$ and S is advantaged for $\theta^* < \theta < \overline{\theta}$ for some $\overline{\theta}$, while L is advantaged for $\theta_0 < \theta < \theta^*$ for some θ_0.

Since C is incrementally concave in $\theta^* < \theta < \overline{\theta}$ and S is advantaged there, it follows that S follows a strategy that FOSD that of Ly. Hence in the limit at θ^* the strategy of the small party either FOSD that of the large party or is the same as that of the large party. However, for $\theta > \theta^*$, S plays $\eta_L \underline{y}$ with probability zero while L plays it with probability

$$1 - \frac{C((b_L/\eta_S))}{\eta_S V} + \frac{C((\eta_L/\eta_S)\underline{y})}{\eta_S V} \to \frac{C((\eta_L/\eta_S)\underline{y})}{\eta_S V} > 0$$

so in the limit the two strategies are not identical. Since at θ^* the strategy of S FOSD that of L, it has a strictly higher probability of winning and strictly higher expected turnout. Since the probability of winning and expected turnout are continuous functions of the strategies which are continuous in θ it follows that this remains true in an open neighborhood of θ^*. □

Proposition 1. *The measures* $\underline{\gamma}$ and $\overline{\gamma}$ *satisfy* $0 \leq \underline{\gamma}, \overline{\gamma} \leq 1/2$. *If* $g(c)$ *is weakly decreasing then* $\underline{\gamma} = 1/2$. *If* $g(c)$ *is weakly increasing then* $\overline{\gamma} = 1/2$. *If the density* $g(c)$ *shifts to the right then* $\overline{\gamma}$ *is constant and* $\underline{\gamma}$ *decreases; if the density shifts to the right holding fixed* $c(1)$ *then* $\overline{\gamma}$ *increases. Furthermore, increasing dispersion by a change of scale around the mode increases both* $\underline{\gamma}$ *and* $\overline{\gamma}$.

Proof. Recall from the text that $G(c)$ is the cdf of costs for an individual so that $c(\varphi) = G^{-1}(\varphi)$, $\varphi = G(c)$ and the support is $[c(0), c(1)]$. We denote the density of $G(c)$ by $g(c)$, and we assume it is continuously differentiable, strictly positive, and has a single "top" in the sense that it is either single peaked or a it is a limiting case such as the uniform where the density is flat at the top. To define $\underline{\gamma}, \overline{\gamma}$ we first defined

$$\mu(c) = \frac{(g(c))^2}{2\,(g(c))^2 + (1 - G(c))\,g'(c)},$$

then $\underline{\gamma} = \min_{c \geq 0} \mu(c)$ and $\overline{\gamma} = \max\{0, 1 - \max_{c \geq 0} \mu(c)|\}$. For the purpose of deriving the properties of $\underline{\gamma}, \overline{\gamma}$ it will be convenient instead to define

$$\lambda(c) = -\frac{(1 - G(c))\,g'(c)}{(g(c))^2}$$

and observe that $\mu(c) = 1/(2 - \lambda(c))$ so that $\mu(c)$ and $\lambda(c)$ share the same monotonicity properties. We take $\underline{\lambda} = \min_{c \geq 0} \lambda(c) \leq 0$ the smallest possible value of $\lambda(c)$ and $\overline{\lambda} = \max_{c \geq 0} \lambda(c) \geq 0$ the largest. In the uniform case $g'(c) = 0$ so $\lambda(c) = 0$. If the density is increasing then $\lambda(c) \leq 0$ so $\overline{\lambda} = 0$ and if it is decreasing $\underline{\lambda} = 0$. Equivalent to the definition in the text

$$\underline{\gamma} = \frac{1}{2 - \underline{\lambda}}$$

and define $\overline{\gamma} = 0$ if $\overline{\lambda} > 1$ and

$$\overline{\gamma} = 1 - \frac{1}{2 - \overline{\lambda}}$$

otherwise. Hence $\underline{\gamma}$ is an increasing function of $\underline{\lambda}$ and $\overline{\gamma}$ is a decreasing function of $\overline{\lambda}$. Since $\underline{\lambda} \leq 0$ and $\overline{\lambda} \geq 0$ we have $0 \leq \underline{\gamma}, \overline{\gamma} \leq 1/2$. The properties of $\underline{\gamma}, \overline{\gamma}$ for the uniform, increasing and decreasing cases can be read directly from the results for $\underline{\lambda}, \overline{\lambda}$: both $1/2$ for the uniform case, $\overline{\gamma} = 1/2$ in the increasing case, and $\underline{\gamma} = 1/2$ in the decreasing case. For the single-peaked case we now prove

