Leverage and Deepening Business Cycle Skewness: Online Appendix

Henrik Jensen* Ivan Petrella†

Søren H. Ravn‡ Emiliano Santoro§

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*University of Copenhagen and CEPR. Department of Economics, University of Copenhagen, Øster Farimagsgade 5, Bld. 26, 1353 Copenhagen, Denmark. *E-mail*: Henrik.Jensen@econ.ku.dk.

†University of Warwick and CEPR. Warwick Business School, University of Warwick, Scarman Rd, CV4 7AL Coventry, United Kingdom. *E-mail*: Ivan.Petrella@wbs.ac.uk.

‡University of Copenhagen. Department of Economics, University of Copenhagen, Øster Farimagsgade 5, Bld. 26, 1353 Copenhagen, Denmark. *E-mail*: Soren.Hove.Ravn@econ.ku.dk.

§University of Copenhagen. Department of Economics, University of Copenhagen, Øster Farimagsgade 5, Bld. 26, 1353 Copenhagen, Denmark. *E-mail*: Emiliano.Santoro@econ.ku.dk.

Appendix A. Assets and liabilities in the US

Figure 2 shows the ratio of liabilities to assets for households and firms in the United States, respectively. All data are taken from FRED (Federal Reserve Economic Data), Federal Reserve Bank of St. Louis. The primary source is Flow of Funds data from the Board of Governors of the Federal Reserve System. For business liabilities we use the sum of debt securities and loans of nonfinancial corporate and noncorporate businesses. For assets we follow Liu *et al.* (2013) and use data on both sectors' equipment and software as well as real estate at market value. For households and nonprofit organizations, we again use the sum of debt securities and loans as data for liabilities and use as assets both groups' real estate at market value and equipment and software of nonprofit organizations.

The ratios reported in Figure 2 are aggregate measures, and may therefore not reflect actual loan-to-value (LTV) requirements for the marginal borrower. Nonetheless, we report these figures since the flow of funds data deliver a continuous measure of LTV ratios covering the entire period 1952–2016. For households, the aggregate ratio of credit to assets in the economy is likely to understate the actual downpayment requirements faced by households applying for a mortgage loan, since loans and assets are not evenly distributed across households. In our model we distinguish between patient and impatient households, and we assume that only the latter group is faced with a collateral constraint. In the data we do not make such a distinction, so that the LTV ratio for households reported in Figure 2 represents an average of the LTV of patient households (savers), who are likely to have many assets and small loans, and that of impatient households (borrowers), who on average have larger loans and fewer assets. Justiniano *et al.* (2014) use the Survey of Consumer Finances and identify borrowers as households with liquid assets of a value less than two months of their income. Based on the surveys from 1992, 1995, and 1998, they arrive at an average LTV ratio for this group of around 0.8, while our measure fluctuates around 0.5 during the 1990s. Following Duca *et al.* (2011), an alternative approach is to focus on first-time home-buyers, who are likely to fully exploit their borrowing capacity. Using data from the American Housing Survey, these authors report LTV ratios approaching 0.9 towards the end of the 1990s; reaching a peak of almost 0.95 before the onset of the recent crisis. While these alternative approaches are likely to result in higher *levels* of LTV ratios, we are especially interested in the development of these ratios over a rather long time span. While we believe the Flow of Funds data provide the most comprehensive and consistent time series evidence in this respect, substantial increases over time in the LTV ratios faced by households have been extensively documented; see, e.g., Campbell and Hercowitz (2009), Duca *et al.* (2011), Favilukis *et al.* (2017), and Boz and Mendoza (2014). It should be noted that for households, various government-sponsored programs directed at lowering the down-payment requirements faced by low-income or first-time home buyers have been enacted by different administrations (Chambers *et al.*, 2009). These are likely to have contributed to the increase in the ratio of loans to assets illustrated in the left panel of Figure 2.

Likewise, the aggregate ratio of business loans to assets in the data may cover for a disparate distribution of credit and assets across firms. In general, the borrowing patterns and conditions of firms are more difficult to characterize than those of households, as their credit demand is more volatile, and their assets are less uniform and often more difficult to assess. Liu *et al.* (2013) also use Flow of Funds data to calibrate the LTV ratio of the entrepreneurs, and arrive at a value of 0.75. This ratio is based on the assumption that commercial real estate enters with a weight of 0.5 in the asset composition of firms. The secular increase in firm leverage over the second half of the 20th century has also been documented by Graham *et al.* (2014)

using data from the Compustat database.[41,42] These authors report loan-to-asset ratios that are broadly in line with those we present. More generally, an enhanced access of firms to credit markets over time has been extensively documented in the literature, as also discussed in the main text.

A1. Long-run properties of the LTV ratios

In this subsection we investigate the low-frequency properties of the LTV ratios of households and the corporate sector over the 1952:I-2016:II time window. We follow the approach of Müller and Watson (2018), who develop methods to investigate the long-run comovement of two time series.

Since it has been argued that the amplitude of the financial cycle can potentially be much longer than the business cycle (Borio, 2014), we focus on the very low-frequency movements in the LTV ratios. In our baseline specification, we focus on fluctuations over periods longer than 30 years. Table A1 reports the long-run correlation coefficients, as well as the slope coefficient of a linear regression relating household to corporate debt, together with the 68% confidence interval. Figure A1 reports the two LTV ratios, together with their low-frequency components. The two series display strong comovement at the very low frequency, with the slope coefficient containing 1 in the confidence interval. Between 1984 and 2016, this component increased by 20 and 23 basis points for households and firms, respectively. It is also worth emphasizing that, once we remove low-frequency variation in the LTV ratios, their 'cyclical' variations are strongly correlated (about 65%). This evidence supports our modelling choice for the behavior of the LTV ratios, with the trend components for the household and the corporate sector rising in tandem by 23 basis points in Section 6.2, and a common cyclical component. The spread between the household and entrepreneurial steady-state LTV ratios is set to match the average difference in the low-frequency components over the entire sample, which is roughly equal to the average difference in the original series.

Table A1 also reports additional robustness results for different choices of the minimum-length period of the low-frequency component. The results of the baseline specification are quite robust for reasonable variations of the cut-off choice.

[41] It should be mentioned that they also show a Flow of Funds-based measure of debt to total assets at historical cost (or book value) for firms. The increase over time in this measure is smaller. However, we believe that the ratio of debt to *pledgeable* assets at market values (as shown in Figure 2) is the relevant measure for firms' access to collateralized loans, and hence more appropriate for our purposes.

[42] We emphasize that Figure 2 reports a *gross* measure of firm leverage. Bates *et al.* (2009) report that firm leverage *net of* cash holdings has been declining since 1980, but that this decline is entirely due to a large increase in cash holdings.

Table A1. Household and corporate LTV ratios in the long run

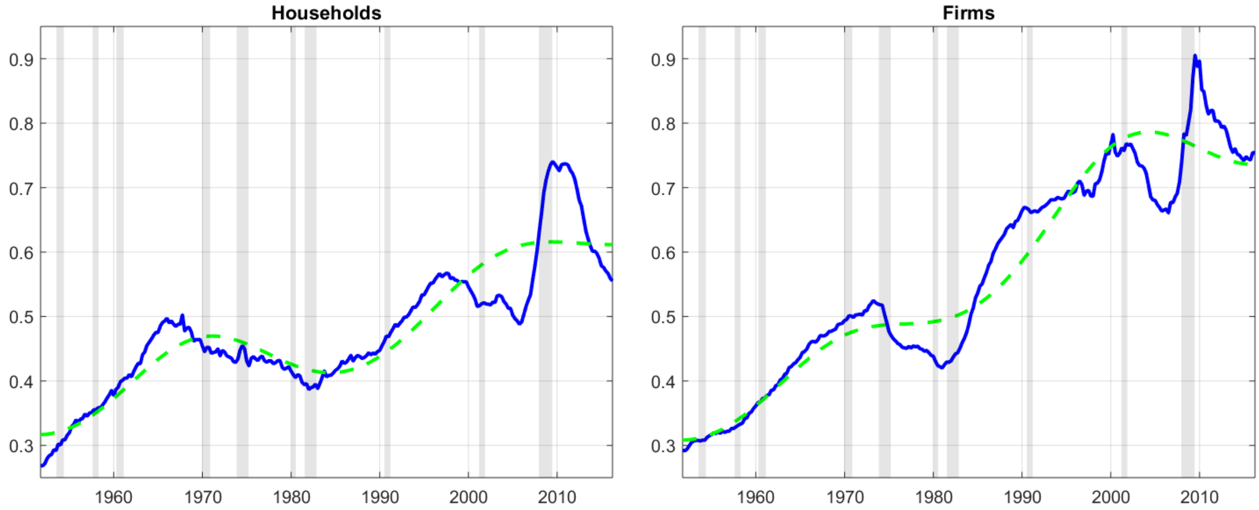
Periods longer than 30 years	
$\widehat{\rho}$	$\widehat{\beta}$
0.847	1.224
[0.511 - 0.947]	[0.680 - 1.618]

Periods longer than 35 years	
$\widehat{\rho}$	$\widehat{\beta}$
0.776	1.317
[0.250 - 0.950]	[0.439 - 1.952]

Periods longer than 25 years	
$\widehat{\rho}$	$\widehat{\beta}$
0.892	1.160
[0.703 - 0.957]	[0.769 - 1.605]

Notes: Table A1 summarizes the long-run covariance ($\hat{\rho}$ denotes the correlation coefficient and $\hat{\beta}$ denotes the slope coefficient of the linear relationship) and the 68% confidence set (in brackets) for the household and the corporate LTV ratios.

