# Online Appendix for
# “Raising Capital from Heterogeneous Investors”

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This Online Appendix provides the proof of Proposition 5 and supplementary results that we discuss in the paper. We abbreviate Nash equilibrium by NE.

## B Proof of Proposition 5

We consider the firm's problem with initial capital $W > 0$. As stated, suppose $1/F(x)$ is convex for $x \in [0, X]$, $X > 0$, and there exists an optimal return schedule guaranteeing investments $(x_n)_{n \in S}$ with $W + X_N \leq X$. We show that an optimal such schedule specifies a permutation $\pi^* = (n_1^*, \ldots, n_N^*)$ and returns $(r_i^*, k_i^*)_{i \in S}$ as described in the proposition. Since the claim is trivially satisfied for $W \geq \theta X_N$ (as noted in the text), we assume below that $W < \theta X_N$.

**Optimal returns.** We begin by showing that an optimal schedule specifies returns $(r_i^*, k_i^*)_{i \in S}$ for some permutation $\pi = (n_1, \ldots, n_N)$. Observe that the result in Lemma 1 applies to this setting without change. Hence, any optimal schedule specifies some permutation $\pi = (n_1, \ldots, n_N)$ and returns $(r_i, k_i)_{i \in S}$ which satisfy, for each $i \in S$ and each $j \in \{i, \ldots, N\}$,

$$r_i F(W + X_j) + k_i (1 - F(W + X_j)) \geq \theta. \tag{25}$$

Suppose first that $r_i \geq k_i$ for some $i \in S$. By the arguments in the proof of Proposition 1, we must then have $r_i \geq \theta$ and (25) holding with equality at $j = i$:

$$r_i F(W + X_i) + k_i (1 - F(W + X_i)) = \theta. \tag{26}$$

Suppose next that $r_i < k_i$ for some $i \in S$. Then analogous arguments now yield $k_i > \theta$ and (25) holding with equality at $j = N$:

$$r_i F(W + X_N) + k_i (1 - F(W + X_N)) = \theta. \tag{27}$$

Let us define

$$\eta_i \equiv \frac{1 - F\left(W + X_i\right)}{F\left(W + X_i\right)}$$

and

$$\widetilde{\eta}_i \equiv \begin{cases} \eta_i \text{ if } r_i \geq k_i \\ \eta_N \text{ if } r_i < k_i. \end{cases}$$

Note that by (26) and (27), if $r_i \neq k_i$, changing $k_i$ by $\varepsilon > 0$ arbitrarily small and $r_i$ by $-\varepsilon\widetilde{\eta}_i$ preserves agent $i$'s incentives to participate.

The following four claims yield that the returns $(r_i^*, k_i^*)_{i \in S}$ described in the proposition are optimal.

*Claim 1:* There is an optimal return schedule satisfying $r_i^* \geq k_i^*$ for all $i \in S$.

*Proof:* Suppose by contradiction that $k_i > r_i$ for some $i \in S$ in any optimal return schedule. Take any such schedule and $i \in S$. By (25), $k_i > \theta$, and by ($\mathrm{BC}_W$) and (25), $k_j < \theta < r_j$ for some $j \neq i$. Consider a perturbation in which we increase $k_j$ by $\varepsilon > 0$ arbitrarily small and reduce $r_j$ by $\varepsilon\widetilde{\eta}_j$, while at the same time reducing $k_i$ by $\varepsilon\frac{x_{n_j}}{x_{n_i}}$ and increasing $r_i$ by $\varepsilon\widetilde{\eta}_i\frac{x_{n_j}}{x_{n_i}}$. Note that $\widetilde{\eta}_i = \eta_N$ and $\widetilde{\eta}_j = \eta_j \geq \eta_N$. The perturbed schedule therefore continues to satisfy the firm's budget constraint ($\mathrm{BC}_W$) and, by (26) and (27), it preserves the agents' incentives to participate. Moreover, the perturbation changes the firm's expected payoff by

