Online Appendix

A Bargaining-Based Model of Security Design

Matan Tsur

1 Appendix A

In this section, we characterize the equilibrium of a sequential bargaining game and show that our main results do not depend on the details of the bargaining model.

Bilateral Bargaining

A firm with a single project negotiates with a single buyer. We consider the sequential bargaining game of Rubinstein (1982). The firm and the buyer alternate offers in the usual way and following a rejected offer, the game continues with probability \( p < 1 \) and a breakdown occurs with probability \( 1 - p \). The game ends when either an offer is accepted or a breakdown occurs. The payoffs are given by \( G(S) \), and the bargaining outcomes \( \mathcal{F}(G(S)) \) are the subgame perfect equilibrium agreements when the probability of a breakdown vanishes.\(^1\)

Lemma 1. In any outcome in \( \mathcal{F}(G(S)) \) the firm’s payoff is no more than the buyer’s payoff.

Proof. Assume without loss of generality that \( x - S(x) \) is weakly increasing. Let \( \bar{x}_F \) and \( \bar{x}_B \) be the maximal SPE prices offered by the firm and the buyer, respectively. An equilibrium price \( x \) offered by the buyer satisfies \( x - S(x) \leq p(\bar{x}_F - S(\bar{x}_F)) - (1 - p)S(0) \) and a price accepted by the buyer satisfies \( p(v - \bar{x}_B) \leq v - x \). Since \( S \) is positive, weakly increasing, and \( \bar{x}_B \leq \bar{x}_F \), we have that \( \bar{x}_B \leq p \bar{x}_F + (1 - p)S(\bar{x}_B) \) and \( \bar{x}_F \leq (1 - p)v + p \bar{x}_B \). These imply that \( p(\bar{x}_F - S(\bar{x}_F)) \leq v - \bar{x}_F \) and\(^2\) \( \bar{x}_B - S(\bar{x}_B) \leq p(v - \bar{x}_B) \). If player \( i = B \), \( F \) proposes, the firm’s equilibrium payoff will be at most \( \bar{x}_i - S(\bar{x}_i) \), and the buyer’s at least \( v - \bar{x}_i \). \( \square \)

\(^1\)We let \( p \to 1 \) in order to eliminate a first-move advantage; the analysis trivially extends to any \( p > 0 \).

\(^2\)To see it, we inject the latter inequality, \( \bar{x}_F \leq (1 - p)v + p \bar{x}_B \), into the former, \( \bar{x}_B \leq p \bar{x}_F + (1 - p)S(\bar{x}_B) \).
It immediately follows from Lemma 3 that a defaultable debt contract is optimal. If investors get at least \( c \), no more than \( v - c \) is left, and the firm’s maximal payoff is therefore \( \frac{1}{2}(v - c) \). Finally, the bargaining game with debt \( S(x) = \min \{ x, D \} \) and \( D = c \), has a subgame perfect equilibrium that achieves the firm’s maximal payoff, and investors break even.\(^3\)

**Remark 4.** The sequential bargaining game of Rubinstein (1982) generally has a subgame perfect equilibrium that is close to (and when offers are frequent converges to) the Nash solution (see, e.g., Herrero 1989). But when the payoff set is not convex, the bargaining game may also have outcomes that do not converge to the Nash solution. For example, the security \( S(x) = \min \{ \lambda x, D \} \), with \( \lambda < 1 \), is natural to consider: it specifies that only a fraction \( \lambda \) of the proceeds go towards repaying the debt and the rest goes to the firm. If \( \frac{v}{2} < D < v \), then for \( \lambda < 1 \) sufficiently large both \( x_1 = \frac{v}{2} \) and \( x_2 = \frac{v + D}{2} \) can be supported as an SPE of the corresponding bargaining game, but the former does not maximize the Nash product.\(^4\)

**Remark 5.** The sequential bargaining game with debt \( S(x) = \min \{ x, D \} \) has a class of degenerate equilibria with low prices \( x < D \). These equilibria arise because the firm gets nothing from agreements that fall short of the debt, and is therefore indifferent over the range of low prices. However, these outcomes will not survive simple refinements (such as trembling hand). If degenerate outcomes are ruled out, whether by refinements or by an indifference and so \( x_B \leq p((1-p)v + p x_B) + (1-p)S(x_B) \Rightarrow x_B - S(x_B) \leq p(v - x_B) \). Likewise, we can inject the former inequality, \( x_B \leq p x_F + (1-p)S(x_B) \leq p x_F + (1-p)S(x_F) \), into the latter \( x_F \leq (1-p)v + p x_B \), and so \( x_F \leq (1-p)v + p (p x_F + (1-p)S(x_F)) \Rightarrow p(x_F - S(x_F)) \leq v - x_F \).