Figure A1. Low-frequency components of the LTV ratios



Notes: Each plot reports the LTV ratio (solid-blue line) and its low-frequency component (dashed-green line). The left panel reports data for the household sector, whereas the right panel refers to the corporate sector.

Appendix B. Additional empirical evidence

B1. Time-varying volatility and skewness

In the main text we report evidence on the skewness of real GDP growth being different before and during the Great Moderation. The choice of a cut-off date is inspired by a large literature that has documented a drop in the volatility over the two samples. This exercise entails a possible drawback: The estimates of the skewness can be biased by the first and second moment of the business cycle changing over time. In particular: i) There is now ample evidence that the volatility of the business cycle displays a cyclical behavior (see, e.g., Kim and Nelson, 1999; and McConnell and Perez-Quiros, 2000) and ii) the long-run growth rate of the economy since around 2000 is substantially lower than the average for the entire sample (see, e.g., Antolin-Diaz *et al.*, 2017). To account for these issues we report a measure of time-varying skewness of real GDP growth for the entire sample, relying on a nonparametric estimator. To this end, take a generic time series, y_t, so that its variance and skewness can be respectively calculated as

$$\begin{aligned}
\sigma^2 &= Var\left(y_t\right) = \frac{1}{T}\sum_{t=1}^{T}\left(y_t - \mu\right)^2, \\
\varrho &= Skew\left(y_t\right) = \left\{\frac{1}{T}\sum_{t=1}^{T}\left(y_t - \mu\right)^2\right\}^{-3/2}\left\{\frac{1}{T}\sum_{t=1}^{T}\left(y_t - \mu\right)^3\right\},
\end{aligned}$$

where T denotes the number of observations in the sample and $\mu = E\left(y_t\right) = T^{-1}\sum_{t=1}^{T} y_t$ is the sample average. Define the sample autocovariance and autocorrelation as

$$\begin{aligned}
\gamma_\tau &= \frac{1}{T}\sum_{t=1}^{T-|\tau|}\left(y_{t-|\tau|} - \mu\right)\left(y_t - \mu\right), \\
\rho_\tau &= \frac{\gamma_\tau}{\sigma^2}.
\end{aligned}$$

When y_t is a Gaussian process with absolutely summable autocovariances, it can be shown that the standard errors associated with the two measures are:[43]

$$\begin{aligned}
Var\left(\sigma^2\right) &= \frac{2}{T}\left(\sum_{\tau=-\infty}^{\infty}\gamma_\tau\right)^2, \\
Var\left(\varrho\right) &= \frac{6}{T}\sum_{\tau=-\infty}^{\infty}\rho_\tau^3.
\end{aligned}$$

In practice the two summations are truncated at some appropriate (finite) lag k.

The framework we follow in order to account for time-variation in the variance and skewness has a long pedigree in statistics, starting with the work of Priestley (1965), who introduced the concept of slowly varying process. This work suggests that time series may have time-varying spectral densities which change slowly over time, and proposed to describe those changes as the result of a non-parametric process. This work has more recently been followed up by Dahlhaus (1996), as well as Kapetanios (2007) and Giraitis *et al.* (2014) in the context of time-varying regression models and economic forecasting, respectively. Specifically, the time-varying variance and skewness are calculated as

$$\begin{aligned}
\sigma_t^2 &= Var_t\left(y_t\right) = \sum_{j=1}^{t}\omega_{j,t}\left(y_j - \mu_t\right)^2, \\
\varrho_t &= Skew_t\left(y_t\right) = \left\{\sum_{j=1}^{t}\omega_{j,t}\left(y_j - \mu_t\right)^2\right\}^{-3/2}\left\{\sum_{j=1}^{t}\omega_{j,t}\left(y_j - \mu_t\right)^3\right\},
\end{aligned}$$

[43] The first expression computes the variance as the Newey-West variance of the squared residuals, in order to account for the autocorrelation of the errors. The second equality follows from Gasser (1975) and Psaradakis and Sola (2003).

where $\mu_t = \sum_{j=1}^{t} \omega_{j,t} y_j$. Thus, the sample moments are discounted by the function $\omega_{t,T}$:

$$\omega_{j,t} = cK\left(\frac{t-j}{H}\right),$$

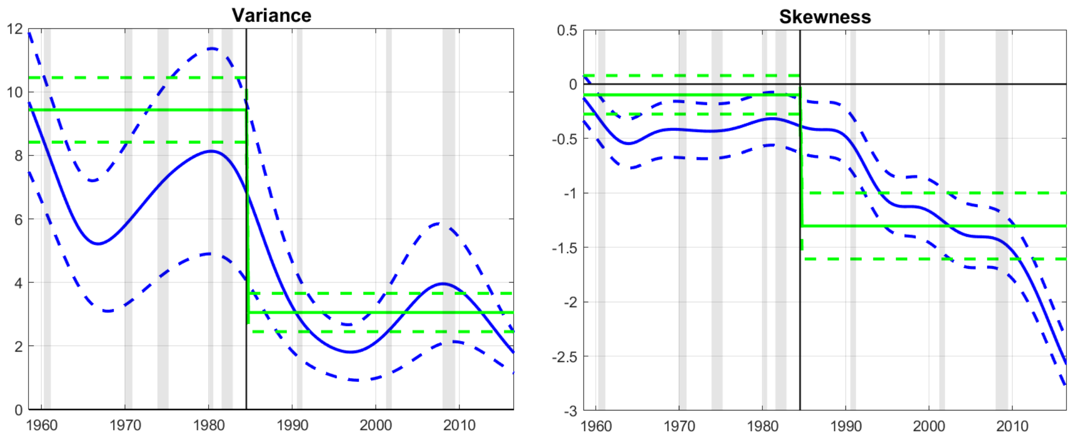
where c is an integration constant and $K\left(\frac{T-t}{H}\right)$ is the kernel function determining the weight of each observation j in the estimation at time t. This weight depends on the distance to t normalized by the bandwidth H. Giraitis *et al.* (2014) show that the estimator has desirable frequentist properties. They suggest using Gaussian kernels with the optimal bandwidth value $H = T^{1/2}$.

Similarly, we can compute the time-varying standard deviation of variance and skewness estimates using time-varying estimates of the sample autocovariance and autocorrelations:

$$\begin{aligned}\gamma_{\tau,t} &= \sum_{j=1}^{t-|\tau|} \omega_{j,t}\left(y_{j-|\tau|} - \mu_t\right)\left(y_j - \mu_t\right), \\ \rho_{\tau,t} &= \frac{\gamma_{\tau,t}}{\sigma_t^2}.\end{aligned}$$

Based on this, Figure B1 reports time-varying measures of volatility and skewness of GDP growth. The left panel confirms the widely documented decline in volatility. From the right panel, it is clear that skewness drops in the second subsample, with a first drop being identified after the 1991 recession and a further one after the Great Recession.

Figure B1. Time-varying volatility and skewness



Notes. Figure B1 reports the time-varying variance and skewness of year-on-year growth of real GDP (solid-blue lines)—obtained by using a nonparametric estimator in the spirit of Giraitis *et al.* (2014)—as well as the associated 68% confidence interval (dashed-blue lines). We also report the variance and skewness of real GDP growth computed over the pre- and post-Great Moderation sample (solid-green lines), as well as the associated 68% confidence interval (dashed-green lines). The vertical shadowed bands denote the NBER recession episodes. Sample: 1947:I-2016:II. The first 10 years of data are dropped to initialize the algorithm. Data source: FRED.

