In this online appendix, we provide the detailed proofs for the results in the paper. We also present an general equilibrium extension in the end and show that the distortion from private information leads to lower equilibrium price of capital compared to full information.

**Deriving the Lagrangian**

In this appendix, we present the derivation of the planner given in (10). First we start with the following lemma that characterizes the set of incentive compatible contracts.

**Lemma 1:** A continuous and piecewise differentiable contract $y(\theta, A), k(\theta, A), B(\theta, A)$ is incentive compatible if and only if $y$ is weakly decreasing in $\theta$ and the envelope condition is satisfied.

**Proof:**
To simplify the notations, we suppress the dependence on $A$ of the contract. First we show that incentive compatibility implies that $y(\theta)$ is decreasing in $\theta$ (it is standard to show that local incentive compatibility constraint implies Envelope Condition). Let

$$\phi(\theta', \theta) = u(\theta' + y(\theta)) + v(k(\theta)) + \beta V(A(\theta)).$$

Fixing any $\theta$, from the incentive constraint,

$$\Delta(\theta') = \phi(\theta', \theta') - \phi(\theta', \theta) \geq 0$$

and by definition of $\phi$,

$$\Delta(\theta) = 0.$$

Therefore,

$$\Delta'(\theta) = 0 \quad \text{and} \quad \Delta''(\theta) \geq 0.$$

From the definition of $\Delta(.)$, we have:

$$\Delta'(\theta') = u'(\theta' + y(\theta')) - u'(\theta' + y(\theta))$$

and

$$\Delta''(\theta') = u''(\theta' + y(\theta')) \frac{dy(\theta')}{d\theta'} + u''(\theta' + y(\theta')) - u''(\theta' + y(\theta)).$$
Since $u'' < 0$, this implies, from $\Delta''(\theta) \geq 0$,
\[
\frac{dy(\theta)}{d\theta} \leq 0.
\]

Now if $\frac{dy(\theta)}{d\theta} \leq 0$ for all $\theta \in [\bar{\theta}, \tilde{\theta}]$. We show that the incentive compatibility constraint is satisfied. Indeed, for $\theta > \theta'$:
\[
\begin{align*}
&u(\theta + y(\theta)) + v(k(\theta)) + \beta V(A(\theta)) - u(\theta + y(\theta')) - v(k(\theta')) - \beta V(A(\theta')) \\
&= \int_{\theta'}^{\theta} \left( u'(\bar{\theta} + y(\bar{\theta})) \frac{dy(\bar{\theta})}{d\bar{\theta}} + v'(k(\bar{\theta})) \frac{dk(\bar{\theta})}{d\bar{\theta}} + \beta V'(A(\bar{\theta})) \frac{dA(\bar{\theta})}{d\bar{\theta}} \right) d\bar{\theta} \\
&\geq \int_{\theta'}^{\theta} \left( u'(\tilde{\theta} + y(\tilde{\theta})) \frac{dy(\tilde{\theta})}{d\tilde{\theta}} + v'(k(\tilde{\theta})) \frac{dk(\tilde{\theta})}{d\tilde{\theta}} + \beta V'(A(\tilde{\theta})) \frac{dA(\tilde{\theta})}{d\tilde{\theta}} \right) d\tilde{\theta} \\
&= 0,
\end{align*}
\]
where the inequality comes from $0 < u'(\theta + y(\tilde{\theta})) \leq u'(\tilde{\theta} + y(\tilde{\theta}))$ and $\frac{dy(\tilde{\theta})}{d\tilde{\theta}} \leq 0$, and the last equality comes from the Envelope Condition. The proof for $\theta < \theta'$ is identical.

Lemma 1 allows us to replace the IC constraint, (4), by the Envelope Condition, (11) and the constraint that $y(\theta)$ is decreasing. We use the integral form of the Envelope Condition:
\[
\begin{align*}
&u(\theta + y(\theta)) + v(k(\theta)) + \beta V(A(\theta)) \\
&= u(\theta + y(\theta)) + v(k(\theta)) + \beta V(A(\theta)) \\
&\quad + \int_{\theta}^{\theta} u'(\tilde{\theta} + y(\tilde{\theta})) d\tilde{\theta},
\end{align*}
\]
and let $\mu(\theta)$ denote the multiplier on this constraint.

The monotonicity of $y(.)$ can be written as:
\[
y(\theta) = y(\bar{\theta}) - \int_{\theta}^{\bar{\theta}} j(\tilde{\theta}) d\tilde{\theta},
\]
where $j = -\frac{dy}{d\theta} \geq 0$. Let $\eta(\theta)$ denote the multiplier on this constraint and $\gamma(\theta)$ denote the multiplier on the positivity constraint on $j(\theta)$. 
The Lagrangian of the planner problem is then:

\[
\mathcal{L} = \int_{\theta}^{\bar{\theta}} (u(\theta + y(\theta)) + v(k(\theta)) + \beta V(A(\theta))) dF(\theta) \\
+ \int_{\theta}^{\bar{\theta}} \mu(\theta) (u(\theta + y(\theta)) + v(k(\theta)) + \beta V(A(\theta))) d\theta \\
- \left( \int_{\theta}^{\bar{\theta}} \mu(\theta) d\theta \right) (u(\theta + y(\theta)) + v(k(\theta)) + \beta V(A(\theta))) \\
- \int_{\theta}^{\bar{\theta}} \mu(\theta) \int_{\theta}^{\bar{\theta}} u'(\bar{\theta} + y(\bar{\theta})) d\bar{\theta} d\theta \\
+ \int_{\theta}^{\bar{\theta}} \eta(\theta) \left( y(\theta) - y(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} j(\bar{\theta}) d\bar{\theta} \right) d\theta + \int_{\theta}^{\bar{\theta}} \gamma(\theta) j(\theta) d\theta \\
+ \lambda \left( A - \int_{\theta}^{\bar{\theta}} \left( y(\theta) + \hat{q}k(\theta) + \frac{1}{R} A(\theta) \right) dF(\theta) \right).
\]

Using Fubini Theorem to switch the order of integrals:

\[
\int_{\theta}^{\bar{\theta}} \mu(\theta) \int_{\theta}^{\bar{\theta}} u'(\bar{\theta} + y(\bar{\theta})) d\bar{\theta} d\theta = \int_{\theta}^{\bar{\theta}} u'(\theta + y(\theta)) \left( \int_{\theta}^{\bar{\theta}} \mu(\bar{\theta}) d\bar{\theta} \right) d\theta
\]

and

\[
\int_{\theta}^{\bar{\theta}} \eta(\theta) \int_{\theta}^{\bar{\theta}} j(\bar{\theta}) d\bar{\theta} d\theta = \int_{\theta}^{\bar{\theta}} j(\theta) \left( \int_{\theta}^{\bar{\theta}} \eta(\bar{\theta}) d\bar{\theta} \right) d\theta,
\]

and plugging them into the expression for \( \mathcal{L} \), we arrive at (10).

**DISTORTION IN THE GENERAL STATIC PROBLEM**

First, we consider a more general static optimal contracting problem with two good \( c \) and \( k \) in which the utility function \( U(c, k) \) is not necessarily separable in the two goods. Instead, we assume only that it is twice continuously differentiable, strictly increasing, strictly concave, and both goods are normal goods:

\[(C1) \quad \max_{y(\cdot), k(\cdot)} \int_{\theta}^{\bar{\theta}} U(\theta + y(\theta), k(\theta)) dF(\theta) \]

subject to

\[\int_{\theta}^{\bar{\theta}} (y(\theta) + qk(\theta)) dF(\theta) \leq A\]
and

\( (C2) \quad U(\theta + y(\theta), k(\theta)) \geq U(\theta + y(\hat{\theta}), k(\hat{\theta})) \forall \theta, \hat{\theta} \in [\theta, \bar{\theta}] \).

It is standard to show that that the normality of \( c \) and \( k \) is equivalent to a weak complementarity condition:

\( (C3a) \quad \frac{\partial^2 U}{\partial c \partial k} > \max \left\{ \frac{\partial U / \partial c}{\partial U / \partial k} \frac{\partial^2 U}{\partial k^2}, \frac{\partial U / \partial k}{\partial U / \partial c} \frac{\partial^2 U}{\partial c^2} \right\}, \)

or equivalently,

\( (C3b) \quad \frac{\partial U^2}{\partial c^2} \frac{\partial U}{\partial k} - \frac{\partial^2 U}{\partial c \partial k} \frac{\partial U}{\partial c} < 0 \)

and

\( (C3c) \quad \frac{\partial U^2}{\partial k^2} \frac{\partial U}{\partial c} - \frac{\partial^2 U}{\partial c \partial k} \frac{\partial U}{\partial k} < 0. \)

The condition \( (C3b) \) is equivalent to the Strict Single Crossing Condition (SSCC) for

\[ \hat{U}(y,k,\theta) \equiv U(\theta + y, k). \]

Indeed,

\( (C3d) \quad \frac{\partial}{\partial \theta} \left( \frac{\partial U}{\partial k} \right) = \frac{\partial^2 U}{\partial c \partial k} \frac{\partial U}{\partial c} - \frac{\partial^2 U}{\partial c^2} \frac{\partial U}{\partial k} > 0. \)

The following lemma characterizes the distortion in the solution to problem \( (C1) \).