\(^3\)The strategies are stationary: the buyer and the firm always offer prices \( x_B = D + \frac{p}{1-p}(v-D) \) and \( x_F = D + \frac{1}{1-p}(v-D) \), and on the assumption that an offer will be accepted in the next period, each player accepts offers only if the payoff is no less than the continuation payoff. As \( p \to 1 \), the prices converge to \( x = \frac{v + D}{2} \), which is the unique Nash solution. When \( D = c \), investors break even and the firm and the buyer split the social surplus, each getting \( \frac{1}{2}(v - c) \).

\(^4\)See Herrero (1989) for more details on the relationship between the axiomatic and the strategic models in a bargaining problem with a non-convex payoff set. Also, a technical method for convexifying the payoff set is to allow for randomization. But this is highly unrealistic in our setting and completely changes the interpretation and trade-offs of the model. For instance, under randomization, a bargaining game with debt is equivalent to a bargaining game with equity.
assumption,\textsuperscript{5} then the strategic bargaining game with debt has a unique equilibrium that converges to the price $x = \frac{v + D}{2}$, which is the unique Nash solution.

**Multilateral Bargaining**

We now consider a multilateral bargaining game. Let us first mention well-known results in bargaining theory. Several papers provide a strategic foundation for bilateral stability (or the stronger property of consistency) in different settings (e.g., Krishna and Serrano (1996), Hart and Mas-Colell (1996), Collard-Wexler et al. (2014)). That is, each paper specifies a sequential bargaining game and shows that the equilibrium outcomes converge, as offers become frequent, to the bilaterally stable (or consistent) outcomes. The latter two papers use equilibrium refinements.

However, applying these results to our model is not straightforward because the bargaining games are different. The feasible payoff sets in our model are not necessarily convex, and hence bilateral bargaining games may have multiple equilibria. The same is true for multilateral bargaining games. The proofs of the above-mentioned results are inductive, relying heavily on convexity to guarantee a unique equilibrium in bilateral bargaining games. Thus the proof techniques do not apply to our model. In what follows we consider a sequential bargaining game and, using refinements, characterize the equilibrium for all concave securities.

**The Bargaining Procedure.** The buyers arrive at the same time and the procedure goes as follows. The firm makes offers simultaneously and privately to each buyer. Each buyer then simultaneously and independently either accept or reject her offer. Any buyer who accepts pays and leaves. The game continues with the remaining buyers. These buyers simultaneously and independently make offers to the firm which chooses which offers, if any, to accept. There is a probability of a breakdown between rounds: following any rejected

\textsuperscript{5}It is reasonable to assume that the firm would rather wait than reach an agreement from which it gains nothing.
offer, the game continues with probability \( p \) and a breakdown occurs with probability \( 1 - p \). The game ends when all the goods are sold or a breakdown occurs. When the game ends, the proceeds from the sales are divided with the investors. The payoffs are given by \( G(S) \) and \( G(S_1, \ldots, S_N) \). Past transactions are observable but offers are private. The solution concept is Perfect Bayesian Equilibrium (Fudenberg and Tirole (1991)).

**Equilibrium Refinements.** For tractability, we will use equilibrium refinements: 1) strategies are *stationary* in that actions depend only on the payoff-relevant state variables: the revenues from previous agreements, and the buyers who have not reached agreements; and 2) the off-equilibrium beliefs are restricted. Since offers are private, each buyer who receives an off-equilibrium offer makes conjectures about the offers made to the other players and the off-equilibrium beliefs can have a large effect on the continuation game generating multiplicity. A buyer has *passive beliefs* if she believes that the other buyers received their equilibrium price. The need for such refinements is common when agreements among pairs of players affect the payoffs of the other players (for instance in vertical contracting when there are several downstream firms; see e.g., McAfee and Schwartz (1994)).

Let the bargaining outcomes \( \hat{F}(G(S)) \) be the agreements that are a limit, as \( p \to 1 \), of \( PBE \) agreements with passive beliefs and stationary strategies.

**Proposition 1.** If the bargaining solution is \( \hat{F} \), financing each project with debt separately is optimal within the class of concave securities.

The idea of the proof is to bound the equilibrium payoffs of the large game using only the outcomes of bilateral subgames. We can then apply Lemmata 2 and 3 to bound the equilibrium payoffs.