B2. Normality tests

Table B1. Normality tests

	GDP growth (QoQ)	
	1947:I-1984:II	1984:III-2016:II
KS	0.638	0.002
AD	0.534	0.000
SW	0.507	0.000
JB	$\gg$0.50	$\ll$0.001

	GDP growth (YoY)	
	1947:I-1984:II	1984:III-2016:II
KS	0.289	0.004
AD	0.060	0.000
SW	0.091	0.000
JB	$\gg$0.50	$\ll$0.001

Notes. Table B1 reports the p-values of a battery of tests assuming the null hypothesis that real GDP growth is normally distributed in a given sample. KS refers to Kolmogorov-Smirnov test with estimated parameters (see Liliefors, 1967); AD refers to the test of Anderson and Darling (1954); SW refers to the Shapiro-Wilk test (Shapiro and Wilk, 1965) with p-values calculated as outlined by Royston (1992); JB refers to the Jarque-Bera test for normality (Jarque and Bera, 1987). Data source: FRED.

B3. Additional evidence on the standardized violence of the US business cycle

Table B2. Standardized violence of U.S. recessions (Robustness)

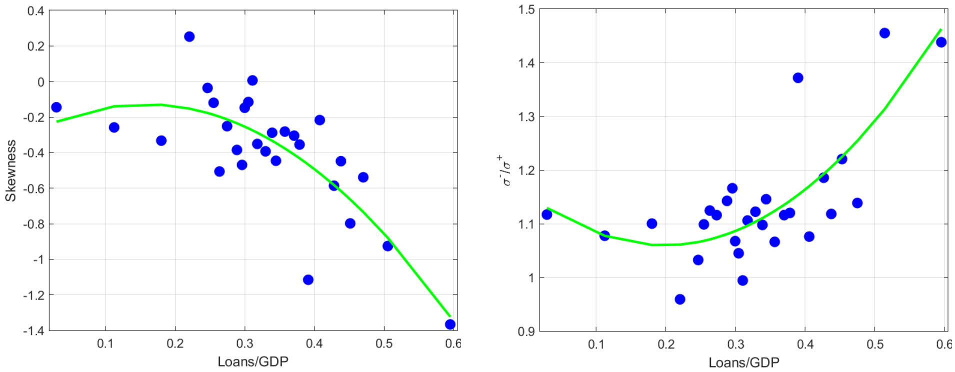
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1953:II – 1954:II	0.635	1.127	0.734	0.731	0.599	0.664	0.584
1957:III – 1958:II	1.629	2.415	1.572	1.767	1.716	1.688	1.785
1960:II – 1961:I	0.351	0.595	0.387	0.429	0.307	0.464	0.304
1969:IV – 1970:IV	0.163	0.156	0.101	0.248	0.155	0.240	0.158
1973:IV – 1975:I	0.662	0.836	0.544	0.847	0.542	0.833	0.509
1980:I – 1980:III	0.999	1.454	0.947	1.239	0.972	1.234	0.863
1981:III – 1982:IV	0.598	0.885	0.576	0.645	0.448	0.621	0.400
1990:III – 1991:I	1.910	1.527	1.132	1.363	1.255	1.229	1.301
2001:I – 2001:IV	0.730	0.730	0.541	0.755	0.463	0.701	0.419
2007:IV – 2009:II	1.847	1.665	1.234	2.020	1.607	1.915	1.571
Average							
Pre-84	0.720	1.067	0.695	0.844	0.677	0.821	0.657
Post-84	1.495	1.307	0.969	1.380	1.108	1.282	1.0967

Notes: Table B2 reports different measures of standardized violence that change depending on the business cycle volatility employed in the denominator. Column (1) follows the same procedure employed to obtain standardized violence in Table 3, though the volatility measure is retrieved from quarter-on-quarter growth rates of real GDP. In the remaining computations, even column numbers report violence statistics that are standardized by volatility measures retrieved from quarter-on-quarter growth rates or real GDP, while in odd column numbers the standardization is operated through volatility measures obtained from year-on-year growth rates. Columns (2) and (3) calculate the volatility by splitting the data between pre- and post-Great Moderation. In columns (4) and (5) the standardization is operated by considering the following stochastic volatility model for real GDP growth: $y_t = \rho_0 + \rho_1 y_{t-1} + \rho_2 y_{t-2} + \sigma_t \varepsilon_t$, where $\sigma_t^2 = \sigma_{t-1}^2 + \kappa\sigma_t^2\left(\varepsilon_t^2 - 1\right)$ and $\varepsilon_t \sim N\left(0,1\right)$. In columns (6) and (7) the standardization is operated by considering a time-varying AR model for real GDP growth with stochastic volatility similar to that of Stock and Watson (2005), where all the time-varying parameters follow random walk laws of motion (as in Delle Monache and Petrella, 2017). Data source: NBER.

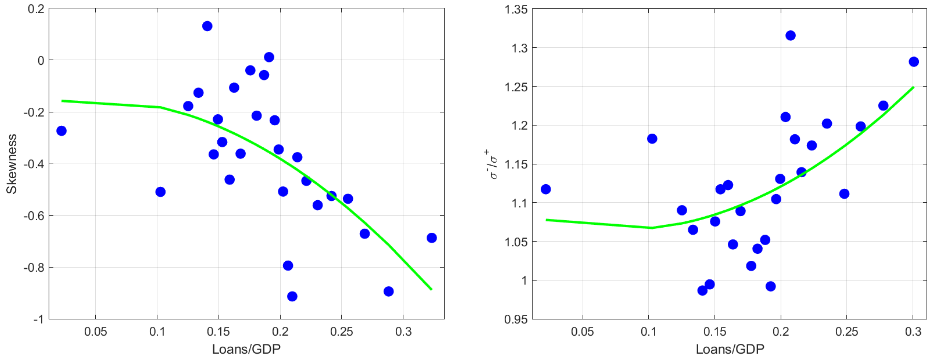
B4. Leverage and asymmetry (G7 countries)

Figure B2. Leverage and asymmetry: G7 countries

Households



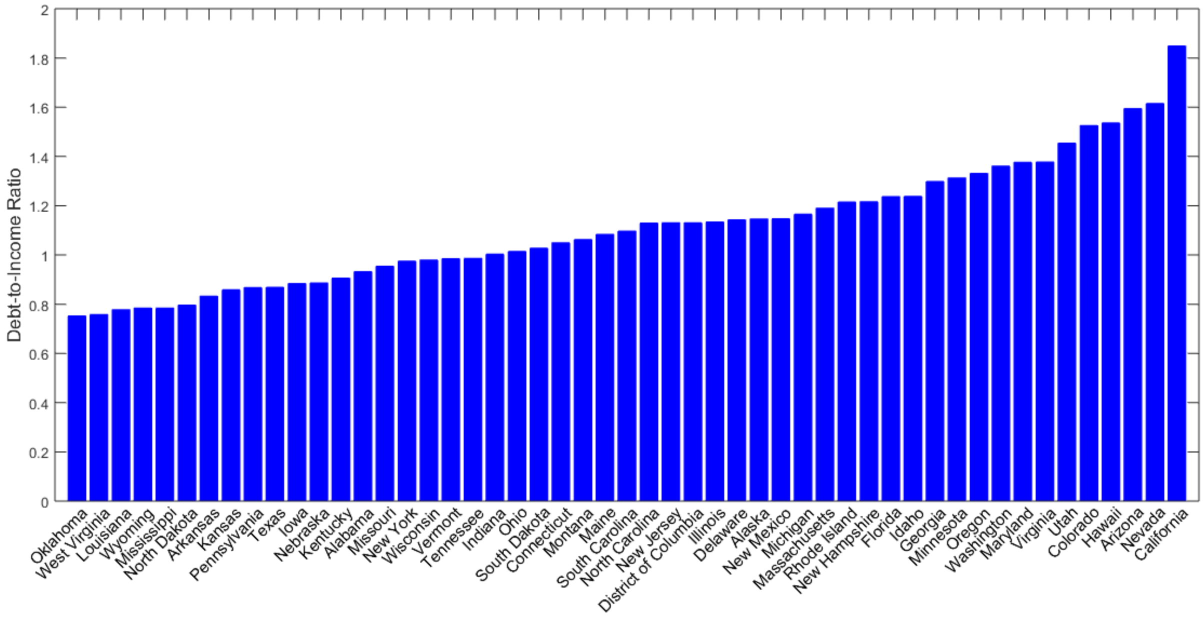
Firms



Notes: The top panels refer to the household sector, while the bottom panels refer to the corporate sector. The left panel of each line reports the skewness of GDP growth, computed for each G7 country, against the loan-to-GDP ratio of a specific sector. In the right panels we replace the skewness with the ratio between the downside and the upside semivolatility of business fluctuations. The regression line is obtained by assuming a quadratic relationship between the two variables (accounting for sector-specific fixed effects). Data source: OECD and Jordà-Schularick-Taylor Macrohistory Database.

B5. Household leverage in the US

Figure B3. U.S. States ordered by households' average debt-to-income ratio



Notes. U.S. States ordered by the average debt-to-income ratio in the household sector, over the period 2003-2007. Data source: State Level Household Debt Statistics produced by the New York Fed.