$$x_{n_j}\varepsilon F(W + X_N)\left(\eta_j - \eta_N\right).$$

If we can pick $j < N$, the perturbation strictly increases the firm's expected payoff, contradicting the optimality of the original schedule. So suppose that in the original schedule, $k_\ell \geq \theta$ for all $\ell \neq N$. Then we can perform the perturbation for $j = N$ without affecting the firm's expected payoff. Moreover, we can continue performing this perturbation until we obtain $k_\ell = \theta = r_\ell$ for all $\ell \neq N$. Since the perturbation keeps $\sum_{\ell \in S} k_\ell x_{n_\ell}$ unchanged and we end up with $\sum_{\ell \neq N} k_\ell x_{n_\ell} = \theta \sum_{\ell \neq N} x_{n_\ell}$, the fact that we must have started with $\sum_{\ell \in S} k_\ell x_{n_\ell} \leq W < \theta X_N$ implies that we end up with $k_N x_{n_N} < \theta x_{n_N}$. Thus, we obtain $k_N < \theta < r_N$, and this completes the construction of an optimal schedule with $r_\ell \geq k_\ell$ for all $\ell \in S$.

*Claim 2:* There is an optimal return schedule satisfying $r_i^* \geq k_i^* \geq 0$ for all $i \in S$.

*Proof:* By Claim 1 and (26), there is an optimal return schedule satisfying $r_i^* \geq k_i^*$ and

$$r_i^* F(W + X_i) + k_i^* (1 - F(W + X_i)) = \theta \tag{28}$$

for all $i \in S$. Claim 2 then follows from analogous arguments to those used in the proof of Proposition 1.

*Claim 3:* There is an optimal return schedule satisfying $r_i^* \geq k_i^* \geq 0$ for all $i \in S$ and $\sum_{i \in S} k_i^* x_{n_i} = W$.
*Proof:* By Claim 2, there is an optimal return schedule satisfying $r_i^* \geq k_i^* \geq 0$ and (28) for all $i \in S$. Suppose towards a contradiction that $\sum_{i \in S} k_i x_{n_i} < W$ in any such schedule, and take any one of them. Note that there must exist $j \in S$ with $k_j < \theta < r_j$. Then consider a perturbation in which we increase $k_j$ by $\varepsilon > 0$ arbitrarily small and reduce $r_j$ by $\varepsilon \eta_j$. Since $\sum_{i \in S} k_i x_{n_i} < W$, the perturbed schedule continues to satisfy the firm's budget constraint ($\text{BC}_W$), and by (28) it preserves the agents' incentives to participate. Moreover, the perturbation changes the firm's expected payoff by

$$x_{n_j} \varepsilon \frac{(F(W + X_N) - F(W + X_j))}{F(W + X_j)},$$

which is positive (and strictly positive if $j \in \{1, \ldots, N-1\}$). Since we can perform this perturbation until $\sum_{i \in S} k_i x_{n_i} = W$, and we continue to satisfy $r_i^* \geq k_i^* \geq 0$ and (28) for all $i \in S$, we obtain a contradiction, proving the claim.

*Claim 4:* In any optimal return schedule, $k_i^* \in (0, \theta)$ for at most one agent $n_i \in S$.
*Proof:* Suppose by contradiction that there exists an optimal return schedule specifying $k_i, k_j \in (0, \theta)$ for some $i, j \in S$, $i \neq j$. Without loss, take $i > j$. Then we can perform a perturbation like the one considered in Claim 1: we increase $k_j$ by $\varepsilon > 0$ arbitrarily small, reduce $r_j$ by $\varepsilon \widetilde{\eta}_j$, reduce $k_i$ by $\varepsilon \frac{x_{n_j}}{x_{n_i}}$, and increase $r_i$ by $\varepsilon \widetilde{\eta}_i \frac{x_{n_j}}{x_{n_i}}$. Since $\widetilde{\eta}_j = \eta_j > \eta_i = \widetilde{\eta}_i$, the perturbation satisfies the firm's budget constraint, preserves the agents' incentives to participate, and strictly increases the firm's expected payoff.