a. If the density shifts to the right then $\overline{\lambda}$ is constant and $\underline{\lambda}$ decreases ($\underline{\gamma}$ decreases); if the density shifts to the right holding fixed $c(1)$ then $\overline{\lambda}$ decreases ($\overline{\gamma}$ increases);

b. Increasing dispersion by a change of scale around the mode increases $\underline{\lambda}$ ($\underline{\gamma}$ increases) and decreases $\overline{\lambda}$ ($\overline{\gamma}$ increases).

(a) We consider first the case of shifting the density to the right holding fixed $c(1)$. The only interesting case is when the peak c_g is interior, that is, satisfies $c(1) > c_g > 0$. Consider a $h(c)$ also with upper support $c(1)$ with mode c_h. Suppose that for some positive constants Δ, ζ we have $c_h > c_g + \Delta$ and for $c > c_h$ we have $h(c) = \zeta g(c - \Delta)$ (density shifts right). Notice the scaling factor ζ is needed since holding fixed the upper bound $c(1)$ mass is lost as we shift g to the right. We prove that $\overline{\lambda}_h < \overline{\lambda}_g$.

Notice that since by assumption of a single peak $g'(c_g) = h'(c_h) = 0$, we can define $\overline{\lambda}$ without loss of generality only for values of c to the right of the mode. Hence we have that

$$\overline{\lambda}_h = \max_{c(1)\geq c\geq c_h} -\frac{\int_c^{c(1)} h(\xi)d\xi h'(c)}{\left(h(c)\right)^2} = \max_{c(1)\geq c\geq c_g+\Delta} -\frac{\int_c^{c(1)} \zeta g(\xi-\Delta)d\xi\zeta g'(c-\Delta)}{\left(\zeta g(c-\Delta)\right)^2}$$

and after a change of variable $\tilde{c} = c - \Delta$ we have

$$= \max_{c(1)-\Delta\geq\tilde{c}\geq c_g} -\frac{\int_c^{c(1)-\Delta} g(\tilde{\xi})d\tilde{\xi}g'(\tilde{c})}{\left(g(\tilde{c})\right)^2} < \max_{c(1)\geq c\geq c_g} -\frac{\int_c^{c(1)} g(\tilde{\xi})d\tilde{\xi}g'(\tilde{c})}{\left(g(\tilde{c})\right)^2} = \overline{\lambda}_g.$$

This gives the result for fixed $c(1)$. Focus on the key result

$$\overline{\lambda}_h = \max_{c(1)-\Delta\geq\tilde{c}\geq c_g} -\frac{\int_c^{c(1)-\Delta} g(\tilde{\xi})d\tilde{\xi}g'(\tilde{c})}{\left(g(\tilde{c})\right)^2}; \overline{\lambda}_g = \max_{c(1)\geq c\geq c_g} -\frac{\int_c^{c(1)} g(\tilde{\xi})d\tilde{\xi}g'(\tilde{c})}{\left(g(\tilde{c})\right)^2}$$

For $\overline{\lambda}$ there are two effects of a right shift: the range over which the integral of $g(\tilde{\xi})d\tilde{\xi}g'(\tilde{c})$ in the numerator is taken is shorter for h and the maximum is taken over a narrower range. There is no analogous result for $\underline{\lambda}$. For $\underline{\lambda}$ the range of the integral remains the same, but rather than a maximum over $c(1) - \Delta \geq \tilde{c} \geq c_g$ we take minimum over $c_g \geq \tilde{c} \geq c(0) - \Delta$. Hence the minimum is taken over a larger range, offsetting the effect of the shorter range of the integral and the combination of the two is ambiguous.

For an ordinary right shift (that is, not holding fixed $c(1)$) the range of the integral does not change. For $\overline{\lambda}$ the range over which the maximum is is taken does not change, so the right shift is neutral. For $\underline{\lambda}$ the range over which the minimum is taken increases so the minimum becomes more negative.

(b) We first prove the result for $\overline{\lambda}$. Consider a $h(c)$ also with upper support $c(1)$ with mode $c_h = c_g$. Suppose that for some positive constants $\sigma > 1, \zeta$ for $c > c_h$ we have $h(c) = \zeta g(c_g + (c - c_g)/\sigma)$ (greater dispersion to the right of the mode).