Appendix C. Details on the solution of the two-period model

Here, we provide details on the computation of the competitive equilibrium of the two-period model discussed in Section 3. The notation is explained in the main text.

Optimality

We first derive the optimality conditions. Rewrite the maximand with the budget constraints and the definition of capital accumulation to get

$$\begin{aligned}\widetilde{U} &= \log\left[r_1^K K_0 + W_1 - RB_0 + B_1 - K_1 + (1-\delta) K_0\right] \\ &\quad + \beta \log\left[\left(1+r_2^K\right) K_1 + W_2 - RB_1\right].\end{aligned}$$

We maximize $\widetilde{U}$ w.r.t. K_1 and B_1, subject to (5). Factor payments are taken as given, as these are co-determined by the demands of all firms in the economy.

We get the first-order conditions

$$-\frac{1}{r_1^K K_0 + W_1 - RB_0 + B_1 - K_1 + (1-\delta) K_0} + \beta \frac{1+r_2^K}{\left(1+r_2^K\right) K_1 + W_2 - RB_1} + \mu \frac{s}{R} = 0, \quad (31)$$

$$\frac{1}{r_1^K K_0 + W_1 - RB_0 + B_1 - K_1 + (1-\delta) K_0} - \beta \frac{R}{\left(1+r_2^K\right) K_1 + W_2 - RB_1} - \mu = 0, \quad (32)$$

$$\mu \left(B_1 - s\frac{K_1}{R}\right) = 0, \qquad \mu \geq 0, \quad (33)$$

where μ is the multiplier applying to the collateral constraint.

Otherwise, the production factors are remunerated at their marginal product:

$$r_t^K = \alpha A_t K_{t-1}^{\alpha-1} L_t^{1-\alpha}, \tag{34}$$

$$W_t = (1-\alpha) A_t K_{t-1}^{\alpha} L_t^{-\alpha}, \qquad t=1,2. \tag{35}$$

The case of an equilibrium with a non-binding constraint

In this case, $\mu = 0$, so that (31) and (32) become

$$\frac{1}{C_1} = \beta \frac{1+r_2^K}{C_2},$$

$$\frac{1}{C_1} = \beta \frac{R}{C_2},$$

and no-arbitrage implies

$$1 + r_2^K = R. \tag{36}$$

This pins down K_1 from (34):

$$K_1 = \left[\frac{\alpha A}{R-1}\right]^{\frac{1}{1-\alpha}}. \tag{37}$$

From (35) we can also recover the wage rate in period 2:

$$W_2 = (1-\alpha) A \left[\frac{\alpha A}{R-1}\right]^{\frac{\alpha}{1-\alpha}}. \tag{38}$$

Thus, total income amounts to

$$\left(1+r_2^K\right) K_1 + W_2 = \frac{1}{\alpha}\left[\frac{\alpha A}{R-1}\right]^{\frac{1}{1-\alpha}} (\alpha + R - 1),$$

so that we retrieve

$$C_2 = \Gamma - RB_1, \tag{39}$$

where $\Gamma \equiv \frac{1}{\alpha}\left[\frac{\alpha A}{R-1}\right]^{\frac{1}{1-\alpha}}(\alpha+R-1)$. Plugging (39) into (32), together with (2), returns

$$\frac{1}{\underbrace{r_1^K K_0 + W_1 - RB_0 + B_1 - K_1 + (1-\delta) K_0}_{=C_1}} = \beta \frac{R}{\Gamma - RB_1},$$

and, therefore:

$$\Gamma - RB_1 = \beta R \left[r_1^K K_0 + W_1 - RB_0 + B_1 - K_1 + (1-\delta) K_0\right].$$

Plugging in the solutions for W_1, r_1^K and K_1 results into

$$\Gamma - RB_1 = \beta R \left[A_1 K_0^{\alpha} - RB_0 + B_1 - \left(\frac{\alpha A}{R-1}\right)^{\frac{1}{1-\alpha}} + (1-\delta) K_0\right],$$

from which we can characterize B_1 as

$$B_1 = \frac{\Gamma}{R(1+\beta)} - \frac{\beta}{1+\beta}\left[A_1 K_0^\alpha - RB_0 - \left(\frac{\alpha A}{R-1}\right)^{\frac{1}{1-\alpha}} + (1-\delta) K_0\right]. \tag{40}$$

We can then derive I_1, C_1 and C_2. We have, by solution of K_1, that

$$I_1 = \left(\frac{\alpha A}{R-1}\right)^{\frac{1}{1-\alpha}} - (1-\delta) K_0. \tag{41}$$

We find C_2 by combining (40) with (39):

$$C_2 = \frac{\beta\Gamma}{1+\beta} + \frac{\beta R}{1+\beta}\left[A_1 K_0^\alpha - RB_0 - \left(\frac{\alpha A}{R-1}\right)^{\frac{1}{1-\alpha}} + (1-\delta) K_0\right].$$

Finally, using $1/C_1 = \beta R/C_2$ in the unconstrained case, we get

$$C_1 = \frac{\Gamma}{R(1+\beta)} + \frac{1}{1+\beta}\left[A_1 K_0^\alpha - RB_0 - \left(\frac{\alpha A}{R-1}\right)^{\frac{1}{1-\alpha}} + (1-\delta) K_0\right].$$

The case of a binding constraint

In this case, $\mu > 0$. We first use (31) and (32):

$$-\frac{1}{C_1} + \beta\frac{1+r_2^K}{C_2} + \mu\frac{s}{R} = 0, \tag{42}$$

$$\frac{1}{C_1} - \beta\frac{R}{C_2} - \mu = 0. \tag{43}$$

Adding the left- and the right-hand side terms gives

$$\beta\frac{1+r_2^K - R}{C_2} = \mu\left(1 - \frac{s}{R}\right) > 0. \tag{44}$$

This shows how a binding borrowing constraint induces a wedge between the return on borrowing and capital; i.e., (36) ceases to hold. Specifically, investment is depressed, which drives the gross marginal return of capital above R.

C_2 depends on K_1 and W_2 as before, but B_1 and K_1 are now linked by the credit constraint. However, r_2^K does not pin down K_1 as in the unconstrained case, as $\mu > 0$; cf. (44). Using (42) and (43) eliminate μ:

$$\frac{1}{C_1} = \beta\frac{1+r_2^K - s}{C_2\left(1-\frac{s}{R}\right)}.$$

Thus, using (2) and (3):

$$\frac{1}{r_1^K K_0 + W_1 - RB_0 + B_1 - K_1 + (1-\delta) K_0} = \beta\frac{1+r_2^K - s}{\left[\left(1+r_2^K\right) K_1 + W_2 - RB_1\right]\left(1-\frac{s}{R}\right)}.$$

We can now use the expressions for r_1^K, r_2^K, W_1 and W_2 to get

$$\frac{1}{A_1 K_0^\alpha - RB_0 + B_1 - K_1 + (1-\delta) K_0} = \beta\frac{1+\alpha A K_1^{\alpha-1} - s}{\left[K_1 + AK_1^\alpha - RB_1\right]\left(1-\frac{s}{R}\right)}.$$

Finally, we use the binding credit constraint,

$$B_1 = s\frac{K_1}{R},$$

to eliminate B_1:

$$\frac{1}{A_1K_0^\alpha - RB_0 - \left(1-\frac{s}{R}\right)K_1 + (1-\delta)K_0} = \beta\frac{1+\alpha AK_1^{\alpha-1}-s}{\left[(1-s)K_1 + AK_1^\alpha\right]\left(1-\frac{s}{R}\right)}. \tag{45}$$

This provides a non-linear characterization of K_1 (and, thus, investment). The expression above can be reshuffled to get

$$\Psi(K_1; A_1) = 0,$$

where

$$\begin{aligned}\Psi(K_1;A_1) \equiv\ & \beta\left(1+\alpha AK_1^{\alpha-1}-s\right)\left[A_1K_0^\alpha - RB_0 - \left(1-\frac{s}{R}\right)K_1 + (1-\delta)K_0\right] \\ & -\left[(1-s)K_1 + AK_1^\alpha\right]\left(1-\frac{s}{R}\right).\end{aligned}$$

Appendix D. Details on the design and solution of the DSGE model

This appendix reports further information on the design and solution of the DSGE model. We first provide some details on the modeling and calibration of the debt contracts. We then proceed to state the first-order conditions, the steady state, and the log-linearization of the model.