**Optimal permutation.** Given the characterization of the optimal returns, we next show that the permutation $\pi^* = (n_1^*, \ldots, n_N^*)$ described in the proposition is optimal. Consider a return schedule specifying some $\pi = (n_1, \ldots, n_N)$ and $(r_i^*, k_i^*)_{i \in S}$. Note that for some $i_W \in S$, we have $(r_i^*, k_i^*) = \left(\frac{\theta}{F(W+X_i)}, 0\right)$ for all $i < i_W$ and $(r_i^*, k_i^*) = (\theta, \theta)$

for all $i > i_W$. It then follows from Proposition 2 and $1/F(x)$ convex that an optimal ranking of agents $n_i$ for $i < i_W$ satisfies

$$x_{n_1} \geq \ldots \geq x_{n_{i_W-1}}.$$

Furthermore, by Proposition 3, any increase in the dispersion of investments $(x_{n_i})_{i<i_W}$ (in the sense of majorization) lowers the firm's cost. Instead, for agents $n_i$ for $i > i_W$, neither the ranking of these agents nor the distribution of their capital affects the firm's cost. The reason is that the firm's cost of raising $(x_{n_i})_{i>i_W}$ is simply equal to $\sum_{i>i_W} \theta x_{n_i}$. Consequently, it follows that it is optimal for the firm to specify a permutation satisfying

$$x_{n_1} \geq \ldots \geq x_{n_{i_W-1}} \geq x_{n_{i_W+1}} \geq \ldots \geq x_{n_N}.$$

To complete the proof, we next show that an optimal permutation also satisfies $x_{n_{i_W-1}} \geq x_{n_{i_W}} \geq x_{n_{i_W+1}}$. Note that this follows immediately if $i_W = 1$ or $\sum_{i>i_W} \theta x_{n_i} = W$. Suppose that neither of these holds. The firm's cost of raising $x_{n_{i_W}}$ is equal to

$$F(W + X_N) x_{n_{i_W}} r^*_{i_W} + (1 - F(W + X_N)) x_{n_{i_W}} k^*_{i_W}.$$

Substituting with the optimal returns, taking into account that $\min\{\theta x_{n_{i_W}}, W_{i_W}\} = W_{i_W}$, yields

$$\frac{F(W + X_N)}{F(W + X_{i_W})} \left[ x_{n_{i_W}} \theta - W_{i_W}(1 - F(W + X_{i_W}))) \right] + (1 - F(W + X_N)) W_{i_W}.$$

Rearranging terms yields

$$F(W + X_N) \frac{\theta}{F(W + X_{i_W})} \left[ x_{n_{i_W}} - \frac{W_{i_W}}{\theta} \right] + \theta \frac{W_{i_W}}{\theta}.$$

This expression shows that the firm's cost of raising $x_{n_{i_W}}$ is equal to the cost of paying net returns $(r_{i_W}, k_{i_W}) = (\theta, \theta)$ on the portion of capital $W_{i_W}/\theta$ and net returns $(r_{i_W}, k_{i_W}) = \left( \frac{\theta}{F(W+X_{i_W})}, 0 \right)$ on the remaining portion $x_{n_{i_W}} - W_{i_W}/\theta$. By Proposition 3 and $1/F(x)$ convex, it follows that a permutation satisfying $x_{n_{i_W-1}} \geq x_{n_{i_W}} \geq x_{n_{i_W+1}}$ is optimal.