Let \( S \) be a continuous and concave security. To characterize the equilibrium payoffs of the multilateral bargaining game, we assume without loss of generality that \( Y - S(Y) \) is weakly
increasing.\footnote{For any security that does not satisfy this restriction, there exists a security that does satisfy it and implements the same equilibrium outcomes.} Let $x^F_j(Y, p) \text{ and } x^B_j(Y, p)$ be the maximal SPE prices in a subgame game where only buyer $j$ remains, the firm or buyer makes the first offer, the previous proceeds are $Y$, and the continuation probability is $p < 1$. The functions are well defined (see Lemma 5 below).

**Lemma 2.** Let $\epsilon > 0$. If the agreements $y_1, \ldots, y_N$ are achieved in an equilibrium with passive beliefs and stationary strategies of the multilateral bargaining game, then $x^F_j(Y_{-j}, p) > y_j - \epsilon$, where $Y_{-j} = \sum_{i \neq j}^N y_i$, $\forall j$, provided that $p < 1$ is sufficiently large.

**Proof.** Let $Y = \sum_{i=1}^N y_i$ and $Y_{-j} = \sum_{i \neq j}^N y_i$. To ease notation, we write $x^B_j(Y_{-j})$ and $x^F_j(Y_{-j})$ instead of $x^B_j(Y_{-j}, p)$ and $x^F_j(Y_{-j}, p)$; this should not cause confusion. We focus on equilibria with positive profits,\footnote{The proof can be extended to the case where $Y - S(Y) = 0$, but these equilibria (if they exist) are irrelevant because we are looking for an upper bound on the firm’s payoffs.} $Y - S(Y) > 0$.

In an equilibrium with passive beliefs, the firm’s offers are accepted without delay. Therefore, if the firm offers $y_1, \ldots, y_N$, since buyer $j$ can reject the offer and guarantee a payoff of at least $p \left( v_j - x^B_j(Y_{-j}) \right)$, it must be that

$$v_j - y_j \geq p \left( v_j - x^B_j(Y_{-j}) \right) = v_j - x^F_j(Y_{-j}) \implies x^F_j(Y_{-j}) \geq y_j$$

The equality follows from standard arguments: the maximal prices are achieved without delay; $p(v - x^B_j(Y)) \geq v_j - x^F_j(Y)$ because the buyer will not accept a higher price; $v_j - x^F_j(Y) \geq p(v - x^B_j(Y))$ because otherwise $x^F(Y)$ is not maximal.

When the buyers propose, the argument is more intricate. Suppose that buyer $j$’s offer is accepted.\footnote{If buyer $j$’s offer is rejected, then by the previous step $x^F_j(Y_{-j}) \geq y_j$ and we are done.} Following a deviation by buyer $j$ to a lower price, the firm will reject this and possibly other offers. Let us assume that offers $y_1, \ldots, y_K$ are accepted and the others are rejected, where $j > K$. From the previous step, the game will end in the next period with the prices $y'_{K+1}, \ldots, y'_N$. The firm’s payoff is $(1 - p) \left( Y_K - S(Y_K) \right) + p \left( Y' - S(Y') \right)$, where
$Y_K = \sum_{i=1}^{K} y_i$ and $Y' = Y_K + \sum_{i=K+1}^{N} y'_i$, and the equilibrium conditions imply

$$v_j - y_j \geq p(v_j - y'_j)$$  \hspace{1cm} (1.1)$$

$$Y' \geq Y - \epsilon,$$  \hspace{1cm} (1.2)

where (1.2) holds for $p$ sufficiently large.\(^9\) If either condition (1.1) or (1.2) does not hold, there is a profitable deviation for buyer $j$.

We let $Y'_{-j} = Y' - y'_j$ and there are two cases to consider: the more difficult one is when $Y_{-j} > Y'_{-j}$. The key step is given in Lemma 5, which proves that all bilateral subgames satisfy a basic monotonicity property: the firm’s payoff $Y + x^F_j(Y)$ is increasing. Therefore, in this case,

$$Y_{-j} + x^F_j(Y_{-j}) > Y'_{-j} + x^F_j(Y'_{-j}) \geq Y' > Y - \epsilon \implies x^F_j(Y_{-j}) > y_j - \epsilon$$

where the weak inequality is true because $x^F_j(Y'_{-j}) \geq y'_j$ (previous step) and the third inequality is (1.2).

For the second case, suppose that $Y'_{-j} \geq Y_{-j}$. From the previous step $x^F_j(Y'_{-j}) \geq y'_j$ and (1.1), we get

$$px^F_j(Y'_{-j}) + (1 - p)v_j \geq y_j$$  \hspace{1cm} (1.3)

The function $x^F_j(Y)$ is weakly decreasing for many securities.\(^10\) Observe that $x^F_j(Y_{-j}) \geq x^F_j(Y'_{-j})$ and (1.3) implies $px^F_j(Y_{-j}) + (1 - p)v_j \geq y_j$, and we are done. But for some securities, the price $x_j(Y)$ may increase with $Y$. For the case where $x^F_j(Y'_{-j}) > x^F_j(Y_{-j})$, we will show that the difference $x^F_j(Y'_{-j}) - x^F_j(Y_{-j})$ is arbitrarily small for $p < 1$ sufficiently large.