D1. Debt contracts

Impatient households and entrepreneurs take up debt with maturity greater than one period. The borrowing constraints presented in the main text, (18) and (24), are rationalized in line with Kydland *et al.* (2016). Let L_t^i denote the flow of lending to agent $i = \{I, E\}$ in period t. This consists of two elements: Agent i's share of existing, non-amortized debt that is refinanced in period t, $\vartheta^i(1-\xi^i)B_{t-1}^i$, and new 'net' lending, $L_t^{i,net}$. Thus:

$$L_t^i = L_t^{i,net} + \vartheta^i(1-\xi^i)B_{t-1}^i. \tag{46}$$

The flow of lending is related to the stock of debt via the following law of motion:

$$B_t^i = \left(1-\vartheta^i\right)\left(1-\xi^i\right)B_{t-1}^i + L_t^i, \tag{47}$$

or, using (46):

$$B_t^i = \left(1-\xi^i\right)B_{t-1}^i + L_t^{i,net}.$$

When taking on new debt, borrowers can pledge as collateral only the fraction of their assets not already used to secure the existing stock of debt. Since ϑ^i denotes the fraction of existing debt that is refinanced, the remaining share $1-\vartheta^i$ of existing debt is collateralized by the same fraction of the borrower's assets. This implies the upper bounds on new lending, for each agent:

$$L_t^I \leq \vartheta^I s_t^I \frac{\mathrm{E}_t\left\{Q_{t+1}\right\}H_t^I}{R_t},$$

$$L_t^E \leq \vartheta^E s_t^E \mathrm{E}_t \left\{ \frac{Q_{t+1}^K K_t + Q_{t+1} H_t^E}{R_t} \right\}.$$

Combining these two expressions with the law of motion for debt, (47), we obtain the borrowing constraints presented in the main text:

$$B_t^I \leq \vartheta^I s_t^I \frac{\mathrm{E}_t \left\{ Q_{t+1} \right\} H_t^I}{R_t} + \left(1 - \vartheta^I\right) \left(1 - \xi^I\right) B_{t-1}^I,$$

$$B_t^E \leq \vartheta^E s_t^E \mathrm{E}_t \left\{ \frac{Q_{t+1}^K K_t + Q_{t+1} H_t^E}{R_t} \right\} + \left(1 - \vartheta^E\right) \left(1 - \xi^E\right) B_{t-1}^E.$$

Steady state and calibration of the debt contracts

It is useful to introduce Λ_t^i (for $i = \{I, E\}$) to denote the fraction of total lending that goes into the refinancing of old debt. From (46), it follows that:

$$\Lambda_t^i \equiv \frac{L_t^i - L_t^{i,net}}{L_t^i} = \vartheta^i (1 - \xi^i) \frac{B_{t-1}^i}{L_t^i}.$$

In the steady state, this becomes:

$$\Lambda^i = \vartheta^i (1 - \xi^i) \frac{B^i}{L^i}.$$

We can obtain an expression for $\frac{B^i}{L^i}$ from the steady-state version of the debt-accumulation equation (47):

$$\frac{B^i}{L^i} = \frac{1}{1 - \left(1 - \vartheta^i\right) \left(1 - \xi^i\right)},$$

which can be inserted into the previous expression to obtain:

$$\Lambda^i = \frac{\vartheta^i (1 - \xi^i)}{1 - \left(1 - \vartheta^i\right) \left(1 - \xi^i\right)}.$$

This pins down the steady-state value of the refinancing parameter, ϑ^i, for given values of the amortization rate, ξ^i, and the share of refinancing to total loans, Λ^i. Solving for ϑ^i, we obtain:

$$\vartheta^i = \frac{\Lambda^i \xi^i}{\left(1 - \Lambda^i\right) \left(1 - \xi^i\right)}. \tag{48}$$

As discussed in Section 5.1.1, this expression is employed in our calibration strategy: For both household and corporate debt, we set empirical values of Λ^i and ξ^i. We then use (48) to calibrate the refinancing parameter for each of the two agents. For households, we set $\xi^I = 0.014$ and $\Lambda^I = 0.39$, thus obtaining $\vartheta^I = 0.009$. For firms, we set $\xi^I = 0.125$ and $\Lambda^I = 0.83$, so that $\vartheta^I = 0.698$.

D2. First-order conditions

Here we report the first-order conditions from the optimization problems faced by the three types of agents in the model.

Patient households

Patient households' optimal behavior is described by the following first-order conditions:

$$\frac{1}{C_t^P - \theta^P C_{t-1}^P} - \frac{\beta\theta^P}{\mathrm{E}_t\left\{C_{t+1}^P\right\} - \theta^P C_t^P} = \lambda_t^P, \tag{49}$$

$$\nu^P\left(1 - N_t^P\right)^{-\varphi^P} = \lambda_t^P W_t^P, \tag{50}$$

$$\lambda_t^P = \beta^P R_t \mathrm{E}_t\left\{\lambda_{t+1}^P\right\}, \tag{51}$$

$$Q_t = \frac{\varepsilon_t}{\lambda_t^P H_t^P} + \beta^P \mathrm{E}_t\left\{\frac{\lambda_{t+1}^P}{\lambda_t^P} Q_{t+1}\right\}, \tag{52}$$

where λ_t^P is the multiplier associated with (15).

Impatient households

The first-order conditions of the impatient households are given by:

$$\frac{1}{C_t^I - \theta^I C_{t-1}^I} - \frac{\beta\theta^I}{\mathrm{E}_t\left\{C_{t+1}^I\right\} - \theta^I C_t^I} = \lambda_t^I, \tag{53}$$

$$\nu^I\left(1 - N_t^I\right)^{-\varphi^I} = \lambda_t^I W_t^I, \tag{54}$$

$$\lambda_t^I - \mu_t^I = \beta^I R_t \mathrm{E}_t\left\{\lambda_{t+1}^I\right\} - \beta^I\left(1 - \vartheta^I\right)\left(1 - \xi^I\right)\mathrm{E}_t\left\{\mu_{t+1}^I\right\}, \tag{55}$$

$$Q_t = \frac{\varepsilon_t}{\lambda_t^I H_t^I} + \beta^I \mathrm{E}_t\left\{\frac{\lambda_{t+1}^I}{\lambda_t^I} Q_{t+1}\right\} + \vartheta^I s_t^I \frac{\mu_t^I}{\lambda_t^I}\frac{\mathrm{E}_t\left\{Q_{t+1}\right\}}{R_t}, \tag{56}$$

where λ_t^I is the multiplier associated with (17), and μ_t^I is the multiplier associated with (18). Additionally, the complementary slackness condition

$$\mu_t^I\left(B_t^I - \vartheta^I s_t^I \frac{\mathrm{E}_t\left\{Q_{t+1}\right\} H_t^I}{R_t} - \left(1 - \vartheta^I\right)\left(1 - \xi^I\right) B_{t-1}^I\right) = 0, \tag{57}$$

must hold along with $\mu_t^I \geq 0$ and (18).

Entrepreneurs

The optimal behavior of the entrepreneurs is characterized by:

$$\frac{1}{C_t^E - \theta^E C_{t-1}^E} - \frac{\beta\theta^E}{\mathrm{E}_t\left\{C_{t+1}^E\right\} - \theta^E C_t^E} = \lambda_t^E, \tag{58}$$

$$\lambda_t^E - \mu_t^E = \beta^E R_t \mathrm{E}_t\left\{\lambda_{t+1}^E\right\} - \beta^E\left(1 - \vartheta^E\right)\left(1 - \xi^E\right)\mathrm{E}_t\left\{\mu_{t+1}^E\right\}, \tag{59}$$

$$\lambda_t^E = \psi_t^E\left[1 - \frac{\Omega}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2 - \Omega\frac{I_t}{I_{t-1}}\left(\frac{I_t}{I_{t-1}} - 1\right)\right] + \beta^E \Omega \mathrm{E}_t\left\{\psi_{t+1}^E\left(\frac{I_{t+1}}{I_t}\right)^2\left(\frac{I_{t+1}}{I_t} - 1\right)\right\}, \tag{60}$$

$$\psi_t^E = \beta^E r_t^K \mathrm{E}_t\left\{\lambda_{t+1}^E\right\} + \beta^E(1-\delta)\mathrm{E}_t\left\{\psi_{t+1}^E\right\} + \vartheta^E\mu_t^E s_t^E \frac{\mathrm{E}_t\left\{Q_{t+1}^K\right\}}{R_t}, \tag{61}$$

$$Q_t = \beta^E r_t^H \mathrm{E}_t\left\{\frac{\lambda_{t+1}^E}{\lambda_t^E}\right\} + \beta^E \mathrm{E}_t\left\{\frac{\lambda_{t+1}^E}{\lambda_t^E}Q_{t+1}\right\} + \vartheta^E s_t^E \frac{\mu_t^E}{\lambda_t^E}\frac{\mathrm{E}_t\left\{Q_{t+1}\right\}}{R_t}, \tag{62}$$

where λ_t^E, ψ_t^E, and μ_t^E are the multipliers associated with (22), (23), and (24), respectively. Moreover,

