Pricing Network Effects: Competition

Itay P. Fainmesser and Andrea Galeotti

Online Appendix A.

Consumption Stage

Proof of Proposition 1. We first assume that a consumption equilibrium exists and is unique and we characterize it. We then consider an auxiliary game and show that: (a) the set of equilibria of the two games coincides, and (b) the auxiliary game has a unique equilibrium.

Consumption equilibrium characterization. Assume that a consumption equilibrium exists and is unique. We now prove that if $(\mathbf{p}^X, \mathbf{p}^Y)$ satisfies condition A then the equilibrium probability that consumer i purchases product X is given by

$$D_i^X(\mathbf{p}^X, \mathbf{p}^Y) = \frac{1}{2}\left(1 - \frac{1}{\tau}\mathsf{p}_i - \frac{\gamma k}{\tau\left(\tau - \gamma k\right)}\overline{\mathsf{p}}\right).$$

Consumer i prefers X to Y if, and only if, $\theta_i - \mathsf{p}_i + \gamma k \mathsf{A}\left(\mathbf{x}_{-i}\right) > 0$ $(\theta_i > \mathsf{p}_i - \gamma k \mathsf{A}\left(\mathbf{x}_{-i}\right))$ which happens with probability $x_i = \frac{\tau - \mathsf{p}_i + \gamma k \mathsf{A}(\mathbf{x}_{-i})}{2\tau}$; note that our model implies that θ_i is uniformly distributed in the support $[-\tau, \tau]$. Noting that $A^X\left(\mathbf{x}_{-i}\right) = 1 - A^Y\left(\mathbf{x}_{-i}\right)$ this can be re-written as $x_i = \frac{\tau - \mathsf{p}_i + 2\gamma k A^X(\mathbf{x}_{-i}) - \gamma k}{2\tau}$.

Let $x\left(l\right) = E\left[x_i | l_i = l\right]$ and $y\left(l\right) = E\left[y_i | l_i = l\right]$; solving for $\mathsf{A}\left(\mathbf{x}_{-i}\right)$ we obtain:

$$\begin{aligned}
\mathsf{A}\left(\mathbf{x}_{-i}\right) &= \sum_l \overline{H}\left(l\right)\left(x\left(l\right) - y\left(l\right)\right) = \sum_l \overline{H}\left(l\right)\left(2x\left(l\right) - 1\right) \\
&= \sum_l \overline{H}\left(l\right)\left(E\left[\frac{\tau - \mathsf{p}_i + \gamma k \mathsf{A}\left(\mathbf{x}_{-i}\right)}{\tau} | l_i = l\right] - 1\right) = -\frac{\overline{\mathsf{p}}}{\tau} + \frac{\gamma k}{\tau}\mathsf{A}\left(\mathbf{x}_{-i}\right)
\end{aligned}$$

so that

$$\mathsf{A}\left(\mathbf{x}_{-i}\right) = -\frac{\overline{\mathsf{p}}}{\tau - \gamma k}$$

and the probability that i buys from firm X is

$$D^X(p_i) = \frac{1}{2}\left(1 - \frac{1}{\tau}\mathsf{p}_i - \frac{\gamma k}{\tau(\tau - \gamma k)}\overline{\mathsf{p}}\right).$$

An auxiliary game. We prove existence and uniqueness by proving existence and uniqueness in an auxiliary game, and then showing that the set of equilibria of the two games coincide.

Consider an economy with the same timeline as in our model with the following two differences:

1. There is only one firm (X) producing a divisible good, and charging linear prices—that is, consumer i is charged p_i per one unit of the good.

2. Consumers are heterogeneous only with respect to their levels of influence, and consumer i's ex post utility function is:

$$u_i = -\frac{1}{2}x_i^2 - \frac{1}{2}(1-x_i)^2 + \frac{\gamma}{\tau}\sum_{j \in N_j} x_i x_j + \frac{\gamma}{\tau}\sum_{j \in N_j}(1-x_i)(1-x_j) - \frac{1}{\tau}p_i x_i.$$

Recall that this is an auxiliary utility function used only for the purpose of the proof, and that there isn't necessarily any product that will correspond to the function. In this auxiliary game, the expected utility of consumer i is given by

$$U_i = -\frac{1}{2}x_i^2 - \frac{1}{2}(1-x_i)^2 + \frac{\gamma k}{\tau}x_i A^X(x_{-i}) + \frac{\gamma k}{\tau}(1-x_i)(1-A^X(x_{-i})) - \frac{1}{\tau}p_i x_i.$$

Differentiating w.r.t x_i we get

$$\begin{aligned}
\frac{\partial U_i}{\partial x_i} &= -2x_i + 1 + \frac{\gamma k}{\tau}\left(2A^X(x_{-i}) - 1\right) - \frac{1}{\tau}p_i \\
\frac{\partial^2 U_i}{(\partial x_i)^2} &= -2
\end{aligned}$$

and the first-order condition yields

$$x_i = \frac{1}{2}\left(1 + \frac{\gamma k}{\tau}\left(2A^x(\mathbf{x}_{-i}) - 1\right) - \frac{1}{\tau}p_i\right).$$

We note that this is equivalent to the equilibrium conditions in the consumption stage of this paper (with $p_i = p_i^X - p_i^Y$) and thus if we prove that for any price schedule there exists a unique equilibrium in the auxiliary game that satisfies the first-order conditions, then it is also the case that there exists a unique equilibrium satisfying the equilibrium condition of this paper. We next show that for any price schedule there exists a unique equilibrium in the auxiliary game.

Existence. When applied to the auxiliary utlity function, Proposition 1 in Glaeser and Scheinkman (2002) implies that a sufficient condition for existence of equilibrium is $\forall_{p\in\mathbb{R}}\exists_{\overline{x}\geq 0}\forall_{x\leq\overline{x}}\frac{\partial u\left(\overline{x},(x)_{j\neq i}\right)}{\partial x_i}\leq 0$ or $\forall_{p\in\mathbb{R}}\exists_{\overline{x}\geq 0}\forall_{x\leq\overline{x}} - 2\overline{x}+1+\frac{\gamma k}{\tau}\left(2A^X\left((x)_{j\neq i}\right)-1\right)-\frac{1}{\tau}p\leq 0$. To see that this holds when $\gamma k<1$, note that $-2\overline{x}+1+\frac{\gamma k}{\tau}\left(2A^X\left((x)_{j\neq i}\right)-1\right)=-2\overline{x}+1+\frac{\gamma k}{\tau}(x-1)\leq -2\overline{x}+1+\frac{\gamma k}{\tau}(\overline{x}-1)$. Therefore, it is sufficient to show that $\forall_{p\in\mathbb{R}}\exists_{\overline{x}\geq 0}\left(\frac{\gamma k}{\tau}-2\right)\overline{x}+1-\frac{\gamma k}{\tau}-p\leq 0$, which is true for any $\frac{\gamma k}{\tau}<2$.

