Non-dogmatic Social Discounting
Online Appendix

Antony Millner*[1]

[1]Department of Economics, University of California, Santa Barbara.

Contents

*Millner: `millner@ucsb.edu`.

A Proof of Lemma 1

Lemma 1. *The system (12) defines a unique bounded set of time preferences, which are non-decreasing in all utilities, if*

$$\max_i \left\{ \sum_{s=1}^{\infty} \sum_{j=1}^{N} f_s^{ij} < 1 \right\}.$$

Proof. The system of time preferences (12) can be written as a single matrix equation as follows:

$$\begin{pmatrix} V_\tau^1 \\ \vdots \\ V_\tau^N \\ V_{\tau+1}^1 \\ \vdots \\ V_{\tau+1}^N \\ \vdots \end{pmatrix} = \begin{pmatrix} U^1(c_\tau) \\ \vdots \\ U^N(c_\tau) \\ U^1(c_{\tau+1}) \\ \vdots \\ U^N(c_{\tau+1}) \\ \vdots \end{pmatrix} + \begin{pmatrix} \vec{0}_N & f_1^{11} & \dots & f_1^{1N} & f_2^{11} & \dots & f_2^{1N} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vec{0}_N & f_1^{N1} & \dots & f_1^{NN} & f_2^{N1} & \dots & f_2^{NN} & \dots \\ \vec{0}_N & \vec{0}_N & f_1^{11} & \dots & f_1^{1N} & f_2^{11} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vec{0}_N & \vec{0}_N & f_1^{N1} & \dots & f_1^{NN} & f_2^{N1} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} V_\tau^1 \\ \vdots \\ V_\tau^N \\ V_{\tau+1}^1 \\ \vdots \\ V_{\tau+1}^N \\ \vdots \end{pmatrix}$$

where $\vec{0}_N$ is an $1 \times N$ vector of zeros. Letting $\vec{X}_\tau$ denote the vector on the left hand side of this expression, $\mathbf{\Lambda}$ the infinite dimensional square matrix on the right hand side, and $\vec{U}_\tau$ denote the vector of Us on the right hand side, we have

$$\begin{aligned} \vec{X}_\tau &= \vec{U}_\tau + \mathbf{\Lambda}\vec{X}_\tau \\ \Rightarrow \vec{X}_\tau &= (\mathbf{1}_\infty - \mathbf{\Lambda})^{-1}\vec{U}_\tau, \end{aligned}$$

where $\mathbf{1}_\infty$ is the infinite dimensional identity matrix, and we have assumed that the relevant matrix inverse exists.

In general infinite dimensional matrices do not have unique inverses. However, Lemma 1 in Bergstrom (1999) shows that $\mathbf{1}_\infty - \mathbf{\Lambda}$ has a unique bounded inverse with non-negative elements if and only if $\mathbf{1}_\infty - \mathbf{\Lambda}$ is a dominant diagonal matrix. A denumerably infinite matrix $\mathbf{1}_\infty - \mathbf{\Lambda}$ with $\mathbf{\Lambda} \geq 0$ is said to be dominant diagonal if there exists a bounded diagonal matrix $\mathbf{D} \geq 0$ such that the infimum of the row sums of $(\mathbf{1}_\infty - \mathbf{\Lambda})\mathbf{D}$ is positive. Clearly, a sufficient condition for $\mathbf{1}_\infty - \mathbf{\Lambda}$ to be dominant diagonal is if $\sum_{s=1}^{\infty} \sum_{j=1}^{N} f_s^{ij} < 1$ for all i. □

Although this lemma focusses on providing a sufficient condition that is easy to check, the proof also provides a necessary and sufficient condition: $\mathbf{1}_\infty - \mathbf{\Lambda}$ must be dominant diagonal. This is equivalent to requiring the spectral radius of the linear operator $\mathbf{\Lambda}$ to be less than 1, as this guarantees that the sequence $(\mathbf{1}_\infty - \mathbf{\Lambda})^{-1} = \mathbf{1}_\infty + \mathbf{\Lambda} + \mathbf{\Lambda}^2 + \dots$ converges (Duchin & Steenge, 2009). Checking this condition is however difficult in practice given the infinite dimensionality of $\mathbf{\Lambda}$. I will thus work with the simpler sufficient condition throughout, but the results do not depend on this simplification. The proof of the main proposition in Appendix C only requires the spectral radius of $\mathbf{\Lambda}$ to be bounded above by 1.

B Proof of Lemma 2

We wish to prove that non-dogmatic planners' with preferences (12) have consistent beliefs iff the intratemporal weights w_s^{ij} satisfy (14). In the notation established in the text,

Lemma 2.

$$Prob_\tau(i \to j; s) = \sum_{k=1}^{N} Prob_\tau(i \to k; t) Prob_{\tau+t}(k \to j; s-t) \tag{A.1}$$

for all $\tau \in \mathbb{N}, s \geq 2, 1 \leq t < s$ *if and only if there exists an* $N \times N$ *stochastic matrix* $\mathbf{P}$ *such that*

$$w_s^{ij} = (\mathbf{P}^s)_{i,j}.$$

Let the beliefs of planners at time τ about the probability of a future self who subscribes to theory i at time $\tau+s-1$ switching to theory j at time $\tau+s$ be $T_s^{ij,(\tau)}$. Denote the matrix of these transition probabilities by $\mathbf{T}_s^{(\tau)}$. Let $\mathbf{W}_s^{(\tau)}$ be the matrix of time τ planners' beliefs about which theory they will subscribe to at time $\tau+s$, whose i,j element is $\mathrm{Prob}_\tau(i \to j; s)$. Then we have

$$\mathbf{W}_s^{(\tau)} = \mathbf{T}_s^{(\tau)} \mathbf{T}_{s-1}^{(\tau)} \dots \mathbf{T}_1^{(\tau)}.$$

Using this relation, (A.1) can be written as the requirement that

$$\mathbf{T}_s^{(\tau)} \mathbf{T}_{s-1}^{(\tau)} \dots \mathbf{T}_1^{(\tau)} = \mathbf{T}_{s-t}^{(\tau+t)} \mathbf{T}_{s-t-1}^{(\tau+t)} \dots \mathbf{T}_1^{(\tau+t)} \mathbf{T}_t^{(\tau)} \mathbf{T}_{t-1}^{(\tau)} \dots \mathbf{T}_1^{(\tau)}, \tag{A.2}$$

for all τ, t, s. It is clear that a sufficient condition for this to be satisfied is

$$\mathbf{T}_s^{(\tau)} = \mathbf{P}$$

for all τ, s, where $\mathbf{P}$ is an $N \times N$ stochastic matrix. To prove necessity, put $s = 2, t = 1$ in (A.2) to find

$$\mathbf{T}_2^{(\tau)}\mathbf{T}_1^{(\tau)} = \mathbf{T}_1^{(\tau+1)}\mathbf{T}_1^{(\tau)}$$

which implies

$$\mathbf{T}_2^{(\tau)} = \mathbf{T}_1^{(\tau+1)}. \tag{A.3}$$

Putting $s = 3, t = 1$ in (A.2), we find

$$\begin{aligned} \mathbf{T}_3^{(\tau)}\mathbf{T}_2^{(\tau)}\mathbf{T}_1^{(\tau)} &= \mathbf{T}_2^{(\tau+1)}\mathbf{T}_1^{(\tau+1)}\mathbf{T}_1^{(\tau)} \\ \Rightarrow \mathbf{T}_3^{(\tau)}\mathbf{T}_2^{(\tau)} &= \mathbf{T}_2^{(\tau+1)}\mathbf{T}_1^{(\tau+1)}. \end{aligned}$$

and using (A.3) this reduces to

$$\mathbf{T}_3^{(\tau)} = \mathbf{T}_2^{(\tau+1)}.$$

Repeating this process of substitution, we find that a necessary condition for (A.2) to be satisfied is

$$\mathbf{T}_{s+1}^{(\tau)} = \mathbf{T}_s^{(\tau+1)}.$$

Since non-dogmatic planners' preferences are time invariant, it must be the case that

$$\mathbf{T}_s^{(\tau+1)} = \mathbf{T}_s^{(\tau)}.$$

Substituting this relation into the previous equation shows that

$$\mathbf{T}_{s+1}^{(\tau)} = \mathbf{T}_s^{(\tau)}$$

for all τ, s. This implies that the matrix of planners' beliefs $\mathbf{W}_s^{(\tau)}$ must be of the form

$$\mathbf{W}_s^{(\tau)} = (\mathbf{P})^s$$

for all τ.

C Proof of Proposition 1

We prove a more general version of the result in Proposition 1. The proof has two main steps. First we find conditions under which all planners' utility weights a_s^{ij} are proportional to a common discount factor $\hat{\mu}^s$ for large s. We then show that when these conditions are satisfied all non-dogmatic planners' long-run SDRs are the same.

STEP 1:

Begin by defining the sequence of $N \times N$ matrices

$$\mathbf{F}_s := \begin{pmatrix} f_s^{11} & f_s^{12} & \dots & f_s^{1N} \\ f_s^{21} & f_s^{22} & \dots & f_s^{2N} \\ \vdots & \vdots & \vdots & \vdots \\ f_s^{N1} & f_s^{N2} & \dots & f_s^{NN} \end{pmatrix} \tag{A.4}$$

and the sequences of $N \times 1$ vectors

$$\vec{V}_\tau = \begin{pmatrix} V_\tau^1 \\ V_\tau^2 \\ \vdots \\ V_\tau^N \end{pmatrix}, \quad \vec{U}_\tau = \begin{pmatrix} U^1(c_\tau) \\ U^2(c_\tau) \\ \vdots \\ U^N(c_\tau) \end{pmatrix}. \tag{A.5}$$

Our general model (12) can be written as:

$$\vec{V}_\tau = \vec{U}_\tau + \sum_{s=1}^{\infty} \mathbf{F}_s \vec{V}_{\tau+s}. \tag{A.6}$$

We seek an equivalent representation of this system of the form

$$\vec{V}_\tau := \sum_{s=0}^{\infty} \mathbf{A}_s \vec{U}_{\tau+s}, \tag{A.7}$$

where $\mathbf{A}_s$ is a sequence of $N \times N$ matrices of the form,

$$\mathbf{A}_s := \begin{pmatrix} a_s^{11} & a_s^{12} & \dots & a_s^{1N} \\ a_s^{21} & a_s^{22} & \dots & a_s^{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_s^{N1} & a_s^{N2} & \dots & a_s^{NN} \end{pmatrix} \tag{A.8}$$

where a_s^{ij} is the weight planner i at time τ assigns to theory j's utility function at time $\tau + s$, i.e., $U^j(c_{\tau+s})$.

We now prove the following:

Proposition A.I. *Assume that the condition (13) is satisfied, and that $f_s^{ii} > 0$ for all*

$i = 1 \dots N$, $s = 1 \dots \infty$. Construct a directed graph G with N nodes labelled $1, 2, \dots, N$. Draw an edge from node i to node $j \neq i$ iff $f_s^{ij} > 0$ for at least one $s \geq 1$. If G contains a directed cycle of length N, then there exists a $\hat{\mu} \in (0, 1)$ such that

$$\lim_{s \to \infty} \frac{a_s^{ij}}{\hat{\mu}^s} = K_{ij} > 0$$

where the K_{ij} are finite constants.

Notice that the definition of non-dogmatic time preferences in (12) automatically implies that the directed cycle condition in this proposition is satisfied (the graph G is complete in this case, i.e., all edges exist). However, the directed cycle condition itself is considerably weaker than is assumed in this definition.

Proof. Substitute (A.7) into (A.6) to find

$$\sum_{s=0}^{\infty} \mathbf{A}_s \vec{U}_{\tau+s} = \vec{U}_\tau + \sum_{p=1}^{\infty} \mathbf{F}_p \left(\sum_{q=0}^{\infty} \mathbf{A}_q \vec{U}_{\tau+p+q} \right) \tag{A.9}$$

Equating coefficients of $\vec{U}_{\tau+s}$ in this expression, we see that $\mathbf{A}_s$ must satisfy

$$\mathbf{A}_0 = \mathbf{1}_N \tag{A.10}$$

$$\mathbf{A}_s = \sum_{p=1}^{s} \mathbf{F}_p \mathbf{A}_{s-p} \text{ for } s > 0. \tag{A.11}$$

where $\mathbf{1}_N$ is the $N \times N$ identity matrix. The solution of this recurrence relation determines the utility weights a_s^{ij}. It will be convenient to split this matrix recurrence relation into a set of N vector recurrence relations as follows. Let $\vec{A}_s^j$ be the j-th column vector of $\mathbf{A}_s$, i.e.,

$$\vec{A}_s^j = \begin{pmatrix} a_s^{1j} \\ a_s^{2j} \\ \vdots \\ a_s^{Nj} \end{pmatrix}. \tag{A.12}$$

Define $\vec{e}^j$ to be the unit vector with elements

$$(\vec{e}^j)_i = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Then (A.11) is equivalent to the N vector recurrence relations

$$\begin{aligned} \vec{A}^j_0 &= \vec{e}^j \\ \vec{A}^j_s &= \sum_{p=1}^{s} \mathbf{F}_p \vec{A}^j_{s-p} \text{ for } s > 0. \end{aligned} \tag{A.13}$$

for $j = 1 \ldots N$.

The proof now has the following steps. We consider finite order models, i.e., $\mathbf{F}_{M'} = 0$ for all M' greater than some finite M. We show that if a certain augmented matrix constructed from the matrices $\mathbf{F}_1, \ldots, \mathbf{F}_M$ is *primitive*, all planners will have a common long-run pure time discount factor. A square matrix $\mathbf{B}$ is primitive if there exists an integer $k > 0$ such that $\mathbf{B}^k > 0$. We then extend this result to infinite order models by taking an appropriate limit of finite order models. Finally, we show that primitivity of the required matrices in the infinite order case is ensured by the graph theoretic condition in the statement of the proposition.

Begin with the finite order case. Let $M = \max\{s | \exists i, j \; f^{ij}_s > 0\} < \infty$. In this case, for all $s > M$, (A.13) reduces to

$$\vec{A}^j_s = \sum_{p=1}^{M} \mathbf{F}_p \vec{A}^j_{s-p}. \tag{A.14}$$

Define the $NM \times NM$ matrix

$$\mathbf{\Phi}_M = \begin{pmatrix} \mathbf{F}_1 & \mathbf{F}_2 & \ldots & \mathbf{F}_{M-1} & \mathbf{F}_M \\ \mathbf{1}_N & 0 & \ldots & 0 & 0 \\ 0 & \mathbf{1}_N & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \mathbf{1}_N & 0 \end{pmatrix} \tag{A.15}$$

where $\mathbf{1}_N$ is the $N \times N$ identity matrix. In addition, define the 'stacked' vector

$$\vec{Y}_s^j = \begin{pmatrix} \vec{A}_s^j \\ \vec{A}_{s-1}^j \\ \vdots \\ \vec{A}_{s-M+1}^j \end{pmatrix}$$

Then we can rewrite the Mth order recurrence (A.14) as a first order recurrence as follows:

$$\begin{aligned} \vec{Y}_s^j &= \mathbf{\Phi}_M \vec{Y}_{s-1}^j \\ \Rightarrow \vec{Y}_{M+s}^j &= (\mathbf{\Phi}_M)^s \vec{Y}_M^j. \end{aligned} \tag{A.16}$$

We now assume that $\mathbf{\Phi}_M$ is a primitive matrix. By the Perron-Frobenius theorem for primitive matrices (Sternberg, 2014), this implies

1. $\mathbf{\Phi}_M$ has a positive eigenvalue, which we label as $\mu(M)$.
2. All other eigenvalues of $\mathbf{\Phi}_M$ have complex modulus strictly less than $\mu(M)$.
3. There exists a matrix $\mathbf{C} > 0$ such that
$$\lim_{s \to \infty} \frac{\mathbf{\Phi}_M^s}{[\mu(M)]^s} = \mathbf{C}$$
4. $\mu(M)$ increases when any element of $\mathbf{\Phi}_M$ increases.
5. $$\mu(M) < \max_i \sum_j \phi_{ij}. \tag{A.17}$$
   where ϕ_{ij} is the ijth element of $\mathbf{\Phi}_M$.

Since the first N elements of $\vec{Y}_s^j$ coincide with a_s^{ij}, the third of these conclusions implies that

$$\forall i, j, \lim_{s \to \infty} \frac{a_s^{ij}}{[\mu(M)]^s} = \mathbf{C}\vec{Y}_M^j > 0.$$

To bound the value of $\mu(M)$, note that from point 5 of the Perron-Frobenius theorem in (A.17), and the definition of $\mathbf{\Phi}_M$ in (A.15), we have

$$\mu(M) < \max_i \left\{ \sum_{s=1}^{M} \sum_{j=1}^{N} f_s^{ij} \right\} \tag{A.18}$$

Thus, if

$$\sum_{s=1}^{\infty}\sum_{j=1}^{N} f_s^{ij} < 1 \tag{A.19}$$

for all i, $\mu(M) < 1$, and hence $\lim_{s\to\infty} a_s^{ij} = 0$. Thus (13) guarantees that the time preferences (12) are complete (i.e., finite on bounded consumption streams, and hence able to rank arbitrary pairs of bounded consumption streams) for all finite M. This concludes the finite M case.

We now extend this result to the case of infinite M. Assume that there exists an $M' > 0$ such that the matrix $\mathbf{\Phi}_M$, defined in (A.15), is primitive for all $M > M'$. For $M > M'$, define

$$\vec{V}_\tau(M) = \vec{U}_\tau + \sum_{s=1}^{M} \mathbf{F}_s \vec{V}_{\tau+s}(M)$$

and let

$$\hat{\vec{V}}_\tau = \lim_{M\to\infty} \vec{V}_\tau(M).$$

Define the equivalent representations of these preferences by

$$\vec{V}_\tau(M) = \sum_{s=0}^{\infty} \mathbf{A}_s(M)\vec{U}_{\tau+s} \tag{A.20}$$

$$\hat{\vec{V}}_\tau = \sum_{s=0}^{\infty} \hat{\mathbf{A}}_s \vec{U}_{\tau+s} \tag{A.21}$$

In addition, let $\mu(M)$ be the Perron-Frobenius eigenvalue of $\mathbf{\Phi}_M$. We begin by proving that:

Lemma 3.

$$\hat{\mu} := \lim_{M\to\infty} \mu(M) \textit{ exists.} \tag{A.22}$$

Proof. Consider the eigenvalue $\mu(M+1)$, where $M > M'$. This is the Perron-Frobenius eigenvalue of $\mathbf{\Phi}_{M+1}$. The M-th order preferences $\vec{V}_\tau(M)$ are equivalent to an $M+1$th order model, with $\mathbf{F}_{M+1} = 0$. The matrix $\mathbf{\Phi}_M$, which controls the asymptotic behavior of $\vec{V}_\tau(M)$ can thus be thought of as an $N \times (M+1)$ matrix, where the last M rows and columns are zeros. Call this matrix $\tilde{\mathbf{\Phi}}_{M+1}$. The matrix $\mathbf{\Phi}_{M+1}$, associated with the asymptotic behavior of $\vec{V}_\tau(M+1)$, has entries that are strictly larger than than those of $\tilde{\mathbf{\Phi}}_{M+1}$ in at least some elements. Thus, by point 4 in our statement of the Perron-Frobenius theorem, $\mu(M+1) > \mu(M)$. We also know that $\mu(M) < 1$ for all M. Since the sequence

$\mu(M)$ is increasing and bounded above, the monotone convergence theorem implies that $\hat{\mu}$ exists. $\square$

We have thus proved that if the matrices $\mathbf{\Phi}_M$ are primitive for $M > M'$,

$$\lim_{M\to\infty} \lim_{s\to\infty} \frac{a^{ij}_{s+1}(M)}{a^{ij}_s(M)} = \lim_{M\to\infty} \mu(M) = \hat{\mu}. \tag{A.23}$$

Note that since (A.17) and (A.19) are strict inequalities, $\hat{\mu} < 1$. We now wish to know whether it is also true that:

$$\lim_{s\to\infty} \lim_{M\to\infty} \frac{a^{ij}_{s+1}(M)}{a^{ij}_s(M)} = \hat{\mu}. \tag{A.24}$$

That is, can we change the order of the limits in (A.23)? For limit operations to be interchangeable we require the sequence of functions they operate on to be uniformly convergent. The functions in question here are $V^i_\tau(M)$ and $\hat{V}^i_\tau$, which we can think of as linear functions from the infinite dimensional space $\mathbb{R}^\infty \times \mathbb{R}^N = \{(\vec{U}_\tau, \vec{U}_{\tau+1}, \vec{U}_{\tau+2}, \ldots)\}$ to $\mathbb{R}$. If the sequence of functions $V^i_\tau(M)$ converges uniformly to $\hat{V}^i_\tau$ on any bounded subset of $\mathbb{R}^\infty \times \mathbb{R}^N$, then (A.24) will be satisfied. We now prove a second lemma:

Lemma 4. *Let B be a compact subset of $\mathbb{R}^\infty \times \mathbb{R}^N$, and assume that (13) is satisfied. Then $V^i_\tau(M)$ converges uniformly to $\hat{V}^i_\tau$ on B.*

Proof. Equation (A.13) shows that for all $s \leq M$, $a^{ij}_{\tau+s}(M) = \hat{a}^{ij}_{\tau+s}$. Let $\bar{U} = \max_j\{\sup_s\{U^j(c_{\tau+s})\}\}$ be the largest component of any $\vec{U} \in B$. For any $\vec{U} \in B$,

$$\begin{aligned}
\sup_{\vec{U}\in B} \left| V^i_\tau(M) - \hat{V}^i_\tau \right| &= \sup_{\vec{U}\in B} \left| \sum_{s=1}^{\infty} \sum_{j=1}^{N} a^{ij}_{\tau+M+s}(M) U^j(c_{\tau+M+s}) - \sum_{s=1}^{\infty} \sum_{j=1}^{N} \hat{a}^{ij}_{\tau+M+s} U^j(c_{\tau+M+s}) \right| \\
&\leq \sum_{s=1}^{\infty} \sum_{j=1}^{N} \left[\left| a^{ij}_{\tau+M+s}(M) \right| + \left| \hat{a}^{ij}_{\tau+M+s} \right| \right] \bar{U}
\end{aligned}$$

By Lemma 3, $\hat{\mu} < 1$ also implies $\mu(M) < 1$ for all M, so we know that $\lim_{M\to\infty} a^{ij}_{\tau+M+s}(M) = 0 = \lim_{M\to\infty} \hat{a}^{ij}_{\tau+M+s}$ for all i, j. Thus

$$\lim_{M\to\infty} \sup_{\vec{U}\in B} \left| V^i_\tau(M) - \hat{V}^i_\tau \right| = 0.$$

Hence $V^i_\tau(M)$ converges uniformly to $\hat{V}^i_\tau$. $\square$

This concludes the infinite order case.

The final step of the proof is to show that if the graph G, defined in the statement of the proposition, has a directed cycle of length N, then there exists an $M' > 0$ such that for all $M > M'$ the matrix $\mathbf{\Phi}_M$ is primitive. We demonstrate this using a graphical argument.

Consider an aribtrary $R \times R$ matrix B_{ij}, and form a directed graph $H(B)$ on nodes $1 \ldots R$, where there is an edge from node i to node j iff $B_{ij} > 0$. The matrix B_{ij} is primitive if there exists an integer $k \geq 1$ such that there is a path of length k from each node i to every other node j in $H(B)$. If $H(B)$ is *strongly connected*, i.e., there exists a path from every node to every other node, then a sufficient condition for B_{ij} to be primitive is for there to be at least one node that is connected to itself.

Now consider our $NM \times NM$ matrices $\mathbf{\Phi}_M$. To construct the directed graph $H(\mathbf{\Phi}_M)$ associated with $\mathbf{\Phi}_M$ in a convenient form, follow the following procedure: Construct an $M \times N$ grid of nodes (where N is the number of planners), with node (m, n) representing planner n at time $\tau + m$. For all $m > 1, n$, construct a directed edge from node (m, n) to node $(m-1, n)$. In addition, construct a directed edge from node $(1, n)$ to node (m', n') if $f_{m'}^{nn'} > 0$.

As an example, take the case $M = N = 3$, i.e., a third order model with three planners. In this case $\mathbf{\Phi}_M$ is a 9×9 matrix. Assume that $f_s^{ii} > 0$ for all $i, s = 1 \ldots 3$, that $f_1^{12}, f_1^{23}, f_1^{31} > 0$, and that $f_s^{ij} = 0$ otherwise. Figure F.1 represents the directed graph associated with the matrix $\mathbf{\Phi}_3$ in this case.

Examination of the figure shows that since $f_s^{ii} > 0$, each of the 'column' subgraphs $\{(m, 1)\}, \{(m, 2)\}, \{(m, 3)\}, m = 1 \ldots 3$ is strongly connected. Moreover, the cycle between columns (the red dashed edges) connects the columns to each other, and causes the entire graph to be strongly connected. Since each node in the first row is connected to itself, the matrix $\mathbf{\Phi}_3$ in this example is primitive.

Returning to the general case, suppose that $f_s^{ii} > 0$ for all i and s. From the example in Figure F.1 it is clear that this implies that for each fixed i the subgraph $\{(m, i) | m = 1 \ldots \infty\}$ is strongly connected, with each of the nodes $(1, i)$ connected to itself. Thus, if there is a directed cycle between all of the 'columns' of the graph $H(\mathbf{\Phi}_{M'})$ for some M', then for all $M > M'$, $H(\mathbf{\Phi}_M)$ is strongly connected, and contains nodes that are connected to themselves. Hence for all $M > M'$, $\mathbf{\Phi}_M$ is a primitive matrix. This concludes the proof. $\square$

STEP 2:

We now show that when the conditions of Proposition A.I are satisfied, all non-dogmatic theories yield the same long-run SDR, and we compute an explicit formula for this con-

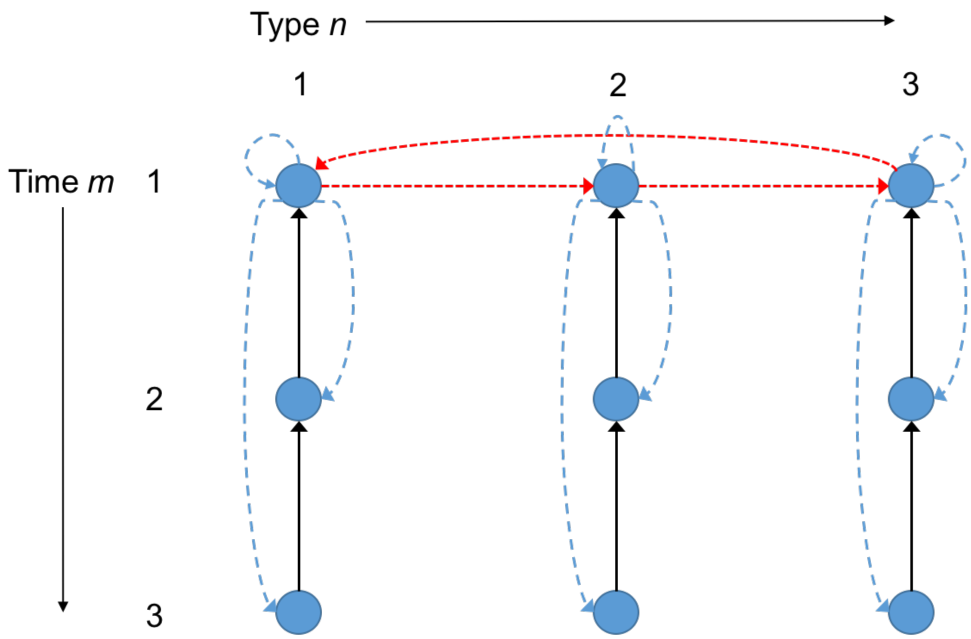


Figure F.1: The directed graph $H(\mathbf{\Phi}_3)$ associated with the matrix in our example. The vertical black edges arise from the identity matrices in the definition of $\mathbf{\Phi}_M$ (see (A.15)). The dashed blue edges arise from $f_s^{ii} > 0$, and the dashed red edges from $f_1^{12}, f_1^{23}, f_1^{31} > 0$.

sensus discount rate.

Begin by defining

$$\hat{\rho} = -\ln \hat{\mu},$$

where $\hat{\mu}$ is defined in (A.22). When the conditions of Proposition A.I hold we know that

$$a_s^{ij} \sim K_{ij}(s) e^{-\hat{\rho} s} \tag{A.25}$$

where $\sim$ denotes asymptotic behaviour as $s \to \infty$, and the multiplicative factors $K_{ij}(s)$ satisfy $\lim_{s\to\infty} \frac{1}{s} \ln K_{ij}(s) = 0$.

Now integrate the definition of $\eta^j(c)$ in (17) to find[1]

$$(U^j)'(c) = \exp\left(-\int_0^c \frac{\eta^j(x)}{x} dx \right).$$

Make the change of variables $x = c_\tau e^{gs'}$ in the integral in the exponent (recall that g is the

[1] In other words, solve the differential equation $-c(U^j)''/(U^j)' = \eta^j(c)$ for $(U^j)'(c)$.

long-run consumption growth rate), and evaluate $(U^j)'(c)$ at $c = c_\tau e^{gs}$ to find

$$(U^j)'(c_\tau e^{gs}) = \exp\left(-g \int_0^s \eta^j(c_\tau e^{gs'})ds'\right).$$

Defining

$$\hat{\eta^j} = \begin{cases} \lim_{c\to\infty} \eta^j(c) & g > 0 \\ \lim_{c\to 0} \eta^j(c) & g < 0 \end{cases} \tag{A.26}$$

we see that the $s \to \infty$ asymptotic behaviour of marginal utility is given by

$$(U^j)'(c_\tau e^{gs}) \sim L_j(s) e^{-g\hat{\eta^j}s} \tag{A.27}$$

for some functions $L_j(s)$ that satisfy $\lim_{s\to\infty} \frac{1}{s} \ln L_j(s) = 0$. Combining (A.25) and (A.27), we find

$$\begin{aligned} r^i(s) &= -\frac{1}{s} \ln\left(\frac{1}{(U^i)'(c_\tau)} \sum_{j=1}^N a_s^{ij} (U^j)'(c_{\tau+s})\right) \\ &\sim -\frac{1}{s} \ln\left(\sum_j K_{ij}(s) L_j(s) e^{-\hat{\rho}s} e^{-\eta^j gs}\right) \\ &\sim \hat{\rho} - \frac{1}{s} \ln\left(\sum_j K_{ij}(s) L_j(s) e^{-\eta^j gs}\right) \end{aligned}$$

Define $\tilde{K}_{ij}(s) = K_{ij}(s)L_j(s)$, and let q be the index of the planner with the lowest (highest) value of $\hat{\eta}^j$ when $g > 0$ ($g < 0$). Then

$$\begin{aligned} \sum_j K_{ij}(s) L_j(s) e^{-\eta^j gs} &= \sum_j \tilde{K}_{ij}(s) e^{-\eta^j gs} \\ &= \tilde{K}_{iq}(s) e^{-\eta^q gs} \left(1 + \sum_{j\neq q} \frac{\tilde{K}_{ij}(s)}{\tilde{K}_{iq}(s)} e^{-(\eta^j - \eta^q)gs}\right) \end{aligned}$$

Since $\eta^j - \eta^q > 0$ for all $j \neq q$ when $g > 0$, and $\eta^j - \eta^q < 0$ for all $j \neq q$ when $g < 0$,

$$\sum_j K_{ij}(s) L_j(s) e^{-\eta^j gs} \sim \tilde{K}_{iq}(s) e^{-\hat{\eta}gs},$$

where $\hat{\eta}$ is given by (18). Thus

$$\begin{aligned} r^i(s) &\sim \hat{\rho} - \frac{1}{s}\ln\left(\tilde{K}_{iq}(s)e^{-\hat{\eta}gs}\right) \\ \Rightarrow \lim_{s\to\infty} r^i(s) &= \hat{\rho} + \hat{\eta}g. \end{aligned}$$

D Consensus long-run SDRs under uncertainty

It is straightforward to extend the proof of Proposition 1 to the case where future consumption is uncertain. If consumption is uncertain non-dogmatic planners' IWFs are simply the expectation over their deterministic IWFs, i.e.,

$$V_\tau^i = \mathsf{E}_{c_{\tau+1},c_{\tau+2},\ldots} \sum_{s=0}^{\infty}\sum_{j=1}^{N} a_s^{ij} U^j(c_{\tau+s})$$

where $\mathsf{E}_{c_{\tau+1},c_{\tau+2},\ldots}$ denotes the expectation over future consumption values, and the coefficients a_s^{ij} are determined by the dynamical system in (A.11), as in the deterministic case.

The analysis of the consensus long-run SDR now proceeds in close analogy to the second part of the proof of Proposition 1. The consensus long-run pure rate of social time preference is unchanged, however examination of the proof shows that we need to account for the effect of expectations on the growth terms in the Ramsey rule.

Under uncertainty planners' marginal rates of substitution between consumption today and consumption s years from now are given by:

$$e^{-r^i(s)s} = MRS_s^i = \frac{\sum_{j=1}^N a_s^{ij}\mathsf{E}_{c_{\tau+s}}(U^j)'(c_{\tau+s})}{(U^i)'(c_\tau)} \tag{A.28}$$

Define a planner specific 'certainty equivalent' long-run growth rate $\hat{g}_j$ by requiring that

$$(U^j)'(e^{\hat{g}_j s}c_\tau) \equiv \mathsf{E}_g(U^j)'(e^{gs}c_\tau) \tag{A.29}$$

as $s \to \infty$, i.e.,

$$\hat{g}_j \equiv \lim_{s\to\infty}\frac{1}{s}\log\left[((U^j)')^{-1}\left(\mathsf{E}_g(U^j)'(e^{gs}c_\tau)\right)\right]. \tag{A.30}$$

The long-run consumption growth rate g is uncertain in this expression, and E_g denotes expectations over the value of g. In analogy with (A.26), define

$$\hat{\eta}_j(\hat{g}_j) = \begin{cases} \lim_{c\to\infty} \eta^j(c) & \hat{g}_j > 0 \\ \lim_{c\to 0} \eta^j(c) & \hat{g}_j < 0. \end{cases}$$

Then for large s, we know from (A.27) that

$$\mathsf{E}_{c_{\tau+s}}(U^j)'(c_{\tau+s}) = (U^j)'(e^{\hat{g}_j s}c_\tau) \sim e^{-\hat{g}_j\hat{\eta}_j(\hat{g}_j)s}$$

where $\sim$ denotes $s\to\infty$ asymptotic behaviour, as before.

As in the deterministic case, we see from (A.28) that planner i's long-run elasticity of marginal utility is determined by the term that dominates the sum

$$\sum_{j=1}^{N} a_s^{ij}\mathsf{E}_{c_{\tau+s}}(U^j)'(c_{\tau+s}) \sim \sum_j a_s^{ij} e^{-\hat{g}_j\hat{\eta}_j(\hat{g}_j)s}$$

as $s\to\infty$. This sum is dominated by the exponential with the minimum value of $\hat{g}_j\hat{\eta}_j(\hat{g}_j)$ (which may be negative), for all i. We thus conclude that the consensus long-run SDR under uncertainty is given by

$$\hat{\rho} + \min_i\{\hat{g}_i\hat{\eta}_i(\hat{g}_i)\} \tag{A.31}$$

As an example of the application of this formula suppose that planners' utility functions are iso-elastic with elasticities of marginal utility η_i, i.e., $(U^i)'(c) = c^{-\eta_i}$. In addition, assume that consumption growth is asymptotically log-normally distributed, i.e.,

$$\log g \sim \mathcal{N}(\mu, \sigma^2).$$

From (A.29) planner i's certainty equivalent long-run growth rate $\hat{g}_i$ is thus defined by requiring that at large s,

$$e^{-\eta_i\hat{g}_i s}(c_\tau)^{-\eta_i} \equiv \mathsf{E}_g e^{-\eta_i g s}(c_\tau)^{-\eta_i} = e^{-(\eta_i\mu - \frac{1}{2}\eta_i^2\sigma^2)s}(c_\tau)^{-\eta_i}$$
$$\Rightarrow \hat{g}_i = \mu - \frac{1}{2}\eta_i\sigma^2$$

Since elasticities of marginal utility are constant by assumption we know that $\hat{\eta}_i(\hat{g}_i) = \eta_i$, and thus the consensus long-run SDR in this example is given by

$$\hat{\rho} + \min_i\{\mu\eta_i - \frac{1}{2}\eta_i^2\sigma^2\}.$$

E Proof of Proposition 2

Part 1 of the proposition is immediate from point 4 in our statement of the Perron-Frobenius theorem in Proposition A.I. Part 2 of the proposition follows from the fact that the eigenvalues of a matrix are continuous in its entries. Consider a set of N 'dogmatic' models, in which each planner assigns weight only to her own theory in future periods. This set of N independent planners' time preferences can be represented as a single non-dogmatic set of N planners as in (12), but where $f_s^{ij} = 0$ if $j \neq i$. As in the proof of Proposition A.I, begin by considering a model of finite order M, so that no planner places any weight on any IWF more than M years ahead. Equation (A.16) shows that the asymptotic behaviour of such a model can be described by first order difference equations of the form:

$$\vec{Y}_s^j = \mathbf{\Phi}_M^0 \vec{Y}_{s-1}^j.$$

In this case however, the matrix $\mathbf{\Phi}_M^0$, defined in (A.15), is reducible. The largest eigenvalue of $\mathbf{\Phi}_M^0$ is the rate of decline of the utility weights of the most patient dogmatic planner in the long-run. As $M \to \infty$, the set of eigenvalues of $\mathbf{\Phi}_M^0$ contains $\hat{\mu}_1^i$, the long-run utility discount factor of planner i, and all eigenvalues of $\mathbf{\Phi}_M^0$ are less than or equal to $\max_i\{\hat{\mu}_1^i\}$.

Now consider the continuous set of models with weights $f_s^{ij}(\epsilon)$, where $\epsilon > 0$. Let $\mathbf{\Phi}_M(\epsilon)$ be the corresponding $\mathbf{\Phi}_M$ matrix for this set of models, where by assumption $\lim_{\epsilon \to 0^+} \mathbf{\Phi}_M(\epsilon) = \mathbf{\Phi}_M^0$. The consensus long-run discount factor in model ϵ of order M, denoted $\mu_1(\epsilon, M)$ is the largest eigenvalue of $\mathbf{\Phi}_M(\epsilon)$. Define

$$\hat{\mu}_1(\epsilon) = \lim_{M \to \infty} \mu_1(M, \epsilon).$$

We know that this limit exists, due to the proof of Proposition A.I. Since the matrix $\mathbf{\Phi}_M(\epsilon)$ is continuous in $\epsilon > 0$, and in the limit as $M \to \infty$ the largest eigenvalue of $\mathbf{\Phi}_M(0) = \mathbf{\Phi}_M^0$ is equal to $\max_i\{\hat{\mu}_1^i\}$, we must have

$$\lim_{\epsilon \to 0^+} \hat{\mu}_1(\epsilon) = \max_i\{\hat{\mu}_1^i\}.$$

Since $\hat{\rho}(\epsilon) = -\ln \hat{\mu}_1(\epsilon)$ by definition, the result follows.

F Comparative statics of the consensus long-run pure rate of social time preference

It is naturally of interest to ask how the consensus long-run pure rate of social time preference $\hat{\rho}$ depends on the intertemporal weights f_s^{ij}. Unfortunately strong comparative statics results on this question are likely out of reach. Technically, we need to understand how the spectral radius (i.e., largest eigenvalue) of the matrices $\mathbf{\Phi}_M$ from Proposition A.I behaves when we spread out or contract the distribution of weights f_s^{ij}. In order to sign the effect of a spread in the weights we require something akin to a convexity property for the spectral radius. Unfortunately, it is known that the spectral radius of a matrix is a convex function of its diagonal elements, but not of the off-diagonal elements (Friedland, 1981).[2]

This section describes a special case of the model in which clean comparative statics are possible. Assume that planner i's intertemporal weights f_s^{ij} depend on a parameter $\lambda_i \subset \mathbb{R}^+$, i.e., $f_s^{ij} = f_s^{ij}(\lambda_i)$. Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N)$ be the vector of planners' λ parameters, and assume that $\vec{\lambda}$ takes values in a convex subset of $\mathbb{R}^{N+}$. Using the notation of Proposition A.I we write the matrix of weights f_s^{ij} at a fixed value of s as $\mathbf{F}_s(\vec{\lambda})$, where we now emphasize the dependence of these weights on the parameter vector $\vec{\lambda}$. We will say that preferences are **symmetric in** $\vec{\lambda}$ iff for all permutation matrices[3] $\mathbf{P}$,

$$\mathbf{F}_s(\mathbf{P}\vec{\lambda}) = \mathbf{P}\mathbf{F}_s(\vec{\lambda})\mathbf{P}^T \tag{A.32}$$

for all s, where $\mathbf{P}^T$ is the transpose of $\mathbf{P}$. Intuitively, if preferences are symmetric in $\vec{\lambda}$, switching any two planners' values of λ is equivalent to switching their entire set of intertemporal weights, as this induces a permutation of the weight matrix $\mathbf{F}_s(\vec{\lambda})$. The parameters λ_i are thus 'sufficient statistics' for planners' intertemporal weights, and switching $\lambda_i \leftrightarrow \lambda_j$ is equivalent to relabelling $i \leftrightarrow j$.

As an example of preferences that are symmetric in $\vec{\lambda}$ consider the following:

$$f_s^{ij} = \begin{cases} \beta(s, \lambda_i) x_s & j = i \\ \beta(s, \lambda_i) \frac{1-x_s}{N-1} & j \neq i \end{cases} \tag{A.33}$$

[2]Similarly, it is not possible to sign the effect of premultiplying $\mathbf{\Phi}_M$ by a doubly stochastic matrix, as the spectral radius of a product of two matrices is not sub-multiplicative in general. Gelfand's formula shows that the spectral radius of a matrix product is sub-multiplicative if the matrices in question commute, but this is not much use for our purposes.

[3]A square matrix is a permutation matrix if each of its rows and each of columns contains exactly one entry of 1, and zeros elsewhere.

where $x_s \in [1/N, 1)$ for all $s = 1 \dots \infty$, and $\sum_{s=1}^{\infty} \beta(s, \lambda) < 1$ for all $\lambda \in I \subset \mathbb{R}^+$. In this model the time dependence of planners' intertemporal weights f_s^{ij} has a common functional form, given by a discount function $\beta(s, \lambda)$ on the IWF of selves s years in the future, where $\lambda > 0$ is a parameter. Variations in planners' attitudes to time are solely due to differences in their values of λ. The parametric model defined in (22), which we used in Section IV of the paper, is of this form if $\gamma_i = \gamma$ for all i.

Let $\hat{\rho}(\vec{\lambda})$ be the consensus long-run pure rate of time preference in a model that is characterized by the parameter vector $\vec{\lambda}$.

Proposition A.II. *Assume that planners' time preferences are symmetric in $\vec{\lambda}$ and that $f_s^{ij}(\lambda)$ is strictly log-convex in $\lambda > 0$ for all i, j, s. Then if the parameter vector $\vec{\lambda}^A$ majorizes*[4] *$\vec{\lambda}^B$,*

$$\hat{\rho}(\vec{\lambda}^A) < \hat{\rho}(\vec{\lambda}^B).$$

In words, this result says that if preferences are symmetric in $\vec{\lambda}$, intertemporal weights are log-convex functions of λ, and planners in group A disagree more about the parameter λ than planners in group B, the consensus long-run pure rate of time preference will be *lower* in group A than in group B.

I will provide some interpretation of the log-convexity condition in examples below, but first we turn to the proof.

Proof. The proof relies on the following result due to Kingman (1961): Let $b_{ij}(\theta) \geq 0$ be the elements of a non-negative matrix $\mathbf{B}$, where $\theta \in \mathbb{R}$ is a parameter. If $b_{ij}(\theta)$ is log-convex in θ for all i, j, the spectral radius of $\mathbf{B}$ is a log-convex function of θ. Remark 1.3 in Nussbaum (1986) observes that Kingman's result can be extended as follows: Let $\vec{\theta}$ be a vector of parameters that takes values in a convex set, and assume that the elements $b_{ij}(\vec{\theta}) \geq 0$ of a matrix $\mathbf{B}$ are log-convex functions of $\vec{\theta}$. Then the spectral radius of $\mathbf{B}$ is log-convex is $\vec{\theta}$.

We will employ the usual trick of working with finite order models first (i.e., setting f_s^{ij} to zero for $s > M$), and taking a limit as $M \to \infty$ at the end. The consensus long-run pure rate of time preference in a model of order M is determined by the largest eigenvalue of $\mathbf{\Phi}_M$, defined in (A.15). Denote this eigenvalue by $\hat{\mu}_M(\vec{\lambda})$. If the matrix elements $f_s^{ij}(\lambda)$ are log-convex functions of the scalar variable λ, then $f_s^{ij}(\vec{\lambda}) = f_s^{ij}(\lambda_i)$ is also a log-convex

[4] $\vec{\lambda}^A$ majorizes $\vec{\lambda}^B$ iff there exists a doubly stochastic matrix $\mathbf{H}$ such that $\vec{\lambda}^B = \mathbf{H}\vec{\lambda}^A$. Intuitively, the elements of $\vec{\lambda}^A$ are 'more spread out' than those of $\vec{\lambda}^B$, and the sums of their elements are equal. See e.g. Marshall (2010) for a discussion of majorization and its relationship to e.g. stochastic orders and inequality measures.

function of the vector of parameters $\vec{\lambda}$. Thus, if $f_s^{ij}(\lambda)$ is log-convex (or identically zero) for all i, j, s, $\hat{\mu}_M(\vec{\lambda})$ is a log-convex function of $\vec{\lambda}$.

The final step of the proof is to observe that because of the symmetry of the set of intertemporal weights in (A.32) the spectral radius must be a symmetric function of $\vec{\lambda}$, i.e., any permutation of the elements of $\vec{\lambda}$ will leave the spectral radius unchanged. This follows since the eigenvalues of a matrix are invariant under the permutations (A.32). Since $\hat{\mu}_M(\vec{\lambda})$ is a log convex, symmetric function of $\vec{\lambda}$, its log is Schur-convex. Since $\hat{\mu}_M(\vec{\lambda}) = e^{-\hat{\rho}_M(\vec{\lambda})}$, this implies that $\hat{\rho}_M(\vec{\lambda})$ is Schur-concave in $\vec{\lambda}$. Thus by the properties of Schur-concave functions, if $\vec{\lambda}^A$ majorizes $\vec{\lambda}^B$ we must have

$$\hat{\rho}_M(\vec{\lambda}^A) < \hat{\rho}_M(\vec{\lambda}^B).$$

The final result follows by taking the limit as $M \to \infty$. □

As an initial example of the application of this result, consider a model in which the discount function $\beta(s, \lambda)$ in the example in (A.33) declines exponentially, i.e.,

$$\beta(s, \lambda) = (1 + \lambda)^{-s}.$$

This discount function satisfies $\log \beta(s, \lambda) = -s \log(1 + \lambda)$, which is strictly convex in λ. Thus the result applies – more disagreement about the parameter λ *decreases* the consensus long-run pure rate of social time prefenence.

We can extend this finding to a more general class of models by assuming that $\beta(s, \lambda) = \tilde{\beta}(\lambda s)$, i.e., the parameter λ acts to rescale the time variable s. Following Prelec (2004) we will say that $\tilde{\beta}(s)$ exhibits *decreasing impatience* if $\log \tilde{\beta}(s)$ is a convex function of s for $s > 0$. Discount functions that exhibit decreasing impatience have the form $\tilde{\beta}(s) = e^{-h(s)}$ where $h(s)$ is a concave function. The rate of increase of $h(s)$ (which measures impatience) slows as the time horizon s increases.

Corollary 1. *Assume that $\tilde{\beta}(s)$ exhibits decreasing impatience, and that the parameter vector $\vec{\lambda}^A$ majorizes $\vec{\lambda}^B$. Then*

$$\hat{\rho}(\vec{\lambda}^A) < \hat{\rho}(\vec{\lambda}^B).$$

Thus, for example, in a hyperbolic model (see e.g. Prelec, 2004) we would have

$$\tilde{\beta}(s) = (1 + s)^{-(1+p)} \Rightarrow \beta(s, \lambda) = \tilde{\beta}(\lambda s) = (1 + \lambda s)^{-(1+p)}$$

where $p > 0$ is a parameter. $\tilde{\beta}(s)$ is log convex in s, so more disagreement about λ reduces the consensus long-run pure rate of time preference in this model.

G Details of calibration

The data I use to calibrate the model and generate the results in Figures 1a and 1b are taken from a recent survey by Drupp et. al. (2018). They surveyed expert economists who have published papers on social discounting, asking for their opinions on, amongst other things, the appropriate values of the pure rate of social time preference and the elasticity of marginal social utility. The distribution of respondents' views on these two parameters is plotted in Figure F.2.

The calibration assumption I use is that the data in Figure F.2 correspond to 'dogmatic' views on the IWF, and in particular that these data correspond to the parameters of a discounted utilitarian IWF with iso-elastic utility function. This assumption is consistent both with the survey authors' description of what they aim to elicit in their survey, and with the participants' responses. See footnote 19 of the main text for further explanation.

The calibration is made slightly delicate by the fact that there is no version of the model in (12) in which planners place non-zero weight on *all* future selves that reduces to a discounted utilitarian IWF. I calibrate the parametric model in (22) so that when the weight on own preferences $x = 1$, planners' time preferences can be represented by a function that is a close approximation to a discounted utilitarian IWF, but still assigns non-zero weight to all future selves.

To calibrate the values of γ_i, α_i in (22), I use the fact that when $x = 1$ the model reduces to a set of N independent intertemporal preferences of the form:

$$V_\tau^i = U^i(c_\tau) + \gamma_i \sum_{s=1}^{\infty} (\alpha_i)^s V_{\tau+s}^i, \tag{A.34}$$

where $\alpha_i \in (0, 1)$ and $\gamma_i \in (0, \frac{1-\alpha_i}{\alpha_i})$. These time preferences have been studied by Saez-Marti & Weibull (2005), and axiomatized by Galperti & Strulovici (2017). It is straightforward to show that they have the following equivalent representation:

$$V_\tau^i = U^i(c_\tau) + \sum_{s=1}^{\infty} \kappa_i^s \left(\frac{1+\gamma_i}{\gamma_i} \right)^{s-1} U^i(c_{\tau+s}), \text{ where } \kappa_i = \alpha_i \gamma_i. \tag{A.35}$$

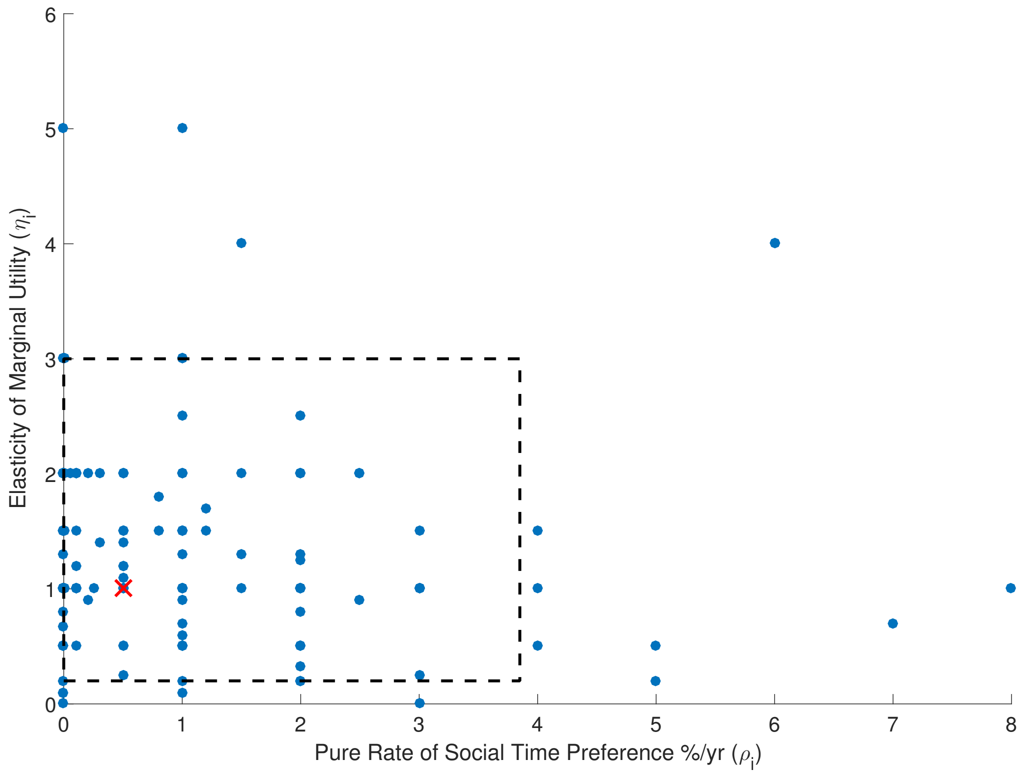


Figure F.2: Experts' recommended values for the pure rate of social time preference (ρ_i), and the elasticity of marginal utility (η_i) for appraisal of long-run public projects, from the Drupp et. al. (2018) survey. 173 responses were recorded. The dashed box depicts data points that fall inside the $5-95\%$ ranges of both parameters. The red cross indicates the location of the median values of ρ_i and η_i.

Writing out the sequence of intertemporal utility weights in this model explicitly,

$$1, \kappa_i, \left(\frac{1+\gamma_i}{\gamma_i}\right)\kappa_i^2, \left(\frac{1+\gamma_i}{\gamma_i}\right)^2\kappa_i^3, \left(\frac{1+\gamma_i}{\gamma_i}\right)^3\kappa_i^4, \ldots, \tag{A.36}$$

it is clear that if we take the limit as $\gamma_i \to \infty$ of this model holding κ_i fixed, we recover discounted utilitarian time preferences with discount factor κ_i. For any finite γ_i the preferences in (A.35) are quasi-hyperbolic, with a short run pure time discount factor given by κ_i, and a long-run pure time discount factor given by $\left(\frac{1+\gamma_i}{\gamma_i}\right)\kappa_i$.

Recall that the data in Figure F.2 correspond to the parameters of a discounted utilitarian IWF, and that our calibration assumption is that these data correspond to the

$x \to 1$ limit of the non-dogmatic model (22). The sequence in (A.36) shows that to ensure consistency with the calibration assumption we must calibrate κ_i so that

$$\kappa_i = e^{-\rho_i}, \tag{A.37}$$

where ρ_i is survey respondent i's recommended value for the pure rate of social time preference. In addition, we must choose γ_i sufficiently large that the model closely approximates discounted utilitarian time preferences. Notice from (A.36) that the discount factor of planner i for $s > 1$ is given by

$$(1 + \gamma_i^{-1})\kappa_i \approx e^{-(\gamma_i^{-1} + \rho_i)}$$

when γ_i^{-1} is small. Thus $\gamma_i^{-1} = 1\%$, for example, corresponds to an additional 1%/yr discount rate on the long-run future, over and above the short run discount rate ρ_i. Thus if γ_i^{-1} is too large, the model will provide a poor fit to a discounted utilitarian IWF when $x = 1$, since non-dogmatic planners will exhibit sharply quasi-hyperbolic time preferences in this case. To ensure that the model is a close approximation to discounted utilitarianism when $x = 1$, but also that all planners place non-zero weight on all future selves' IWFs (which requires γ_i be finite), we must pick γ_i^{-1} to be small but non-zero for all i, i.e., $\gamma_i^{-1} \approx 0.1\%$. The numerical results presented in the paper are robust to heterogeneity in γ_i^{-1}, provided that none of these parameters is too large relative to respondents' pure rates of social time preference. As stated, γ_i^{-1} *must* be small if the calibrated model is to provide a good approximation to discounted utilitarian IWFs at $x = 1$.

In addition, I assume in line with Drupp et. al. (2018) that planners' utility functions are iso-elastic, i.e.,

$$U^i(c) = \frac{c^{1-\eta_i}}{1 - \eta_i} \tag{A.38}$$

for some $\eta_i > 0$. This implies that the elasticity of marginal utility is constant and equal to η_i, and I simply calibrate η_i to be each respondent's preferred value of this elasticity.

The requirement that the calibrated model provide a close approximation to discounted utilitarian IWFs in an appropriate 'dogmatic' limit implies that the results depicted in Figure 1a are robust to alternative specifications of the weights w_s^{ij} for $s > 1$. The reason for this is that, as discussed above (and as is evident from (A.36)), in order for the model to closely approximate discounted utilitarian IWFs at $x = 1$, the calibrated values of γ_i must be large, which in turn implies that the values of α_i must be correspondingly small since $\kappa_i = e^{-\rho_i} = \gamma_i \alpha_i$, where ρ_i is the observed pure time preference rate recommendation of respondent i. Now notice that the models in (22) can be written as

$$V_\tau^i = U^i(c_\tau) + \gamma_i \left[\alpha_i \sum_{j=1}^{N} w_1^{ij} V_{\tau+1}^j + (\alpha_i)^2 \sum_{j=1}^{N} w_2^{ij} V_{\tau+2}^j + \mathcal{O}((\alpha_i)^3) \right].$$

Since $(\alpha_i)^s \ll \alpha_i$ for all $s \geq 2$ if $\alpha_i \ll 1$, it does not much matter how the weights w_s^{ij} behave for $s \geq 2$. Even if a weight x is given to current preferences at *every* future maturity, i.e.,

$$w_s^{ij} = \begin{cases} x & i = j \\ \frac{1-x}{1-N} & i \neq j \end{cases} \tag{A.39}$$

for all $s \geq 1$, the results of the simulations hardly change.[5]

H Changing the model's time step

This section of the appendix describes how to transform the parameters of the model used in Figure 1 when the time step is changed.

For the version of the model in question planners' time preferences took the form

$$V_\tau^i = U^i(c_\tau) + \gamma_i \left[\alpha_i \sum_{j=1}^{N} (\mathbf{P})_{i,j} V_{\tau+1}^j + (\alpha_i)^2 \sum_{j=1}^{N} (\mathbf{P}^2)_{i,j} V_{\tau+2}^j + \mathcal{O}((\alpha_i)^3) \right]$$

where $\mathbf{P}$ is the annual transition probability matrix defined in (22), which depends on the parameter x, i.e., the chance of a preference change in a year.

If the model's time step is changed from 1 year to $\Delta T > 0$ years the values of all its dynamical parameters must change as well. Consumption growth rates are multiplied by ΔT, and, as in the calibration methodology set out in Section G above, the values of α_i and γ_i must be recalibrated so that:

$$\kappa_i = \alpha_i \gamma_i = e^{-\rho_i \Delta T},$$
$$\gamma_i^{-1} \approx 0.1\% \times \Delta T$$

Transforming the matrix $\mathbf{P}$ is more complex. To make the version of the model with time

[5] Planners with beliefs (A.39) do not obey the consistency condition (14), but this has no relevance for this discussion.

step ΔT comparable to the original annual model, we need to find a stochastic matrix $\mathbf{Q}$ such that

$$\mathbf{Q} = \mathbf{P}^{\Delta T}. \tag{A.40}$$

When ΔT is not a positive integer (e.g., if $\Delta T = 1/12$ for a monthly time step) such matrix equations may have no solution, or multiple non-negative solutions. However, in our case the structure of the model ensures that there is a natural 'ΔTth power' of $\mathbf{P}$ for any $\Delta T > 0$, and for all interesting values of the parameter x.

Begin by observing that the eigenvalues of $\mathbf{P}$ are 1 (with algebraic multiplicity 1) and $\frac{Nx-1}{N-1}$ (with algebraic multiplicity $N-1$), and are thus positive provided that $x > 1/N$.[6] Matrices with positive eigenvalues have a unique 'principal power' that satisfies the equation (A.40) and itself has positive eigenvalues (see e.g., Horn & Johnson, 2013). It is essential that transforming the time step of the model does not change the signs of the eigenvalues of the model's transition probability matrix. If this were not the case the qualitative dynamics of preference change would not be preserved under a change of time step. One could, for example, find that planner's intratemporal weights w_s^{ij} oscillate with maturity s, where no such behaviour existed before.

Since $\mathbf{P}$ is diagonalizable, it can be written as

$$\mathbf{P} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & -1 & -1 & \dots & -1 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{pmatrix}$$

is a matrix whose jth column corresponds to the jth eigenvector of $\mathbf{P}$, and $\mathbf{D}$ is a diagonal matrix of corresponding eigenvalues, i.e., $(\mathbf{D})_{1,1} = 1$, $(\mathbf{D})_{j,j} = \frac{Nx-1}{N-1}$ for $j \neq 1$. The principal ΔTth power of $\mathbf{P}$ is given by

$$\mathbf{Q} = \mathbf{V}\mathbf{D}^{\Delta T}\mathbf{V}^{-1}.$$

for any $\Delta T > 0$.

Consider the case $\Delta T = 1/12$, corresponding to a model with a monthly time step. It

[6] The case $x < 1/N$ is not plausible.

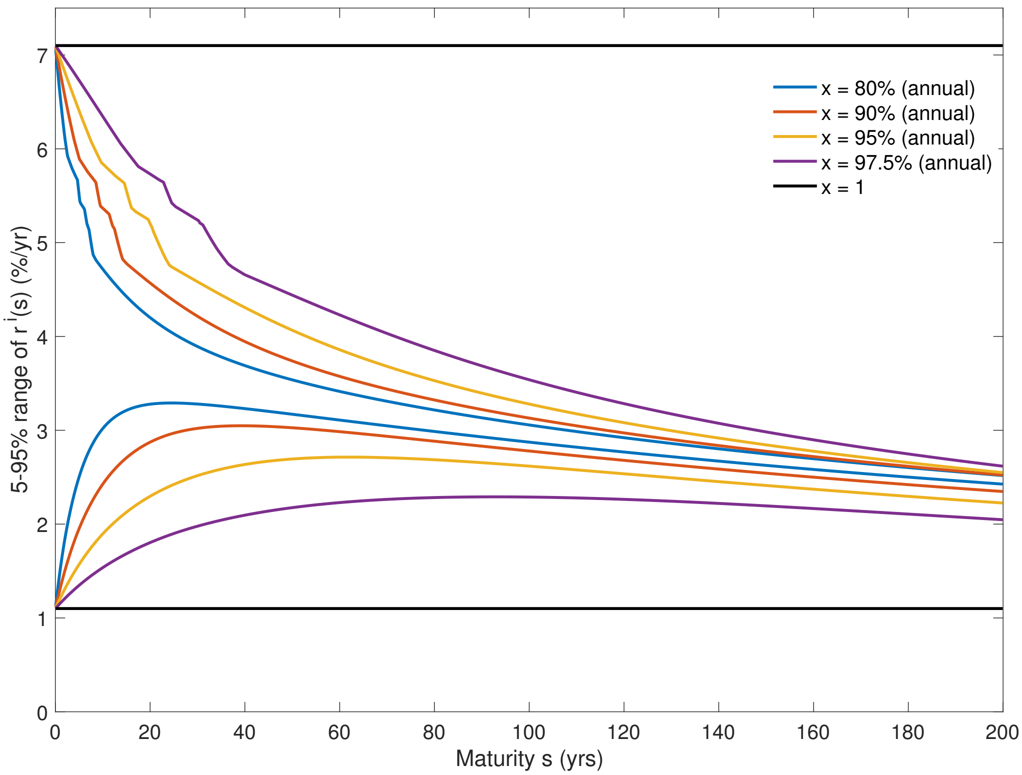


Figure F.3: Replication of Figure 1a in the paper for a monthly time step. To facilitate comparison with Figure 1a monthly discount rates have been converted to annual equivalents (vertical axis), and the horizontal axis is scaled to years, rather than months.

is clear from the definition in the previous equation that raising $\mathbf{Q}$ to the twelfth power yields the original matrix $\mathbf{P}$, and that $\mathbf{Q}$ has positive eigenvalues. The matrix $\mathbf{Q}$ is the only 12th root of $\mathbf{P}$ that has these properties.[7]

Figure F.3 presents an analogue of Figure 1a in the paper, however this time I have calibrated the model with a monthly time step using the procedure outlined above. The figure shows that there is no appreciable difference between versions of the model defined at different time steps provided that the model parameters are adjusted to reflect the change in time step.

Finally, I note that any version of the model defined with a discrete time step can be

[7]Other solutions of (A.40) have the same basic form as $\mathbf{Q}$ however we may replace any of the entries on the diagonal of $\mathbf{D}^{1/12}$ with any of the twelve complex roots of the corresponding eigenvalue of $\mathbf{P}$. As there is only one way of choosing these roots so that they are all positive (and real), there is a unique 'principal power' of $\mathbf{P}$.

thought of as an approximation to an underlying continuous model. Preferences could change at any instant, and there is some underlying infinitesimal transition probability matrix that could describe this continuous Markov process. But any discrete approximation of this process, at any temporal resolution, is legitimate – any behaviour of the continuous process, when aggregated up to a discrete time step ΔT by exponentiating the infinitesimal transition matrix, can be replicated by an 'ab initio' discrete model with time step ΔT. We lose nothing (at resolution ΔT) in this discrete approximation, although the entries of the discrete transition probability matrix (and hence the weight x) will differ according to the magnitude of ΔT.

I Decomposing non-dogmatic SDRs

This section studies the resolution of disagreement about the two components of the SDR – pure time preference and the consumption growth/inequality aversion term – separately. It shows that much of the rapid convergence of SDRs with maturity shown in Figure 1a is due to exponential convergence in the consumption growth term.

Section C of the appendix showed that the set of IWFs consistent with (12) can be represented by

$$V_\tau^i = \sum_{s=0}^{\infty}\sum_{j=1}^{N} a_s^{ij} U^j(c_{\tau+s}),$$

where the coefficients a_s^{ij} are determined by the difference equations in (A.11), and $a_0^{ii} = 1, a_0^{ij} = 0$ if $i \neq j$. Planner i's SDR at maturity s is

$$r^i(s) = -\frac{1}{s}\ln\left(\frac{\sum_{j=1}^{N} a_s^{ij}(U^j)'(c_{\tau+s})}{(U^i)'(c_\tau)}\right)$$

We decompose this expression into a pure time preference term and a consumption growth term. Defining

$$\tilde{\rho}^i(s) = -\frac{1}{s}\ln\left(\sum_{j=1}^{N} a_s^{ij}\right) \tag{A.41}$$

$$G^i(s) = -\frac{1}{s}\ln\left(\frac{\sum_{j=1}^{N} a_s^{ij}(U^j)'(c_{\tau+s})}{\left(\sum_{j=1}^{N} a_s^{ij}\right)(U^i)'(c_\tau)}\right). \tag{A.42}$$

we have

$$r^i(s) = \tilde{\rho}^i(s) + G^i(s). \tag{A.43}$$

To understand the meaning of $\tilde{\rho}^i(s)$, notice that $\sum_{j=1}^N a_s^{ij}$ is the total weight on utilities at maturity s in IWF i, i.e., it is a pure time discount factor. Hence $\tilde{\rho}^i(s)$ is IWF i's pure rate of social time preference at maturity s. To interpret $G^i(s)$ it is helpful to consider the case where the utility functions $U^i(c)$ are iso-elastic as in (A.38). Denoting the compound annual consumption growth rate at maturity s by g_s, we have[8]

$$G^i(s) = -\frac{1}{s} \ln \left(\frac{\sum_{j=1}^N a_s^{ij} e^{-\eta_j g_s s}}{\sum_{j=1}^N a_s^{ij}} \right). \tag{A.44}$$

Consider a hypothetical case in which planners have no normative insecurity, i.e., $a_s^{ij} = 0$ for all $j \neq i$; in this case we see that $G^i(s) = \eta_i g_s$, and we recover the familiar consumption growth term in the Ramsey rule. $G^i(s)$ is the generalization of this term to the non-dogmatic case, i.e., it is the contribution to the discount rate from consumption growth and inequality aversion. Figure F.4 plots the range of values for $\tilde{\rho}(s)$ and $G(s)$ as a function of maturity for the model calibration described in Section G of the appendix. The figure shows two important things. First, disagreements about the consumption growth term are significantly larger, and thus quantitatively more important, than disagreements about the pure rate of social time preference.[9] Second, although the range of values for $G(0)$ is larger than that for $\tilde{\rho}(0)$, disagreements about this term reduce substantially faster as maturity s increases. The expression for $G^i(s)$ in (A.44) suggests why this occurs. The argument of the log in this expression is a weighted sum of exponential functions, and thus converges exponentially fast to $e^{-\min_j\{\eta_j g_s\}s}$ as s increases. For example, if consumption growth is a constant 2%/yr, and we take $\eta = 2$ as a modal value of η, and $\eta = 0.05$ as the smallest value of η, at a maturity of 50 years we have $e^{-2\times 0.02\times 50} = 0.13$, and $e^{-0.05\times 0.02\times 50} = 0.95$. Thus values of $\eta_i g_s$ that differ substantially from $\min_j\{\eta_j g_s\}$ receive little weight at long maturities, causing the values of $G(s)$ to converge rapidly.

To relate variation in the components $\tilde{\rho}(s)$ and $G(s)$ back to variation in the SDR

[8]For convenience in this calculation we have chosen units so that current consumption $c_\tau = 1$. This is without loss of generality.

[9]The reader may wonder why the ranges for $\tilde{\rho}(s)$ and $G(s)$ depicted in Figure F.4 do not sum to the range for $r(s)$ in Figure 1a. The answer is that the ranges in Figure F.4 are properties of the *marginal* distributions of $\tilde{\rho}(s)$ and $G(s)$, while the range of their sum $r(s)$ depends on the *joint* distribution of these two quantities. Figure F.4 demonstrates how disagreements about these two independently meaningful quantities reduce as a function of maturity.

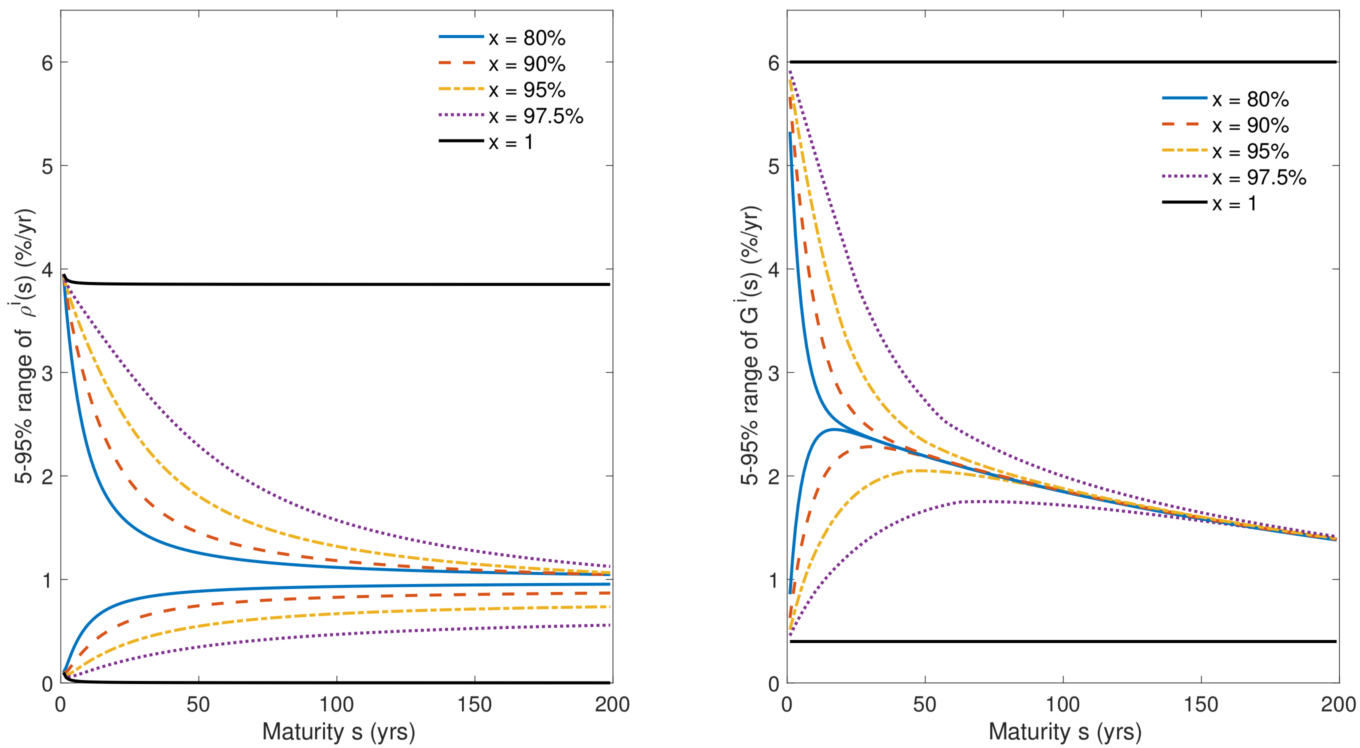
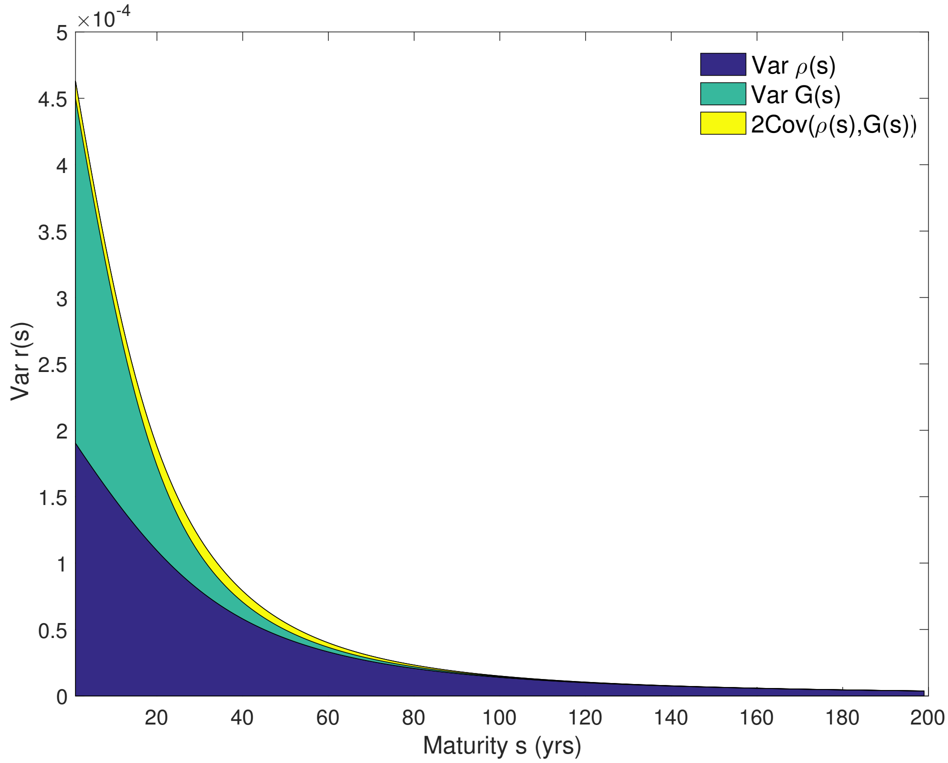


Figure F.4: Range of values for the two components of the SDR – the pure rate of social time preference ($\tilde{\rho}(s)$, left), and the consumption growth term ($G(s)$, right) – as a function of maturity s. The model calibration is the same as in Figure 1a.

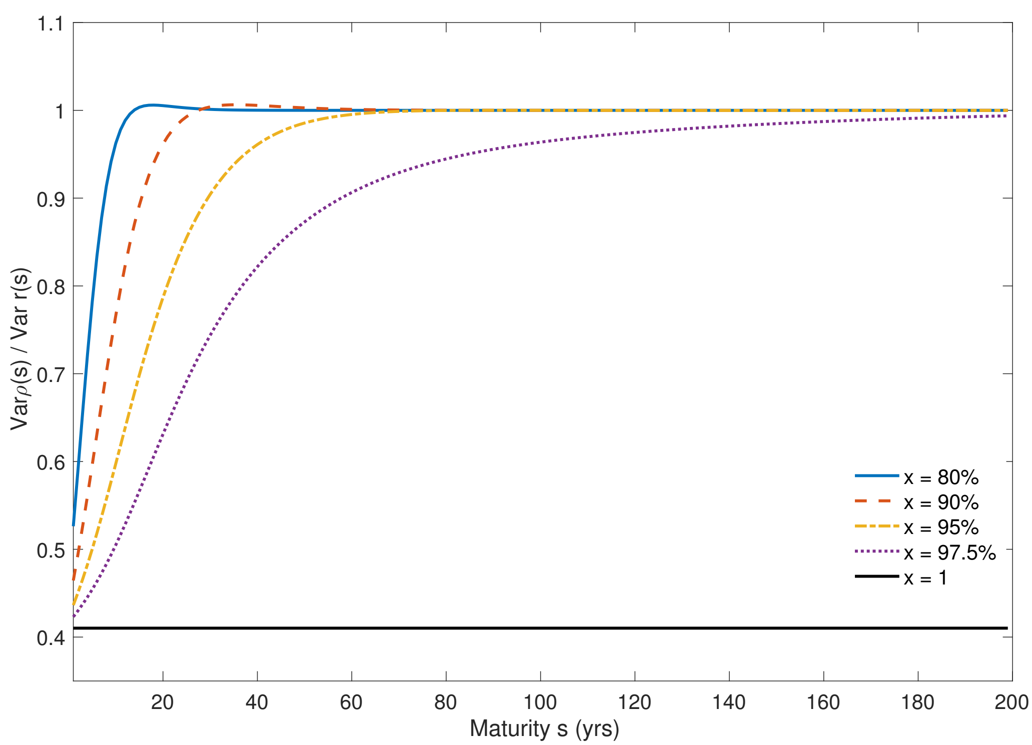
$r(s) = \tilde{\rho}(s) + G(s)$, we make use of the fact that

$$\text{Var } r(s) = \text{Var } \tilde{\rho}(s) + \text{Var } G(s) + 2\text{Cov}\{\tilde{\rho}(s), G(s)\}. \tag{A.45}$$

Figure F.5a breaks the total variance in $r(s)$ into each of these three components at each maturity, for the illustrative case $x = 97.5\%$. This figure confirms that much of the variation in $r(0)$ derives from variation in the growth term $G(0)$, but that as maturities increase disagreements about this term rapidly evaporate. Figure F.5b plots the ratio $\frac{\text{Var}_{\tilde{\rho}(s)}}{\text{Var}_{r(s)}}$ as a function of s for a range of values of x, showing that for all these parameter values almost all the remaining variation in $r(s)$ for $s > 50$ is attributable to variation in $\tilde{\rho}(s)$ – we have almost complete convergence on the dominant $G(s)$ term at these maturities.



(a) Components of the variance of $r(s)$ (see equation (A.45)). $x = 97.5\%$.



(b) Share of the variance of $r(s)$ due to the variance of $\tilde{\rho}(s)$.

Figure F.5: Decomposition of the variance of $r(s)$.

J References

Bergstrom, Theodore C. 1999. "Systems of Benevolent Utility Functions." *Journal of Public Economic Theory*, 1(1): 71–100.

Drupp, Moritz, Mark C. Freeman, Ben Groom, and Frikk Nesje. 2018. "Discounting Disentangled." *American Economic Journal: Economic Policy*, 10(4): 109–134.

Duchin F., Steenge B. 2009. "Mathematical Models in Input-Output Economics", in W. Zhang ed., *Mathematical Science, UNESCO Encyclopedia of Life Support Systems.*

Friedland, Shmuel. 1981. "Convex spectral functions." *Linear and Multilinear Algebra*, 9(4): 299–316.

Galperti, Simone, and Bruno Strulovici. 2017. "A Theory of Intergenerational Altruism." *Econometrica*, 85(4): 1175–1218.

Horn, Roger A., Charles R. Johnson 2013. "Matrix Analysis." *Cambridge University Press*, New York, 2nd edition.

Kingman, J.F.C. 1961. "A convexity property of positive matrices." *Q J Math*, 12(1): 283–284.

Marshall, Albert W., Ingram Olkin and Barry C. Arnold 2010. "Inequalities: Theory of Majorization and its applications." *Springer*, New York, 2nd edition.

Nussbaum, Roger D., 1986. "Convexity and log-convexity for the spectral radius." *Linear algebra and its applications*, 73: 59–122.

Prelec, Drazen. 2004. "Decreasing Impatience: A Criterion for Non-stationary Time Preference and "Hyperbolic" Discounting." *Scandinavian Journal of Economics*, 106(3): 511–532.

Saez-Marti, Maria, and Jörgen W. Weibull. 2005. "Discounting and altruism to future decision-makers." *Journal of Economic Theory*, 122(2): 254–266.

Sternberg, Shlomo. 2014. *Dynamical Systems.* Dover Publications.