Online Appendix for
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B Online Appendix

This online appendix contains additional proofs for results in Daley and Green (2019).

PROOF OF PROPOSITION 2:

From the expression in (13), β is decreasing in u_1. Clearly u_1 decreases with ϕ, which implies (i). The remaining comparative static results will be shown with respect to u_1. For (ii), using the expression in (19) we have that

$$\frac{d}{du_1}q(z) = \frac{rV_L}{e^{u_1 z}(u_1-1)^2u_1^2(K_H-V_L)}\zeta^{u_1}\left(1+u_1(z-2)-u_1^2 z+(u_1-1)u_1\ln(\zeta)\right)$$

where $\zeta \equiv \frac{u_1(K_H-V_L)}{(u_1-1)(V_H-K_H)} = e^{\beta} > 0$. The expression above is strictly positive (negative) for $z > (<) \beta - \frac{2u_1-1}{u_1(u_1-1)}$, which implies (ii). For (iii), it is sufficient to show that F_B is decreasing in u_1 below β. To do so, plug in the expression for $C_1 = C_1^*$ into F_B and differentiate with respect to u_1 to get that

$$\begin{aligned}\frac{d}{du_1}F_B(z) &= \frac{1}{1+e^z}e^{u_1 z}\left(\frac{\partial C_1^*}{\partial u_1}+zC_1^*\right)\\ &= \frac{1}{1+e^z}e^{u_1 z}\left(\frac{K_H-V_L}{u_1-1}\right)\zeta^{-u_1}(z-\ln(\zeta))\\ &< 0,\end{aligned}$$

where the inequality follows from noting that $\ln(\zeta) = \beta$. For (iv), note that for $z < \beta$,

$$\begin{aligned}\frac{d}{du_1}F_L(z) &= e^{u_1 z}\left((1+(u_1-1)z)C_1^* + (u_1-1)\frac{\partial C_1^*}{\partial u_1}\right)\\ &= e^{u_1 z}\left(\frac{K_H-V_L}{u_1-1}\right)\zeta^{-u_1}\left(1+(u_1-1)(z-\ln(\zeta)\right).\end{aligned}$$

Noting that $e^{u_1 z}\left(\frac{K_H-V_L}{u_1-1}\right)\zeta^{-u_1}>0$, we have that $F_L(z)$ increases with u_1 for $z\in(\beta-\frac{1}{u_1-1},\beta)$ and decreases in u_1 for $z<\beta-\frac{1}{u_1-1}$, which proves (iv). For (v), note that $\Pi(z)=F_B(z)+(1-p(z))F_L(z)$ and therefore

$$\begin{aligned}\frac{d}{du_1}\Pi(z) &= \frac{d}{du_1}F_B(z)+(1-p(z))\frac{d}{du_1}F_L(z)\\ &= \frac{1}{1+e^z}e^{u_1 z}\left(\frac{K_H-V_L}{u_1-1}\right)\zeta^{-u_1}\left(1+u_1(z-\ln(\zeta))\right),\end{aligned}$$

which is positive for $z\in(\beta-\frac{1}{u_1},\beta)$ and negative for $z<\beta-\frac{1}{u_1}$, implying (v).

PROOF OF PROPOSITION 3:

First, note that taking the limit as $\phi\to\infty$ is equivalent to taking the limit as $u_1\to 1$ from above (denoted $u_1\to 1^+$). For (i), using the expression for β in (13), we have that

$$\lim_{u_1\to 1^+}\beta = \underline{z} + \lim_{u_1\to 1^+}\ln\left(\frac{u_1}{u_1-1}\right)=\infty.$$

For (ii), using the expressions for C_1^* and q from (15) and (19) respectively,

$$q(z)=\frac{rV_L e^{-u_1 z}}{C_1^* u_1(u_1-1)}=\frac{rV_L e^{-u_1 z}\left(\frac{u_1(K_H-V_L)}{(u_1-1)(V_H-K_H)}\right)^{u_1}}{u_1(K_H-V_L)},$$

which, for all $z<\beta$, tends to ∞ as $u_1\to 1^+$. Incorporating the expression for β yields:

$$\lim_{u_1\to 1^+}q(\beta-x)=\lim_{u_1\to 1^+}\frac{rV_L e^{u_1 x}}{u_1(K_H-V_L)}=\frac{rV_L e^x}{K_H-V_L}.$$

For (iii), from plugging the expression for C_1^* from (15) into (14), we have

$$F_B(z)=\begin{cases} V(z)-K_H & \textit{if } z\ge\beta\\ \frac{e^{u_1 z}(V_H-K_H)\left(\frac{u_1(K_H-V_L)}{(u_1-1)(V_H-K_H)}\right)^{1-u_1}}{(1+e^z)u_1} & \textit{if } z<\beta\end{cases}$$

As $u_1 \to 1^+$, $\beta \to \infty$, meaning for any $z \in \mathbb{R}$,

$$\lim_{u_1 \to 1^+} F_B(z) = \lim_{u_1 \to 1^+} \frac{e^{u_1 z}(V_H - K_H)\left(\frac{u_1(K_H - V_L)}{(u_1-1)(V_H - K_H)}\right)^{1-u_1}}{(1+e^z)u_1}$$
$$= \frac{e^z}{1+e^z}(V_H - K_H) = p(z)(V_H - K_H).$$

Further, since $F_B(z)$ is continuous in z and nondecreasing in ϕ (Proposition 2), the convergence is uniform by Dini's Theorem.[1] For (iv), using the expression for $W(z)$ in (17) and the fact that $F_L = W$,

$$F_L(z) = \begin{cases} K_H & \textit{if } \; z \geq \beta \\ V_L + e^{u_1 z}(V_H - K_H)^{u_1}(K_H - V_L)\left(\frac{u_1(K_H - V_L)}{u_1 - 1}\right)^{-u_1} & \textit{if } \; z < \beta \end{cases}$$

As $u_1 \to 1^+$, $\beta \to \infty$, meaning for any $z \in \mathbb{R}$,

$$\lim_{u_1 \to 1^+} F_L(z) = V_L + \lim_{u_1 \to 1^+} e^{u_1 z}(V_H - K_H)^{u_1}(K_H - V_L)\left(\frac{u_1(K_H - V_L)}{u_1 - 1}\right)^{-u_1} = V_L.$$

Finally, for (v),

$$0 \leq \mathcal{L}(z) = \frac{\Pi^{FB}(z) - \Pi(z)}{\Pi^{FB}(z)} = \frac{p(z)(V_H - K_H) - F_B(z) + (1 - p(z))(V_L - F_L(z))}{\Pi^{FB}(z)}$$
$$\leq \frac{p(z)(V_H - K_H) - F_B(z)}{\Pi^{FB}(z)}, \tag{1}$$

where the last inequality follows from $F_L(z) \geq V_L$ for all z (regardless of ϕ). By (iii), the term in (1) uniformly converges to 0 as $u_1 \to 1^+$, implying $\mathcal{L}$ does as well.

PROOF OF PROPOSITION 4:

First, note that taking the limit as $\phi \to 0$ is equivalent to taking the limit as $u_1 \to \infty$. For (i), using the expression for β in (13), we have that

$$\lim_{u_1 \to \infty} \beta = \underline{z} + \lim_{u_1 \to \infty} \ln\left(\frac{u_1}{u_1 - 1}\right) = \underline{z} + \ln(1) = \underline{z}.$$

[1] To apply Dini's Theorem, the function's domain must be compact. However, simply transform log-likelihood states, z, back into probability states, $p \in [0,1]$, and, for all ϕ-values, extend the function to $p = 0, 1$ to preserve continuity.

From (19), we have that $q(z) = \frac{rV_L}{C_1^* u_1(u_1-1)e^{u_1 z}}$. Therefore, to prove (ii) it suffices to show that $\lim_{u_1\to\infty} C_1^* u_1(u_1-1)e^{u_1 z} = 0$ for $z < \underline{z}$ and $\lim_{u_1\to\infty} C_1^* u_1(u_1-1)e^{u_1 \underline{z}} = \infty$. Using the expression for C_1^* in (15), we obtain

$$C_1^* u_1(u_1-1)e^{u_1 z} = (K_H - V_L) \times \left(\frac{u_1-1}{u_1}\right)^{u_1} \times \left(\frac{V_H - K_H}{K_H - V_L} e^z\right)^{u_1} u_1$$

The first term on the right hand side is positive and independent of u_1. The second term limits to e^{-1} as $u_1 \to \infty$. Thus, the remaining term determines the limiting properties. It can be written as $u_1 y^{u_1}$, where $y \equiv \frac{V_H - K_H}{K_H - V_L} e^z$. Notice that $z < \underline{z} \implies y < 1 \implies \lim_{u_1\to\infty} u_1 y^{u_1} = 0$, whereas $z = \underline{z} \implies y = 1 \implies \lim_{u_1\to\infty} u_1 y^{u_1} = \lim_{u_1\to\infty} u_1 = \infty$. This completes the proof of (ii).

For (iii), note that for all $z \le \underline{z}$, $0 \le F_B(z) \le C_1^* e^{u_1 z} \le C_1^* e^{u_1 \underline{z}}$. And further, $C_1^* e^{u_1 \underline{z}} = (K_H - V_L)\left(\frac{u_1-1}{u_1}\right)^{u_1} \frac{1}{u_1-1} \to 0$ as $u_1 \to \infty$. Thus, we have obtained uniform bound on $F_B(z)$ below $\underline{z}$, which converges to zero implying the first part of (iii). That $F_B(z) \overset{u}{\to} V(z) - K_H$ for $z \ge \underline{z}$ follows from continuity of F_B, $F_B(z) = V(z) - K_H$ for $z \ge \beta$, and $\beta\ to \underline{z}$.

For (iv), the pointwise convergence above $\underline{z}$ is immediate. For $z \le \underline{z}$,

$$\begin{aligned} 0 \le F_L(z) - V_L &= C_1^*(u_1-1)e^{u_1 z} \\ &= (K_H - V_L)\left(\frac{u_1-1}{u_1}\right)^{u_1}\left(\frac{V_H - K_H}{K_H - V_L} e^z\right)^{u_1} \\ &\to (K_H - V_L)e^{-1} \lim_{u_1\to\infty} y^{u_1}. \end{aligned}$$

The remainder of (iv) follows from $z < \underline{z} \implies y < 1 \implies \lim_{u_1\to\infty} y^{u_1} = 0$ and $z = \underline{z} \implies y = 1 \implies \lim_{u_1\to\infty} y^{u_1} = 1$. Finally, (v) is immediately implied by (iii) and (iv).

PROOF OF PROPOSITION 5:

As shown in DG12 (see the proof of Lemma B.3 therein), $\beta_c > z_H^*$, where z_H^* is the threshold belief at which a high-type seller would stop in a game where $V(z)$ is always offered and beliefs evolve only according to news. Using the closed form expressions for z_H^* (see (41) in DG12) and β_b in (13), it is straightforward to check that $z_H^* > \beta_b$, which proves the lemma.

PROOF OF PROPOSITION 6:

First, $\mathcal{L}_b, \mathcal{L}_c \ge 0$, $\mathcal{L}_b(z) > 0$ if and only $z < \beta_b$, and $\mathcal{L}_c(z) > 0$ if and only $z < \beta_c$. By Proposition 5, $\beta_b < \beta_c$. Hence, by continuity of $\mathcal{L}_c$ and $\mathcal{L}_b$, there exists $z_2 < \beta_b$ such that $\mathcal{L}_b(z) < \mathcal{L}_c(z)$ for all $z \in (z_2, \beta_c)$.

In the bilateral outcome, $F^b_H = 0$, so $\Pi_b(z) = F^b_B(z) + (1 - p(z))\, F^b_L(z)$. In the competitive outcome, $F^c_B = 0$, so $\Pi_c(z) = p(z)F^c_H(z) + (1 - p(z))F^c_L(z)$. Further, in the competitive outcome, for all $z < \alpha_c$, both seller payoffs are constant: $F^c_L(z) = V_L$ and $F^c_H(z) = A \in (0, V_H - K_H)$. Direct calculations then show:

$$\lim_{z\to-\infty} \mathcal{L}_b(z) = \lim_{z\to-\infty} \mathcal{L}_c(z) = 0.$$

Therefore, by L'Hospital's rule:

$$\lim_{z\to-\infty} \left(\frac{\mathcal{L}_b(z)}{\mathcal{L}_c(z)}\right) = \lim_{z\to-\infty} \left(\frac{\mathcal{L}'_b(z)}{\mathcal{L}'_c(z)}\right) = \frac{V_H - K_H}{V_H - K_H - A} > 1.$$

Hence, there exists $z_1 > -\infty$ such that $\mathcal{L}_b(z) > \mathcal{L}_c(z)$ for all $z < z_1$.

LEMMA B.1 *The unique solution to* (23) *is of the form* $\tau = T(\beta_\lambda) = \inf\{t : \hat{Z} \geq \beta_\lambda)\}$, *with* $\underline{z} < \beta_\lambda < \infty$. *For* $z < \beta_\lambda$ *the buyer's value function satisfies*

$$(r + \lambda)F_B(z) = \lambda(V(z) - K(z)) + \frac{\phi^2}{2}\big((2p(z) - 1)F'_B(z) + F''_B(z)\big), \tag{2}$$

where β_λ *and the constants in the buyer's value function are characterized by the following boundary conditions.*

$$\lim_{z\to-\infty} |F_B(z)| < \infty \tag{3}$$

$$F_B(\beta_\lambda) = V(\beta_\lambda) - K_H \tag{4}$$

$$F'_B(\beta_\lambda) = V'(\beta_\lambda). \tag{5}$$

PROOF OF LEMMA B.1:

We proceed by constructing the candidate value function, demonstrate there is a unique β_λ satisfying the boundary conditions, and then verify the candidate policy is indeed optimal.

For $z < \beta_\lambda$, the buyer's value function satisfies (2), which has solution of the form

$$F_B(z) = \frac{\lambda}{r + \lambda}(V(z) - K(z)) + \frac{1}{1 + e^z}\left(C_1 e^{\hat{u}_1 z} + C_2 e^{\hat{u}_2 z}\right)$$

where $(\hat{u}_1, \hat{u}_2) = \frac{1}{2}\left(1 \pm \sqrt{1 + \frac{8(\lambda+r)}{\phi^2}}\right)$. The boundary condition (3) requires $C_2 = 0$. Jointly solving (4) and (5) for C_1 and β_λ yields:

$$\beta_\lambda^* = \ln\left(\frac{\hat{u}_1}{\hat{u}_1 - 1}\frac{(\lambda + r)K_H - rV_L}{r(V_H - K_H)}\right)$$
$$C_{1\lambda}^* = \frac{(\lambda + r)K_H - rV_L}{(r + \lambda)(\hat{u}_1 - 1)}e^{-\hat{u}_1\beta_\lambda}.$$

Thus, there is a unique candidate solution. To verify that the policy $\tau = \inf\left\{t : \hat{Z} \geq \beta_\lambda\right\}$ is optimal, note that by construction, the buyer's value function under the candidate policy is $\mathcal{C}^1$ and satisfies:

$$F_B(z) = \begin{cases} \frac{\lambda}{r+\lambda}(V(z) - K(z)) + \frac{1}{1+e^z}C_1^* e^{\hat{u}_1 z} & z \leq \beta_\lambda^* \\ V(z) - K_H & z \geq \beta_\lambda^* \end{cases}$$

Analogous to the proof of Proposition 1, it suffices to check that (1) $F_B(z) \geq V(z) - K_H$ for all $z \leq \beta_\lambda$, and (2) that $(\mathcal{A} - (r + \lambda))F_B(z) + \lambda(V(z) - K(z)) \leq 0$ for all $z \geq \beta_\lambda$. To verify (1), make a change of variables from z to p (i.e., substitute $\ln\left(\frac{p}{1-p}\right)$ for z into both F_B and V). Note that F_B is convex in p, while V is linear. Given that both the slopes and values match at $p(\beta_\lambda)$, F_B must lie everywhere above to the left. For (2), since $\mathcal{A}F_B = 0$ for $z > \beta_\lambda$, it suffices to show that $V(z) - K_H \geq \frac{\lambda}{\lambda+r}(V(z) - K(z))$ for all $z \geq \beta_\lambda$. Making the same change of variables from z to p, observe that both $V - K_H$ and $\frac{\lambda}{\lambda+r}(V - K)$ are linear in p and that $V - K_H > \frac{\lambda}{\lambda+r}(V - K)$ for all $p > \hat{p} \equiv \frac{(r+\lambda)K_H - rV_L}{r(V_H - V_L) + \lambda K_H}$. The final step is to observe that $\ln\left(\frac{\hat{p}}{1-\hat{p}}\right) = \beta_\lambda - \ln\left(\frac{\hat{u}_1}{\hat{u}_1 - 1}\right) < \beta_\lambda$.

PROOF OF PROPOSITION 7:

First, we must modify equilibrium condition (W.2) to account for the possibility of the fully revealing information arrival as follows:

$$F_B(z) \geq E_z\left[\int_0^\tau \lambda e^{-(r+\lambda)s}(V(\hat{Z}_s) - K(\hat{Z}_s))ds + e^{-(r+\lambda)\tau}F_B(\hat{Z}_\tau)\right]. \quad \text{(B.3")}$$

Next, because $\Gamma(z) = 0$ for all $z < \beta_\lambda$, $F_L = W$ and must again satisfy (16). Substituting the expression for F_B derived in the proof of Lemma B.1 into (16), one can easily verify that F_L is nondecreasing in the proposed equilibrium. Proposition A.1—modified with (B.3") replacing (B.3)—then applies. From here, the proof is analogous to the verification argument in the proof of Theorem 1.

PROOF OF THEOREM 3:

In the proposed equilibrium candidate, for all $z \in \mathbb{R}$, trade is immediate, $W(z) = F_L(z) = K_H$ (which is trivially nondecreasing), and $F_B(z) = V(z) - K_H$. Hence, the equilibrium candidate is of $\Sigma(\beta, q)$ form in which $\beta = -\infty$. As in the proof of Theorem 1, Conditions 3 and 2 are by construction of the Σ-profile. In the candidate, $\beta = -\infty$, so verification of *Seller Optimality* (Condition 1) is trivial: for all z, $W(z) \leq K_H$, so for $\theta \in \{L, H\}$:

$$\sup_{\tau \in \mathcal{T}} E^{\theta}\left[e^{-r\tau}(W(Z_\tau) - K_\theta)\right] \leq K_H - K_\theta = F_\theta(z).$$

Finally, the verification of conditions (B.1)-(B.3) are identical to the ones given for the case of $z > \beta^*$ in the proof of Theorem 1.

To see that no other Σ-equilibrium exists, suppose first that $\Sigma(\beta, q)$ was an equilibrium with $\beta \in \mathbb{R}$. Following the same analysis of necessary conditions from Section III.B yields that

$$\beta = \ln\left(\frac{K_H - V_L}{V_H - K_H}\right) + \ln\left(\frac{u_1}{u_1 - 1}\right),$$

which is not in $\mathbb{R}$ when the SLC fails, contradicting the supposition. Finally, if $\beta = \infty$, then $F_B(z) = 0$ for all $z \in \mathbb{R}$. But then the buyer would improve her payoff by offering K_H (leading to payoff $V(z) - K_H > 0$) for any z. Hence, no other Σ-equilibrium exists.

The argument for uniqueness of equilibrium form follows closely the proof of Theorem 2 with two minor modifications. First, since $\underline{z}$ does not exist when the SLC does not hold, the first statement in Lemma A.5 (i.e., that $\beta > \underline{z}$) is vacuous and no longer required. Second, the proof of Lemmas A.6 and A.7 are immediate if $\beta = -\infty$ and follow the same argument for any $\beta > \infty$.

References

Daley, Brendan and Brett Green, "Bargaining and News," *The American Economic Review (forthcoming)*, 2019.