We have

$$\overline{\lambda}_h = \max_{c(1)\geq c\geq c_g} -\frac{\int_c^{c(1)} h(\xi)d\xi h'(c)}{\left(h(c)\right)^2}$$

$$= \max_{c(1)\geq c\geq c_g} -\frac{\int_c^{c(1)} \zeta g(c_g + (\xi - c_g)/\sigma) d\xi (1/\sigma)\zeta g'(c_g + (c - c_g)/\sigma)}{(\zeta g(c_g + (c - c_g)/\sigma))^2}$$

and after a change of variable $\tilde{c} = c_g + (c - c_g)/\sigma$ we have

$$= \max_{c_g+(c(1)-c_g)/\sigma\geq\tilde{c}\geq c_g} -\frac{\int_c^{c_g+(c(1)-c_g)/\sigma} g(\tilde{\xi})d\tilde{\xi}g'(\tilde{c})}{(g(\tilde{c}))^2}.$$

Furthermore, since $\sigma > 1$ and $c_g < c(1)$ we have $c_g + (c(1) - c_g)/\sigma = [(\sigma - 1)/\sigma]c_g + [1/\sigma]c(1) < c(1)$ so

$$\overline{\lambda}_h < \max_{c(1)\geq c\geq c_g} -\frac{\int_c^{c(1)} g(\tilde{\xi})d\xi g'(\tilde{c})}{(g(\tilde{c}))^2} = \overline{\lambda}_g.$$

Here again there are two effects, a shorter range of integral and a shorter range over which the maximum is taken, both lowering $\overline{\lambda}$. In the case of $\underline{\lambda}$ it is also the case that both the range of the integral and range over which the minimum is taken shrink: hence the minimum must increase. □

Theorem 4. *is equivalent to*

Proposition 2. *a. cost is incrementally convex if and only if* $\theta < 1/(2 - \underline{\lambda})$
b. cost is incrementally concave if and only if $\overline{\lambda} < 1$ *and* $\theta > 1/(2 - \overline{\lambda})$.

Proof. We report expected cost $C(\varphi) = \int_{\underline{y}}^{\varphi} c(y)dy + \theta(1 - \varphi)c(\varphi)$ and its first two derivatives $C'(\varphi) = (1-\theta)c(\varphi)+\theta(1-\varphi)c'(\varphi)$, and $C''(\varphi) = (1-2\theta)c'(\varphi)+\theta(1 - \varphi)c''(\varphi)$. Observe that $c(\varphi) = G^{-1}(\varphi)$ so

$$c'(\varphi) = \frac{1}{g(G^{-1}(\varphi))}$$

$$c''(\varphi) = -\frac{g'(G^{-1}(\varphi))}{(g(G^{-1}(\varphi)))^3}$$

and hence we can rewrite $C''(\varphi)$ as

$$C''(\varphi) = \frac{1 - \theta(2 - \lambda(c))}{g(c)}.$$

Hence $C''(\varphi) > 0$ if and only if $\theta < 1/(2-\lambda(c))$ from which the result follows. $\square$

The value of elections

Theorem 5. *In a high value election the probabilities that the small party concedes and the large party preempts the election increase in V, and approach 1 in the limit. As V increases the bid distribution of the small party declines in FOSD and the bid distribution of the large party increases in FOSD. The expected vote differential increases in V while the expected turnout cost remains constant.*

Proof. In a high value election S is constrained and L is advantaged. The probability L preempts is $F_L^0(\eta_S) = 1 - (\eta_S/V)C(1)$, increasing in V. The probability of concession by S is $F_S^0(\eta_S \underline{y}) = 1 - \eta_L C(\eta_S/\eta_L)/V$ increasing in V.

Since changing V with $\bar{b}_d = \eta_S$ the support and shape of the cost function in the mixing range do not change, so raising V simply lowers the densities by a common factor, meaning that these shifts reflect stochastic dominance as well. The FOSD result implies the increased vote differential.