## C Simple Contracts

Let $x_n^* \in [0, \overline{x}_n]$ be the amount of capital that the firm wishes to induce agent $n$ to invest in the unique NE. A bilateral contract for agent $n$ is $(r_n(x_n), k_n(x_n))$, specifying net returns under success and failure as a function of the amount $x_n$ that agent $n$ chooses to invest. We say that $\{r_n(x_n), k_n(x_n)\}_{n\in S}$ is *simple* if for each $n \in S$, agent $n$'s minimum best response to any investment $X_{-n} \equiv \sum_{\ell \neq n} x_\ell$ by the other agents is either 0 or $x_n^*$ under such returns. Without loss, a simple scheme therefore specifies net returns for each $n \in S$ of the form:

$$(r_n(x_n), k_n(x_n)) = \begin{cases} (r_n^*, k_n^*) & \text{if } x_n = x_n^*, \\ (0,0) & \text{otherwise.} \end{cases}$$

The following proposition shows that, among bilateral contracts, simple contracts are without loss of optimality if two conditions are satisfied. The first condition is that the firm raises the full capital endowment of the agents. This condition is implied by the surplus $A$ from success being large enough and $F'(\overline{X}_N) > 0$. The second condition is that $xF'(x)/F(x)$ is weakly decreasing for $x \in [0, \overline{X}_N]$. One can verify that this condition holds for many commonly used distribution functions, including exponential, log-normal, Pareto, power, and uniform.

**Proposition A1.** *Consider the firm's problem allowing for any self-financing scheme of bilateral contracts* $\{r_n(x_n), k_n(x_n)\}_{n\in S}$. *Suppose that an optimal scheme guarantees investments* $x_n^* = \overline{x}_n$ *for each* $n \in S$. *If* $xF'(x)/F(x)$ *is weakly decreasing for* $x \in [0, \overline{X}_N]$, *a simple scheme is optimal.*

*Proof.* Suppose by contradiction that a simple scheme is not optimal. Then any optimal scheme guaranteeing investments $(x_n^*)_{n\in S}$ has $(r_n(x_n), k_n(x_n)) \neq (0,0)$ for some $n \in S$ and $x_n < x_n^* = \overline{x}_n$, where $x_n > 0$ is agent $n$'s minimum best response to some investment $X_{-n} < X_{-n}^* \equiv \sum_{\ell\neq n} x_\ell^* = \sum_{\ell \neq n} \overline{x}_\ell$ by the other agents. We will consider a perturbation that increases all such best responses from $x_n$ to $x_n^*$. The perturbation therefore yields a simple scheme that guarantees $(x_n^*)_{n\in S}$, and we show that it weakly increases the firm's expected payoff relative to the original scheme.

Specifically, take an optimal scheme and, for each agent $n \in S$, consider the investments $X_{-n} \leq X_{-n}^*$ by the other agents to which agent $n$ has a minimum best response

$x_n \in (0, x_n^*]$. If there are multiple such $X_{-n}$, take the smallest among them. Denote that investment by $X'_{-n}$ and agent $n$'s minimum best response to it by $x'_n$. Note that $(r_n(x'_n), k_n(x'_n))$ must satisfy

$$F(X'_{-n} + x'_n) r_n(x'_n) x'_n + (1 - F(X'_{-n} + x'_n)) k_n(x'_n) x'_n \geq \theta x'_n. \tag{29}$$

By definition of the unique NE, $x_n^*$ is the minimum best response to $X_{-n}^*$, so $(r_n(x_n^*), k_n(x_n^*))$ must satisfy

$$\begin{aligned} F(X_{-n}^* + x_n^*) r_n(x_n^*) x_n^* &+ (1 - F(X_{-n}^* + x_n^*)) k_n(x_n^*) x_n^* \\ &\geq F(X_{-n}^* + x'_n) r_n(x'_n) x'_n + (1 - F(X_{-n}^* + x'_n)) k_n(x'_n) x'_n + (x_n^* - x'_n)\theta. \end{aligned} \tag{30}$$