**Lemma 2:** A continuous and piecewise differential solution to problem \( (C1), y(.), k(.) \) satisfy:

(i) \( y(\theta) \) is strictly decreasing and \( k(\theta) \) is strictly increasing in a neighborhood of \( \hat{\theta} \) and

\[ \frac{\partial U}{\partial c}(\hat{\theta} + y(\hat{\theta}), k(\hat{\theta})) = q \]

(ii) For all \( \theta \in (\theta, \hat{\theta}) \),

\[ \frac{\partial U}{\partial c}(\theta + y(\theta), k(\theta)) > q. \]
To show this result, we first characterize the properties of incentive compatible contracts.

**Lemma 3**: A contract \((y, k)\) satisfies the incentive constraint \((C2)\) if and only if for \(\theta > \theta'\), we have

\[
(C4) \quad y(\theta) \leq y(\theta') \quad \text{and} \quad k(\theta) \geq k(\theta'),
\]

\[
(C5) \quad \frac{dU^-}{d\theta} (\theta + y(\theta), k(\theta)) \geq \frac{\partial U}{\partial c} (\theta + y(\theta), k(\theta)), \quad \text{and}
\]

\[
(C6) \quad \frac{dU^+}{d\theta} (\theta + y(\theta), k(\theta)) \leq \frac{\partial U}{\partial c} (\theta + y(\theta), k(\theta))
\]

where \(dU^-\) and \(dU^+\) are the left and right one-sided derivatives respectively.

**Proof:**

From the IC constraint for type \(\theta\):

\[
U(\theta + y(\theta), k(\theta)) \geq U(\theta + y(\theta'), k(\theta')),
\]

we rule out the possibility that \(y(\theta') \geq y(\theta)\) and \(k(\theta') \geq k(\theta)\), with at least one strictly inequality. Similarly from the IC constraint for type \(\theta'\), we rule out the possibility that \(y(\theta') \leq y(\theta)\) and \(k(\theta') \leq k(\theta)\), with at least one strict inequality. Therefore to obtain this lemma, we just need to eliminate the possibility that \(y(\theta) \geq y(\theta')\) and \(k(\theta) \leq k(\theta')\) with at least one strict inequality.

We show this by contradiction. Suppose that it is true. Let

\[
\tilde{U}(\tilde{\theta}, k, \theta) \equiv U(\theta - \tilde{y}, k).
\]

Then \(\tilde{U}\) satisfies the Strict Single Crossing Condition (SSCC) because it satisfies the Spence-Mirlees condition (see Milgrom and Shannon (1994) for the exact definition of these conditions):

\[
\frac{\partial}{\partial \tilde{\theta}} \left( \frac{\partial \tilde{U}}{\partial \tilde{y}} \right) = \frac{\partial}{\partial \tilde{\theta}} \left( - \frac{\partial U(\theta - \tilde{y}, k) / \partial c}{\partial U(\theta - \tilde{y}, k) / \partial k} \right) > 0,
\]

where the last inequality is equivalent to \((C3b)\). By Milgrom and Shannon (1994, Theorem 3), since

\[
(-y(\theta), k(\theta)) < (-y(\theta'), k(\theta')),
\]

and

\[
\tilde{U}(-y(\theta'), k(\theta'), \theta') \geq \tilde{U}(-y(\theta), k(\theta), \theta'),
\]

we have

\[
\tilde{U}(-y(\theta'), k(\theta'), \theta) > \tilde{U}(-y(\theta), k(\theta), \theta),
\]

is equivalent to the statement that

\[
\tilde{U}(\tilde{x}, k, \theta) < \tilde{U}(\tilde{x}', k, \theta'),
\]

for some \(\tilde{x}, \tilde{x}'\).

Therefore, \(\tilde{U}\) satisfies the Spence-Mirlees condition and the Strict Single Crossing Condition. By the results of Milgrom and Shannon (1994), we conclude that there exists a contract \((y, k)\) that satisfies the incentive constraint \((C2)\).
or equivalently
\[ U(\theta + y(\theta'), k(\theta')) > U(\theta + y(\theta), k(\theta)) , \]
which contradicts the IC constraint for \( \theta \). Therefore by contradiction, we obtain

the monotonicity property.

Now we turn to the envelope conditions. For \( \theta' < \theta \), we write the IC constraint for type \( \theta \) as

\[ U(\theta + y(\theta), k(\theta)) - U(\theta' + y(\theta'), k(\theta')) \geq U(\theta + y(\theta'), k(\theta')) - U(\theta' + y(\theta'), k(\theta')) . \]

Dividing both-side by \( \theta - \theta' \) and take the limit \( \theta' \to \theta \), we obtain

\[ \frac{dU}{d\theta}(\theta + y(\theta), k(\theta)) \geq \frac{\partial U}{\partial c}(\theta + y(\theta), k(\theta)) . \]

The second half of the envelope condition is obtained similarly by considering the IC constraint for type \( \theta' \).

Now going back to the original problem (C1), we consider a “weakly relaxed problem” in which only downward incentive compatibility and monotonicity are imposed, the multiplier \( \xi(\theta) \) on the local downward IC constraint (C5) is positive by assumption. We then show that the optimal solution to this “weakly relaxed problem” also satisfies the global incentive constraint, therefore it is also the optimal solution to the original problem.

Let \( w(\theta) \equiv U(\theta + y(\theta), k(\theta)) \). The local downward IC constraint (C5) can be written as (for simplicity we assume differentiability)

(C7) \[ w'(\theta) - \frac{\partial U}{\partial c}(\theta + y(\theta), k(\theta)) \geq 0. \]

We also require that \( k(\cdot) \) is non-decreasing in \( \theta \):

(C8) \[ k'(\theta) = i(\theta) \geq 0. \]

We denote \( \gamma(\theta) \geq 0 \) the multiplier on this constraint.
We write the Lagrangian of the "weakly relaxed problem" as

\[ L_C = \int_\theta^\beta U(\theta + y(\theta), k(\theta)) \, dF(\theta) \]

\[ + \lambda \left( A - \int_\theta^\beta (y(\theta) + qk(\theta)) \, dF(\theta) \right) \]

\[ + \int_\theta^\beta \xi(\theta) \left( w'(\theta) - \frac{\partial U}{\partial c} (\theta + y(\theta), k(\theta)) \right) d\theta \]

\[ + \int_\theta^\beta \eta(\theta) \left( k(\theta) - k(\theta) - \int_\theta^\beta i(\theta) \, d\theta \right) d\theta + \int_\theta^\beta \gamma(\theta) i(\theta) \, d\theta. \]

Because of the constraints (C7) and (C8), \( \xi \geq 0 \) and \( \gamma \geq 0 \).

Using the derivations similar to the ones in Appendix B and let \( \mu(\theta) = -\xi'(\theta) \), we rewrite \( L_C \) as

\[ L_C = \int_\theta^\beta U(\theta + y(\theta), k(\theta)) (f(\theta) + \mu(\theta)) \, d\theta \]

\[ + \lambda \left( A - \int_\theta^\beta (y(\theta) + qk(\theta)) \, d\theta \right) \]

\[ - \int_\theta^\beta \xi(\theta) \frac{\partial U}{\partial c} (\theta + y(\theta), k(\theta)) \, d\theta + \xi(\theta) w(\theta) - \xi(\theta) w(\theta) \]

\[ + \int_\theta^\beta \eta(\theta) \left( k(\theta) - k(\theta) - \int_\theta^\beta i(\theta) \, d\theta \right) \, d\theta + \int_\theta^\beta \gamma(\theta) i(\theta) \, d\theta. \]

We use shortcuts \( \frac{\partial U}{\partial c}(\theta), \frac{\partial^2 U}{\partial c^2}(\theta) \) to write the F.O.Cs as follow.

F.O.C. in \( y(\theta) \)

\[ \text{(C9)} \quad \frac{\partial U}{\partial c}(\theta) (\mu(\theta) + f(\theta)) - \lambda f(\theta) = \frac{\partial^2 U}{\partial c^2} (\theta) \xi(\theta). \]

F.O.C. in \( k(\theta) \)

\[ \text{(C10)} \quad \frac{\partial U}{\partial k}(\theta) (\mu(\theta) + f(\theta)) + \eta(\theta) - \lambda qf(\theta) = \frac{\partial^2 U}{\partial c \partial k} (\theta) \xi(\theta). \]

F.O.C. in \( i(\theta) \)

\[ \text{(C11)} \quad \gamma(\theta) - \int_\theta^\beta \eta(\theta) \, d\theta = 0. \]
F.O.C. in $k(\bar{\theta})$

\[(C12)\quad \mathcal{U}_k(\bar{\theta}) \bar{\xi}(\bar{\theta}) = 0.\]

F.O.C. in $y(\bar{\theta})$:

\[(C13)\quad \mathcal{U}_c(\bar{\theta}) \bar{\xi}(\bar{\theta}) = 0.\]

F.O.C. in $k(\theta)$:

\[(C14)\quad -\mathcal{U}_k(\bar{\theta}) \bar{\xi}(\bar{\theta}) + \int_{\bar{\theta}}^{\theta} \eta(\theta)d\theta = 0.\]

Because $\mathcal{U}_k(\bar{\theta}) > 0$, (C12) implies that $\bar{\xi}(\bar{\theta}) = 0$. Similarly, because $\mathcal{U}_c(\bar{\theta}) > 0$, (C13) implies $\bar{\xi}(\bar{\theta}) = 0$.