The argument relies on stationarity. Given $\delta > 0$, it must be that $Y'_{-j} - Y_{-j} < \delta$ for $p < 1$ sufficiently large. Otherwise, (1.1) implies $Y' - \delta + (1 - p)v_j > Y$ and since strategies are station-

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\(^9\)If the buyers’ offers are accepted immediately, then $Y' > Y$, for all $p < 1$, because $(1 - p) (Y_K - S(Y_K)) + p (Y' - S(Y')) \geq Y - S(Y)$.

\(^10\)For example, with debt $x^F_j(Y) = \max \left\{ \frac{v_j}{1+p}, \frac{v_j + p(D - Y)}{1+p} \right\}$ is weakly decreasing with $Y$. 

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ary, the firm can deviate and reject $K$ offers and get \((1 - p) (Y_K - S(Y_K)) + p (Y' - S(Y'))\), which is profitable when $p$ is sufficiently large. The function $x^F_j(Y)$ is continuous (see Lemma 5) and therefore $x^F_j(Y'_j) - x^F_j(Y_{-j}) < \frac{1}{p}$ for $p < 1$ sufficiently large. Finally, condition (1.3) implies $px^B_j(Y_{-j}) + \frac{p^2}{2} + (1 - p)v_j \geq y_j$.

**Lemma 3.** For $Y \in [0, V - v_j]$, the functions $x^F_j(Y, p), x^B_j(Y, p)$ are well defined, continuous in $Y$, and $Y + x^B_j(Y, p)$ and $Y + x^F_j(Y, p)$ are (strictly) increasing in $Y$.

**Proof.** Let $p < 1$ and to ease the notation, we write $x^B_j(Y_{-j})$ and $x^F_j(Y_{-j})$ instead of $x^B_j(Y_{-j}, p)$ and $x^F_j(Y_{-j}, p)$.

Let $x_0(Y) = \inf\{x : x - S(Y + x) + S(Y) > 0\}$. Since $S$ is concave, $x_0(Y)$ is decreasing, and the firm’s payoff, $Y + x - S(Y + x)$, is strictly increasing in $x$ when $x > x_0(Y)$. Consider the subgame where only buyer $j$ remains and the previous proceeds are $Y$. First, for $Y$ sufficiently large, $v_j > x_0(Y)$, and hence there are gains from trade in this subgame because there exists an agreement such that $v_j - x > 0$ and $Y + x - S(Y + x) > Y - S(Y)$. We now show, by construction, that there exists an $SPE$ in this subgame. We can find a pair $x_B, x_F$, where $v_j > x_F > x_B > x_0(Y)$, that solve

\[
\begin{align*}
v_j - x_F &= p(v_j - x_B) \quad \text{(1.4)} \\
Y + x_B - S(Y + x_B) &= (1 - p) (Y - S(Y)) + p (Y + x_F - S(Y + x_F)) \quad \text{(1.5)}
\end{align*}
\]

To see this, let

\[
g(x, Y) = x - S(Y + x) + S(Y) - p[px + (1 - p)v_j - S(Y + px + (1 - p)v) + S(Y)]
\]

Since $g(v_j, Y) > 0 > g(x_0(Y), Y)$, there exists a price $x_B \in (x_0(Y), v_j)$ such that $g(x_B, Y) = 0$ and the prices $x_B$ and $x_F = (1 - p)v + px_B$ satisfy (1.4) and (1.5). It is straightforward to construct an $SPE$ that supports these prices.\(^{11}\) The set of equilibrium outcomes is non-

\(^{11}\)The strategies are stationary: the buyer and the firm always offer prices $x_B$ and $x_F$, and on the assumption that an offer will be accepted in the next period, each player accepts offers only if the payoff is no less than the continuation payoff.
empty, closed, and bounded and the maximal prices \(x^B_j(Y)\) and \(x^F_j(Y)\) exist.