Note that $F(X_{-n}^* + x'_n) \geq F(X'_{-n} + x'_n)$ and the firm's budget constraint requires $k_n(x_n) \leq 0$ for all $x_n \in (0, \overline{x}_n]$ and $n \in S$. Thus, among pairs of net returns $(r_n(x'_n), k_n(x'_n))$ that satisfy (29) and the budget constraint, the right-hand side of (30) is minimized under the pair $(r_n(x'_n), k_n(x'_n)) = (\theta / F(X'_{-n} + x'_n), 0)$. It follows that

$$\begin{aligned} F(X_{-n}^* + x_n^*) r_n(x_n^*) x_n^* &+ (1 - F(X_{-n}^* + x_n^*)) k_n(x_n^*) x_n^* \\ &\geq F(X_{-n}^* + x'_n) \frac{\theta}{F(X'_{-n} + x'_n)} x'_n + (x_n^* - x'_n)\theta. \end{aligned} \tag{31}$$

Consider a perturbation in which we replace each agent $n$'s contract $(r_n(x_n), k_n(x_n))$ with a simple contract $(r'_n(x_n), k'_n(x_n))$ defined as follows:

$$(r'_n(x_n), k'_n(x_n)) = \begin{cases} \left(\theta / F(X'_{-n} + x_n^*), 0\right) & \text{if } x_n = x_n^*, \\ (0, 0) & \text{otherwise,} \end{cases}$$

where $X'_{-n}$ is the smallest investment by others to which the agent has a minimum best response $x_n \in (0, x_n^*]$ in the original scheme. By construction, the perturbed scheme guarantees investments $(x_n^*)_{n \in S}$. We next show that the perturbed scheme weakly reduces the firm's costs. Since the perturbation sets the net returns under failure to zero, this will suffice to establish that the perturbed scheme continues to satisfy the firm's budget constraint, and to prove that a simple scheme is optimal.

By (31) and the definition of $(r'_n(x_n), k'_n(x_n))$, a sufficient condition for the firm's costs to decline with the perturbation is

$$\frac{F(X_{-n}^*+x_n^*)}{F(X_{-n}'+x_n^*)}x_n^* \leq \frac{F(X_{-n}^*+x_n')}{F(X_{-n}'+x_n')}x_n' + (x_n^* - x_n').$$

If $X_{-n}' = X_{-n}^*$, then $x_n' = x_n^*$ (by definition of the unique NE) and this inequality holds with equality. If $X_{-n}' < X_{-n}^*$, then the inequality is equivalent to

$$\frac{\big(F(X_{-n}^*+x_n^*) - F(X_{-n}'+x_n^*)\big)}{(X_{-n}^* - X_{-n}')}\frac{x_n^*}{F(X_{-n}'+x_n^*)} \leq \frac{\big(F(X_{-n}^*+x_n') - F(X_{-n}'+x_n')\big)}{(X_{-n}^* - X_{-n}')}\frac{x_n'}{F(X_{-n}'+x_n')},$$

which is implied by $x_n' \leq x_n^*$ and $xF'(x)/F(x)$ being weakly decreasing. □

# D Relaxed Budget Constraint

Consider schemes for the firm specifying investments $(x_n)_{n\in S}$ and returns $(r_n, k_n)_{n\in S}$. Suppose that we replace the firm's budget constraint (BC) with the following relaxed version:

$$\sum_{n=1}^{N} r_n x_n \leq A \quad \text{and} \quad \sum_{n=1}^{N} k_n x_n \leq 0. \qquad (\text{BC}_{\text{on-path}})$$

The original constraint (BC) required that the firm have sufficient final capital to make the payments offered to the agents regardless of agents' choices, i.e. for all profiles $\mathcal{Y} = (y_1, \ldots, y_N)$. Instead, ($\text{BC}_{\text{on-path}}$) only requires the firm to satisfy budget balance on the equilibrium path, i.e. under the profile $\mathcal{Y}^1 \equiv (1, \ldots, 1)$. It is clear that in the case of project success, the constraints imposed by ($\text{BC}_{\text{on-path}}$) and (BC) are the same. In the case of failure, however, ($\text{BC}_{\text{on-path}}$) is a strict relaxation: whereas (BC) required the net returns under failure to satisfy $k_n \leq 0$ for all $n \in S$, ($\text{BC}_{\text{on-path}}$) allows for net returns $k_n > 0$ for some $n \in S$ provided that $\sum_{n=1}^{N} k_n x_n \leq 0$.