Combining this result with (C14) yields

\[(C15)\quad \int_{\theta}^{\bar{\theta}} \eta(\theta)d\theta = 0.\]

Therefore, from (C11),

\[(C16)\quad \gamma(\bar{\theta}) = 0.\]

Also from (C11),

\[(C17)\quad \gamma(\bar{\theta}) = 0.\]

Armed with these properties, Lemma 6 below show that in the optimal solution of the "weakly relaxed problem," the local incentive constraint is satisfied. Lemma 6 uses the following two lemmas.

**LEMMA 4:** In an optimal solution to the "weakly relaxed problem", if $k(\theta)$ is constant over some interval $[\theta_1, \theta_2] \subseteq [\bar{\theta}, \bar{\bar{\theta}}]$ then $y(\theta)$ is constant over the same interval.

**PROOF:**

Assume that $k(\theta) = k^*$ over $[\theta_1, \theta_2] \subseteq [\bar{\theta}, \bar{\bar{\theta}}]$. By downward incentive compatibility, $y(\theta)$ is non-decreasing over $[\theta_1, \theta_2]$.

We show the result in this lemma by contradiction. Assume that $y(\theta)$ is not constant over the same interval because $y$ is continuous, there exists a non-degenerate subinterval $[\theta', \theta''] \subseteq [\theta_1, \theta_2]$ such that $y(.)$ is strictly increasing over this interval.
In this interval

\[ w'(\theta) = \frac{d}{d\theta} \left( U(\theta + y(\theta), k^*) \right) \]

\[ = \frac{\partial U}{\partial c}(\theta + y(\theta), k^*) + \frac{\partial U}{\partial c}(\theta + y(\theta), k^*) \frac{dy}{d\theta} \]

\[ > \frac{\partial U}{\partial c}(\theta + y(\theta), k^*). \]

Therefore \((C7)\) does not bind, i.e., \(\xi(\theta) = 0\) for \(\theta \in [\theta', \theta'']\). Since \(\mu = -\xi'\), \(\mu(\theta) = 0\) for \(\theta \in [\theta', \theta'']\). \((C9)\) then implies

\[ \frac{\partial U}{\partial c}(\theta + y(\theta), k^*) = \lambda \]

for \(\theta \in [\theta', \theta'']\). Differentiate both sides with respect to \(\theta\), we have

\[ \frac{\partial^2 U}{\partial c^2}(\theta + y(\theta), k^*) \left(1 + \frac{dy}{d\theta}\right) = 0. \]

This is a contradiction since \(\frac{\partial^2 U}{\partial c^2} < 0\) and \(\frac{dy}{d\theta} > 0\).

**LEMMA 5:** In an optimal solution to the "weakly relaxed problem," for each \(\theta^* \in (\bar{\theta}, \bar{\theta})\), if \(\xi(\theta^*) = 0\) then \(\gamma(\theta^*) > 0\).

**PROOF:**

We show this result by contradiction. Assume that \(\gamma(\theta^*) = 0\).

We rewrite \((C9)\) as

\[ \xi'(\theta) + \frac{\partial^2 U}{\partial c^2}(\theta) \xi(\theta) = f(\theta) \left(1 - \frac{\lambda}{\frac{\partial U}{\partial c}(\theta)}\right) \]

where \(\frac{\partial U}{\partial c}(\theta), \frac{\partial^2 U}{\partial c^2}(\theta)\) are shortcuts. Using the fact that \(\xi(\theta) = 0\), we obtain

\[ \xi(\theta) = \exp \left(-\int_{\bar{\theta}}^{\theta} \frac{\partial^2 U}{\partial c^2}(\bar{\theta}) d\bar{\theta}\right) g_1(\theta), \]

where

\[ g_1(\theta) = \int_{\bar{\theta}}^{\theta} \exp \left(-\int_{\bar{\theta}}^{\theta} \frac{\partial^2 U}{\partial c^2}(\bar{\theta}) d\bar{\theta}\right) \left(1 - \frac{\lambda}{\frac{\partial U}{\partial c}(\theta)}\right) dF(\theta). \]

Because \(g \geq 0, g_1(\theta) \geq 0\) for all \(\theta \in (\bar{\theta}, \bar{\theta})\). In addition \(g_1(\theta^*) = 0\). Therefore \(\theta^*\) is
a local minimum of \( g_1 \). Therefore, \( g'_1(\theta^*) = 0 \) and \( g''_1(\theta^*) \geq 0 \). By the definition of \( g_1 \), this is equivalent to,

\[
1 - \frac{\lambda}{\frac{\partial U}{\partial c}(\theta^*)} = 0
\]

and

\[
\frac{d}{d\theta} \left( 1 - \frac{\lambda q}{\frac{\partial U}{\partial c}(\theta)} \right) \geq 0
\]

and \( \theta = \theta^* \). Equivalently, at \( \theta = \theta^* \)

\[
(C18) \quad \chi = \frac{d}{d\theta} \left\{ \frac{\partial U}{\partial c}(\theta) \right\} \geq 0.
\]

Similarly, we rewrite (C10) as

\[
\xi'(\theta) + \frac{\partial^2 U}{\partial c \partial k}(\theta) \xi(\theta) = f(\theta) \left( 1 - \frac{\lambda q}{\frac{\partial U}{\partial c}(\theta)} + \frac{\eta(\theta)}{\frac{\partial U}{\partial c}(\theta)} f(\theta) \right).
\]

Again, since \( \xi(\theta) = 0 \), we have

\[
\xi(\theta) = \exp \left( - \int_{\theta}^{\theta^*} \frac{\partial^2 U}{\partial c \partial k}(\theta) \ d\theta \right) g_2(\theta),
\]

where

\[
g_2(\theta) = \int_{\theta}^{\theta^*} \exp \left( \int_{\theta}^{\theta_1} \frac{\partial^2 U}{\partial c \partial k}(\theta) \ d\theta \right) \left( 1 - \frac{\lambda q}{\frac{\partial U}{\partial c}(\theta)} + \frac{\eta(\theta)}{\frac{\partial U}{\partial c}(\theta)} f(\theta) \right) dF(\theta_1).
\]

Because \( g \geq 0, g_2 \geq 0 \). In addition, \( g_2(\theta^*) = 0 \). Therefore \( \theta^* \) is a local minimum of \( g_2 \). Thus, \( g'_2(\theta^*) = 0 \) and \( g''_2(\theta^*) \geq 0 \). By the definition of \( g_2 \), this is equivalent to

\[
(C19) \quad 1 - \frac{\lambda q}{\frac{\partial U}{\partial c}(\theta^*)} + \frac{\eta(\theta)}{\frac{\partial U}{\partial c}(\theta^*) f(\theta^*)} = 0
\]

and at \( \theta = \theta^* \):

\[
(C20) \quad \frac{d}{d\theta} \left( 1 - \frac{\lambda q}{\frac{\partial U}{\partial c}(\theta)} + \frac{\eta(\theta)}{\frac{\partial U}{\partial c}(\theta)} f(\theta) \right) \geq 0.
\]

Notice also that \( \gamma \geq 0 \) and \( \gamma'(\theta) = -\eta(\theta) \) and \( \gamma''(\theta) = -\eta'(\theta) \). In addition
\( \gamma(\theta^*) = 0 \). Therefore \( \theta^* \) is a local minimum of \( \gamma \). Thus \( \gamma'(\theta^*) = -\eta(\theta^*) = 0 \) and \( \gamma''(\theta^*) = -\eta'(\theta^*) \geq 0 \). Plugging the first equality into (C19) implies

\[
1 - \frac{\lambda q}{\partial k(\theta^*)} = 0.
\]

Plugging the second inequality into (C20) implies

\[
\frac{d}{d\theta} \left( 1 - \frac{\lambda q}{\partial k(\theta)} + \frac{\eta(\theta)}{\partial k(\theta)f(\theta)} \right) = \frac{d}{d\theta} \left( 1 - \frac{\lambda q}{\partial c(\theta)} \right) + \frac{\eta(\theta)}{\partial k(\theta)f(\theta)} \frac{d}{d\theta} (\eta(\theta)) \geq 0.
\]

(C21)

Since \( \frac{d}{d\theta} (\eta(\theta)) \leq 0 \), the inequality above implies that \( \frac{d}{d\theta} \left( 1 - \frac{\lambda q}{\partial c(\theta)} \right) \geq 0 \) at \( \theta = \theta^* \). Therefore,

\[
\mathcal{Z} = \frac{d}{d\theta} \left\{ \frac{\partial U}{\partial k}(\theta^*) \right\} \geq 0.
\]

(C22)

However, (C18) and (C22) contradict the normality of \( c \) and \( k \).