Moreover, the maximal \(SPE\) prices \(x^B_j(Y)\) and \(x^F_j(Y)\) satisfy conditions (1.4) and (1.5), the argument is standard: the maximal prices are achieved without delay; \(v_j - x^F_j(Y) \geq p(v - x^B_j(Y))\) because otherwise \(x_F(Y)\) is not maximal; and \(p(v - x^B_j(Y)) \geq v_j - x^F_j(Y)\) because the buyer will not accept a higher price. A similar argument establishes (1.5) (note that the firm’s payoff \(Y + x - S(Y + x)\) is (strictly) increasing in \(x\) for \(x > x_0(Y)\)). Therefore, \(v_j - x^F_j(Y) = p(v_j - x^B_j(Y))\) and \(x^B_j(Y) = \max \{x : g(x, Y) = 0\text{ and } x \leq v_j\}\).

The function \(x^B_j(Y)\) is continuous by the maximum theorem. To show that \(Y + x^B_j(Y)\) is (strictly) increasing, we will assume that \(S\) is a smooth function. There is no loss of generality here. In the case that the security \(S\) has kinks, we can find a smooth security \(\hat{S}\) so that the subgame perfect equilibria agreements in the bargaining games with the securities \(S\) and \(\hat{S}\) are sufficiently close. Observe that \(g(x^B_j(Y), Y) = 0\) and in an \(\epsilon\)-neighborhood of \((x^B_j(Y), Y)\),

\[
\frac{\partial g}{\partial x} > \frac{\partial g}{\partial Y}
\]  

(1.6)

To see this, \(\frac{\partial g}{\partial x} > \frac{\partial g}{\partial Y} \iff 1 + p > S'(Y) + pS'(Y + px + (1 - p)v)\), which is true for \(\epsilon\) sufficiently small because \(1 > S'(Y + x^B_j(Y))\). Additionally,

\[
\frac{\partial g}{\partial x} \bigg|_{x=x^B_j(Y)} \geq 0
\]  

(1.7)

Otherwise, \(x^B_j(Y)\) is not the maximal solution to \(g(x, Y) = 0\) because \(g(v_j, Y) > 0\).

Suppose first that (1.7) holds with equality, i.e., \(\frac{\partial g}{\partial x} \bigg|_{x=x^B_j(Y)} = 0\), then (1.6) implies that \(\frac{\partial g}{\partial Y} < 0\) in the neighborhood around \((x^B_j(Y), Y)\). Therefore, for \(\Delta > 0\) sufficiently small, \(g(x^B_j(Y), Y + \Delta) < 0\), and since \(g(v_j, Y + \Delta) > 0\), it must be that \(x_j(Y + \Delta) > x_j(Y)\). Thus, \(Y + x^B_j(Y)\) is (strictly) increasing.

Otherwise, the inequality in (1.7) is strict, i.e., \(\frac{\partial g}{\partial x} \bigg|_{x=x^B_j(Y)} > 0\), and we can use the implicit
function theorem,
\[
\frac{dx_j^B}{dY} = -\frac{\partial g}{\partial Y} \frac{\partial g}{\partial x}
\]
Since \(\frac{\partial g}{\partial x} > \frac{\partial g}{\partial Y}\) and \(\frac{\partial g}{\partial x}|_{x=x_j^B(Y)} > 0\), we have that \(\frac{dx_j^B}{dY} > -1\), and \(Y + x_j^B(Y)\) is (strictly) increasing.

Finally, consider the case where \(x_0(Y) \geq v_j\). There are no gains from trade in the bilateral subgame because \(x - S(Y + x) + S(Y) \leq 0\) for all \(x \leq v_j\). Therefore, \(x - S(Y + x) - S(Y) = 0\) for all \(x < x_0(Y)\) (because \(S'_+ \leq 1\)). Hence, any price can be supported as an SPE and therefore \(x_j^B(Y) = v_j\). To show continuity, the only non-trivial case is on the boundary: suppose that \(Y \to Y_0\) and \(x_0(Y_0) = v_j\). By definition \(x_j^B(Y_0) = v_j\), and since \(v_j \geq x_j^B(Y) \geq \min \{x_0(Y), v_j\}\) and \(x_0(Y) \to x_0(Y_0) = v_j\), it follows that \(x_j^B(Y) \to x_j^B(Y_0)\) as \(Y \to Y_0\). Finally, \(Y + x_j^B(Y)\) is (strictly) increasing because \(x_j^B(Y)\) is constant.

Note the \(x_j^F(Y)\) is also continuous and \(Y + x_j^F(Y)\) is (strictly) increasing because \(x_j^F(Y) = (1 - p)v_j + px_j^B(Y)\).