Nevertheless, we find that replacing (BC) with ($\text{BC}_{\text{on-path}}$) does not affect the solution to the firm's problem. The reason is that the firm wants to guarantee a unique outcome, and the return an agent expects to receive depends on her conjectures of others' behavior. If agent $n \in S$ expects that only other agents $\ell \in S' \subseteq S \setminus n$ will invest with the firm, then she will expect to receive a return under failure $k_n x_n$ no greater than $-\sum_{\ell \in S'} k_\ell x_\ell$. As a result, we show that while setting $k_n > 0$ for some $n \in S$ is feasible under ($\text{BC}_{\text{on-path}}$), an optimal unique-implementation scheme sets $k_n \leq 0$ for all $n \in S$.

Specifically, observe first that Lemma 1 continues to apply when the firm's budget constraint is given by ($\text{BC}_{\text{on-path}}$) (since the proof of the lemma does not make use of the budget constraint). It follows that an optimal scheme guaranteeing investments $(x_n)_{n \in S}$ specifies some permutation $\pi = (n_1, \ldots, n_N)$ of the set of agents and returns $(r_i, k_i)_{i \in S}$ such that, for each $i \in S$ and each $j \in \{i, \ldots, N\}$, equation (12) in the paper's Appendix is satisfied. Moreover, by arguments analogous to those in the proof of Proposition 5, either agent $n_i$'s returns satisfy $r_i \geq k_i$ and

$$r_i F(X_i) + k_i (1 - F(X_i)) = \theta,$$

or the agent's returns satisfy $r_i < k_i$ and

$$r_i F(X_N) + k_i (1 - F(X_N)) = \theta.$$

Let us define

$$\eta_i \equiv \frac{1 - F(X_i)}{F(X_i)}$$

and

$$\widetilde{\eta}_i \equiv \begin{cases} \eta_i \text{ if } r_i \geq k_i \\ \eta_N \text{ if } r_i < k_i. \end{cases}$$

We prove by induction that an optimal scheme specifies $k_i \leq 0$ for all $i \in S$ and thus satisfies (BC). Consider agent $n_1$ in the permutation $\pi$. This agent must be willing to invest with the firm when expecting no other agent to invest and thus her return under failure to be weakly negative. It follows that the firm cannot benefit from specifying a return $k_1 > 0$. Hence, $k_1 \leq 0$ is optimal.

Now take any $\ell \in \{2, \ldots, N-1\}$ and suppose $k_i \leq 0$ for all $i \in \{1, \ldots, \ell\}$. We show that $k_{\ell+1} \leq 0$ is optimal. Agent $n_{\ell+1}$ must be willing to invest with the firm when expecting only agents $(n_1, \ldots, n_\ell)$ to invest and thus her return under failure to satisfy $k_{\ell+1} x_{n_{\ell+1}} \leq -\sum_{i=1}^{\ell} k_i x_{n_i}$. Suppose by contradiction that $k_{\ell+1} \leq 0$ is not optimal, so the firm specifies $k_{\ell+1} > 0$ and $k_{i'} < 0$ for some $i' \in \{1, \ldots, \ell\}$. Take any such $i'$ and consider the following perturbation: we increase $k_{i'}$ by $\varepsilon > 0$ arbitrarily small, reduce

$r_{i'}$ by $\varepsilon\widetilde{\eta}_{i'}$, reduce $k_{\ell+1}$ by $\varepsilon\dfrac{x_{n_{i'}}}{x_{n_{\ell+1}}}$, and increase $r_{\ell+1}$ by $\varepsilon\widetilde{\eta}_{\ell+1}\dfrac{x_{n_{i'}}}{x_{n_{\ell+1}}}$. By the same logic as in the proof of Proposition 5 and the fact that $\widetilde{\eta}_{i'} = \eta_{i'} > \widetilde{\eta}_{\ell+1}$, this perturbation satisfies the firm's budget constraint ($\text{BC}_{\text{on-path}}$), preserves the agents' incentives, and strictly increases the firm's expected payoff. We thus obtain a contradiction, implying that $k_{\ell+1} \leq 0$ is optimal.