Total surplus is $V - \eta_{-d}C(\bar{b}_d/\eta_{-d})$. Since some party certainly gets the prize this implies the expected turnout cost is $\eta_{-d}C(\bar{b}_d/\eta_{-d})$ and in a high value election $\bar{b}_d$ remains constant at η_S, so expected turnout cost is $\eta_L C(\eta_S/\eta_L)$ independent of V . $\square$

Monitoring Difficulty in High Value Elections

Theorem 6. *In a high value election, an increase in monitoring difficulty θ decreases the turnout of the advantaged (large) party in terms of FOSD. Furthermore, there exists $0 < \underline{\eta} < \overline{\eta} \leq 1/2$ such that for $\underline{\eta} < \eta_S < \overline{\eta}$ the expected turnout of the disadvantaged (small) party decreases in monitoring difficulty in terms of FOSD while the expected vote differential also decreases.*

Proof. If the election is not high value the disadvantaged party is unconstrained. Hence, given the definition of willingness to bid $\eta_k C(\bar{b}_k/\eta_k) - V = 0$, we can apply the implicit function theorem and find that

$$\frac{d\bar{b}_d}{d\theta} = -\frac{\eta_d dC(\bar{b}_d/\eta_d)/d\theta}{C'(\bar{b}_d/\eta_d)} = -\frac{\eta_d\theta(1-\bar{b}_d/\eta_d)c(\bar{b}_d/\eta_d)}{C'(\bar{b}_d/\eta_d)} < 0$$

Hence as θ decreases, that is as monitoring efficiency increases, so it does peak turnout. In a high value election the peak turnout $\bar{b}_d$ is fixed at η_S and S is disadvantaged. Examining the equilibrium bid distributions we have

$$F_S(b) = 1 - \frac{\eta_L C(\eta_S/\eta_L)}{V} + \frac{\eta_L C(b/\eta_L)}{V}$$

$$F_L(b) = \frac{\eta_S C(b/\eta_S)}{V}$$

while $C(\varphi_k) = T(\varphi_k) + \theta(1-\varphi_k)T^{'}(\varphi_k)$. Examining $F_L(b)$ first, we see that $dF_L/d\theta > 0$ which is the condition for a decrease in FOSD. For $F_S(b)$ we have

$$\frac{dF_S}{d\theta} = -\frac{\eta_L}{V}\Big((1-\eta_S/\eta_L)T^{'}(\eta_S/\eta_L) - (1-b/\eta_L)T^{'}(b/\eta_L)\Big)$$

Notice that for φ_k sufficiently close to $\underline{y}$ we must have $(1-\varphi_k)T^{'}(\varphi_k)$ increasing, say for $\underline{y} < \varphi_k < \varphi_0$. Hence for $\eta_S/\eta_L < \varphi_0$ we have $dF_S/d\theta < 0$ for $b \leq \eta_S$. This is the condition for an increase in FOSD. Since F_L stochastically dominates F_S and F_L decreases while F_S increases it follows that the expected vote differential must decrease.

Consider next that as $\eta_S \to 1/2$, it follows that $(1-\eta_S/\eta_L)T^{'}(\eta_S/\eta_L) \to 0$. Hence for any fixed b it is eventually true that $dF_S(b)/d\theta > 0$. It follows that, for sufficiently large η_S, the expected turnout of$S$ must decline with θ. Since the derivative of expected turnout is a continuous function of θ, it follows that there is a value of $\underline{\eta}$ such that expected turnout of S is constant with θ while for larger η_S it declines. At $\underline{\eta}$ the expected vote differential must decline with θ since S expected turnout is constant and L expected turnout declines. Since the derivative of the expected vote differential is also continuous in η_S it follows that for η_S larger than but close enough to $\underline{\eta}$, S expected turnout declines and the expected vote differential does as well. □