Uniqueness. When applied to the auxiliary utlity function, Proposition 3 in Glaeser and Scheinkman (2002) implies that a sufficient condition for uniqueness of equilibrium is $\forall_i\left|\frac{\partial^2 u_i}{(\partial x_i)^2}\right| > \left|\frac{\partial^2 u_i}{\partial x_i\partial A^X(\mathbf{x}_{-i})}\right|$ or $2>2\frac{\gamma k}{\tau}$. ■

Equilibrium pricing and consumption

Lemma 1. *For any firm $J\in\{X,Y\}$, Π^J is concave in $\left(p^J,\left\{p^J(l)\right\}_{l\in\{0,\ldots,l^{\max}\}}\right)$.*

Proof of Lemma 1. Let $\mathcal{H}^X$ denote the negative of the Hessian matrix of firm X's profit function with respect to $\left(p^X,\left\{p^X(l)\right\}_{l\in\{0,\ldots,l^{\max}\}}\right)$. To prove that Π^X is concave in $\left(p^X,\left\{p^X(l)\right\}_{l\in\{0,\ldots,l^{\max}\}}\right)$ it is sufficient to show that $\mathcal{H}^X$ is positive definite. For the purposes of this proof, it will be useful to denote $p^X(-1)=p^X$. Thus, we need to prove that Π^X is concave in $\left\{p^X(l)\right\}_{l\in\{-1,0,\ldots,l^{\max}\}}$ (the proof for $J=Y$ is by symmetry), and the negative of the Hessian matrix of firm X's profit function can be written as follows

$$\mathcal{H}^X=\begin{bmatrix} -\frac{\partial^2\Pi^X}{\partial(p^X(-1))^2} & -\frac{\partial^2\Pi^X}{\partial(p^X(-1))\partial(p^X(0))} & \cdots & -\frac{\partial^2\Pi^X}{\partial(p^X(-1))\partial(p^X(l^{\max}))} \\ -\frac{\partial^2\Pi^X}{\partial(p^X(0))\partial(p^X(-1))} & -\frac{\partial^2\Pi^X}{\partial^2(p^X(0))} & \cdots & -\frac{\partial^2\Pi^X}{\partial(p^X(0))\partial(p^X(l^{\max}))} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial^2\Pi^X}{\partial(p^X(l^{\max}))\partial(p^X(-1))} & -\frac{\partial^2\Pi^X}{\partial(p^X(l^{\max}))\partial(p^X(0))} & \cdots & -\frac{\partial^2\Pi^X}{\partial^2(p^X(l^{\max}))} \end{bmatrix}.$$

Our first step is to prove the following claim:

Claim 1. *There exists $\Gamma\in\mathbb{R}_+$ and $(p_j,b_j,c_j)_{j\in\{-1,0,\ldots,l^{\max}\}}\in\left(\mathbb{R}^3_+\right)^{l^{\max}+2}$ such that for every $s,t\in\{-1,0,\ldots,l^{\max}\}$,*

$$-\frac{\partial^2\Pi^X}{\partial p^X(s)\,\partial p^X(t)}=\frac{1}{\tau}\left(p_sp_t\Gamma\left[b_tc_s+b_sc_t\right]+2p_s\mathbf{1}_{\{s=t\}}\right),$$

where $\mathbf{1}_{\{s=t\}}$ *is the indicator function for when* $s = t$.

Let $\Gamma = \frac{\gamma}{\tau - \gamma k} \in [0, 1)$. To prove the claim note the following:

$$-\frac{\partial^2 \Pi^X}{\partial \left(p^X\right)^2} = \frac{1}{\tau}\left(\left(\frac{1-w^X}{2}\right)^2 \Gamma\left(2k+2k\right) + 2\frac{1-w^X}{2}\right)$$

$$-\frac{\partial^2 \Pi^X}{\partial p^X\left(s\right)^2} = \frac{1}{\tau}\left(\left(\frac{w^X H\left(s\right)}{2}\right)^2 \Gamma\left(2s+2s\right) + 2\frac{w^X H\left(s\right)}{2}\right)$$

$$-\frac{\partial^2 \Pi^X}{\partial p^X \partial p^X\left(s\right)} = \frac{1}{\tau}\left(\frac{w^X H\left(s\right)}{2}\frac{\left(1-w^X\right)}{2}\Gamma\left[2k+2s\right]\right)$$

$$-\frac{\partial^2 \Pi^X}{\partial p^X\left(t\right)\partial p^X\left(s\right)} = \frac{1}{\tau}\left(\frac{w^X H\left(t\right)}{2}\frac{w^X H\left(s\right)}{2}\Gamma\left(2t+2s\right)\right)$$

It is then left to note that $\left(p_j, b_j, c_j\right)_{j\in\{-1,0,\ldots,l^{\max}\}} \in \left(\mathbb{R}^3_+\right)^{l^{\max}+2}$ can be chosen to be the following:

1. $p_{-1} = \frac{1-w^X}{2}$; and for $s \in \{0, 1, \ldots l^{\max}\}$, $p_s = \frac{w^X H(s)}{2}$
2. $b_{-1} = 1$; and for $s \in \{0, 1, \ldots l^{\max}\}$, $b_s = 1$
3. $c_{-1} = 2k$; and for $s \in \{0, 1, \ldots l^{\max}\}$, $c_s = 2s$

Our second step in proving Lemma 1 is to recall Lemma 2 in the Online Appendix of Fainmesser and Galeotti (2016). That is,

Lemma 2. *(Fainmesser and Galeotti 2016) Let* $\mathcal{G} = \left(\left(p_s p_t \Gamma\left[b_t c_s + b_s c_t\right] + 2p_s \mathbf{1}_{\{s=t\}}\right)\right)_{s,t\in\{-1,0,1,\ldots,l^{\max}\}}$. *Then, the determinant of* $\mathcal{G}$ *is given by*

$$\det\left(\mathcal{G}\right) = \left(2^{l^{\max}}_j p_j\right)\left(4 + 4\Gamma\sum_j\left(b_j c_j p_j\right) - \Gamma^2\sum_{i<j}\left(p_i p_j\left[b_j c_i - b_i c_j\right]^2\right)\right)$$

Since for all j, $b_j, c_j, p_j \geq 0$, and since $sign\left\{\det\left(\mathcal{G}\right)\right\} = sign\left\{K\det\left(\mathcal{G}\right)\right\}$ for any $K > 0$, to prove Lemma 1 it is then sufficient to prove that:

$$\Gamma^2\sum_{i<j}\left(p_i p_j\left[b_j c_i - b_i c_j\right]^2\right) < 4 + 4\Gamma\sum_j\left(b_j c_j p_j\right)$$

which is what we prove now in the third and final step in the proof of Lemma 1.

Now note that

$$\Gamma = \gamma \frac{1}{\tau - \gamma k} < \frac{1}{2l^{\max}} 2 = \frac{1}{l^{\max}}.$$

and

$$\begin{aligned}
\sum_{i<j} \left(p_i p_j \left[b_j c_i - b_i c_j\right]^2\right) &\leq \sum_i \sum_j \left(p_i p_j \left[b_j c_i - b_i c_j\right]^2\right) \\
&= \sum_i \sum_j \left(p_i p_j \left[c_i - c_j\right]^2\right) \\
&\leq \sum_i \sum_j \left(p_i p_j \left[2l^{\max}\right]^2\right) \\
&= 4\left[l^{\max}\right]^2 \sum_i p_i \sum_j p_j \\
&= 4\left[l^{\max}\right]^2 \frac{1}{4} \\
&= \left[l^{\max}\right]^2
\end{aligned}$$

so we have that

$$\begin{aligned}
\Gamma^2 \sum_{i<j} \left(p_i p_j \left[b_j c_i - b_i c_j\right]^2\right) &< \left(\frac{1}{l^{\max}}\right)^2 \left(l^{\max}\right)^2 \\
&= 1 \\
&< 4 + 4\Gamma \sum_j \left(b_j c_j p_j\right).
\end{aligned}$$

This complete the proof of the concavity of Π^X. The proof for Π^Y is equivalent. ■