# E General Equilibrium

Consider a simple market setting with two firms, $a$ and $b$, and two investors, 1 and 2, investing capital amounts $x_1$ and $x_2$ respectively, where $x_2 > x_1$. The two firms have the same technology: if $x > 0$ is invested in a firm's project, the probability of success is $F(x) > 0$, for $F$ strictly increasing, and success yields an additional surplus $A > 0$.

The interaction is as follows. First, firms $a$ and $b$ simultaneously offer to the agents publicly observable contracts, specifying net returns under success $(r_1^a, r_2^a)$ and $(r_1^b, r_2^b)$ respectively. Next, the two agents simultaneously decide whether to invest their capital with firm $a$ or with firm $b$. Finally, the uncertainty is resolved and payments are made.

In line with our analysis of the one-firm problem, we solve for equilibria in which the firms' offers yield a unique outcome in the interaction between the agents. For this, assume that if agent 1 (agent 2) is indifferent between investing with firm $a$ and investing with firm $b$ given the firms' offers and her conjecture of the other agent's behavior, then she invests with firm $a$ (firm $b$). Furthermore, assume that if a firm deviates from its equilibrium offer, then the investors play the equilibrium that is worst for the firm. We solve for the equilibrium within this class that maximizes the firms' expected profits.

Given our focus, in equilibrium the two firms must offer different returns, i.e. $(r_1^a, r_2^a) \neq (r_1^b, r_2^b)$, and the two investors must choose to invest with different firms.[35] Equilibrium then requires that, given $(r_1^a, r_2^a)$ and $(r_1^b, r_2^b)$ and the behavior of the other investor, neither investor have an incentive to deviate unilaterally to the other investor's firm. In fact, each investor must be indifferent between deviating and not; otherwise the firm where the investor invests would have an incentive to reduce her return. By our assumption on behavior under indifference, it follows that investor 1 invests with firm

[35] An equilibrium in which the firms' return offers are the same cannot induce a unique outcome in the interaction between the agents. Moreover, such an equilibrium does not exist for $F$ concave.

$a$ and investor 2 invests with firm $b$, where the indifference conditions are

$$F(x_1)r_1^a x_1 = F(x_1 + x_2)r_1^b x_1, \tag{32}$$

$$F(x_2)r_2^b x_2 = F(x_1 + x_2)r_2^a x_2. \tag{33}$$

In addition, equilibrium requires that neither firm have an incentive to deviate unilaterally and change its offers. This requires that both firm $a$ and firm $b$ make non-negative expected profits, namely that their participation constraints be satisfied:

$$F(x_1)(A - r_1^a x_1) \geq 0,$$

$$F(x_2)(A - r_2^b x_2) \geq 0.$$

Moreover, to deter deviations in which a firm seeks to attract both investors, the firms' profits must satisfy:

$$F(x_1)(A - r_1^a x_1) \geq F(x_1 + x_2)(A - r_1^a x_1 - r_2^a x_2), \tag{34}$$

$$F(x_2)(A - r_2^b x_2) \geq F(x_1 + x_2)(A - r_1^b x_1 - r_2^b x_2), \tag{35}$$

(where the right-hand side is zero if the firm's budget constraint binds). Note that these inequalities become tighter (the left-hand side minus the right-hand side decreases) as the returns are lowered, being violated for returns that are low enough.