Proposition 5 follows from Lemmata 2, 3 and 4.

\[\]
Proof. If \((y_1^*, \ldots, y_N^*) \in \hat{\mathcal{F}}G(S),\) there exists a sequence of price vectors \(\{(y_1^p, \ldots, y_N^p)\}_p\) such that: 1) \(y_j^p \to y_j^*\) as \(p \to 1, \forall j\); and 2) for all \(p\), the agreements \(y_1^p, \ldots, y_N^p\) are an equilibrium with passive beliefs and stationary strategies of the bargaining game with a continuation probability \(p\). Let \(Y^p = \sum_{i=1}^N y_i^p\) and \(Y_{-j}^p = \sum_{i \neq j}^N y_i^p\). By definition, \(x_j^B(Y_{-j}^p, p)\) is an equilibrium outcome of a bilateral bargaining game, and by Lemmata 2 and 3,

\[v_j - x_j^B(Y_{-j}^p, p) \geq x_j^B(Y_{-j}^p, p) - S(Y_{-j}^p) + x_j^B(Y_{-j}^p, p) + S(Y_{-j}^p), \forall j\]

(i.e., the firm will gain no more than the buyer). Lemma 4 implies that \(x_j^B(Y_{-j}^p, p) \geq y_j^p - \epsilon_j^p,\)
where $\epsilon_j^p \to 0$ as $p \to 1, \forall j$. Thus,

$$v_j - y_j^p \geq y_j^p - S(Y^p) + S(Y_{-j}^p) - 2\epsilon_j^p, \forall j$$

(because the firm’s payoff is weakly increasing with the price).

Taking $p \to 1$, $v_j - y_j^* \geq y_j^* - S(Y^*) + S(Y_{-j}^*)$, and the rest of the proof follows from Lemma 2: by concavity, \( \frac{y_j^*}{Y^*} (S(Y^*) - S(0)) \geq S(Y^*) - S(Y_{-j}^*) \), and so $v_j - y_j^* \geq y_j^* - \left( \frac{y_j^*}{Y^*} (S(Y^*) - S(0)) \right)$, and in sum, $V - Y^* \geq Y^* - S(Y^*)$. Finally, since $S(Y^*) \geq I$, the firm will not get more than $\frac{1}{2}(V - I)$. □

**Remark 6.** To our knowledge, this is the first paper to characterize the equilibrium of multilateral bargaining games with non-convex payoff sets. The technical derivations may also prove useful in other environments where bargaining agreements are between pairs of players and payoffs across pairs are interdependent. For example, consider the negotiations between a single input supplier and downstream producers who subsequently compete in the downstream market, as in Horn and Wolinsky (1988).

## 2 Appendix B

In this section, we extend the model to a setting in which the firm also chooses how much to invest in each project and the buyers’ surplus is increasing with investment. Since the investment decision is made before the buyers arrive, the firm incurs the entire investment cost and only a fraction of the gain. Thus a hold-up problem curbs the incentive to invest. However, the firm can finance the investment with defaultable debt and make the buyers internalize some of the investment costs. We will show that financing the investment deal-by-deal aligns the investment incentives of the firm with each buyer, achieving the Pareto efficient
allocation. If the firm issues debt on a pool of the proceeds, the investment decision depends on the number of buyers. When there are relatively few buyers, the allocation is Pareto efficient. When there are relatively many buyers, hold-ups lead to inefficient allocations.

More formally, let us consider a firm that produces $N$ units, one for each buyer. The firm can make an investment which increases the quality of the units. The cost of investment is $I$ and the buyers’ values are $v_i = v_j = v(I)$. The function $v$ is increasing, strictly concave, and $v(0) = 0$. The total surplus is $V(I) = Nv(I)$ and the efficient investment level maximizes the social surplus,

$$I^* = \arg\max V(I) - I$$

We assume that $0 < I^* < \infty$. At date 0, the firm makes investment decisions. At date 1, the buyers arrive and negotiate prices with the firm. The bargaining outcomes are bilaterally stable.

As a benchmark, suppose that the firm finances the investment on its own. Since the cost of investment is sunk, each buyer will pay $\frac{v}{2}$, and the firm will invest

$$I_0 = \arg\max \frac{V(I)}{2} - I$$

Since $v$ is strictly concave, $I_0 < I^*$. Although outcomes with higher investments and higher prices are Pareto-improving, the inability of the firm and the buyers to commit makes them infeasible.

Suppose that the firm can finance the investment with a defaultable debt contract. We first consider the case where the firm issues debt on a pool of the proceeds from the sales. The firm chooses the investment $I$ and the debt level $D$, under the restriction that investors at least break even, $D \geq I$.