Symmetry of the Fundamentals

Let $\rho \in [0, 1]$ be a measure of the mix of issues between transfers and laws where $\rho = 0$ mans the election is purely about transfers and $\rho = 1$ means it is purely about laws. Examples of transfers include control over natural resources, the division of government jobs, the division of a fixed budget, taxes and subsidies and limitations on competition such as trade restrictions or occupational licensing. Examples of laws include civil rights, laws concerning abortion, criminal law, defense expenditures, non-trade foreign policy and policies concerning monuments. We suppose that $V_k = v(\eta_k, \rho)$ where $v(1/2, \rho) = V$. We take pure transfers to mean a common prize so that $v(\eta_k, 0) = V$ and pure laws to mean a common per capita prize so that $v(\eta_k, 1) = 2V\eta_k$. We assume that $v(\eta_k, \rho) \geq 0$ twice continuously differentiable with $v_\eta(\eta_k, \rho) \geq 0$. Define the prize elasticity with respect to party size $\gamma(\eta_k, \rho) = d \log v(\eta_k, \rho)/d \log \eta_k) = v_\eta(\eta_k, \rho)\eta_k/v(\eta_k, \rho)$. Then for pure transfers we have $\gamma(\eta_k, 0) = 0$ for for pure laws we have $\gamma(\eta_k, 1) = 1$. It is natural to assume then that $\gamma_\rho(\eta_k, \rho) > 0$: that as the importance of laws as an issue increases the prize elasticity with respect to party size goes up. This implies in addition that $v_\rho(\eta_k, \rho) > 0$ for $\eta_k > 1/2$ and $v_\rho(\eta_k, \rho) < 0$ for $\eta_k < 1/2$. That is, as the importance of laws as an issue increases the value of prize to the large party goes up and to the small party goes down. It follows directly that increasing the importance of laws improves the advantage (positive or negative) of the large party by raising its willingness to bid and lowering that of the small party.

Example. Suppose that the election has a mix of transfer and legal issues so that $v(\eta_k, \rho) = (1-\rho)+2\rho\eta_k$ where $0 \leq \rho \leq 1$ is the relative importance of legal issues. Then $\gamma(\eta_k, \rho) = 2\rho\eta_k/ \left((1-\rho) + 2\rho\eta_k\right)$ and $\gamma(\eta_k, 0) = 0$, $\gamma(\eta_k, 1) = 1$ and the derivative is

$$\gamma_\rho(\eta_k, \rho) = \frac{2\eta_k \left((1-\rho) + 2\rho\eta_k\right) + 2\rho\eta_k(1-2\eta_k)}{\left((1-\rho) + 2\rho\eta_k\right)^2} > 0.$$

Proposition 3. *If $\rho > 1$ then there are cost functions, prize values, party sizes, and monitoring difficulty for which the small party is advantaged.*

Proof. Willingness to bid is $1 - (\eta_k/v(\eta_{k,}\rho))C(b_k/\eta_k) = 0$. Using the implicit function theorem we find

$$db_k/d\eta_k = -\frac{(v(\eta_k,\rho) - v'(\eta_k,\rho)\eta)C(b_k/\eta_k)/v(\eta_k,\rho)^2 - (1/v(\eta_k,\rho))C'(b_k/\eta_k)(b_k/\eta_k)}{C'(b/\eta)/v(\eta_k,\rho)}$$

so that the sign determined by $C'(\varphi_k)\varphi_k - (1-\gamma(\eta_k,\rho))C(\varphi_k)$. If the parties are of near equal size and this is positive or $V > (1/2)C(1)$ then L is advantaged, if the parties are of near equal size, $V < (1/2)C$ and this is negative, S is advantaged. If $\rho = 0$ so the election is purely about transfers then this is $C'(\varphi_k)\varphi_k - C(\varphi_k)$ so which party is advantaged depends on whether average cost is increasing or decreasing as we know. If $\rho = 1$ so the election is purely about laws this is $C'(\varphi_k)\varphi_k$ which is always positive so L is always advantaged. In the intermediate cases there are always parameter values for which S is advantaged. Take the quadratic case with no committed voters where $C(\varphi_k) = (1-2\theta)\varphi_k^2 + 2\theta\varphi_k$ At $\theta = 1$ this is $C(\varphi_k) = -\varphi_k^2 + 2\varphi_k$ and $C'(\varphi_k) = -2\varphi_k + 2$. Hence

$$C'(\varphi_k)\varphi_k - (1-\gamma(\eta_k,\rho))C(\varphi_k) = (-2\varphi_k + 2)\,\varphi_k - (1-\gamma(\eta_k,\rho))\left(-\varphi_k^2 + 2\varphi_k\right).$$

$$= -\varphi_k^2 + \gamma(\eta_k,\rho))\left(-\varphi_k^2 + 2\varphi_k\right)$$

$$= -(1 + \gamma(\eta_k,\rho))\varphi_k^2 + \gamma(\eta_k,\rho))2\varphi_k.$$

Notice that for positive $\gamma(\eta_k,\rho)$ and small φ_k this is necessarily positive. However, as $\varphi_k \to 1$ this approaches $-(1-\gamma(\eta_k,\rho))$ which is strictly negative for $\rho < 1$, so also for $\varphi_k < 1$ but close to 1. □

The proof shows that with quadratic cost given $\rho < 1$ if there are sufficiently few committed voters, if $V < (1/2)C(1)$ but close enough (intermediate size prize), parties of similar enough size (small party not too small) and θ near enough 1 (high monitoring costs) the small party is advantaged. This is the same qualitatively as in the $\rho = 0$ case: however, quantitatively the criteria are much more stringent.

Endogenous versus Exogenous Uncertainty

Suppose that the probability of winning the election for party k is given by $P(b_k, b_{-k})$ non-decreasing in b_k. This must satisfy the identity $P(b_k, b_{-k}) = 1 - P(b_{-k}, b_k)$. Suppose there is a common prize the value of which we may normalize to 1 and common cost $C(\varphi)$. The objective function of party k is therefore $P(b_k, b_{-k}) - \eta_k C(b_k/\eta_k)$.

Proposition 4. *In any pure strategy equilibrium b_k, b_{-k} (if one exists) if $C''(\varphi) > 0$ then $b_L > b_S$ and the large party receives strictly greater utility than the small party; if $b_L \leq \eta_S$ and $C''(\varphi < 0)$ then $b_S > b_L$ and the small party receives strictly greater utility than the large party.*

Proof. In the convex case if $b_L > \eta_S$ then certainly L turns out more than Sy, so in both cases we may assume $b_L \leq \eta_S$. Consider that the utility to party k from playing b_{-k} rather than b_k must not yield an improvement in utility. That is

$$P(b_k, b_{-k}) - \eta_k C(b_k/\eta_k) \geq (1/2) - \eta_k C(b_{-k}/\eta_k)$$

or

$$P(b_k, b_{-k}) - (1/2) \geq \eta_k C(b_k/\eta_k) - \eta_k C(b_{-k}/\eta_k).$$

For party $-k$ this reads

$$P(b_{-k}, b_k) - (1/2) \geq \eta_{-k} C(b_{-k}/\eta_{-k}) - \eta_{-k} C(b_k/\eta_{-k})$$

and using $P(b_{-k}, b_k) = 1 - P(b_k, b_{-k})$

$$(1/2) - P(b_k, b_{-k}) \geq \eta_{-k} C(b_{-k}/\eta_{-k}) - \eta_{-k} C(b_k/\eta_{-k})$$

or

$$P(b_k, b_{-k}) - 1/2 \leq \eta_{-k} C(b_k/\eta_{-k}) - \eta_{-k} C(b_{-k}/\eta_{-k})$$

so the inequalities for the two parties are

$$\eta_k C(b_k/\eta_k) - \eta_k C(b_{-k}/\eta_k) \leq \eta_{-k} C(b_k/\eta_{-k}) - \eta_{-k} C(b_{-k}/\eta_{-k}).$$

Suppose without loss of generality that $b_k \geq b_{-k}$ so both sides are non-negative.

We work through the convex case. If $k = S$ we see that we must have

$$\eta_S C(b_k/\eta_S) - \eta_S C(b_{-k}/\eta_S) \leq \eta_L C(b_k/\eta_L) - \eta_L C(b_{-k}/\eta_L).$$

Consider the function $\eta_k C(b_k/\eta_k) - \eta_k C(b_{-k}/\eta_k)$ and differentiate it with respect to η_k to find

$$C(b_k/\eta_k) - C(b_{-k}/\eta_k) - ((b_k/\eta_k)C'(b_k/\eta_k) - (b_{-k}/\eta_k)C'(b_{-k}/\eta_k))$$

which may also be written as

$$C(b_k/\eta_k) - (b_k/\eta_k)C'(b_k/\eta_k) - (C(b_{-k}/\eta_k) - (b_{-k}/\eta_k)C'(b_{-k}/\eta_k)).$$

Consider the function $C(\varphi) - \varphi C'(\varphi)$ and differentiate with respect to φ to find

$$-\varphi C''(\varphi) < 0.$$

This implies

$$C(b_k/\eta_k) - (b_k/\eta_k)C'(b_k/\eta_k) - (C(b_{-k}/\eta_k) - (b_{-k}/\eta_k)C'(b_{-k}/\eta_k)) < 0$$

which in turn implies

$$\eta_L C(b_k/\eta_L) - \eta_L C(b_{-k}/\eta_L) < \eta_S C(b_k/\eta_S) - \eta_S C(b_{-k}/\eta_S)$$

a contradiction, so we conclude that $k = L$, that is, $b_L > b_S$.

If $b_L > b_S$ suppose that L were to lower its bid to b_S. It would then have a 1/2 chance of winning - at least the equilibrium utility of S - and a cost lower than the equilibrium cost of S, so bidding b_S yields L more than the equilibrium utility of S. Hence the equilibrium utility of L must be larger than that of S.

Finally if $C(\varphi)$ is concave then the role of the two parties in determining the equilibrium bids is reversed, so we conclude that $b_S > b_L$. □

Bounded Costs

We compare two participation cost functions: $c(y)$, $\xi(y)$ where for some $\eta_S/\eta_L < \overline{y} < 1$ and $y \leq \overline{y}$ we have $c(y) = \xi(y)$ while for $\overline{y} < y \leq 1$ we have $c(y) < \xi(y)$. The cost function $c(y)$ is bounded, but we allow $\xi(1) = \infty$. It follows that the corresponding expected cost functions $C(y), \Xi(y)$ share the same property that $y \leq \overline{y}$ we have $C(y) = \Xi(y)$ while for $\overline{y} < y \leq 1$ we have $C(y) < \Xi(y)$ and $C(y)$ is bounded while $\Xi(y)$ need not be

Proposition 5. *If c is such that $V > \max\{\eta_L C(\eta_S/\eta_L), \eta_S C(1)\}$ and ξ has high costs $\Xi(1) > V/\eta_S$ then the large party is advantaged. The equilibrium strategies and payoffs of the small party are the same for c, ξ. For the large party for low bids $b \leq \eta_S\overline{y}$ the strategies are the same for c, ξ. The probability of a high bid under ξ is approximately the same as the atom at η_S under c*

$$\left[1 - F_L^{\xi}(\eta_S\overline{y})\right] - F_L^{0c}(1) = \eta_S \left[C(1) - C(\overline{y})\right]/V$$

as are the equilibrium payoffs

$$\eta_L \left(C(1) - C(\overline{y})\right) > U_L^{\xi} - U_L^{c} > 0.$$

Proof. As L never bids more than η_S and $\overline{y} > \eta_S/\eta_L$ only c is relevant for computing the payoffs of L; this implies in particular that the strategy of S is the same for c or ξ. Moreover, L is advantaged for both c, ξ. This follows from $V > \eta_L C(\eta_S/\eta_L)$ meaning L is willing to bid more than η_S which is the most S can bid. Since L is advantaged for c, ξ, S gets 0 in either case. For L bids below $\eta_S\overline{y}$ we have $F_L^c(b) = \eta_S C(b/\eta_S)/V = \eta_S\Xi(b/\eta_S)/V = F_L^{\xi}(b)$.

Under c, S is willing to bid η_S (by high stakes) while under χ, S is willing to bid $\eta_S\overline{y} < \overline{b}_S < \eta_S$. The first part $\eta_S\overline{y} < \overline{b}_S$ follows from $V - \eta_S\Xi(\overline{y}) = V - \eta_S C(\overline{y}) > V - \eta_S C(1) > 0$ and the second part $\overline{b}_S < \eta_S$ follows from the high cost assumption $V - \eta_S\Xi(1) < 0$.

We now compute the probability L makes a high bid $1 - F_L^{\xi}(\eta_S\overline{y})$. Since $F_L^{\xi}(\eta_S\overline{y})V - \eta_S D(\overline{y}) = 0$ we have $1 - F_L^{\xi}(\eta_S\overline{y}) = 1 - \eta_S C(\overline{y})/V$. By contrast $F_L^{0c}(1)$ satisfies $(1 - F_L^{0c}(1))V - \eta_S C(1) = 0$ so $F_L^{0c}(1) = 1 - \eta_S C(1)/V$. These two give the desired result

$$\left[1 - F_L^{\xi}(\eta_S \overline{y})\right] - F_L^{0c}(1) = \eta_S \left[C(1) - C(\overline{y})\right] / V$$

Finally we compute $U_L^{\xi} - U_L^{c} = \eta_L \left(C(1) - C(\bar{b}_S/\eta_S)\right)$. Hence indeed

$$\eta_L \left(C(1) - C(\overline{y})\right) > U_L^{\xi} - U_L^{c} > 0$$

□