Lemma 3. *There exists $\overline{\mu} > 0$ such that for all $\mu > \overline{\mu}$ and $J \in \{X, Y\}$, if $p^J, \left\{p_l^J\right\}_{l \in \{0,1,\ldots,l^{\max}\}} \in [-\mu, \mu]$, then*

$$\arg \max_{p^{-J}, \left\{p_l^{-J}\right\}_{l \in \{0,1,\ldots,l^{\max}\}}} \Pi^{-J} \in (-\mu, \mu)^{l^{\max}+2}.$$

Proof of Lemma 3. Recall that Firms' prices are strategic complements, whereas different prices of the same firm are strategic substitutes. As a result, to prove the Lemma is it sufficient to show that there exists $\overline{\mu} > 0$ such that for all $\mu > \overline{\mu}$ the following inequalities hold:

$$\varphi_1 = \left(\frac{\partial \Pi^X}{\partial p^X(l)} | p^X, \left\{p^X(s)\right\}_{s \neq l} = -\mu, \text{ and } p^Y, \left\{p^Y(s)\right\}_s, p^X(l) = \mu\right) < 0$$

$$\varphi_2 = \left(\frac{\partial \Pi^X}{\partial p^X(l)} | p^X, \left\{p^X(s)\right\}_{s \neq l} = \mu, \text{ and } p^Y, \left\{p^Y(s)\right\}_s, p^X(l) = -\mu\right) > 0$$

$$\varphi_3 = \left(\frac{\partial \Pi^X}{\partial p^X} | \left\{ p^X(s) \right\}_s = -\mu, \text{ and } p^Y, \left\{ p^Y(s) \right\}_s, p^X = \mu \right) < 0$$

$$\varphi_4 = \left(\frac{\partial \Pi^X}{\partial p^X} | \left\{ p^X(s) \right\}_s = \mu, \text{ and } p^Y, \left\{ p^Y(s) \right\}_s, p^X = -\mu \right) > 0.$$

We begin with φ_1, which can be written as follows

$$\varphi_1 = \frac{w^X H(l)}{2} \left(1 - \frac{1}{\tau}\mu \left[1 + \frac{\gamma k}{\tau - \gamma k} \left[2\frac{l}{k} H(l) w^X - \left(2 + \frac{l}{k} \right) \left(1 - w^X \right) - w^Y - w^X \sum_{s \neq l} \frac{s+l}{k} H(s) \right] \right] \right).$$

Thus, it is sufficient to show that

$$\frac{\gamma k}{\tau - \gamma k} \left[2\frac{l}{k} H(l) w^X - \left(2 + \frac{l}{k} \right) \left(1 - w^X \right) - w^Y - w^X \sum_{s \neq l} \frac{s+l}{k} H(s) \right] > -1.$$

And in fact

$$\begin{aligned} & \frac{\gamma k}{\tau - \gamma k} \left[2\frac{l}{k} H(l) w^X - \left(2 + \frac{l}{k} \right) \left(1 - w^X \right) - w^Y - w^X \sum_{s \neq l} \frac{s+l}{k} H(s) \right] \\ > \; & \frac{\gamma k}{\tau - \gamma k} \left[- \left(2 + \frac{l}{k} \right) - 1 \right] = -3\frac{\gamma k}{\tau - \gamma k} - \frac{\gamma l}{\tau - \gamma k} > -\frac{3}{4} - \frac{1}{4} = -1 \end{aligned}$$

where the first inequality holds because $2\frac{l}{k} H(l) w^X - \left(2 + \frac{l}{k}\right) \left(1 - w^X\right) - w^Y - w^X \sum_{s \neq l} \frac{s+l}{k} H(s)$ is increasing in w^X and decreasing in w^X, andthe second inequality holds because $\gamma l^{\max} < 1/2$.

We now turn to φ_2, which can be reduced to:

$$\varphi_2 = \frac{w^X H(l)}{2} \left[1 + \frac{1}{\tau}\mu \left(1 - 2\frac{\gamma k}{\tau - \gamma k} \right) \right].$$

Thus, it is sufficient to show that

$$\varphi_2 = 1 > 2\frac{\gamma k}{\tau - \gamma k}$$

which holds because $\gamma k < 1/2$.

We now turn to φ_3, which can be reduced to:

$$\varphi_3 = \frac{\left(1 - w^X\right)}{2}\left[1 + \frac{1}{\tau}\mu\left(-1 + \frac{\gamma k}{\tau - \gamma k}\left[4w^X - 1\right]\right)\right].$$

Thus, it is sufficient to show that

$$\frac{\gamma k}{\tau - \gamma k}\left[4w^X - 1\right] < 1$$

and indeed

$$\frac{\gamma k}{\tau - \gamma k}\left[4w^X - 1\right] \leq 3\frac{\gamma k}{\tau - \gamma k} < \frac{1}{4}$$

where the first inequality holds because $w^X \leq 1$ and the second inequality holds because $\gamma k < 1/2$.

Finally, we turn to φ_4, which can be reduced to:

$$\varphi_4 = \frac{\left(1 - w^X\right)}{2}\left[1 + \frac{1}{\tau}\mu\left(1 - \frac{\gamma k}{\tau - \gamma k}\right)\right].$$

Thus, it is sufficient to show that

$$1 - \frac{\gamma k}{\tau - \gamma k} > 0$$

which holds because $\gamma k < 1/2$. This completes the proof of Lemma 3. ■

Lemma 4. *Equilibrium prices are such that condition A holds, i.e., for all* i, $\frac{1}{2}\left(1 - \frac{1}{\tau}\mathsf{p}_i - \frac{\gamma k}{\tau(\tau - \gamma k)}\overline{\mathsf{p}}\right) \in (0, 1)$.

Proof of Lemma 4. We focus, without loss of generality, on the case that $w^X \geq w^Y$, so that $\overline{\mathsf{p}} \leq 0, \mathsf{p} > 0, \mathsf{p}\left(l^{\max}\right) > 0, \mathsf{p}\left(0\right) < 0$. Therefore,

$$\begin{aligned} \max\left\{p_i\right\} &= p_X - p_Y\left(l^{\max}\right) > 0 \\ \inf\left\{p_i\right\} &= p_X - p_Y\left(0\right) < 0 \end{aligned}$$

and

$$\begin{aligned} \max\left\{\frac{1}{2}\left(1 - \frac{1}{\tau}\mathsf{p}_i - \frac{\gamma k}{\tau\left(\tau - \gamma k\right)}\overline{\mathsf{p}}\right)\right\} &\leq \frac{1}{2}\left(1 - \frac{1}{\tau}\inf\left\{\mathsf{p}_i\right\} - \frac{\gamma k}{\tau\left(\tau - \gamma k\right)}\overline{\mathsf{p}}\right) \\ \min\left\{\frac{1}{2}\left(1 - \frac{1}{\tau}\mathsf{p}_i - \frac{\gamma k}{\tau\left(\tau - \gamma k\right)}\overline{\mathsf{p}}\right)\right\} &= \frac{1}{2}\left(1 - \frac{1}{\tau}\max\left\{\mathsf{p}_i\right\} - \frac{\gamma k}{\tau\left(\tau - \gamma k\right)}\overline{\mathsf{p}}\right) \end{aligned}$$

Part 1: We first show that

$$\frac{1}{2}\left(1-\frac{1}{\tau}\inf\left\{\mathsf{p}_i\right\}-\frac{\gamma k}{\tau\left(\tau-\gamma k\right)}\overline{\mathsf{p}}\right)<1.$$

Note that,

$$\begin{aligned}
\frac{1}{2}\left(1-\frac{1}{\tau}\inf\left\{\mathsf{p}_i\right\}-\frac{\gamma k}{\tau\left(\tau-\gamma k\right)}\overline{\mathsf{p}}\right) &= \frac{1}{2}\left(1-\frac{1}{\tau}\left(p^X-p^Y\left(0\right)\right)-\frac{\gamma k}{\tau\left(\tau-\gamma k\right)}\overline{\mathsf{p}}\right)\\
&= \frac{1}{2}\left(1+\frac{\gamma(2p^Y+w^Xp^X)}{\tau(\tau-\gamma k)(4-w^Xw^Y)}k+\frac{2k\gamma}{\tau(3\tau-2k\gamma)}\overline{\mathsf{p}}-\frac{\gamma k}{\tau\left(\tau-\gamma k\right)}\overline{\mathsf{p}}\right)\\
&= \frac{1}{2}\left(1+\frac{\gamma(2p^Y+w^Xp^X)}{\tau(\tau-\gamma k)(4-w^Xw^Y)}k-\frac{\gamma k}{\left(3\tau-2k\gamma\right)\left(\tau-\gamma k\right)}\overline{\mathsf{p}}\right)<1
\end{aligned}$$

if, and only if,

$$\frac{2p_Y+w^Xp^X}{\tau(4-w^Xw^Y)}-\frac{1}{3\tau-2k\gamma}\overline{\mathsf{p}}<\frac{\tau-\gamma k}{\gamma k}.$$

or, equivalently,

$$\frac{2+w_X}{\tau(4-w_Xw_Y)}+\left(\frac{\left(2-w_X\right)k\gamma}{\tau(4-w_Xw_Y)}-1\right)\frac{1}{\left(3\tau-2k\gamma\right)\left(\tau-\gamma k\right)}\overline{p}<\frac{1}{\gamma k}\tag{1}$$

The first observation is that if $\gamma k\to 0$ then condition 1 because the RHS of 1 tends to $+\infty$. Next, note that the RHS of 1 decreases in γ, whereas the LHS weakly increases in γ. Therefore, it is sufficient that 1 holds for large γ (recall that $\gamma l^{\max}\leq 1/2$). A useful way of rewriting inequality 1 is by substituting $\overline{\mathsf{p}}$ and rearranging:

$$\frac{(2+w^X)\gamma k}{\tau(4-w^Xw^Y)}+\left(\frac{\tau(4-w^Xw^Y)-\left(2-w^X\right)k\gamma}{\tau(4-w^Xw^Y)}\right)\left(\frac{2\sigma^2\gamma^2(w^X-w^Y)}{3\tau(\tau-\gamma k)(4-w^Xw^Y)-2\gamma^2\sigma^2[w^X+w^Y-w^Xw^Y]}\right)<1.$$

This can be verified by using the following bounds. Note that $\frac{w^X-w^Y}{3\tau(\tau-\gamma k)(4-w^Xw^Y)-2\gamma^2\sigma^2[w^X+w^Y-w^Xw^Y]}$ and $\frac{\tau(4-w^Xw^Y)-\left(2-w^X\right)k\gamma}{\tau(4-w^Xw^Y)}$ are decreasing in w^Y and increasing in w^X, and therefore we can find the following upper bounds:

$$\begin{aligned}
\frac{2\sigma^2\gamma^2(w^X-w^Y)}{3\tau(\tau-\gamma k)(4-w^Xw^Y)-2\gamma^2\sigma^2[w^X+w^Y-w^Xw^Y]}\leq\frac{2\sigma^2\gamma^2}{12\tau(\tau-\gamma k)-2\gamma^2\sigma^2} &< \frac{1}{11}\\
\frac{\tau(4-w^Xw^Y)-\left(2-w^X\right)k\gamma}{\tau(4-w^Xw^Y)} &\leq \frac{4\tau-k\gamma}{4\tau}<1
\end{aligned}$$

and, in addition, we have that

$$\frac{2+w^X}{\tau(4-w^Xw^Y)}\gamma k < \frac{1}{2}.$$

Part 2: Next we turn to prove that $\frac{1}{2}\left(1-\frac{1}{\tau}\mathsf{p}_i-\frac{\gamma k}{\tau(\tau-\gamma k)}\overline{\mathsf{p}}\right)>0$. We do this by proving that $\frac{1}{2}\left(1-\frac{1}{\tau}\max\{\mathsf{p}_i\}-\frac{\gamma k}{\tau(\tau-\gamma k)}\overline{\mathsf{p}}\right)>0$ or

$$p_X-p_Y\left(l^{\max}\right)+\frac{\gamma k}{\tau-\gamma k}\overline{\mathsf{p}}<\tau.$$

Let $\lambda=\frac{\gamma k}{\tau-\gamma k}$ and let $\beta=\frac{2\gamma k}{3\tau-2\gamma k}$. Because $\gamma k<\frac{1}{2}$ and $\tau\geq 0$ it is the case that $\beta\leq\lambda<1$, and

$$p_X-p_Y\left(l^{\max}\right)+\lambda\overline{\mathsf{p}}=\tau-\gamma k-\frac{\beta\overline{p}}{2}-\left(\tau-\gamma k+\frac{\beta\overline{\mathsf{p}}}{2}+\frac{\gamma(2p_Y+w_Xp_X)}{(\tau-\gamma k)(4-w^Xw^Y)}[k-l^{\max}]\right)+\lambda\overline{\mathsf{p}}.$$

Then, letting $\Delta=l^{\max}-k$, we have:

$$\begin{aligned}
p^X-p^Y\left(l^{\max}\right)+\lambda\overline{\mathsf{p}} &= (\lambda-\beta)\,\overline{\mathsf{p}}+\frac{\gamma\Delta}{4-w^Xw^Y}\left(2+w^X+\frac{\left(2-w^X\right)\beta\overline{\mathsf{p}}}{2(\tau-\gamma k)}\right)\\
&= \left(\frac{\lambda\tau}{3\tau-2\gamma k}\right)\overline{\mathsf{p}}+\frac{\gamma\Delta}{4-w^Xw^Y}\left(2+w^X+\left(2-w^X\right)\frac{\lambda\overline{\mathsf{p}}}{(3\tau-2\gamma k)}\right)
\end{aligned}$$

Recall that $\overline{\mathsf{p}}=-\frac{2\sigma^2\gamma^2(w^X-w^Y)(3\tau-2\gamma k)}{\lambda[3\tau(\tau-\gamma k)(4-w^Xw^Y)-2\gamma^2\sigma^2(w^X+w^Y-w^Xw^Y)]}$. Now let

$$\eta=-\overline{\mathsf{p}}\left(\frac{\lambda}{3\tau-2\gamma k}\right)=\frac{2\sigma^2\gamma^2(w^X-w^Y)}{3\tau(\tau-\gamma k)(4-w^Xw^Y)-2\gamma^2\sigma^2\left(w^X+w^Y-w^Xw^Y\right)}$$

which implies that

$$p^X-p^Y\left(l^{\max}\right)+\lambda\overline{\mathsf{p}}=-\eta\tau+\frac{\gamma\Delta}{4-w^Xw^Y}\left(2+w^X-\left(2-w^X\right)\eta\right).$$

Now let $\delta=\frac{2+w^X}{4-w^Xw^Y}$ and note that $\delta\leq 1$. Hence,

$$p^X-p^Y\left(l^{\max}\right)+\lambda\overline{\mathsf{p}}=\gamma\Delta\left(\delta-\frac{2-w^X}{4-w^Xw^Y}\eta\right)-\eta\tau.$$

As a result, it is sufficient to prove that

$$\gamma\Delta\left(\delta-\frac{2-w^X}{4-w^Xw^Y}\eta\right)<(1+\eta)\,\tau.$$

Let us expand the expression on the LHS. It becomes:

$$\frac{\gamma\Delta}{4-w^Xw^Y}\frac{3\tau\left(2+w^X\right)(\tau-\gamma k)(4-w^Xw^Y)-2\gamma^2\sigma^2\left[\left(2+w^X\right)\left(w^X+w^Y-w^Xw^Y\right)+\left(2-w^X\right)(w^X-w^Y)\right]}{3\tau(\tau-\gamma k)(4-w^Xw^Y)-2\gamma^2\sigma^2\left(w^X+w^Y-w^Xw^Y\right)}$$

and

$$1+\eta=\frac{3\tau(\tau-\gamma k)(4-w^Xw^Y)-2\gamma^2\sigma^2\left[\left(2-w^X\right)w^Y\right]}{3\tau(\tau-\gamma k)(4-w^Xw^Y)-2\gamma^2\sigma^2\left(w^X+w^Y-w^Xw^Y\right)}$$

Observe that $(2+w^X)(w^X+w^Y-w^Xw^Y)+(2-w^X)(w^X-w^Y)=w^X(4-w^Xw^Y)$. It is therefore sufficient to show that

$$\gamma\Delta\left(3\tau\left(2+w^X\right)(\tau-\gamma k)-2\gamma^2\sigma^2w^X\right)<\left(3\tau(\tau-\gamma k)(4-w^Xw^Y)-2\gamma^2\sigma^2\left(2-w^X\right)w^Y\right)\tau.$$

Now,

$$\frac{\partial(L.H.S)}{\partial w^X}=\gamma\Delta\left(3\tau(\tau-\gamma k)-2\gamma^2\sigma^2\right)>0$$

and

$$\frac{\partial(R.H.S)}{\partial w^X}=\left(-3\tau(\tau-\gamma k)+2\gamma^2\sigma^2\right)w^Y\tau<0$$

where both inequalities hold because $3(1-\gamma k)>2\sigma^2\gamma^2$. Moreoever, $\frac{\partial(L.H.S)}{\partial w^Y}=0$ and

$$\frac{\partial(R.H.S)}{\partial w^Y}=\left(-3\tau(\tau-\gamma k)w^X-2\gamma^2\sigma^2\left(2-w^X\right)\right)\tau<0$$

Therefore it is sufficient to show the inequality is satisfied when $w^X=w^Y=1$ which it does, given that $\gamma\Delta<1$ and $\tau\geq 1$ ■

Online Appendix B: Additional results on pricing and consumption.

We provide additional results on equilibrium pricing and equilibrium consumption, for the case where firms have different levels of information with regard to the level of influence of consumers, i.e., $w^X > w^Y$.

When the two firms have different levels of information, they charge different prices.

Corollary 1. *Suppose that $w^X > w^Y$. Then firm X charges non-targeted consumers a higher price than firm Y. Furthermore, the price premium-discount per influence charged by firm X is lower than the one charged by firm Y. Overall, $p^X(l)$ is higher than $p^Y(l)$ if and only if*

$$4\gamma\sigma_H^2(\tau - \gamma k) > [k - l][3\tau(\tau - \gamma k) - 2\sigma_H^2\gamma^2].$$

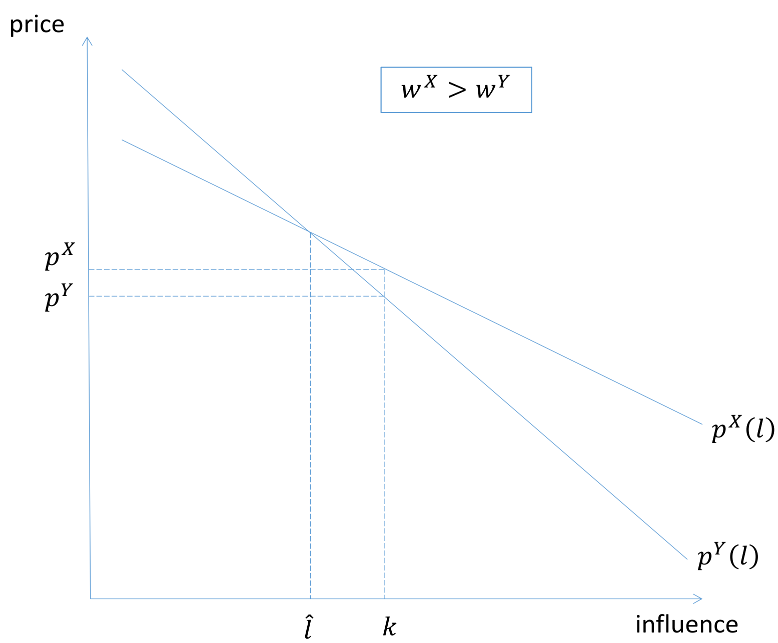

Figure 1 – Equilibrium price schedules when $w^X > w^Y$.

Figure 1 illustrates Corollary 1; we let $\hat{l}$ be such that $4\gamma\sigma_H^2(\tau - \gamma k) = [k - \hat{l}][3\tau(\tau - \gamma k) - 2\sigma_H^2\gamma^2]$. To understand the intuition of Corollary 1 suppose that we start from an equilibrium in which both firms have the same amount of information and let us increase firm X's information. First, keeping everything else constant, such an increase leads to a shift in network effects in favor of firm X and therefore firm X's demand increases and firm Y's demand declines. This *adoption-externality* effect implies that firm X's demand becomes less elastic and firm Y's demand more elastic, and therefore firm X reacts by pricing less aggressively and firm Y more aggressively. Formally, the adoption-externality effect is reflected in the observation that an increase in w^X leads to a decrease in the relative expected price of product X, for a randomly selected neighbor of a consumer. That is:

$$\frac{\partial \overline{\mathsf{p}}}{\partial w^X}|Eq. = \sum_l \overline{H}(l)p^X(l) - p^X < \sum_l H(l)p^X(l) - p^X = 0,$$

where the first inequality follows because the distribution of influence of a randomly selected consumer's neighbor *first order stochastic dominates* the distribution of influence of a randomly selected consumer, and because in equilibrium $p^X(l)$ is a decreasing function of influence.

A second effect is related to the change in competition that the two firms face towards targeted consumers. Firm X now targets more consumers, which implies that, holding all else equal, the average price that highly influential consumers are charged by firm X is lower than the price they are charged by firm Y, and vice versa for the less influential consumers. This *price-targeting effect* implies that to compete with firm X, firm Y must increase the discount it offers to targeted highly influential consumers, and at the same time firm Y can charge higher premia to less influential targeted consumers. Formally, note that an increase in w_X leads to a first-order increase in the price premium-discount per influence charged by firm Y

$$\frac{\partial^2 \Pi^Y}{\partial p^Y(l)\partial w^X}|_{Eq.} = \frac{w^Y}{2\tau}H(l)\left[\underbrace{\frac{\gamma k}{\tau - \gamma k}\left(\sum \overline{H}(s)p^X(s) - p^X\right)}_{\text{adoption-externality effect}} + \underbrace{p^X(l) - p^X}_{\text{price-targeting effect}}\right]$$

The first term captures the adoption-externality effect which, as described above, is negative for firm Y and leads to a shift down of $p^Y(l)$. The second term $p^X(l) - p^X$ is the price-targeting effect. It is positive for below-average influential consumers and negative for above-average consumers. As a result, firm Y increases the premia for low influence consumers and the discounts for high influence consumers.

In the symmetric case captured by Corollary 2, the adoption-externality effect is muted by the symmetry, and the price-targeting effect translates onto the observation that when both firms obtain additional information, the price premia-discounts per influence charged by both firms increase.

We next evaluate how equilibrium pricing depends on the variance of the distribution of influence when firms sample different fractions of consumers. The findings of Proposition 1 are illustrated in

Figure 2.

Proposition 1. *Suppose $w^X > w^Y$. If σ_H^2 increases then firm X prices less aggressively–p^X and $p^X(l)$ increase for all l– whereas firm Y prices more aggressively– p^Y and $p^Y(l)$ decrease for all l.*

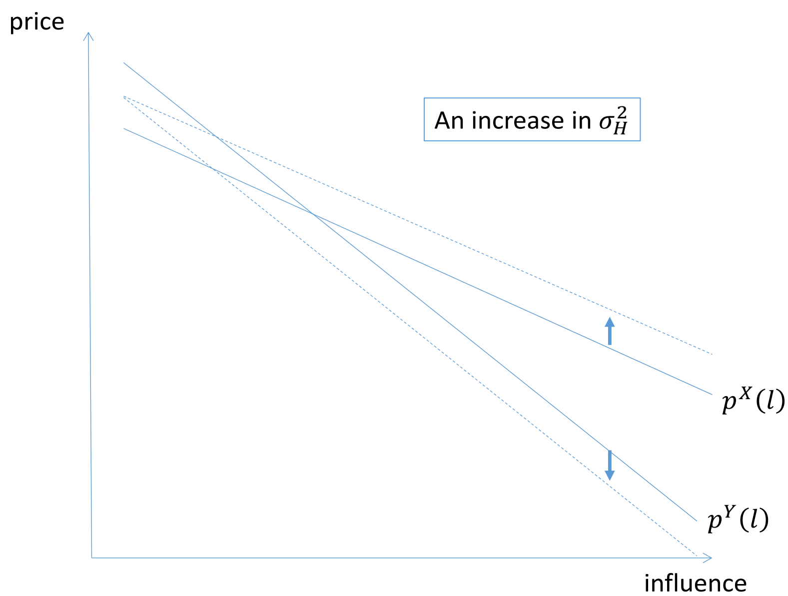

Figure 2 – An increase in the dispersion of influence when $w^X > w^Y$.

That is, an increase in the dispersion of influence amplifies the adoption externality effect. In particular, when the distribution of influence becomes more dispersed then the expected price that a randomly selected neighbor of a consumer observes from firm $J = X, Y$ declines, *ceteris paribus.* To see this note that

$$\sum_l \overline{H}(l)p^X(l) \propto \sum \overline{H}(l)[k-l] \propto k^2 - \sum H(l)l^2 = -\sigma_H^2$$

So, whether $\overline{\mathsf{p}}$ increases or decreases with an increase in σ_H^2 depend on whether this effect is stronger or weaker for firm X relative to firm Y. In turn, this effect for firm J is stronger the larger is w^J and the larger is the price premium/discount. In view of the above result, we have two contrasting effects. On the one hand, since $w^X > w^Y$ the effect is stronger for firm X. On the other hand, because $w^X > w^Y$, the slope of $p^X(l)$ is lower than the one of $p^Y(l)$ and so the effect is stronger for firm Y.

Intuitively, the steeper slope of firm Y's price schedule is a competitive reaction to firm X's informational advantage. The direct effect on firm X's information is, therefore, a first-order one, and, therefore, when $w^X > w^Y$, an increase in σ_H^2 generates higher aggregate demand for firm X. Once this is established, we follow the same intuition we have developed above. Firm X benefits more from the adoption-externality effect and can price less aggressively as compared to firm Y, who, instead, gains demand only by charging low prices.

Consumption

If firms have the same amount of information, their pricing strategy is the same (see Corollary 2) and, therefore, the demand for product X equals the demand for product Y. In contrast, a firm with more information is more effective in leveraging network effects, and as a consequence faces a higher demand.

Proposition 2. *The expected probability that a consumer with level of influence l purchases product X at the equilibrium price schedules is:*

$$x(l) = \frac{1}{2} + \frac{\gamma\left[\gamma\sigma_H^2 + 3(\tau - \gamma k)(l-k)\right]}{\left[3\tau(\tau-\gamma k)(4 - w^X w^Y) - 2\gamma^2\sigma_H^2(w^X + w^Y - w^X w^Y)\right]}(w^X - w^Y)$$

Suppose $w^X > w^Y$, then whether a consumer with influence l is more likely to buy product X relative to product Y is determined by the sign of $\gamma\sigma_H^2 + 3(\tau - \gamma k)(l - k)$. The first part of this expression, $\gamma\sigma_H^2$, captures the overall increase in demand for product X due to increased network effects attributed to better price targeting. The second part, $3(\tau - \gamma k)(l - k)$, captures the slope of the demand facing by firm X. Firm X can target many consumers, thus, relative to firm Y, there are more influential consumers who receive discounted price offers and more non-influential consumers who receive price offers above average. The result is that there is a threshold $\tilde{l} = k - \frac{\gamma\sigma_H^2}{3(\tau-\gamma k)}$ so that consumers with influence $l > \tilde{l}$ are more likely to purchase product X.

Despite consumers with a lower level of influence are more likely to purchase product Y, the demand of firm X is, in aggregate, larger than the demand of firm Y.

Corollary 2. *Suppose that $w^X > w^Y$. Then, the aggregate demand for product X is larger than the aggregate demand for product Y—i.e., $\sum_l H(l)x(l) > 1/2$. Moreover, an increase in the dispersion of influence, σ_H^2, a decrease in the compatibility of the two products, $1/\gamma$, and a decrease in the degree of product differentiation, τ, increase the aggregate demand for product X.*

An increase in the dispersion of influence and/or an increase in γ allows the firm with more information to increase network effects in its favor, and, in turn, it makes it more likely that consumers purchase its product. Likewise, a decrease in the degree of product differentiation makes network effects more important in determining adoption decisions, and this gives an advantage to the firm with more information on network effects.

Proofs

Proof of Corollary 1. If $w^X > w^Y$ then $\overline{\mathsf{p}} < 0$ which implies that $p^X > p^Y$. Next, we sign $p_X(l) - p_Y(l)$. Note that when $l = k$, $p^X(k) - p^Y(k) = p^X - p^Y > 0$. More generally, $p^X(l) - p^Y(l) > 0$ iff:

$$p^X - p^Y + \frac{\gamma[k-l]}{(\tau - \gamma k)(4 - w^X w^Y)}[2(p^X - p^Y) + w^Y p^Y - w^X p^X] > 0$$

and using the expression for p^X and p^Y the condition is equivalent to

$$-\frac{2k}{3\tau - 2k\gamma}\overline{\mathsf{p}} + -\frac{\gamma k[k-l]\left(4 - w^Y - w^X\right)}{(\tau - \gamma k)(4 - w^X w^Y)\left(3\tau - 2k\gamma\right)}\overline{\mathsf{p}} - \frac{[k-l]\left(w^X - w^Y\right)}{(4 - w^X w^Y)} > 0.$$

Now let $G = 3\tau(\tau - \gamma k)(4 - w^X w^Y) - 2\gamma^2\sigma^2[w^X + w^Y - w^X w^Y]$ so that $\overline{\mathsf{p}} = -\frac{2\sigma^2\gamma(\tau - \gamma k)(3\tau - 2\gamma k)(w^X - w^Y)}{kG}$. Then, the above condition becomes

$$\frac{(w^X - w^Y)\gamma}{G}\left[4\gamma\sigma^2(\tau - \gamma k) - [k-l]\left[3\tau(\tau - \gamma k) - 2\gamma^2\sigma^2\right]\right] > 0.$$

Since $w^X > w^Y$ and $G > 0$, this is equivalent to $4\gamma\sigma^2(\tau - \gamma k) - [k-l]\left[3\tau(\tau - \gamma k) - 2\gamma^2\sigma^2\right] > 0$.

To see that the price premium-discount per unit of influence is lower for firm X than for firm Y, note that $\left|p^X(l+1) - p^X(l)\right| < \left|p^Y(l+1) - p^Y(l)\right|$ if and only if $2\left[p^X - p^Y\right] < w^X p^X - w^Y p^Y$, and using the expressions for p^X and p^Y we get that this is equivalent to

$$\frac{(w^X - w^Y)(4 - w^X w^Y)(\tau - \gamma k)}{G}\left[3\tau(\tau - \gamma k) - 2\gamma^2\sigma^2\right] > 0$$

and since $w^X > w^Y$ and $G > 0$, this is equivalent to $3\tau(\tau - \gamma k) - 2\gamma^2\sigma^2 > 0$, which holds for any $\gamma l^{\max} < 1/2$. ■

Proof of Proposition 1. We first prove that if $w^X > w^Y$ then $sign\frac{\partial\left(\frac{1}{\overline{\mathsf{p}}}\right)}{\partial\sigma_H^2} > 0$ (and thus $\frac{\partial\overline{\mathsf{p}}}{\partial\sigma_H^2} < 0$). This follows because

$$\frac{1}{\overline{\mathsf{p}}} = -\frac{3\tau(4 - w^X w^Y)k}{2\sigma_H^2\gamma(3\tau - 2\gamma k)(w^X - w^Y)} + \frac{\gamma k[w^X + w^Y - w^X w^Y]}{(\tau - \gamma k)(3\tau - 2\gamma k)(w^X - w^Y)}$$

which implies that $sign \frac{\partial(\frac{1}{\overline{\mathsf{p}}})}{\partial \sigma_H^2} = sign(w_X - w_Y) > 0$, where the inequality follows because $w^X > w^Y$.

Since p^X declines in $\overline{\mathsf{p}}$ and p^Y increases in $\overline{\mathsf{p}}$ it follows that $\partial p^X/\partial\sigma_H^2 > 0$ and $\partial p^Y/\partial\sigma_H^2 < 0$.

We now study the effect of a change in σ_H^2 on $p^X(l)$ and $p^Y(l)$. First,

$$\frac{\partial p^X(l)}{\partial \sigma_H^2} = \frac{k\gamma}{3\tau - 2k\gamma}\frac{\partial \overline{\mathsf{p}}}{\partial \sigma_H^2}\left(-1 + \frac{\gamma[k-l]}{(\tau-\gamma k)(4-w^X w^Y)}\left[w^Y - 2\right]\right)$$

and therefore $\frac{\partial p^X(l)}{\partial \sigma_H^2} > 0$ if, and only if,

$$\frac{\partial \overline{\mathsf{p}}}{\partial \sigma^2}\left[-1 + \frac{\gamma[k-l]}{(\tau-\gamma k)(4-w^X w^Y)}(w^Y - 2)\right] > 0.$$

When $w^X > w^Y$ we know that $\frac{\partial \overline{\mathsf{p}}}{\partial \sigma_H^2} < 0$ and therefore $\frac{\partial p^X(l)}{\partial \sigma_H^2} > 0$ if, and only if,

$$\left[1 + \frac{\gamma[k-l]}{(\tau-\gamma k)(4-w^X w^Y)}(2 - w^Y)\right] > 0.$$

This condition holds because:

$$\begin{aligned}
& 1 + \frac{\gamma[k-l]}{(\tau-\gamma k)(4-w^X w^Y)}(2-w^Y) > 1 + \frac{\gamma[k-l^{\max}]}{(\tau-\gamma k)(4-w^X w^Y)}(2-w^Y) \\
> \; & 1 - \frac{\gamma l^{\max}}{(\tau-\gamma k)(4-w^X w^Y)}(2-w^Y) > 1 - \frac{1}{2(4-w^X w^Y)}2 > 1 - \frac{1}{4} > 0
\end{aligned}$$

The same arguments are used to prove that $p^Y(l)$ declines with σ_H^2. ■

Proof of Proposition 2. The ex-ante probability that a consumer with level of influence l buys from firm X is

$$x(l) = \frac{1}{2} - \frac{1}{2\tau}E[\mathsf{p}|l] - \frac{\gamma k}{2\tau}\frac{\overline{\mathsf{p}}}{\tau - \gamma k},$$

where $E[\mathsf{p}|l] = w^X p^X(l) + (1-w^X)p^X - w^Y p^Y(l) - (1-w^Y)p^Y$. Using equilibrium pricing we obtain that

$$E[p|l] = -\frac{2k\gamma}{3\tau - 2k\gamma}\overline{p} + \frac{2\gamma[k-l](w^X - w^Y)3\tau(\tau-\gamma k)}{G}$$

where we recall that $G = 3\tau(\tau-\gamma k)(4-w^X w^Y) - 2\gamma^2\sigma_H^2[w^X + w^Y - w^X w^Y]$. Therefore,

$$x(l) = \frac{1}{2} + \frac{\gamma\left[\gamma\sigma_H^2 + 3(\tau-\gamma k)[l-k]\right]}{[3\tau(\tau-\gamma k)(4-w^X w^Y) - 2\gamma^2\sigma_H^2(w^X + w^Y - w^X w^Y)]}(w^X - w^Y).$$

■

Proof of Corollary 2. The aggregate demand of firm X is therefore

$$\sum H(l)x(l) = \frac{1}{2} + \frac{\gamma^2\sigma_H^2(w^X - w^Y)}{[3\tau(\tau-\gamma k)(4-w^X w^Y) - 2\gamma^2\sigma_H^2(w^X + w^Y - w^X w^Y)]} > 1/2.$$

where the inequality follows because $w^X > w^Y$. It is then easy to verify that aggregate demand for product X increases with an increase in σ_H^2, an increase in γ, and a decrease in τ. ■

Online Appendix C: Restrictions on ability to price discriminate.

Let $D = \{l, h\}$, with $l < h$ and $H(l) = q$; hence $k = lq + h(1-q)$, $\overline{H}(l) = lq/k$ and $\overline{H}(h) = h(1-q)/k$. Denote $\beta = \frac{\gamma k}{(\tau-\gamma k)}$. Consider the following strategy of firm $J = X, Y$: Firm J charges a price p^J to non targeted consumers and to consumers with influence l and a price p_h^J to consumers with degree h. The following definitions are used when we write the payoffs and derive the equilibrium conditions:

- $\mathsf{p} = p^X - \hat{p}^Y$ where $\hat{p}^Y = (1-w+wq)p^Y + w(1-q)p_h^Y$;
- $\mathsf{p}(h) = p_h^X - (1-w)p^Y - wp_h^Y$;
- $\mathsf{p}(l) = p^X - p^Y(l) = p^X - p^Y$;
- $\overline{\mathsf{p}} = [1-w+w\overline{H}(l)][p^X - p^Y] + w\overline{H}(h)[p_h^X - p_h^Y]$.

The profit function of firm X is then:

$$\Pi^X(\mathbf{p}^X, \mathbf{p}^Y) = \frac{1}{2\tau}\left[[1-w]p^X\left(\tau - \mathsf{p} - \beta\overline{\mathsf{p}}\right) + wqp^X\left(\tau - \mathsf{p}(l) - \beta\overline{\mathsf{p}}\right) + w(1-q)p_h^X\left(\tau - \mathsf{p}(h) - \beta\overline{\mathsf{p}}\right)\right],$$

and we can focus on maximizing:

$$2\tau\Pi^X(\mathbf{p}^X, \mathbf{p}^Y) = [1-w]p^X\left(\tau - \mathsf{p} - \beta\overline{\mathsf{p}}\right) + wqp^X\left(\tau - \mathsf{p}(l) - \beta\overline{\mathsf{p}}\right) + w(1-q)p_h^X\left(\tau - \mathsf{p}(h) - \beta\overline{\mathsf{p}}\right).$$

The first order condition with respect to p^X is:

$$\begin{aligned}\frac{d2\tau\Pi^X(\mathbf{p}^X,\mathbf{p}^Y)}{dp^X} &= [1-w]\left(\tau-\mathsf{p}-\beta\overline{\mathsf{p}}-p^X-\beta p^X(1-w+w\overline{H}(l))\right)+\\ &+ wq\left(\tau-\mathsf{p}(l)-\beta\overline{\mathsf{p}}-p^X-\beta p^X(1-w+w\overline{H}(l))\right)-\beta w(1-q)p_h^X(1-w+w\overline{H}(l))=0.\end{aligned}$$

Imposing symmetry $p^X=p^Y=p$ and $p_h^X=p_h^Y=p_h$ implies that $\mathsf{p}=w(1-q)(p-p_h)$, $\mathsf{p}(h)=-(1-w)(p-p_h)$, $\mathsf{p}(l)=0$ and $\overline{\mathsf{p}}=0$; we then rewrite the above first order condition and obtain the following equilibrium condition:

$$[1-w+wq]\left(\tau-p-\beta p(1-w+w\overline{H}(l))\right)-w(1-w)(1-q)(p-p_h)-\beta w(1-q)p_h^X(1-w+w\overline{H}(l))=0. \tag{2}$$

We repeat these steps when we take the first order condition with respect to p_h^X; we obtain that $\frac{d2\tau\Pi^X(\mathbf{p}^X,\mathbf{p}^Y)}{dp_h^X}=0$ if and only if:

$$-[1-w+wq]p\beta w\overline{H}(h)+w(1-q)\left(\tau+(1-w)(p-p_h)-p_h-\beta p_h w\overline{H}(h)\right)=0. \tag{3}$$

So the equilibrium is given by p and p_h that solves equilibrium conditions 2-3. We now solve this system. First, we develop:

$$\frac{d\Pi^X}{dp^X}+\frac{d\Pi^X}{dp_h^X}=0,$$

and solve for p as a function of p_h. After some algebra we obtain:

$$p=\frac{\tau-w(1-q)(1+\beta)p_h}{(1+\beta)(1-w(1-q))}. \tag{4}$$

Second, we take equilibrium condition 3, substitute expression 4 and solve for p_h. After some algebra we obtain:

$$p_h=\frac{\tau}{1+\beta}\left[1+\beta\left(\frac{1-w+wq}{2(1-w)+wq}\right)\left(1-\frac{h}{k}\right)\right].$$

Using the definition of β, we can rewrite p_h as:

$$p_h=\tau-\gamma k-\left(\frac{3(1-w)+wq}{2(1-w)+wq}\right)\gamma(h-k). \tag{5}$$

Finally, we substitute expression 5 for p_h into expression 4 and obtain that:

$$p = \tau - \gamma k + \left(\frac{wq}{2(1-w)+wq}\right)\gamma(k-l). \tag{6}$$

This provides the equilibrium characterization in Section 3.4 of the paper.

We now derive an expression for equilibrium misallocation, profits and consumer surplus. For misallocation, note that the only event in which misallocation occurs is when: a) a consumer is targeted by one firm, say firm X, and not the other firm, say firm Y; b) the consumer prefers firm Y products, $\theta < 0$; and c) the consumer has high influence so that she receives a lower price offer from firm X. This is summarized by:

$$MS = 2w(1-w)\Pr\left[\theta \in [p_h - p, 0]\right] \left|E\left[|\theta| \,|\theta \in [p_h - p, 0]\right]\right|,$$

where θ, as before, is uniformly distributed between $[-\tau, \tau]$. Note that

$$\Pr\left[\theta \in [p_h - p, 0]\right] = \frac{1}{2\tau}[p - p_h] \text{ and } \left|E\left[|\theta| \,|\theta \in [p_h - p, 0]\right]\right| = \frac{1}{2}[p - p_h].$$

Let μ_p be the average price offered by any firm across all consumers (that is, $\mu_p = (1-w+wq)p + w(1-q)p_h$) and γ_p^2 be the variance of the prices offered by any firm (that is, $\sigma_p^2 = (1-w+wq)[p-\mu_p]^2 + w(1-q)[p_h-\mu_p]^2 = (1-w+wq)w(1-q)[p-p_h]^2$). We get that:

$$MS = \frac{w(1-w)}{2\tau}[p-p_h]^2 = \frac{(1-w)}{2\tau(1-w+wq)(1-q)}\sigma_p^2.$$

In equilibrium firms obtain the same expected profit. Firm X's equilibrium profit is:

$$\begin{aligned} \Pi^X &= p\left[\frac{(1-w)^2}{2} + \frac{w^2 q}{2} + \frac{2w(1-w)q}{2} + (1-w)w(1-q)\Pr\left(\theta > p - p_h\right)\right] + \\ &+ p_h\left[\frac{w^2(1-q)}{2} + w(1-w)(1-q)\Pr\left(\theta > p_h - p\right)\right]. \end{aligned}$$

Note that

$$\Pr\left(\theta > p - p_h\right) = \frac{1}{2} - \frac{1}{2\tau}[p - p_h] \text{ and } \Pr\left(\theta > p_h - p\right) = \frac{1}{2} + \frac{1}{2\tau}[p - p_h].$$

Hence

$$\begin{aligned}\Pi^X &= \frac{p}{2}[1-w+wq] + \frac{p_h}{2}[1-q]w - \frac{(1-w)w(1-q)}{2\tau}[p-p_h]^2 \\ &= \frac{1}{2}\mu_p - \frac{(1-w)}{(1-w+wq)2\tau}\sigma_p^2 = \frac{1}{2}\mu_p - (1-q)MS.\end{aligned}$$

Finally, using the identity

$$CS + 2\Pi = W^{\max} - MS \to CS = W^{\max} - MS - \mu_p + 2(1-q)MS,$$

or

$$CS = W^{\max} - \mu_p - MS[1-2(1-q)].$$

References

Fainmesser, I. P. and A. Galeotti, 2016. Pricing Network Effects, *Review of Economic Studies*,83 (1), 165-198.

Glaeser E. and J. A. Scheinkman, 2002. Non-Market Interactions, in Hansen, L. and Turnovsky, S. (eds), Advances in Economics and Econometrics: Theory and Applications. (Cambridge, MA: Cambridge University Press).