**Proposition 2.** When the firm issues debt on the pool, there exists a finite number $N_0 \geq 2$ such that if the number of buyers $N \leq N_0$, the firm invests $I^*$. Otherwise, the firm invests
Proof. To see this, recall that for a given debt level $D$ and values $v$, the maximal bilaterally stable outcome is for each buyer to pay

$$x = \begin{cases} 
\frac{v}{2} & \text{if } (N-1)\frac{v}{2} \geq D \\
\frac{v+D}{N+1} & \text{if } (N-1)\frac{v}{2} < D 
\end{cases}$$

(see Example 1), and the firm’s profit $\pi = \max \{ \frac{V^2}{2} - D, \frac{1}{N+1} (V - D) \}$. Since the firm’s profit is decreasing with $D$ and $D \geq I$ (because investors at least break even), the firm optimally sets $D = I$ and chooses an investment level to maximize $\pi(I) = \max \{ \frac{V(I)}{2} - I, \frac{1}{N+1} (V(I) - I) \}$.

For a sufficiently large $N$, $I_0 > 0$, and hence there exists a finite number $N_0 \geq 2$ such that

$$\frac{1}{N_0 + 1} (V(I^*) - I^*) \geq \frac{V(I_0)}{2} - I_0 > \frac{1}{N_0 + 2} (V(I^*) - I^*)$$

We have that $\arg \max \pi(I) = I_0$ whenever $N > N_0$; and $\arg \max \pi(I) = I^*$ whenever $N \leq N_0$.

To provide intuition, recall that for a given debt level $D$ and surplus $V$, equation (1) in the text showed that the firm’s profit is

$$\pi = \begin{cases} 
\frac{V}{2} - D & \text{if } (N-1)\frac{v}{2} \geq D \\
\frac{1}{N+1} (V - D) & \text{if } (N-1)\frac{v}{2} < D 
\end{cases}$$

Since the profit is decreasing in the debt level and investors break even, the firm sets $D = I$. When there are relatively few buyers, each buyer internalizes some of the investment.

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\(^{12}\) We restrict attention to $V(I) > D$ without loss of generality. Also, since each price is decreasing with the number of units, it should be verified that the firm indeed prefers to sell all the units. It is not hard to check that if the firm sells $m < N$ units, then the gains from selling another unit outweigh the losses from previous units.

\(^{13}\) When $N = N_0$, the firm may be indifferent between $I^*$ and $I_0$. 

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costs, the firm receives a fraction of the social surplus and therefore invests to maximize
social surplus. However, each buyer fails to internalize the investment cost when there are
sufficiently many other buyers, and hold-ups lead to under-investment.

We now consider the case where the firm finances the investment deal-by-deal taking out
multiple loans, each one tied to distinct transactions.

**Corollary 1.** When the investment is financed deal-by-deal, the firm invests $I^\ast$.

The argument is identical to the previous one, we assume without loss of generality that
each loan is defaultable (otherwise, the loan does not affect the bargaining) and that each
loan specifies the repayment\(^{14}\) $d_i < v_i$. The break-even condition implies that the sum of the
payments to investors is no less than the initial investment, $D \equiv \sum_{i=1}^{N} d_i \geq I$. The firm’s
profit is $\frac{1}{2}(V(I) - D)$, and so it chooses $D = I$ and invests to maximize social surplus.

### 3 Appendix C

This section contains an extension to a setting with more general contracts, and a proof of
existence of bilaterally stable prices.

#### 3.1 More General Contracts

The analysis focused on joint and separate contracts, but more general contracts can be
written if securities depend on the vector of agreements. Formally, a security $S : \mathbb{R}^N \to \mathbb{R}_+$
indicates the payment to the investors as a function of the vector of agreements with the
buyers. Following the agreements $(x_1, \ldots, x_N)$, the investors will receive $S(x_1, \ldots, x_N)$ and
the firm will keep the remainder $X - S(x_1, \ldots, x_N)$, where $X = \sum_{i=1}^{N} x_i$. While joint and
separate contracts are obviously special cases, we will show that financing each project with
debt separately remains optimal within a large class of these securities.

\(^{14}\)Otherwise, $d_i \geq v_i$, and any price between 0 and $v_i$ is an equilibrium of the bargaining game.
Vectors are denoted by bold symbols. Given a price vector \( \mathbf{x} = (x_1, \ldots, x_N) \), let \( \hat{\mathbf{x}}_j = (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_N) \) denote the vector that results when replacing the \( j \)-th element of \( \mathbf{x} \) with 0, and the *marginal payment* to the investors from reaching an agreement with buyer \( j \) is therefore \( S(\mathbf{x}) - S(\hat{\mathbf{x}}_j) \). The key property is that the sum of the marginal payments to the investors does not exceed the total payment:

\[
\sum_{i=1}^{N} S(\mathbf{x}) - S(\hat{\mathbf{x}}_i) \leq S(\mathbf{x}) \text{ for all } \mathbf{x} \geq 0 \quad (3.1)
\]

**Proposition 3.** *Financing each project with debt separately is optimal within the class of securities satisfying* \((3.1)\).