Indeed by total differentiation

\[
\mathcal{Z} = \frac{\partial^2 U}{\partial k \partial c} \frac{dc}{d\theta} + \frac{\partial^2 U}{\partial c^2} \frac{d^2 \theta}{d\theta}
\]

and

\[
\mathcal{X} = \frac{\partial^2 U}{\partial c \partial k} \frac{dc}{d\theta} + \frac{\partial^2 U}{\partial k^2} \frac{d^2 \theta}{d\theta}
\]

So

\[
\begin{bmatrix}
\frac{dc}{d\theta} \\
\frac{dk}{d\theta}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial^2 U(c,k)}{\partial c^2} & \frac{\partial^2 U(c,k)}{\partial c \partial k} \\
\frac{\partial^2 U(c,k)}{\partial k \partial c} & \frac{\partial^2 U(c,k)}{\partial k^2}
\end{bmatrix}^{-1} \begin{bmatrix}
\mathcal{X} \\
\mathcal{Z}
\end{bmatrix}.
\]

Besides,

\[
\frac{d}{d\theta} \{ U(c(\theta),k(\theta)) \} = \left[ \frac{\partial U}{\partial c} \frac{dc}{d\theta} \right] + \left[ \frac{\partial U}{\partial k} \frac{dk}{d\theta} \right] = \frac{\partial U(c(\theta),k(\theta))}{\partial c} > 0.
\]
On the other hand

\[
\begin{bmatrix}
\frac{\partial U}{\partial c} & \frac{\partial U}{\partial k} \\
\frac{\partial^2 U(c,k)}{\partial c\partial k} & \frac{\partial^2 U(c,k)}{\partial c\partial k}
\end{bmatrix}
\begin{bmatrix}
dc \\
dk
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial U}{\partial c} & \frac{\partial U}{\partial k} \\
\frac{\partial^2 U(c,k)}{\partial c\partial k} & \frac{\partial^2 U(c,k)}{\partial c\partial k}
\end{bmatrix}^{-1}
\begin{bmatrix}
X' \\
Z
\end{bmatrix} \leq 0,
\]

since, by (C3a)

\[
\begin{bmatrix}
\frac{\partial U}{\partial c} & \frac{\partial U}{\partial k} \\
\frac{\partial^2 U(c,k)}{\partial c\partial k} & \frac{\partial^2 U(c,k)}{\partial c\partial k}
\end{bmatrix}^{-1} < 0
\]

and \( X, Z \geq 0 \). This is the desired contradiction.

Given Lemma 4 and Lemma 5, it is relatively straightforward to show the main result.

**Lemma 6:** In the optimal solution to the “weakly relaxed problem,” for all \( \theta \in (\hat{\theta}, \bar{\theta}) \),

\[
\frac{w'(\theta)}{\partial c} (\theta + y(\theta), k(\theta)) = 0.
\]

**Proof:**

We show this result by contradiction. If there exists \( \theta^* \) such that this is not true:

\[
(C23) \quad \frac{w'(\theta^*)}{\partial c} (\theta^* + y(\theta^*), k(\theta^*)) > 0.
\]

Then \( \tilde{\zeta}(\theta^*) = 0 \). By Lemma 5, \( \gamma(\theta^*) > 0 \), therefore by continuity \( \gamma(\theta) > 0 \) in some neighborhood of \( \theta^* \). So \( k(\theta) = k^* \) in this neighborhood. By Lemma 4, \( y(\theta) = y^* \) in this neighborhood. This however contradicts (C23).

We have established that the optimal solution to the “weakly relaxed problem” satisfies the local incentive compatibility constraint (envelope condition), therefore it is also an optimal solution to the original problem, (C1).\(^{32}\) This implies that the optimal solution to (C1) is characterized by the F.O.C.s (C9)-(C17) with \( \xi \geq 0 \). With this property, we are now ready to prove Lemma 2.

**Proof of Lemma 2, Part I:**

Assume by contradiction that the first property does not hold, i.e. there is pooling at the top. Let \([\theta^*, \bar{\theta}]\) denote the maximum pooling interval. If \( \theta^* > \bar{\theta} \), by the definition of \( \theta^* \), \( \gamma(\theta^*) = 0 \). If \( \theta^* = \bar{\theta} \), we also have \( \gamma(\theta^*) = \gamma(\theta) = 0 \) by (C16). Evaluating (C9) and (C10) at \( \bar{\theta} \), we have

\[
\frac{\partial U}{\partial c} (\bar{\theta}) (\mu(\bar{\theta}) + f(\bar{\theta})) - \lambda f(\bar{\theta}) = 0.
\]

\(^{32}\)It is standard to show that local incentive constraint and monotonicity implies global incentive constraint, e.g., Tirole (1988).
This implies \((\mu(\bar{\theta}) + f(\bar{\theta})) > 0\). And
\[
\frac{\partial U}{\partial k}(\theta) (\mu(\bar{\theta}) + f(\bar{\theta})) + \eta(\bar{\theta}) - \lambda q f(\theta) = 0.
\]
Since \(\gamma(\theta) = 0\) and \(\gamma(\theta) \geq 0\) for all \(\theta\), \(\eta(\bar{\theta}) = -\gamma(\bar{\theta}) \geq 0\). Therefore
\[(C24) \quad \frac{\partial U}{\partial k}(\bar{\theta}) \leq q.
\]
From \((C9)\) and \((C10)\) at \(\theta^*\)
\[
\xi(\theta^*) \left( \frac{\partial^2 U}{\partial \theta \partial c}(\theta^*) - \frac{\partial^2 U}{\partial k \partial c}(\theta^*) \right)
= \lambda f(\theta^*) \left( \frac{\partial U}{\partial k}(\theta^*) - q \frac{\partial U}{\partial c}(\theta^*) \right) + \eta(\theta^*) \frac{\partial U}{\partial c}(\theta^*).
\]
Because \(\gamma(\theta^*) = 0\) and \(\gamma(\theta) \geq 0\) for all \(\theta \geq \theta^*\), \(\eta(\theta^*) = -\gamma'(\theta^*) \leq 0\). In addition we have \(\xi(\theta^*) \geq 0\) and \((C3b)\) at \(\theta^*\), so
\[(C25) \quad \frac{\partial U}{\partial k}(\theta^*) \geq q.
\]
Combining \((C24)\) and \((C25)\), we obtain:
\[(C26) \quad \frac{\partial U}{\partial k}(\theta^*) \geq \frac{\partial U}{\partial k}(\bar{\theta}).
\]
This is a contradiction since there is pooling over \([\theta^*, \bar{\theta}]\) and therefore
\[
\frac{d}{d\bar{\theta}} \left( \frac{\partial U}{\partial k}(\theta) \right) = \frac{\partial}{\partial \bar{\theta}} \left( \frac{\partial U}{\partial k}(\theta) \right) > 0,
\]
for \(\theta \in [\theta^*, \bar{\theta}]\), which implies
\[
\frac{\partial U}{\partial k}(\theta^*) < \frac{\partial U}{\partial k}(\bar{\theta}),
\]
contradicting \((C26)\).

Therefore, the optimal contract is separating in a neighborhood of \(\bar{\theta}\). So \(\eta(\bar{\theta}) =\)
\(-\gamma'(\bar{\theta}) = 0\). In addition \(\xi(\bar{\theta}) = 0\). Equations (C9) and (C10) at \(\bar{\theta}\) then imply that

\[
\frac{\partial U}{\partial k}(\bar{\theta} + y(\bar{\theta}), k(\bar{\theta})) = q.
\]

PROOF OF LEMMA 2, PART II:

Consider \(\theta^* \in (\underline{\theta}, \bar{\theta})\). There are two cases:

Case 1: \(\gamma(\theta^*) > 0\) then there is pooling at \(\theta^*\). Let \(\theta^{**}\) denote the left most point, such that there is pooling from \(\theta^{**}\) to \(\theta^*\). Formally

\[
\theta^{**} = \inf \{ \theta \in [\underline{\theta}, \theta^*] : \gamma(\theta) > 0 \text{ for all } \theta_1 \in (\theta, \theta^*) \}.
\]

Then \(\gamma(\theta^{**}) = 0\) and \(\gamma(\theta) > 0\) for all \(\theta \in (\theta^{**}, \theta^*)\) (this comes from definition of \(\theta^{**}\) if \(\theta^{**} > \underline{\theta}\). If \(\theta^{**} = \underline{\theta}\), then this is also true since \(\gamma(\underline{\theta}) = 0\). Consequently,

\[
\gamma'(\theta^{**}) = -\eta(\theta^{**}) \geq 0.
\]

First, we show that

(C27) \[
\frac{\partial^2 U}{\partial c \partial k}(\theta^{**} + y(\theta^{**}), k(\theta^{**})) \leq q.
\]

Indeed, from (C9) and (C10) at \(\theta^{**}\)

\[
\xi(\theta^{**}) \left( \frac{\partial U}{\partial c}(\theta^{**}) \frac{\partial^2 U}{\partial c \partial k}(\theta^{**}) - \frac{\partial U}{\partial k}(\theta^{**}) \frac{\partial^2 U}{\partial c^2}(\theta^{**}) \right)
\]

\[
= \lambda f(\theta^{**}) \left( \frac{\partial U}{\partial k}(\theta^{**}) - q \frac{\partial U}{\partial c}(\theta^{**}) \right) + \eta(\theta^{**}) \frac{\partial U}{\partial c}(\theta^{**}).
\]

Together with \(\xi(\theta^{**}) \geq 0, \eta(\theta^{**}) \leq 0\) and (C3b) at \(\theta^{**}\), we obtain (C27).

Now, since there is pooling over \((\theta^{**}, \theta^*)\), and because of (C3d) we have

\[
\frac{\partial^2 U}{\partial c \partial k}(\theta^* + y(\theta^*), k(\theta^*)) \geq \frac{\partial^2 U}{\partial c \partial k}(\theta^{**} + y(\theta^{**}), k(\theta^{**})) \geq q.
\]

Case 2: \(\gamma(\theta^*) = 0\). By Lemma 5 \(\xi(\theta^*) > 0\).

Since \(\gamma(\theta^*) = 0\) and \(\gamma(\theta) \geq 0\) for all \(\theta\),

\[
\gamma'(\theta^*) = -\eta(\theta^*) \geq 0.
\]
From (C9) and (C10) at $\theta^*$

$$
\bar{\zeta}(\theta^*) \left( \frac{\partial U}{\partial c}(\theta^*) \frac{\partial^2 U}{\partial c \partial k}(\theta^*) - \frac{\partial U}{\partial k}(\theta^*) \frac{\partial^2 U}{\partial c^2}(\theta^*) \right) \\
= \lambda f(\theta^*) \left( \frac{\partial U}{\partial k}(\theta^*) - \eta(\theta^*) \frac{\partial U}{\partial c}(\theta^*) \right) + \eta(\theta^*) \frac{\partial U}{\partial c}(\theta^*).
$$

Together with $\bar{\zeta}(\theta^*) > 0$, $\eta(\theta^*) \leq 0$ and (C3b) at $\theta^*$, we obtain

$$
\frac{\partial U}{\partial k}(\theta^* + y^*(\theta^*), k(\theta^*)) > q.
$$

PROOF OF PROPOSITION 1

Part i is a direct application of the general result in Appendix C to $U(c, \tilde{A}) = u(c) + W(\tilde{A})$ as explained in the body of the paper. In this appendix, we present the proof for Part ii. We observe that

$$
\frac{d\Gamma^*(\theta)}{d\theta} = -1 - \frac{dy^*}{d\theta} - \hat{q} \frac{dk^*}{d\theta} - \frac{1}{R} \frac{dA^*}{d\theta}.
$$

From the incentive constraint,

$$
u'(\theta + y^*) \frac{dy^*}{d\theta} + v'(k^*) \frac{dk^*}{d\theta} + \beta V'(A^*) \frac{dA^*}{d\theta} = 0.
$$

From the F.O.C. in $k$ and $A'$, we have

$$v'(k^*) = \hat{q} R \beta V'(A^*).$$

Plugging this back into the incentive constraint, we obtain

$$\hat{q} \frac{dk^*}{d\theta} + \frac{1}{R} \frac{dA^*}{d\theta} = -\frac{\hat{q} u'(\theta + y^*)}{v'(k^*)} \frac{dy^*}{d\theta}.$$

Therefore

$$\frac{d\Gamma^*(\theta)}{d\theta} = -1 - \frac{dy^*}{d\theta} \left( 1 - \frac{\hat{q} u'(\theta + y^*)}{v'(k^*)} \right).$$

In Part i, we show that $1 - \frac{\hat{q} u'(\theta + y^*)}{v'(k^*)} = 0$ at $\theta = \bar{\theta}$, therefore, by continuity, $1 - \frac{\hat{q} u'(\theta + y^*)}{v'(k^*)}$ is close to 0 in a neighborhood of $\bar{\theta}$. Consequently, $\frac{d\Gamma^*(\theta)}{d\theta} < 0$ in that

---

33 Conditions (C3) are satisfied because $U$ is separable.
neighborhood.

Similarly, if the monotonicity constraint on $y^*$ does not bind at $\theta$, $1 - \frac{\tilde{q}u'(\theta + y^*)}{v'(k^*)} = 0$ at $\theta = \theta$, and $\frac{d\Gamma^*(\theta)}{d\theta} < 0$ in a neighborhood of $\theta$. If the monotonicity constraint binds, then $y^*, k^*, A^*$ are constraint in a neighborhood of $\theta$. In this case $\frac{d\Gamma^*(\theta)}{d\theta} = -1 < 0$ in that neighborhood.

**Proof of Proposition 2**

Let $H(x)$ and $G(z)$ denote the inverses of $u'$ and $v'$, respectively. First we derive the dynamics of $x$ and $z$, captured in the phase diagram, Figure 3, from the original first order conditions (12). From the definition of $x$ and $z$, we have:

$$c(\theta) = H(x(\theta))$$
$$k(\theta) = G(x(\theta)).$$

Notice that

$$u'(H(x)) = x$$

so

$$u''(H(x)) = \frac{1}{H'(x)}.$$  

Because $u$ is strictly concave $H' < 0$.

For later derivations, we also use the fact that

$$(E1) \quad H''(x) = -\frac{u'''(H(x))}{u''(H(x)) (u''(H(x)))^2} \geq 0,$$

since $u''' \geq 0$ and $u'' < 0$.

In addition, we use $\zeta'(\theta) = \int_\theta^\bar{\theta} \mu(\tilde{\theta})d\tilde{\theta}$ and, thus, $\zeta''(\theta) = -\mu(\theta)$.

With these results, we rewrite the original system (12) as

$$(E2a) \quad x(\theta) (f(\theta) - \tilde{\zeta}'(\theta)) - \lambda f(\theta) = \frac{1}{H'(x(\theta))} \xi(\theta)$$

and

$$(E2b) \quad z(\theta) (f(\theta) - \tilde{\zeta}'(\theta)) - \lambda \hat{q} f(\theta) = 0.$$  

Plugging $(f(\theta) - \tilde{\zeta}'(\theta))$ from (E2b) into (E2a), we obtain:

$$H'(x(\theta)) \left( \frac{\frac{x(\theta)}{z(\theta)}}{\hat{q} - 1} \right) \lambda f(\theta) = \xi(\theta).$$
Differentiate both sides with respect to $\theta$, we obtain:

$$
\xi'(\theta) = H'(x(\theta)) \left( \frac{x(\theta)}{z(\theta)} \hat{q} - 1 \right) \lambda f'(\theta) + H''(x(\theta)) x'('\theta) \left( \frac{x(\theta)}{z(\theta)} \hat{q} - 1 \right) \lambda f(\theta)
$$

$$
+ H'(x(\theta)) \left( \frac{x'(\theta)}{z(\theta)} \hat{q} - \frac{x(\theta) z'(\theta)}{z(\theta) z(\theta)} \hat{q} \right) \lambda f(\theta).
$$

From (E2b), we also have

$$
\xi'(\theta) = f(\theta) - \frac{\lambda \hat{q} f(\theta)}{z(\theta)}.
$$

Therefore

$$
H'(x(\theta)) \left( \frac{x(\theta)}{z(\theta)} \hat{q} - 1 \right) \lambda f'(\theta) + H''(x(\theta)) x'('\theta) \left( \frac{x(\theta)}{z(\theta)} \hat{q} - 1 \right) \lambda f(\theta)
$$

$$
+ H'(x(\theta)) \left( \frac{x'(\theta)}{z(\theta)} \hat{q} - \frac{x(\theta) z'(\theta)}{z(\theta) z(\theta)} \hat{q} \right) \lambda f(\theta)
$$

$$
= f(\theta) - \frac{\lambda \hat{q} f(\theta)}{z(\theta)}.
$$

Dividing both sides by $f(\theta)$, noting that $f'(\theta) = \psi f(\theta)$, and regrouping the terms on $x'(\theta)$ and $z'(\theta)$, we obtain

$$
\left( H''(x(\theta)) \left( \frac{x(\theta)}{z(\theta)} \hat{q} - 1 \right) + H'(x(\theta)) \frac{\hat{q}}{z(\theta)} \right) \lambda x'(\theta)
$$

$$
- H'(x(\theta)) \frac{\hat{q}}{z(\theta)} \frac{z'(\theta)}{z(\theta)} \lambda
$$

$$
= 1 - \frac{\lambda \hat{q}}{z(\theta)} - \psi H'(x(\theta)) \left( \frac{x(\theta)}{z(\theta)} \hat{q} - 1 \right) \lambda.
$$

Now the local incentive compatibility reads

$$
u'(c(\theta)) (c'(\theta) - 1) + v'(k(\theta)) k'(\theta) = 0.
$$

In terms of $x$ and $z$, we have

$$(E3) \quad x(\theta) (H'(x(\theta)) x'(\theta) - 1) + z(\theta) G'(z(\theta)) z'(\theta) = 0.
$$

Therefore

$$
x'(\theta) = \frac{x(\theta) - z(\theta) G'(z(\theta)) z'(\theta)}{x(\theta) H'(x(\theta))}.
$$
Plugging this expression for $x'(\theta)$ into the previous equation, we obtain:

\[
\left(H''(x(\theta)) \left(\frac{x(\theta)}{z(\theta)} \hat{\lambda} - 1\right) + H'(x(\theta)) \frac{\hat{\lambda}}{z(\theta)} \right) \lambda \frac{x(\theta) - z(\theta) G'(z(\theta)) z'(\theta)}{x(\theta) H'(x(\theta))} \\
- H'(x(\theta)) \frac{x(\theta)}{z(\theta)} \frac{z'(\theta)}{z(\theta)} \lambda \hat{\lambda} \\
= 1 - \frac{\lambda \hat{\lambda}}{z(\theta)} - \psi H'(x(\theta)) \left(\frac{x(\theta)}{z(\theta)} \hat{\lambda} - 1\right) \lambda.
\]

This can be simplified as:

(E4) \quad \mathcal{A}(x(\theta), z(\theta)) z'(\theta) = \mathcal{B}(x(\theta), z(\theta))

where

\[
\mathcal{A}(x, z) = - \left( H''(x) \left(\frac{\hat{\lambda} x}{z} - 1\right) + H'(x) \frac{\hat{\lambda}}{z} \right) \lambda \frac{z G'(z)}{x H'(x)} - H'(x) \frac{x}{z^2} \lambda \hat{\lambda}
\]

and

\[
\mathcal{B}(x, z) = 1 - \frac{2 \lambda \hat{\lambda}}{z} + \left(1 - \frac{\hat{\lambda} x}{z}\right) \left(\lambda \frac{H''(x)}{H'(x)} + \psi H'(x)\right).
\]

Notice that over \( z > \hat{\lambda} x \)

and because \( H'' \geq 0 \), by (E1), and \( H' < 0 \), we have

\( \mathcal{A}(x, z) > 0 \).

Using this property, we can rewrite (E4) as

\[
z'(\theta) = F^z(x, z) = \frac{\mathcal{B}(x, z)}{\mathcal{A}(x, z)}.
\]

Combining (E3) and (E4), we obtain

\[
x' = F^x(x, z) = \frac{1}{H'(x)} - \frac{z G'(z) \mathcal{B}(x, z)}{x H'(x) \mathcal{A}(x, z)}.
\]

We can rewrite the necessary conditions for \( x \) and \( z \) as

\[(x', z') = F(x, z) = (F^x(x, z), F^z(x, z)),\]
where the expressions for $F^x$ and $F^z$ are given above.

**Lemma 7:** Consider a fully separating optimal contracts, $(y(\theta), k(\theta))_{\theta \in [\hat{\theta}, \tilde{\theta}]}$. We must have

$$v'(k(\theta)) \geq \lambda \geq v'(k(\tilde{\theta})).$$

**Proof:**

Let $(x(\theta), z(\theta)) = (u'(c(\theta)), v'(k(\theta)))$. Because we assume that the contract is fully separating, we have

$$z(\theta) = \hat{q} x(\theta) \text{ when } \theta \in \{\hat{\theta}, \tilde{\theta}\},$$

and by Lemma 2, $z(\theta) > \hat{q} x(\theta)$ for all $\theta \in (\hat{\theta}, \tilde{\theta})$. Therefore, $z'(\theta) \geq \hat{q} x'(\theta)$ and $z'(\tilde{\theta}) \leq \hat{q} x'(\tilde{\theta})$. By the Intermediate Value Theorem, the exists $x^* \in [x(\hat{\theta}), x(\tilde{\theta})]$ such that at $(x^*, z^*) = (x^*, \hat{q} x^*)$, we have

$$F^z(x^*, z^*) = \hat{q} F^x(x^*, z^*).$$

Now

$$F^z(x^*, z^*) = \frac{B(x^*, z^*)}{A(x^*, z^*)} = \frac{1 - \frac{2\hat{q}}{z^*}}{-\hat{q} \lambda G'(z^*)/x^* - \lambda \hat{q} H'(x) \frac{1}{z^*}}$$

and

$$F^x(x^*, z^*) = \frac{1}{H'(x^*)} - \frac{z^* G'(z^*)}{x^* H'(x^*)} - \hat{q} \lambda G'(z^*)/x^* - \lambda \hat{q} H'(x) \frac{1}{z^*}$$

After simplifications, we arrive at:

$$\frac{z^* - \lambda \hat{q}}{z^*} = \hat{q} (\lambda - z^*) \frac{G'(z^*)}{x^* H'(x^*)}$$

which implies $z^* = \lambda \hat{q}$ and $x^* = \lambda$.

Therefore,

$$z(\theta) = \hat{q} x(\theta) \geq \hat{q} \lambda \geq \hat{q} x(\tilde{\theta}) = z(\tilde{\theta}),$$

which corresponds to the desired inequality.

Armed with these derivations, now we are ready to prove the result stated in Proposition 2.

**Proof of Proposition 2:**

First we show that there exist $\theta^*, \tilde{\theta}^*$ such that the optimal contract is fully separating and

$$v'(k^*(\theta^*)) = 2\lambda \hat{q}.$$ 

Indeed, for $z < 2\lambda \hat{q}$ (and $z > \hat{q} x$), from the expression for $B(x, z)$ above, we have
\[ B(x, z) < 0 \text{ if} \]
\[ \lambda \frac{H''(x)}{H'(x)} + \psi H'(x) \leq 0. \]

This inequality holds because of (E1) and \( H' < 0 \) and \( \psi \geq 0 \). Therefore,
\[ z' = F^z(x, z) = \frac{B(x, z)}{A(x, z)} < 0. \]

Starting from an arbitrary \( \theta^* \) and \((x(\theta^*), z(\theta^*)) = (2\lambda, 2\lambda q^*)\), and following the phase diagram - the red solid curve in Figure 3 - \( z \) is decreasing. Lengthy algebras shows that the points \((x, z) = (x, 0), x \geq 0\) are singular and repulsive, therefore, the trajectory must intersect the diagonal \( z = q x \) at some \( \bar{\theta}^* > \theta^* \) and \( x, z > 0 \), i.e.,
\[ (x(\bar{\theta}^*), z(\bar{\theta}^*)) = (x(\bar{\theta}^*), q x(\bar{\theta}^*)) > 0. \]

The trajectory from \( \theta^* \) to \( \bar{\theta}^* \) forms an optimal, fully separating contract.

In any optimal fully separating contract defined over an support \([\theta, \bar{\theta}]\), we must have \( z'(\theta) \leq 0 \) and \( z(\theta) = q x(\theta) \). From the expression for \( B(x, z) \) above, this implies:
\[ z(\theta) = v'(k(\theta)) \leq 2\lambda q. \]

In Lemma 7, we show that
\[ v'(k(\theta)) \geq \lambda q. \]

Now, for each \( z \in [\lambda q, 2\lambda q] \), let \( \Delta(z) \) denote the time (in \( \theta \)) it takes to move along the phase diagram from \( \left( \frac{z}{q}, \tilde{z} \right) \) to reach the diagonal \( z = q x \) again (at some \( \left( \frac{z}{q}, \tilde{z} \right) \) with \( \tilde{z} \leq \lambda q \)). In addition, let
\[ \bar{\Delta} = \max_{z \in [\lambda q, 2\lambda q]} \Delta(z). \]

Then for all separating optimal contracts, \( \bar{\theta} - \theta \leq \bar{\Delta} \).

**Proof of Proposition 3**

**Proof of Proposition 3 Part I:**
Let \( s = \frac{\bar{\theta} - \theta}{2\lambda (c(\bar{\theta}))} \), we rewrite (21) as:
\[ \frac{\log(1 + s)(1 + \sqrt{1 + s})}{\sqrt{s}} = 2\sqrt{2\lambda (\bar{\theta} - \theta)}. \]

After lengthy algebras, we show that the left hand side is strictly increasing in \( s \) and is equal to 0 at \( s = 0 \) and to \( \infty \) at \( s = \infty \). So there exists a unique solution \( s^* \) to this equation.
Now, from the IC constraint, we have
\[
\frac{1}{k(\theta)} k'(\theta) = \frac{1}{c(\theta)} \left( 1 - \frac{1}{4\lambda c(\theta)} \right).
\]
Therefore, in order for \( k \) to be increasing in \( \theta \), we require \( 4\lambda c(\theta) \geq 1 \). Because \( c \) is increasing in \( \theta \), this is equivalent to \( 4\lambda c(\theta) = 4\lambda qk(\theta) \geq 1 \). From the definition of \( s^* \), \( s^* \leq 8\lambda (\bar{\theta} - \theta) \). Equivalently
\[
\log \left( 1 + 8\lambda (\bar{\theta} - \theta) \right) \left( 1 + \sqrt{1 + 8\lambda (\bar{\theta} - \theta)} \right) > 2\sqrt{2\lambda (\bar{\theta} - \theta)}.
\]
After lengthy algebras, we show that
\[
\frac{\log(1 + t) (1 + \sqrt{1 + t})}{t}
\]
is strictly decreasing in \( t \), and it is equal to 2 at \( t = 0 \) and to 0 at \( t = \infty \). Therefore, the above inequality is equivalent to
\[
8\lambda (\bar{\theta} - \theta) < t^*,
\]
where \( t^* \) is uniquely determined by
\[
(F1) \quad \frac{(1 + \sqrt{1 + t}) \log(1 + t)}{t} = 1.
\]
After some algebra manipulation, it is easy to see that \( t^* = \delta - 1 \), where \( \delta \) is defined in the Proposition. The numerical value of \( t^* \) is 11.3402.

The convexity of \( k^* \) and the concavity of \( y^* \) can be obtained easily by twice differentiating the closed form expressions for (18) and (19). \( \Gamma^* \) is strictly decreasing in \( \theta \) since \( k^*, c^* \) are both strictly increasing in \( \theta \).

PROOF OF PROPOSITION 3 PART II:

As stated in the Proposition, we look for continuous allocations \( k(\cdot), y(\cdot) \) pooling over \([\theta, \bar{\theta}]\) and separating over \([\bar{\theta}, \hat{\theta}]\), together with the multipliers \( \mu(\cdot), \eta(\cdot), \gamma(\cdot) \) that satisfy the F.O.Cs (17) given in Section III.

It is actually easier to look for the solution to an alternative system which corresponds to the F.O.Cs on the Lagrangian in which the monotonicity constraint is impose on \( k(\theta) \) as in the static problem presented in Appendix C. Let \( \bar{\mu}, \bar{\eta}, \bar{\gamma} \) respectively denote the multipliers on the (integral) envelope condition, the condition relating \( k \) to \( dk/d\theta \), and the condition that \( dk/d\theta \geq 0 \). We also use
Using this alternative Lagrangian, the F.O.C in \( y(\theta) \) is:

\[
(F2a) \quad \frac{1}{2(\theta + y)} (\bar{\mu}(\theta) + \bar{f}) - \lambda \bar{f} = -\frac{1}{2(\theta + y)} \xi(\theta)
\]

and the F.O.C in \( k \) is:

\[
(F2b) \quad \frac{1}{2k} (\bar{\mu}(\theta) + \bar{f}) + \bar{\eta}(\theta) - \bar{f} \lambda \hat{q} = 0.
\]

Given \( \bar{\mu}, \bar{\eta}, \bar{\gamma} \), using direct calculations, we can verify that

\[
\mu(\theta) \equiv \bar{\mu}(\theta) + \bar{\eta}(\theta) 2k,
\]

and

\[
(F3) \quad \eta(\theta) \equiv -\bar{\eta}(\theta) \frac{k}{\theta + y} - \frac{k}{(\theta + y)^2} \int_{\theta}^{\theta} \bar{\eta}(\theta) d\theta = \frac{d}{d\theta} \left( \frac{k}{\theta + y} \int_{\theta}^{\theta} \bar{\eta}(\theta) \right).
\]

solve (17).

Now, going back to (F2). First, this system implies that \( \bar{\mu}, \bar{\eta} \) are continuous in \( \theta \). The first equation, (F2a) is equivalent to:

\[
\frac{1}{2(\theta + y)} \bar{f} - \lambda \bar{f} = \frac{d}{d\theta} \left( \frac{1}{2(\theta + y)} \xi(\theta) \right)
\]

So

\[
\frac{1}{2(\theta + y)} \xi(\theta) = \frac{\bar{f}}{2} \left( \log \left( \theta + y \right) - \log \left( \theta + y \right) \right) - \lambda \bar{f} (\theta - \theta)
\]

or

\[
(F4) \quad \xi(\theta) = \bar{f} \left( \theta + y \right) \left( \log \left( \theta + y \right) - \log \left( \theta + y \right) \right) - 2\lambda \bar{f} (\theta - \theta) \left( \theta + y \right)
\]

and

\[
\bar{\mu}(\theta) = -\bar{\xi}'(\theta) = -\bar{f} \left( \log \left( \theta + y \right) - \log \left( \theta + y \right) \right) - \bar{f} + 2\lambda \bar{f} \left( \theta + y \right) + 2\lambda \bar{f} (\theta - \theta)
\]

At \( \theta = \hat{\theta} \), \( \bar{\eta}(\hat{\theta}) = -d\bar{\gamma}(\hat{\theta}) / d\theta = 0 \), so the second equation (F2b) yields:

\[
\bar{\mu}(\hat{\theta}) + \bar{f} - 2\bar{f} \lambda \hat{q} k = 0.
\]
Therefore, from the earlier expression for $\tilde{\mu}(\theta)$:

\((F5)\quad - \log (c^*) + \log (c^* - (\hat{\theta} - \theta)) + 2\lambda c^* + 2\lambda (\hat{\theta} - \theta) - 2\lambda \hat{q}_k = 0,\)

where $c^* = y + \hat{\theta}$.

Integrating \((F2b)\) from $\theta$ to $\hat{\theta}$, we obtain

\[
\frac{1}{2\lambda} \left( \tilde{\xi}(\theta) - \tilde{\xi}(\hat{\theta}) + \tilde{f}(\hat{\theta} - \theta) \right) + \tilde{\gamma}(\theta) - \tilde{\gamma}(\hat{\theta}) - \tilde{f} \lambda \hat{q} (\hat{\theta} - \theta) = 0.
\]

Given that $\tilde{\gamma}(\hat{\theta}) = \tilde{\gamma}(\theta) = 0$ and $\tilde{\xi}(\theta) = 0$ and from the earlier expression for $\tilde{\xi}$, this is equivalent to:

\((F6)\quad - c^* (\log c^* - \log (c^* - (\hat{\theta} - \theta))) + 2\lambda (\hat{\theta} - \theta) c^* + (\hat{\theta} - \theta) - 2\lambda \hat{q}_k (\hat{\theta} - \theta) = 0\)

Lastly, using the assumption that the contract is fully separating over $[\hat{\theta}, \bar{\theta}]$ and, thus there is no distortion at the top, we obtain:

\[
\log \left( (c^*)^2 + \frac{\hat{\theta} - \bar{\theta}}{2\lambda} \right) = \log(\hat{q}_k) + 4\lambda \sqrt{(c^*)^2 + \frac{\hat{\theta} - \bar{\theta}}{2\lambda} - 4\lambda c^* + \log c^*}. \quad (F7)
\]

We show that there is a solution \((y, k, \hat{\theta})\) to \((F5)-(F7)\).

From the first two equations, \((F5)\) and \((F6)\), we have

\[
c^* - \log c^* + \log (c^* - (\hat{\theta} - \theta)) = - \log (c^*) + \log (c^* - (\hat{\theta} - \theta)) + 2\lambda c^* + 2\lambda (\hat{\theta} - \theta).
\]

Let $\zeta = \frac{\theta - \hat{\theta}}{c^*}$, this expression simplifies to:

\[
\frac{\log(1 - \zeta)}{\zeta} + 1 = \log(1 - \zeta) + 2\lambda c^* \zeta
\]

So $c^*$ is a function of $\zeta$:

\[
c^* = \hat{c}(\zeta) \equiv \frac{\log(1 - \zeta)}{\zeta} - \frac{\log(1 - \zeta) + 1}{2\lambda \zeta}.
\]
Using Taylor expansion

\[
\hat{c}(\zeta) = \frac{1}{2\lambda} \sum_{n=0}^{\infty} \frac{\zeta^n}{(n + 1)(n + 2)},
\]

which is strictly increasing in \(\zeta\).

Notice that \(\lim_{\zeta \to 0} \hat{c}(\zeta) = \frac{1}{4\lambda}\) and \(\lim_{\zeta \to 1} \hat{c}(\zeta) = \frac{1}{2\lambda}\). Because \(8\lambda(\bar{\theta} - \theta) > t^* > 4\),

\[
1 < 2\lambda(\bar{\theta} - \theta).
\]

So

\[
\lim_{\zeta \to 1} \zeta \hat{c}(\zeta) = \frac{1}{2\lambda} < (\bar{\theta} - \theta),
\]

which implies at \(\zeta = 1\), \(\hat{\theta} = \bar{\theta} + \lim_{\zeta \to 1} \zeta \hat{c}(\zeta) < \bar{\theta}\).

Solving for \(q_k\) from (F5) and (F7), we obtain another equation:

\[
\left( (c^*)^2 + \frac{\tilde{\theta} - \theta}{2\lambda} - \frac{c^* \zeta}{2\lambda} \right) \exp \left( -4\lambda \sqrt{(c^*)^2 + \frac{\tilde{\theta} - \theta}{2\lambda} - \frac{c^* \zeta}{2\lambda}} \right) \frac{\exp(4\lambda c^*)}{c^*} = \log \left( \frac{2\lambda}{1 - \zeta} \right) + c^* + c^* \zeta.
\]

From the earlier expression for \(c^*\), (F8), this corresponds to one equation, and one unknown in \(\zeta\). Let \(\Delta(\zeta)\) denote the difference between the RHS and LHS of this equation.

We will show that \(\Delta(0) < 0\) and \(\Delta(1^-) > 0\).

It is easy to show that \(\lim_{\zeta \to 1} \Delta(\zeta) = +\infty > 0\), since \(\lim_{\zeta \to 1} \hat{c}(\zeta) = \frac{1}{2\lambda} < \infty\) and \(\lim_{\zeta \to 1} \log(1 - \zeta) = -\infty\).

Now at \(\zeta = 0\), \(\hat{c}(0) = \frac{1}{4\lambda}\), so

\[
\Delta(0) = \left( (\hat{c}(0))^2 + \frac{\tilde{\theta} - \theta}{2\lambda} \right) \exp \left( -4\lambda \sqrt{(\hat{c}(0))^2 + \frac{\tilde{\theta} - \theta}{2\lambda}} \right) \frac{\exp(4\lambda \hat{c}(0))}{\hat{c}(0)} - \hat{c}(0).
\]

Let \(\tilde{s} = \frac{\tilde{\theta} - \theta}{2\lambda (\hat{c}(0))^2}\), after lengthy algebra, we desired inequality \(\Delta(0) < 0\) as:

\[
\frac{\log (1 + \tilde{s}) (1 + \sqrt{1 + \tilde{s}})}{\sqrt{\tilde{s}}} < 2\sqrt{\frac{2\lambda}{(\tilde{\theta} - \theta)}}.
\]

Consider the solution to the relaxed problem in Part 1 of this Proposition, \(c_R(\theta)\).
and \( k_R(\theta) \) (\( k_R \) might not be increasing). At \( \theta \),

\[
\frac{\bar{\theta} - \theta}{2\lambda (c_R(\bar{\theta}))^2} = S^* = \bar{\theta} - \theta
\]

and

\[
\frac{\log(1 + S^*) (1 + \sqrt{1 + S^*})}{\sqrt{S^*}} = 2\sqrt{2\lambda (\bar{\theta} - \theta)}.
\]

As shown in the first part of this Proposition, the RHS of (F10) is strictly increasing in \( \tilde{s} \). Therefore (F10) is equivalent to

\[
\tilde{s} < S^*
\]

or, by definition of \( \tilde{s} \) and \( S^* \),

\[
\hat{c}(0) = \frac{1}{4\lambda} > c_R(\theta).
\]

Indeed this is the case since \( 8\lambda (\bar{\theta} - \theta) > t^* \).

So there exists \( \zeta^* \in (0, 1) \) such that \( \Delta(\zeta^*) = 0 \). We determine \( \tilde{\theta}, \tilde{y} \) and \( \tilde{k} \) as

\[
\tilde{\theta} = \hat{c}(\zeta^*)\zeta^* + \tilde{\theta}, \tilde{y} = \hat{c}(\zeta^*) - \tilde{\theta} \text{ and } \tilde{k} \text{ is a function of } c^* \text{ and } \zeta^* \text{ as in either (F5) or (F7).}
\]

It is easy to verify that \( (\tilde{\theta}, \tilde{y}, \tilde{k}) \) solves (F5)-(F7).

We can also verify that for all \( \theta \in (\bar{\theta}, \bar{\theta}) \):

\[
\tilde{\gamma}(\theta) = \frac{1}{2\tilde{k}}(-\tilde{\gamma}(\theta) + \tilde{f}(\theta - \tilde{\theta})) - \tilde{f}\lambda \tilde{q}(\theta - \tilde{\theta}) > 0.
\]

We first show by contradiction that there is no local minimum of \( \tilde{\gamma} \) in \((\bar{\theta}, \bar{\theta})\).

Assume by contradiction that there exists a local minimum \( \tilde{\theta} \in (\bar{\theta}, \bar{\theta}) \). Then \( \tilde{\gamma}'(\tilde{\theta}) = 0 \) and \( \tilde{\gamma}''(\tilde{\theta}) \geq 0 \).

From the expression for \( \tilde{\gamma} \):

\[
\tilde{\gamma}''(\theta) = -\frac{1}{2\tilde{k}}\tilde{\gamma}''(\theta) = -\frac{1}{2\tilde{k}}\tilde{f} \left( \frac{1}{\theta + \tilde{\gamma}} - \lambda \tilde{q} \right)
\]

is increasing in \( \theta \). Therefore \( \tilde{\gamma}''(\theta) > \tilde{\gamma}''(\tilde{\theta}) \geq 0 \) for \( \theta \in (\bar{\theta}, \bar{\theta}) \). From the definition of \( \tilde{\theta} \), \( \tilde{\gamma}'(\tilde{\theta}) = -\hat{q}(\tilde{\theta}) = 0 \). So \( \tilde{\gamma}'(\tilde{\theta}) < 0 \) which contradicts the property that \( \tilde{\gamma}'(\tilde{\theta}) = 0 \). So \( \tilde{\gamma} \) does not have a local minimum in \((\bar{\theta}, \bar{\theta})\).

At \( \theta \), \( \tilde{\gamma}(\theta) = 0 \) and by construction, at \( \tilde{\theta} \), \( \tilde{\gamma}(\tilde{\theta}) = 0 \). So \( \tilde{\gamma} \geq 0 \) for all \( \theta \in [\bar{\theta}, \tilde{\theta}] \). In addition, \( \tilde{\gamma}(\theta) > 0 \) for all \( \theta \in (\bar{\theta}, \tilde{\theta}) \) (otherwise, \( \tilde{\gamma} \) would have a local minimum
From the construction of $\eta(\theta)$ in (F3), we also have
\[
\gamma(\theta) = -\int_{\hat{\theta}}^{\theta} \eta(\tilde{\theta}) d\tilde{\theta} = \frac{k}{\theta + y} \hat{\gamma}(\theta) > 0.
\]

Given $\hat{\theta}, y, k$, we determine the allocation over $[\hat{\theta}, \bar{\theta}]$ using the derivation in Section III:
\[
c(\theta) = \sqrt{(c^*)^2 + \frac{\theta - \hat{\theta}}{2\lambda}}
\]
and
\[
\log(k(\theta)) = \log(k) + 4\lambda \sqrt{(c^*)^2 + \frac{\theta - \hat{\theta}}{2\lambda}} - 4\lambda c^* - \frac{1}{2} \log \left( (c^*)^2 + \frac{\theta - \hat{\theta}}{2\lambda} \right) + \log(c^*).
\]

Since $c^* = \ell(\zeta^*) > \ell(0) = \frac{1}{4\lambda}$, $k(.)$ is strictly increasing over $(\hat{\theta}, \bar{\theta})$.

All the first order conditions are satisfied, therefore $\{y(.), k(.)\}$ as constructed above is an optimal contract. The results on the shape of the allocation and distortion are derived similarly to the fully separating case in Part i.

**General Equilibrium**

We have shown in Subsection II.A and Appendix C that the capital good in the optimal collateralized contract is relatively under-consumed. This holds for a fixed exogenous price $q$. In a general equilibrium model with large numbers of lenders and borrowers, $q$ is endogenously determined. We show in a weighted log model that $q$ may be lower than the full information price. Markets, in essence, partially compensate for the distortion. We use the explicit solution for the log-log uniform model in Section III to compare the variation of capital prices in the two models across income.

To simplify the analysis we focus on the static model. Consumers are endowed with one unit of capital which, for concreteness, we refer to as housing. The banking sector is perfectly competitive. Banks offer contracts to the consumers in exchange for consumers’ deposits before income shocks are realized. Banks maximize profit subject to an endogenous outside options of the consumers. Both consumers and banks have access to a market for housing at unit price
Each bank solves
\[ \Pi(q, U) = \max_{\{y, k\}} \left\{ q \cdot 1 - \int_{\theta}^{\bar{\theta}} (y(\theta) + qk(\theta)) dF(\theta) \right\} \]
subject to
\[ \int_{\theta}^{\bar{\theta}} U(\theta + y(\theta), k(\theta)) dF(\theta) \geq U, \]
and
\[ U(\theta + y(\theta), k(\theta)) \geq U(\theta + y(\theta'), k(\theta')), \]
for all \( \theta, \theta' \in [\underline{\theta}, \bar{\theta}] \).

We define a competitive equilibrium as following.\textsuperscript{34}

**Definition.** A competitive equilibrium is a price \( q \), an endogenous outside option \( U \) and an optimal contract \( \{y^*(\theta), k^*(\theta)\}_{\theta \in [\underline{\theta}, \bar{\theta}]} \) such that at this optimal contract, banks make zero profit
\[ \Pi(q, U) = 0 \]
and the housing and composite good markets clear: \( \int_{\theta}^{\bar{\theta}} k^*(\theta) dF(\theta) = 1 \), and \( \int_{\theta}^{\bar{\theta}} y^*(\theta) dF(\theta) = 0 \).

It is easy to show that the outside option and the equilibrium optimal contracts solves
\[ U = \max_{\{y, k\}} \int_{\theta}^{\bar{\theta}} U(\theta + y(\theta), k(\theta)) dF(\theta) \]
subject to
\[ q = \int_{\theta}^{\bar{\theta}} (y(\theta) + qk(\theta)) dF(\theta) \]
and
\[ U(\theta + y(\theta), k(\theta)) \geq U(\theta + y(\theta'), k(\theta')). \]

The following proposition shows that in general equilibrium the distortion towards under consumption of housing shown in Lemma 2 leads to lower house price.

**PROPOSITION 4:** Assume that \( U(c, k) = \alpha \log(c) + (1 - \alpha) \log(k) \). In general equilibrium under asymmetric information the house price is strictly less than the equilibrium house price under full information.

**PROOF:**

\textsuperscript{34}The competitive equilibrium concept here is similar to the “self-confirming policy equilibrium” concept developed in Rothschild and Scheuer (2013, 2016) in which the optimal incentive compatible Mirleesian tax policies are designed taking wages as given, and in equilibrium wages are determined to clear the labor markets.
In the competitive equilibrium under full information

$$\frac{1-\alpha}{\alpha}(\theta + y^F(\theta)) = q^F k^F(\theta)$$

for all $\theta$. Integrating this equality from $\hat{\theta}$ to $\tilde{\theta}$ and using the market clearing conditions, we arrive at

$$q^F = \frac{1-\alpha}{\alpha} E[\theta].$$

However, under asymmetric information, by the distortion result in Lemma 2

$$\frac{1-\alpha}{\alpha}(\theta + y^*(\theta)) > q^* k^*(\theta)$$

for all $\theta \in (\hat{\theta}, \tilde{\theta})$. After integrating this inequality from $\hat{\theta}$ to $\tilde{\theta}$ and using the market clearing conditions, we obtain

$$q^* < \frac{1-\alpha}{\alpha} E[\theta] = q^F.$$

When $\alpha = 0.5$ and the distribution of types is uniform, Section III provides a closed form solution to the optimal contract given any house price. The following example uses this closed form solution to calculate numerically the house price as a function of the mean income.

Example 3 (General Equilibrium): Consider the case with log-log utility function with equal weight on composite goods and housing and uniform distribution as in Section III. Figure G1 shows the competitive equilibrium house price, compared to the full information competitive equilibrium price, when we vary $E[\theta]$ while keeping $\tilde{\theta} - \hat{\theta} = 2$. We observe that the competitive equilibrium price always lie below the full information price and the difference between the two prices is larger at lower levels of $E[\theta]$. This result might speak to the low house price during the last financial crisis 2007-2009 and the subsequent recession.
Figure G1. General Equilibrium