We use the conditions above to derive the firm's equilibrium offers. First, note that combined with (32) and (33), the inequalities in (34) and (35) require that in equilibrium both firms offer strictly positive returns to both investors. In turn, this implies that for $\varepsilon > 0$ small enough, an available deviation to firm $a$ is to attract only investor 2 by offering $(r_1^a, r_2^a) = (0, r_2^b + \varepsilon)$, and an available deviation to firm $b$ is to attract only investor 1 by offering $(r_1^b, r_2^b) = (r_1^a + \varepsilon, 0)$. For these deviations not to be profitable, it must be that in equilibrium the firms' expected profits are the same:

$$F(x_1)(A - r_1^a x_1) = F(x_2)(A - r_2^b x_2). \tag{36}$$

Next, we show that if both (34) and (35) hold as strict inequalities given the firms' equilibrium offers $(r_1^a, r_2^a)$ and $(r_1^b, r_2^b)$, then there is another equilibrium yielding strictly larger profits to both firms. Specifically, for $\varepsilon > 0$, consider the following returns:

$$\begin{aligned}(\tilde{r}_1^a, \tilde{r}_2^a) &= \left(r_1^a - \varepsilon, \frac{F(x_2)\tilde{r}_2^b}{F(x_1+x_2)}\right), \\ (\tilde{r}_1^b, \tilde{r}_2^b) &= \left(\frac{F(x_1)\tilde{r}_1^a}{F(x_1+x_2)}, r_2^b - \varepsilon\frac{F(x_1)x_1}{F(x_2)x_2}\right).\end{aligned}$$

Given the original returns $(r_1^a, r_2^a)$ and $(r_1^b, r_2^b)$, these returns are constructed so that the indifference conditions (32) and (33), as well as equation (36), continue to be satisfied. Moreover, if (34) and (35) hold as strict inequalities under $(r_1^a, r_2^a)$ and $(r_1^b, r_2^b)$, these inequalities continue to be satisfied under $(\tilde{r}_1^a, \tilde{r}_2^a)$ and $(\tilde{r}_1^b, \tilde{r}_2^b)$ for $\varepsilon > 0$ small enough. We thus obtain that, for small $\varepsilon > 0$, there is an equilibrium in which firm $a$ and firm $b$ offer $(\tilde{r}_1^a, \tilde{r}_2^a)$ and $(\tilde{r}_1^b, \tilde{r}_2^b)$ respectively. Compared to the original equilibrium, the behavior of the agents is unchanged (agent 1 invests with firm $a$ and agent 2 invests with firm $b$), but both firms make strictly larger profits.

It follows that in an equilibrium that maximizes the firms' profits, at least one of the inequalities in (34)-(35) must hold as an equality. We proceed by taking (34) to hold as an equality, and it can be verified that the same solution is obtained by taking (35) as an equality. Combining (34) as an equality with equation (36) yields

$$F(x_2)(A - r_2^b x_2) = F(x_1 + x_2)(A - r_1^a x_1 - r_2^a x_2).$$

Using (33) to substitute for $r_2^a x_2$ yields

$$F(x_2)(A - r_2^b x_2) = F(x_1 + x_2)\left(A - r_1^a x_1 - \frac{F(x_2)r_2^b x_2}{F(x_1+x_2)}\right),$$

which simplifies to

$$F(x_2)A = F(x_1 + x_2)\left(A - r_1^a x_1\right).$$

Therefore, we obtain

$$r_1^a = \frac{F(x_1+x_2) - F(x_2)}{F(x_1+x_2)x_1}A,$$

and using (36) again,

$$r_2^b = \frac{F(x_1+x_2) - F(x_1)}{F(x_1+x_2)x_2}A.$$

The large investor 2 receives a strictly higher expected per-dollar return than the small investor 1 if and only if $F(x_2)r_2^b > F(x_1)r_1^a$. Using the solution above, this inequality reduces to

$$\frac{F(x_2)}{x_2(F(x_1+x_2)-F(x_2))} > \frac{F(x_1)}{x_1(F(x_1+x_2)-F(x_1))}. \tag{37}$$

Condition (37) holds for many commonly used distribution functions, including uniform and exponential, gamma, log-normal, and Pareto for a subset of their parameters. Therefore, we obtain that under a weak distributional condition, our main qualitative results extend to this competitive market setting.