The proof is identical to that of Proposition 3.

**Proof.** First, if the prices \( x_1, \ldots, x_N \) are bilaterally stable, then the firm’s gain from each trade will not exceed the buyer’s gain:

\[
x_i - (S(\mathbf{x}) - S(\hat{\mathbf{x}}_i)) \leq v_i - x_i, \forall i
\]

Second, if the security satisfies (3.1), the firm’s profit will not exceed the sum of the firm’s gains from each trade:

\[
X - S(\mathbf{x}) \leq \sum_{i=1}^{N} x_i - (S(\mathbf{x}) - S(\hat{\mathbf{x}}_i)), \text{ where } X = \sum_{i=1}^{N} x_i
\]

Thus, the firm’s payoff will not exceed the buyers’, \( X - S(\mathbf{x}) < V - X \), and half the social surplus is an upper bound. \( \square \)

These securities are more general but they also require stronger, and perhaps unrealistic, assumptions on the contracts. Observe that if the firm can shuffle the proceeds from one project to another, then the payment to the investors will only depend on the sum of the proceeds. We therefore focused our analysis on the more common contracts.
3.2 Existence of Bilaterally Stable Outcome

**Lemma 6.** If $S$ is a smooth and concave function, then a bilaterally stable outcome exists.

**Proof.** Assume without the loss of generality that $S'(Y) < 1$ for all $Y \geq 0$ (if $S'_+(0) \geq 1$, then $(0, \ldots, 0)$ is bilaterally stable). Recall in a bilateral bargaining game with payoffs $G_j(S, Y)$ and continuation probability $p < 1$, the maximal SPE price when the buyer (resp. the firm) makes the first offer is $x_j^B(Y, p)$ (resp. $x_j^F(Y, p)$). The functions are well defined (see Lemma 4). We will first show that $x_j^F(Y, p)$ and $x_j^B(Y, p)$ converge, as $p \to 1$, to

$$x_j(Y) := \max \left\{ x : x + \frac{x - S(Y + x)}{1 - S'(Y + x)} = v_j \text{ and } x \leq v_j \right\}$$  \hspace{1cm} (3.2)

Since $S' < 1$, $x_j(Y)$ exists and $v_j > x_j(Y) > 0$. It follows from Lemma 4 that there is a pair of prices $x_B$ and $x_F$, where $v_j > x_F > x_B > 0$, that solves

$$v_j - x_F = p(v_j - x_B)$$  \hspace{1cm} (3.3)

$$Y + x_B - S(Y + x_B) = (1 - p)(Y - S(Y)) + p(Y + x_F - S(Y + x_F))$$  \hspace{1cm} (3.4)

and the prices $x_j^B(Y, p)$ and $x_j^F(Y, p)$ are the maximal prices that solve (3.3) and (3.4). Therefore, if we let

$$g(x, Y, p) = x - S(Y + x) + S(Y) - p(px + (1 - p)v_j - S(Y + px + (1 - p)v) + S(Y))$$

then

$$x_j^B(Y, p) = \max \{ x : g(x, Y, p) = 0 \text{ and } x \leq v_j \}$$  \hspace{1cm} (3.5)

By the intermediate value theorem,
\[ g(x, Y, p) = 0 \iff x + \frac{x - S(Y + x) + S(Y)}{p (1 - S'(Y + \bar{x}_p))} = v_j, \tag{3.6} \]

where \( x < \bar{x}_p < px + (1-p)v_j \). Hence, given \( \delta > 0 \), we have that \( x_j^B(Y, p) \in (x_j(Y) - \delta, x_j(Y) + \delta) \), for all \( p \) sufficiently large, and \( x_j^B(Y, p) \to x_j(Y) \) as \( p \to 1 \). Moreover, \( x_j(Y) \) is continuous by the maximum theorem. Finally, prices \( y_1, \ldots, y_N \) that satisfy \( y_j = x_j(Y_{-j}) \), \( \forall j \), are bilaterally stable. Therefore, define \( \phi : [0, v_1] \times \ldots \times [0, v_N] \to [0, v_1] \times \ldots \times [0, v_N] \) such that
\[
\phi(y_1, \ldots, y_N) = (x_1(Y_{-1}), \ldots, x_N(Y_{-N})).
\]
Since \( \phi \) is a continuous function from a convex and compact set onto itself, there exists a fixed point that is bilaterally stable.

References


McAfee, R. P. and M. Schwartz (1994). Opportunism in multilateral vertical contracting: