Online Appendix for "Attention Management": Existence Proof

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In this supplementary appendix, we provide a formal proof of Lemma 1. In fact, we prove the slightly stronger result, that an optimum exists to the program of Lemma 1 that, if $|\Theta|$ is finite, has affinely independent support. This strengthening of the lemma is not invoked in our paper,[1] but may be of use to future users of the Attention Management framework.[2]

We first introduce some additional notation. Given compact metrizable spaces X and Y, a map $f : X \to \Delta Y, x \in X$, and Borel $B \subseteq Y$, let $f(B|x) := (f(x))(B)$. Define the barycenter map $\beta_X : \Delta\Delta X \to \Delta X$ by $\beta_X(\hat{X}|m) := \int_{\Delta X} \gamma(\hat{X}) dm(\gamma)$, $\forall m \in \Delta\Delta X$, Borel $\hat{X} \subseteq X$. In other words, $\beta_X(m) = \mathbb{E}_{\nu\sim m}(\nu)$ for all $m \in \Delta\Delta X$. Note that $\mathscr{R}(\mu) = \beta_\Theta^{-1}(\mu)$, by definition.

Define $\Phi : \Delta\Delta\Delta\Theta \to (\Delta\Delta\Theta)^2$ by $\Phi(\mathbb{P}) = \left(\beta_{\Delta\Theta}(\mathbb{P}), \mathbb{P} \circ \beta_\Theta^{-1}\right)$. While we offer no specific interpretation to this map, it will be of use in deriving required properties of the Blackwell order.

Define the garbling correspondence $G : \Delta\Delta\Theta \rightrightarrows \Delta\Delta\Theta$ by

$$G(p) := \left\{ q \in \Delta\Delta\Theta : p \succeq^B q \right\}.$$

We can view the principal's problem as a delegation problem in which she offers the agent a delegation set $\hat{G} \in \{G(p)\}_{p\in\mathscr{R}(\mu)}$, and the agent makes a selection $q \in \hat{G}$. Recall, the agent's

1. The strengthened result implies Claim 1, but we instead provide an independent, elementary proof in the Section IV of the paper

2. Given results proven in this online appendix, one could employ results of Harris (1985) to establish existence. We instead prove the result directly, enabling us to strengthen the lemma.

optimal garbling correspondence $G^*: \Delta\Delta\Theta \rightrightarrows \Delta\Delta\Theta$ is given by

$$G^*(p) := \operatorname*{argmax}_{q \in G(p)} \int_{\Delta\Theta} U_A \, \mathrm{d}q.$$

CLAIM OA.1. β_Θ, $\beta_{\Delta\Theta}$ are continuous.

Proof. This follows from Phelps (2001, Proposition 1.1). ■

CLAIM OA.2. Φ is continuous.

Proof. Suppose $\{\mathbb{P}_n\}_n \subseteq \Delta\Delta\Delta\Theta$ converges to $\mathbb{P}$. Since $\Delta\Theta$ is compact metrizable, $\beta_{\Delta\Theta}(\mathbb{P}_n) \to \beta_{\Delta\Theta}(\mathbb{P})$, by Claim OA.1. To show $\mathbb{P}_n \circ \beta_\Theta^{-1} \to \mathbb{P} \circ \beta_\Theta^{-1}$, take any continuous function $f: \Delta\Theta \to \mathbb{R}$. Continuity of β_Θ implies that $f \circ \beta_\Theta$ is continuous. Then,

$$\begin{aligned}\int_{\Delta\Theta} f \, \mathrm{d}\left(\mathbb{P}_n \circ \beta_\Theta^{-1}\right) &= \int_{\Delta\Delta\Theta} f \circ \beta_\Theta \, \mathrm{d}\mathbb{P}_n \\ &\to \int_{\Delta\Delta\Theta} f \circ \beta_\Theta \, \mathrm{d}\mathbb{P} \\ &= \int_{\Delta\Theta} f \, \mathrm{d}\left(\mathbb{P} \circ \beta_\Theta^{-1}\right)\end{aligned}$$

where the second line follows from the weak* convergence of $\mathbb{P}_n$ to $\mathbb{P}$. ■

CLAIM OA.3. The partial order $\succeq^B$ is given by $\succeq^B = \Phi(\Delta\Delta\Delta\Theta)$.

Proof. First, take any $p \succeq^B q$ witnessed by mean-preserving spread $r: \Delta\Theta \to \Delta\Delta\Theta$. Define $\mathbb{P} := q \circ r^{-1} \in \Delta\Delta\Delta\Theta$. We now show that $\Phi(\mathbb{P}) = (p, q)$. Notice that $\mathscr{R}(\nu) \cap \mathscr{R}(\nu') = \emptyset$ for $\nu \neq \nu'$. Therefore, any $\nu \in \Delta\Theta$ satisfies $\beta_\Theta^{-1}(\nu) \cap r(\Delta\Theta) = r(\nu)$. As a result, for any Borel $S \subseteq \Delta\Theta$,

$$\mathbb{P} \circ \beta_\Theta^{-1}(S) = q \circ r^{-1}\left(\beta_\Theta^{-1}(S)\right) = q \circ r^{-1}(r(S)) = q(S),$$

and

$$\beta_{\Delta\Theta}(S|\mathbb{P}) = \int_{\Delta\Delta\Theta} \tilde{p}(S) \, \mathrm{d}\mathbb{P}(\tilde{p}) = \int_{\Delta\Delta\Theta} \tilde{p}(S) \, \mathrm{d}\left[q \circ r^{-1}\right](\tilde{p}) = \int_{\Delta\Theta} r(S|\tilde{p}) \, \mathrm{d}q(\tilde{p}) = p(S).$$

Therefore, $(p, q) = \Phi(\mathbb{P})$.

Next, take any $\mathbb{P} \in \Delta\Delta\Delta\Theta$ and let $(\bar{p}, \bar{q}) := \Phi(\mathbb{P})$. We want to show that $\bar{p} \succeq^B \bar{q}$. Notice that we can view β_Θ as a $(\Delta\Theta)$-valued random variable on the probability space $(\Delta\Delta\Theta, \mathscr{B}(\Delta\Delta\Theta), \mathbb{P})$. Let $\gamma : \Delta\Delta\Theta \to \Delta\Delta\Theta$ be a conditional expectation $\gamma = \mathbb{E}_{q \sim \mathbb{P}}\left[q | \beta_\Theta(q)\right]$, which exists by Chatterji (1960, Theorem 1). So γ is β_Θ-measurable, and $\forall$ Borel $S \subseteq \Delta\Theta$, we have

$$\int_{\Delta\Delta\Theta} q(S) \, \mathrm{d}\mathbb{P}(q) = \int_{\Delta\Delta\Theta} \gamma(S|\cdot) \, \mathrm{d}\mathbb{P}.$$

By Doob's theorem (Kallenberg, 2006, Lemma 1.13), there exists a measurable $r : \Delta\Theta \to \Delta\Delta\Theta$ such that $\gamma = r \circ \beta_\Theta$. Then, $\forall$ Borel $S \subseteq \Delta\Theta$,

$$\int_{\Delta\Theta} r(S|\cdot) \, \mathrm{d}\bar{q} = \int_{\Delta\Delta\Theta} \left(r \circ \beta_\Theta\right)(S|\cdot) \, \mathrm{d}\mathbb{P} = \int_{\Delta\Delta\Theta} \gamma(S|\cdot) \, \mathrm{d}\mathbb{P} = \int_{\Delta\Delta\Theta} q(S) \, \mathrm{d}\mathbb{P}(q) = \beta_{\Delta\Theta}(S|\mathbb{P}) = \bar{p}(S).$$

Now, that β_Θ is affine and continuous implies

$$\beta_\Theta \circ \gamma = \mathbb{E}\left[\beta_\Theta \circ \mathrm{id}_{\Delta\Delta\Theta} | \beta_\Theta\right],$$

which is $\mathbb{P}$-a.s. equal to β_Θ. That is, $\beta_\Theta \circ r \circ \beta_\Theta = \mathrm{id}_{\Delta\Theta} \circ \beta_\Theta$, a.s.-$\mathbb{P}$. Equivalently, $\beta_\Theta \circ r = \mathrm{id}_{\Delta\Theta}$, a.s.-$\bar{q}$. The measurable function

$$\begin{aligned} \bar{r} : \Delta\Theta &\to \Delta\Delta\Theta \\ \nu &\mapsto \begin{cases} r(\nu) & : r(\nu) \in \mathscr{R}(\nu) \\ \delta_\nu & : r(\nu) \notin \mathscr{R}(\nu) \end{cases} \end{aligned}$$

is then $\bar{q}$-a.s. equal to r and satisfies $\beta_\Theta \circ \bar{r} = \mathrm{id}_{\Delta\Theta}$. Thus, $\bar{r}$ is a mean-preserving spread witnessing $\bar{p} \succeq^B \bar{q}$. ■

CLAIM OA.4. $\succeq^B$ is a continuous partial order, i.e. $\succeq^B \subseteq (\Delta\Delta\Theta)^2$ is closed.

Proof. This follows from Claims OA.2 and OA.3, because the continuous image of a compact set is compact. ■

CLAIM OA.5. The garbling correspondence G is continuous and nonempty-compact-valued.

Proof. It is nonempty-valued because $\succeq^B$ is reflexive, and upper hemicontinuous and compact-valued by Claim OA.4. Toward showing G is lower hemicontinuous, fix some open $D \subseteq \Delta\Delta\Theta$. Then,

$$\begin{aligned}\{p \in \Delta\Delta\Theta : G(p) \cap D \neq \emptyset\} &= \left\{p \in \Delta\Delta\Theta : \ p \succeq^B q, \ q \in D\right\} \\ &= \{p : \ (p,q) \in \Phi(\Delta\Delta\Delta\Theta), \ q \in D\} \\ &= \Phi_1 \circ \Phi_2^{-1}(D) \\ &= \beta_{\Delta\Theta}\left(\Phi_2^{-1}(D)\right)\end{aligned}$$

where the second line follows from Claim OA.3, and the last line follows from the definition of Φ_1. By Claim OA.2, since D is open, so is $\Phi_2^{-1}(D)$. In addition, $\beta_{\Delta\Theta}$ is an open map by O'Brien (1976, Corollary 1). So $\beta_{\Delta\Theta}\left(\Phi_2^{-1}(D)\right)$ is open, implying that G is lower hemicontinuous. ■

CLAIM OA.6. The optimal garbling correspondence G^* is upper hemicontinuous and nonempty-compact-valued.

Proof. As the indirect utility function U_A is (by Berge's theorem) continuous, so is $q \mapsto \int_{\Delta\Theta} U_A \, dq$. The result then follows from Claim OA.5 and Berge's theorem. ■

CLAIM OA.7. If $q^* \in \mathscr{R}(\mu)$ is such that (q^*, q^*) solves the principal's problem in (2), then there is a set $\mathscr{P} \subseteq \text{ext}\left[\mathscr{R}(\mu)\right]$ such that $q^* \in \overline{\text{co}}\mathscr{P}$ and (p^*, p^*) solves the principal's problem for every $p^* \in \mathscr{P}$.

Proof. By Choquet's theorem, $\exists \mathbb{Q} \in \Delta\left[\mathscr{R}(\mu)\right]$ such that:

$$\begin{aligned}\mathbb{Q}\left[\text{ext}\mathscr{R}(\mu)\right] &= 1, \\ \beta_{\Delta\Theta}(\mathbb{Q}) &= q^*.\end{aligned}$$

By Claim OA.6 and the Kuratowski-Ryll-Nardzewski Selection Theorem (Aliprantis and Border, 2006, Theorem 18.13), which applies here by Aliprantis and Border (2006, Theorem 18.10), there is some measurable selector g of G^*. The random posterior $q_g := \beta_{\Delta\Theta}(\mathbb{Q} \circ g^{-1})$

is then a garbling of q^*. Moreover, that $q^* \in G^*(q^*)$ implies

$$\begin{aligned} 0 &\le \int_{\Delta\Theta} U_A \, \mathrm{d}q^* - \int_{\Delta\Theta} U_A \, \mathrm{d}q_g \\ &= \int_{\mathrm{ext}\mathscr{R}(\mu)} \left[\int_{\Delta\Theta} U_A \, \mathrm{d}q - \max_{\tilde{q} \in G(q)} \int_{\Delta\Theta} U_A \, \mathrm{d}\tilde{q} \right] \mathrm{d}\mathbb{Q}(q). \end{aligned}$$

Since the latter integrand is everywhere nonpositive and the integral is nonnegative, it must be that the integrand is almost everywhere zero. That is, $q \in G^*(q)$ for $\mathbb{Q}$-almost every q. Then, by Claim OA.6, $q \in G^*(q)$ for every $q \in \mathrm{supp}(\mathbb{Q})$. Therefore, $\mathscr{P} := \mathrm{supp}(\mathbb{Q}) \cap \mathrm{ext}\mathscr{R}(\mu)$ is as desired. ■

CLAIM OA.8. There is some $p^* \in \mathrm{ext}\left[\mathscr{R}(\mu)\right]$ such that (p^*, p^*) solves the principal's problem in (2).

Proof. The principal's objective can be formulated as a mapping $\mathrm{Graph}(G^*) \to \mathbb{R}$ with $(p,q) \mapsto \int_{\Delta\Theta} U_P \, \mathrm{d}q$. It is upper semicontinuous and, by Claim OA.6, has compact domain. Therefore, there is some solution $(\hat{p}, q^*)$ to (2). As $G(q^*) \subseteq G(\hat{p})$, it is immediate that $q^* \in G^*(q^*)$; that is, q^* is IC. Letting $\mathscr{P}$ be as delivered by Claim OA.7, and taking any $p^* \in \mathscr{P}$ completes the claim. ■

CLAIM OA.9. If $|\Theta| < \infty$, then: $p \in \mathrm{ext}\left[\mathscr{R}(\mu)\right]$ if and only if $\mathrm{supp}(p)$ is affinely independent.

Proof. First, we prove the "only if" direction. Take any $p \in \mathscr{R}(\mu)$. Then $\mu \in \overline{\mathrm{co}}[\mathrm{supp}(p)] = \mathrm{co}[\mathrm{supp}(p)]$, where the equality follows from Θ being finite. By Carathéodory's theorem, there exists an affinely independent $S \subseteq \mathrm{supp}(p)$ such that $\mu \in \mathrm{co}(S)$; without loss, let S be a smallest such set. Since Θ is finite, $S \subset \mathbb{R}^{|\Theta|}$, so affine independence implies that S is finite. Therefore, $\exists N : S \rightrightarrows \Delta\Theta$ such that, $\forall \nu \in S$, the set $N(\nu)$ is a closed convex neighborhood of ν with $S \cap N(\nu) = \{\nu\}$. Making $\{N(\nu)\}_{\nu \in S}$ smaller, we may assume for all selectors η of N, $\{\eta(\nu)\}_{\nu \in S}$ is affinely independent.

Now define a specific selector $\eta : S \to \Delta\Theta$ by:

$$\eta(\nu) = \beta_\Theta \left(\frac{p\left(N(\nu) \cap \cdot\right)}{p\left(N(\nu)\right)} \right).$$ [3]

3. Note that $p(N(\nu)) > 0$ for every $\nu \in S \subseteq \mathrm{supp}(p)$, so that $\eta(\nu)$ is well-defined. That $N(\nu)$ is closed and convex for every $\nu \in S$ implies η is a selector of N.

Since $\mu \in \text{co}(S)$, $\exists w \in \Delta S$ such that $\sum_{\nu \in S} w(\nu)\eta(\nu) = \mu$, and ($S$ being minimal) $w(\nu) > 0$ for all $\nu \in S$. Let

$$q := \sum_{\nu \in S} w(\nu) \frac{p(N(\nu) \cap \cdot)}{p(N(\nu))}$$
$$\varepsilon := \min_{\nu \in S} \frac{w(\nu)}{p(N(\nu))}$$

Note that $q \in \mathscr{R}(\mu)$. Therefore, $\frac{p-\varepsilon q}{1-\varepsilon} \in \mathscr{R}(\mu)$ and $p \in \text{co}\left\{q, \frac{p-\varepsilon q}{1-\varepsilon}\right\}$.

Now, if $p \in \text{ext}\left[\mathscr{R}(\mu)\right]$, then it must be that $q = p$, even if we make each neighborhood in $\{N(\nu)\}_{\nu \in S}$ smaller, for otherwise $p \in \text{co}\left\{q, \frac{p-\varepsilon q}{1-\varepsilon}\right\}$ contradicts $p \in \text{ext}\left[\mathscr{R}(\mu)\right]$. But then, $\text{supp}(p) = S$, and since S is affinely independent, so is $\text{supp}(p)$.

Now, we prove the "if" direction. Suppose $p \in \mathscr{R}(\mu)$ has affinely independent support S. Suppose $q, q' \in \mathscr{R}(\mu)$ have $p = (1-\lambda)q + \lambda q'$ for some $\lambda \in (0,1)$. Then the support of q must be contained in S. However, q is Bayes-plausible:

$$\sum_{\nu \in S} q(\nu)\nu = \mu = \sum_{\nu \in S} p(\nu)\nu.$$

But S is affinely independent, implying that $q(\nu) = p(\nu)$ for all $\nu \in S$. That is, $q = p$. As q, q', λ were arbitrary, it must be that p is an extreme point. ■

Proof of Lemma 1. By Claim OA.8, a solution to (2) exists. By Claims OA.8 and OA.9, (2) admits some optimal solution, (q^*, q^*), where $\text{supp}(q^*)$ is affinely independent if Θ is finite. This implies that $q^* \in G^*(q^*)$. Finally, notice that the optimal value of the problem in (3) is no larger than that of (2), since the former is a relaxation of the latter. So (q^*, q^*) is also a solution to (3). ■

REMARK OA.1. In the above work, the only properties of U_A and U_P that we use are that the former is continuous and the latter upper semicontinuous. For this reason, Lemma 1 applies without change to environments in which the principal and the agent have different material motives, to settings in which the principal partially internalizes the agent's attention costs, and more.

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