$$\mu_t^E\left(B_t^E - \vartheta^E s_t^E \mathrm{E}_t\left\{\frac{Q_{t+1}^K K_t + Q_{t+1}H_t^E}{R_t}\right\} - \left(1-\vartheta^E\right)\left(1-\xi^E\right)B_{t-1}^E\right) = 0, \tag{63}$$

holds along with $\mu_t^E \geq 0$ and (24). Finally, the definition of Q_t^K implies that

$$Q_t^K = \psi_t^E/\lambda_t^E. \tag{64}$$

Firms

Firms' first-order conditions determine the optimal demand for the input factors:

$$\alpha\gamma Y_t/N_t^P = W_t^P, \tag{65}$$

$$(1-\alpha)\gamma Y_t/N_t^I = W_t^I, \tag{66}$$

$$(1-\gamma)(1-\phi)\mathrm{E}_t\left\{Y_{t+1}\right\}/K_t = r_t^K, \tag{67}$$

$$(1-\gamma)\phi\mathrm{E}_t\left\{Y_{t+1}\right\}/H_t^E = r_t^H. \tag{68}$$

D3. Steady state

The deterministic steady state of the model is described in the following. Variables without time subscripts indicate their steady-state values. We first consider the implications of the patient households' optimality conditions. From (49) and (50), we get

$$\frac{1-\beta^P\theta^P}{\left(1-\theta^P\right)C^P} = \lambda^P \tag{69}$$

and

$$\nu^P\left(1-N^P\right)^{-\varphi^P} = \lambda^P W^P, \tag{70}$$

respectively. The steady-state gross interest rate on loans is recovered from (51):

$$R = \frac{1}{\beta^P}, \tag{71}$$

emphasizing that it is the time preference of the most patient individual that determines the steady-state rate of interest. From (52) we find

$$H^P = \frac{\varepsilon}{Q\lambda^P\left(1-\beta^P\right)}. \tag{72}$$

Turning to impatient households, (53) and (54) lead to

$$\frac{1-\beta^I\theta^I}{\left(1-\theta^I\right)C^I} = \lambda^I, \tag{73}$$

and

$$\nu^I \left(1 - N^I\right)^{-\varphi^I} = \lambda^I W^I, \tag{74}$$

respectively. From (55) we obtain the steady-state value of the multiplier on the credit constraint:

$$\mu^I = \frac{\lambda^I \left(1 - \beta^I R\right)}{1 - \beta^I \left(1 - \vartheta^I\right)\left(1 - \xi^I\right)},$$

which, by use of (71), yields

$$\mu^I = \frac{\lambda^I \left(1 - \frac{\beta^I}{\beta^P}\right)}{1 - \beta^I \left(1 - \vartheta^I\right)\left(1 - \xi^I\right)}. \tag{75}$$

From (75) we see that, in the steady state, $\mu^I > 0$ provided that $\beta^P > \beta^I$, which implies that the credit constraint (18) is binding. In a similar fashion, from (59) we get

$$\mu^E = \frac{\lambda^E \left(1 - \frac{\beta^E}{\beta^P}\right)}{1 - \beta^E \left(1 - \vartheta^E\right)\left(1 - \xi^E\right)}. \tag{76}$$

Hence, $\mu^E > 0$ provided that $\beta^P > \beta^E$, implying that the entrepreneurs' credit constraint, (24), is also binding in the steady state. From (56) we get

$$H^I = \frac{\varepsilon}{Q\lambda^I \left[1 - \beta^I - \frac{\left(1 - \frac{\beta^I}{\beta^P}\right)}{1 - \beta^I \left(1 - \vartheta^I\right)\left(1 - \xi^I\right)} \vartheta^I s^I \beta^P\right]}, \tag{77}$$

where the last line makes use of (71) and (75).

Turning to the remaining optimality conditions of the entrepreneurs, (58) gives

$$\frac{1 - \beta^E \theta^E}{\left(1 - \theta^E\right) C^E} = \lambda^E, \tag{78}$$

and (60) implies

$$\psi^E \left[1 - \frac{\Omega}{2}\left(\frac{I}{I} - 1\right)^2\right] - \psi^E \Omega \frac{I}{I}\left(\frac{I}{I} - 1\right) + \beta^E \psi^E \Omega \left(\frac{I}{I}\right)^2 \left(\frac{I}{I} - 1\right) = \lambda^E,$$

leading to

$$\psi^E = \lambda^E. \tag{79}$$

This reflects that there are no investment adjustment costs in the steady state. Therefore, the shadow value of a unit of capital equals the shadow value of wealth. Combining (79) with (64), we obtain

$$Q^K = 1. \tag{80}$$

After imposing (71), (76), and (79), (61) returns

$$r^K = \frac{\left[1 - \beta^E \left(1 - \vartheta^E\right)\left(1 - \xi^E\right)\right]\left[1 - \beta^E (1 - \delta)\right] - \left(\beta^P - \beta^E\right) \vartheta^E s^E Q^K}{\beta^E \left[1 - \beta^E \left(1 - \vartheta^E\right)\left(1 - \xi^E\right)\right]}. \tag{81}$$

From (62), instead, we find

$$r^H = \frac{\left(1-\beta^E\right)Q}{\beta^E} - \frac{\mu^E \vartheta^E s^E}{\lambda^E \beta^E}\frac{Q}{R}. \tag{82}$$

We then turn to the remaining equilibrium conditions in the steady state. As we saw above, the two credit constraints are binding in the steady state. Hence,

$$B^I = \frac{\vartheta^I s^I}{1-\left(1-\vartheta^I\right)\left(1-\xi^I\right)}\frac{QH^I}{R}, \tag{83}$$

$$B^E = \frac{\vartheta^E s^E}{1-\left(1-\vartheta^E\right)\left(1-\xi^E\right)}\frac{Q^K K + QH^E}{R}. \tag{84}$$

The production function is

$$Y = \left[\left(N^P\right)^\alpha \left(N^I\right)^{1-\alpha}\right]^\gamma \left[\left(H^E\right)^\phi K^{1-\phi}\right]^{1-\gamma}. \tag{85}$$

The steady-state counterparts of firms' first-order conditions, (65)–(68), are:

$$\begin{aligned} \alpha\gamma\frac{Y}{N^P} &= W^P, && (86)\\ (1-\alpha)\gamma\frac{Y}{N^I} &= W^I, && (87)\\ (1-\gamma)(1-\phi)\frac{Y}{K} &= r^K, && (88)\\ (1-\gamma)\phi\frac{Y}{H^E} &= r^H. && (89) \end{aligned}$$

In the steady state, the law of motion for capital implies

$$I = \delta K. \tag{90}$$

We have the following steady-state resource constraints:

$$Y = C^P + C^I + C^E + I, \tag{91}$$

$$H = H^P + H^I + H^E, \tag{92}$$

$$B^P + B^I + B^E = 0. \tag{93}$$

Also, we have the steady-state versions of the agents' budget constraints:

$$\begin{aligned} C^P &= W^P N^P - (R-1)B^P, && (94)\\ C^I &= W^I N^I - (R-1)B^I, && (95)\\ C^E + I &= r^K K + r^H H^E - (R-1)B^E && (96) \end{aligned}$$

We therefore have that the steady state is characterized by the vector

$$\left[\begin{array}{c} Y, C^P, C^I, C^E, I, H^P, H^I, H^E, K, N^P, N^I, B^P, B^I, B^E, \\ Q, Q^K, R, r^K, r^H, W^P, W^I, \lambda^P, \lambda^I, \lambda^E, \mu^I, \mu^E, \psi^E \end{array}\right].$$

These 27 variables are determined by the 27 equations: (69), (70), (71), (72), (73), (74), (75), (76), (77), (78), (79), (80), (81), (82), (83), (84), (85), (86), (87), (88), (89), (90), (91), (92), (93), (94), and (95).

We now briefly proceed with the characterization of the steady state, finding some variables' equilibrium in a closed form. To this end, we define these variables as a ratio of total output. The resulting system, which comprises seven equations, is then solved numerically. The remaining variables then follow from the characterizations above.

First, combine (81) and (88) to get an expression for capital-output ratio:

$$\frac{K}{Y}=\frac{(1-\gamma)(1-\phi)\beta^E\left[1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)\right]}{\left[1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)\right]\left[1-\beta^E(1-\delta)\right]-\left(\beta^P-\beta^E\right)\vartheta^E s^E}, \tag{97}$$

where we have used $Q^K=1$ from (80). Thus, we combine (82) and (89) to get an expression for entrepreneurs' land-output ratio:

$$\frac{QH^E}{Y}=\frac{(1-\gamma)\phi\beta^E\left[1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)\right]}{\left(1-\beta^E\right)\left[1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)\right]-\left(\beta^P-\beta^E\right)\vartheta^E s^E}, \tag{98}$$

where we have made use of (76). Again, based on $Q^K=1$, the entrepreneurial borrowing constraint can be rewritten as

$$\frac{B^E}{Y}=\frac{\vartheta^E}{1-\left(1-\vartheta^E\right)\left(1-\xi^E\right)}\frac{s^E}{R}\left(\frac{K}{Y}+\frac{QH^E}{Y}\right), \tag{99}$$

where we can insert from (71), (97), and (98). The resulting closed-form solution of the entrepreneurial steady-state loan-to-output ratio is central in setting up a sub-system of seven central variables. First, it can be plugged into the entrepreneurs' budget constraint, (96), so as to obtain:

$$\frac{C^E}{Y}+\frac{I}{Y}=r^K\frac{K}{Y}+r^H\frac{H^E}{Y}-(R-1)\frac{B^E}{Y},$$

which, by use of (90), becomes

$$\frac{C^E}{Y}=\left(r^K-\delta\right)\frac{K}{Y}+r^H\frac{H^E}{Y}-(R-1)\frac{B^E}{Y}.$$

Using (81) and (89), we get

$$\frac{C^E}{Y}=\left(\frac{\left(1-\beta^E\right)\left[1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)\right]-\left(\beta^P-\beta^E\right)\vartheta^E s^E Q^K}{\beta^E\left[1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)\right]}\right)\frac{K}{Y}+(1-\gamma)\phi-(R-1)\frac{B^E}{Y},$$

which, by use of (97), returns the entrepreneurs' consumption-to-output ratio:

$$\begin{aligned}\frac{C^E}{Y} &= \frac{(1-\gamma)(1-\phi)\left[\left(1-\beta^E\right)\left[1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)\right]-\left(\beta^P-\beta^E\right)\vartheta^E s^E\right]}{\left[1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)\right]\left[1-\beta^E(1-\delta)\right]-\left(\beta^P-\beta^E\right)\vartheta^E s^E} \\ &\quad +(1-\gamma)\phi-\frac{1-\beta^P}{\beta^P}\frac{B^E}{Y}.\end{aligned} \tag{100}$$

We then turn to the impatient households. Their budget constraint can be written as

$$\frac{C^I}{Y}=\frac{W^I N^I}{Y}-(R-1)\frac{B^I}{Y},$$

which, by use of (71) and (87), becomes

$$\frac{C^I}{Y} = (1-\alpha)\,\gamma - \frac{1-\beta^P}{\beta^P}\frac{B^I}{Y}.$$

Likewise, patient households' budget constraint can be written as

$$\frac{C^P}{Y} = \frac{W^P N^P}{Y} - (R-1)\,\frac{B^P}{Y},$$

which, by use of (71) and (86), becomes

$$\frac{C^P}{Y} = \alpha\gamma - \frac{1-\beta^P}{\beta^P}\frac{B^P}{Y}.$$

Adding up these constraints gives

$$\frac{C^I + C^P}{Y} = \gamma + \frac{1-\beta^P}{\beta^P}\frac{B^E}{Y}, \tag{101}$$

where (93) has been invoked. Note that the right-hand-side of (101) is known, by virtue of (99).

Combining (69), (70) and (86) gives the steady-state equilibrium condition for patient households' labor:

$$\nu^P \left(1-N^P\right)^{-\varphi^P} C^P \frac{1-\theta^P}{1-\beta^P\theta^P} = \alpha\gamma\frac{Y}{N^P}. \tag{102}$$

Similarly, (73), (74) and (87) characterize impatient households' equilibrium labor:

$$\nu^I \left(1-N^I\right)^{-\varphi^I} C^I \frac{1-\theta^I}{1-\beta^I\theta^I} = (1-\alpha)\,\gamma\frac{Y}{N^I}. \tag{103}$$

Combining the two households' land-demand expressions, (72) and (77), gives

$$\frac{H^I}{H^P} = \frac{\lambda^P\left(1-\beta^P\right)\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]}{\lambda^I\left\{\left(1-\beta^I\right)\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]-\left(\beta^P-\beta^I\right)\vartheta^I s^I\right\}}.$$

Eliminating the multipliers by (69) and (73), and eliminating H^P through (92), we obtain the following land-market equilibrium characterization:

$$\frac{H^I}{H-H^I-H^E}\frac{C^P}{C^I} = \frac{\left(1-\beta^P\theta^P\right)\left(1-\theta^I\right)\left(1-\beta^P\right)\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]}{\left(1-\beta^I\theta^I\right)\left(1-\theta^P\right)\left\{\left(1-\beta^I\right)\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]-\left(\beta^P-\beta^I\right)\vartheta^I s^I\right\}}. \tag{104}$$

We also take the impatient households' borrowing constraint into consideration. Using (83) to eliminate B^I in the budget constraint,

$$\frac{C^I}{Y} = (1-\alpha)\,\gamma - \left(1-\beta^P\right)\frac{\vartheta^I}{1-\left(1-\vartheta^I\right)\left(1-\xi^I\right)}\frac{s^I Q H^I}{Y}. \tag{105}$$

Relying on (77),

$$QH^I = \frac{\varepsilon\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]}{\lambda^I\left\{\left(1-\beta^I\right)\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]-\left(\beta^P-\beta^I\right)\vartheta^I s^I\right\}},$$

and, again, (73), we obtain

$$\begin{aligned} QH^I &= \frac{\varepsilon\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]\frac{\left(1-\theta^I\right)}{1-\beta^I\theta^I}C^I}{\left(1-\beta^I\right)\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]-\left(\beta^P-\beta^I\right)\vartheta^I s^I}, && (106)\\ Q &= \frac{\varepsilon\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]\frac{\left(1-\theta^I\right)}{1-\beta^I\theta^I}C^I}{H^I\left\{\left(1-\beta^I\right)\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]-\left(\beta^P-\beta^I\right)\vartheta^I s^I\right\}}. && (107)\end{aligned}$$

We then use (106) to rewrite the consumption-output ratio for impatient households, (105), as

$$\begin{aligned}\frac{C^I}{Y} &= (1-\alpha)\gamma \\ &\quad -\left(1-\beta^P\right)\frac{\vartheta^I}{1-\left(1-\vartheta^I\right)\left(1-\xi^I\right)}\frac{s^I}{Y}\frac{\varepsilon\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]\frac{\left(1-\theta^I\right)}{1-\beta^I\theta^I}C^I}{\left(1-\beta^I\right)\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]-\left(\beta^P-\beta^I\right)\vartheta^I s^I}.\end{aligned} \quad (108)$$

Likewise, we can use (107) to eliminate Q from (98), and obtain:

$$\begin{aligned}\frac{H^E}{Y} &= \frac{(1-\gamma)\phi\beta^E\left[1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)\right]}{\left(1-\beta^E\right)\left[1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)\right]-\left(\beta^P-\beta^E\right)\vartheta^E s^E}\cdot \\ &\quad \cdot\frac{\left\{\left(1-\beta^I\right)\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]-\left(\beta^P-\beta^I\right)\vartheta^I s^I\right\}}{\varepsilon\left[1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)\right]\frac{\left(1-\theta^I\right)}{1-\beta^I\theta^I}}\frac{H^I}{C^I}.\end{aligned} \quad (109)$$

Thus, the production function (85) is rewritten as a function of the derived ratios:

$$Y^\gamma = A\left[\left(N^P\right)^\alpha\left(N^I\right)^{1-\alpha}\right]^\gamma\left[\left(\frac{H^E}{Y}\right)^\phi\left(\frac{K}{Y}\right)^{1-\phi}\right]^{1-\gamma},$$

Using (97), we finally obtain

$$\begin{aligned} Y &= A^{\frac{1}{\gamma}}\left(N^P\right)^\alpha\left(N^I\right)^{1-\alpha}\cdot \\ &\quad \cdot\left[\left(\frac{H^E}{Y}\right)^\phi\left(\frac{(1-\gamma)(1-\phi)\beta^E\left[1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)\right]}{\left[1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)\right]\left[1-\beta^E(1-\delta)\right]-\left(\beta^P-\beta^E\right)\vartheta^E s^E Q^K}\right)^{1-\phi}\right]^{\frac{1-\gamma}{\gamma}}.\end{aligned} \quad (110)$$

We have now reduced the steady state to a matter of finding the vector

$$\left[Y, C^P, C^I, H^I, H^E, N^P, N^I\right],$$

which satisfies the equations (101), (102), (103), (104), (108), (109) and (110), *given* the solution for B^E/Y, (99), and given all the parameters and exogenous variables of the model. We compute the vector numerically using `fsolve` in MATLAB. The remaining variables then follow analytically from the steady-state equations presented above.

D4. Log-linearization

We log-linearize the model around the steady state found in the previous section. In the following, we let $\widehat{X_t}$ denote the log-deviation of a generic variable X_t from its steady state value X, except for the following variables: For the interest rates, $\widehat{R}_t \equiv R_t - R$, $\widehat{r}_t^H \equiv r_t^H - r^H$ and $\widehat{r}_t^K \equiv r_t^K - r^K$; for debt, $\widehat{B}_t^i \equiv \left(B_t^i - B^i\right)/Y$, $i = \{P, I, E\}$. We first derive the log-linear versions of the agents' optimality conditions and conclude with the expressions for market clearing.

Optimality Conditions of Patient Households

Once log-linearized, equations (49), (50) and (51) become

$$\beta^P\theta^P\mathrm{E}_t\left\{\widehat{C}_{t+1}^P\right\} - \left(1+\beta^P\left(\theta^P\right)^2\right)\widehat{C}_t^P + \theta^P\widehat{C}_{t-1}^P = \left(1-\theta^P\right)\left(1-\beta^P\theta^P\right)\widehat{\lambda}_t^P, \tag{111}$$

$$\varphi^P\frac{N^P}{1-N^P}\widehat{N}_t^P = \widehat{\lambda}_t^P + \widehat{W}_t^P, \tag{112}$$

$$\beta^P\widehat{R}_t + \mathrm{E}_t\left\{\widehat{\lambda}_{t+1}^P\right\} = \widehat{\lambda}_t^P, \tag{113}$$

Log-linearizing (52) yields

$$\frac{\varepsilon}{H^P}\left(\widehat{\varepsilon}_t - \widehat{H}_t^P\right) + \beta^P\lambda^P Q\mathrm{E}_t\left\{\widehat{\lambda}_{t+1}^P + \widehat{Q}_{t+1}\right\} = \lambda^P Q\left(\widehat{\lambda}_t^P + \widehat{Q}_t\right).$$

Now use steady-state equation (72) to get

$$-Q\lambda^P\left(1-\beta^P\right)\widehat{H}_t^P + Q\lambda^P\left(1-\beta^P\right)\widehat{\varepsilon}_t + \beta^P\lambda^P Q\mathrm{E}_t\left\{\widehat{\lambda}_{t+1}^P + \widehat{Q}_{t+1}\right\} = \lambda^P Q\left(\widehat{\lambda}_t^P + \widehat{Q}_t\right),$$

and thereby

$$\beta^P\mathrm{E}_t\left\{\widehat{\lambda}_{t+1}^P + \widehat{Q}_{t+1}\right\} - \left(1-\beta^P\right)\widehat{H}_t^P + \left(1-\beta^P\right)\widehat{\varepsilon}_t = \widehat{\lambda}_t^P + \widehat{Q}_t. \tag{114}$$

Moreover, the log-linearized budget constraint reads as

$$\begin{aligned} & \frac{C^P}{Y}\widehat{C}_t^P + \frac{QH^P}{Y}\left(\widehat{H}_t^P - \widehat{H}_{t-1}^P\right) + \frac{B^P}{Y}\widehat{R}_{t-1} + \frac{1}{\beta^P}\widehat{B}_{t-1}^P \\ = \; & \widehat{B}_t^P + \alpha\gamma\left(\widehat{W}_t^P + \widehat{N}_t^P\right). \end{aligned}$$

where we have used (86).

Optimality Conditions of Impatient Households

From (53), (54) and (55) we obtain

$$\beta^I\theta^I\mathrm{E}_t\left\{\widehat{C}_{t+1}^I\right\} - \left(1+\beta^I\left(\theta^I\right)^2\right)\widehat{C}_t^I + \theta^I\widehat{C}_{t-1}^I = \left(1-\theta^I\right)\left(1-\beta^I\theta^I\right)\widehat{\lambda}_t^I, \tag{115}$$

$$\varphi^I\frac{N^I}{1-N^I}\widehat{N}_t^I = \widehat{\lambda}_t^I + \widehat{W}_t^I, \tag{116}$$

and

$$\lambda^I \widehat{\lambda}_t^I - \mu^I \widehat{\mu}_t^I = \beta^I \lambda^I \widehat{R}_t + \beta^I R \lambda^I \mathrm{E}_t \left\{ \widehat{\lambda}_{t+1}^I \right\} - \beta^I \left(1-\vartheta^I\right)\left(1-\xi^I\right)\mu^I \mathrm{E}_t \left\{ \widehat{\mu}_{t+1}^I \right\},$$

respectively. The last expression is rewritten, by means of (75), as

$$\begin{aligned} \widehat{\lambda}_t^I &= \beta^I \widehat{R}_t + \beta^I R \mathrm{E}_t \left\{ \widehat{\lambda}_{t+1}^I \right\} + \frac{\left(1-\frac{\beta^I}{\beta^P}\right)}{1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)} \widehat{\mu}_t^I \\ & -\beta^I \left(1-\vartheta^I\right)\left(1-\xi^I\right) \frac{\left(1-\frac{\beta^I}{\beta^P}\right)}{1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)} \mathrm{E}_t \left\{ \widehat{\mu}_{t+1}^I \right\}. \end{aligned} \tag{117}$$

Furthermore, (56) becomes

$$\begin{aligned} Q\widehat{Q}_t &= \frac{\varepsilon}{H^I \lambda^I}\left(\widehat{\varepsilon}_t - \widehat{\lambda}_t^I - \widehat{H}_t^I\right) + \beta^I Q \mathrm{E}_t \left\{ \widehat{\lambda}_{t+1}^I + \widehat{Q}_{t+1} - \widehat{\lambda}_t^I \right\} \\ & + \frac{\mu^I}{\lambda^I}\frac{\vartheta^I s^I Q}{R}\left[\widehat{\mu}_t^I - \widehat{\lambda}_t^I + \widehat{s}_t + \mathrm{E}_t \left\{ \widehat{Q}_{t+1} \right\} - \beta^P \widehat{R}_t\right], \end{aligned}$$

which, by use of (75) and (77), becomes

$$\begin{aligned} \widehat{Q}_t &= \left[1-\beta^I - \frac{\left(\beta^P-\beta^I\right)}{1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)}\vartheta^I s^I\right]\left(\widehat{\varepsilon}_t - \widehat{\lambda}_t^I - \widehat{H}_t^I\right) + \beta^I \mathrm{E}_t \left\{ \widehat{\lambda}_{t+1}^I + \widehat{Q}_{t+1} - \widehat{\lambda}_t^I \right\} \\ & + \frac{\left(\beta^P-\beta^I\right)}{1-\beta^I\left(1-\vartheta^I\right)\left(1-\xi^I\right)}\vartheta^I s^I \left[\widehat{\mu}_t^I - \widehat{\lambda}_t^I + \widehat{s}_t + \mathrm{E}_t \left\{ \widehat{Q}_{t+1} \right\} - \beta^P \widehat{R}_t\right], \end{aligned} \tag{118}$$

where, again, we have used (71). The budget constraint becomes

$$\frac{C^I}{Y}\widehat{C}_t^I + \frac{QH^I}{Y}\left(\widehat{H}_t^I - \widehat{H}_{t-1}^I\right) + \frac{B^I}{Y}\widehat{R}_{t-1}^{M,I} + \frac{1}{\beta^P}\widehat{B}_{t-1}^I = \widehat{B}_t^I + (1-\alpha)\gamma\left(\widehat{W}_t^I + \widehat{N}_t^I\right), \tag{119}$$

where we have used (87). Finally, the log-linearized version of the collateral constraint is:

$$Y\widehat{B}_t^I \leq \frac{\vartheta^I s^I Q H^I}{R}\left(\widehat{s}_t^I + \mathrm{E}_t \left\{ \widehat{Q}_{t+1} \right\} + \widehat{H}_t^I - \beta^P \widehat{R}_t\right) + \left(1-\vartheta^I\right)\left(1-\xi^I\right) Y\widehat{B}_{t-1}^I. \tag{120}$$

Optimality Conditions of the Entrepreneurs

From (58) and (59) we get

$$\beta^E \theta^E \mathrm{E}_t \left\{ \widehat{C}_{t+1}^E \right\} - \left(1+\beta^E\left(\theta^E\right)^2\right)\widehat{C}_t^E + \theta^E \widehat{C}_{t-1}^E = \left(1-\theta^E\right)\left(1-\beta^E\theta^E\right)\widehat{\lambda}_t^E, \tag{121}$$

$$\lambda^E \widehat{\lambda}_t^E - \mu^E \widehat{\mu}_t^E = \beta^E \lambda^E \widehat{R}_t + \beta^E R \lambda^E \mathrm{E}_t \left\{ \widehat{\lambda}_{t+1}^E \right\} - \beta^E \left(1-\vartheta^E\right)\left(1-\xi^E\right)\mu^E \mathrm{E}_t \left\{ \widehat{\mu}_{t+1}^E \right\},$$

respectively. The latter we can be rewritten using (76):

$$\begin{aligned} \widehat{\lambda}_t^E &= \beta^E \widehat{R}_t + \beta^E R \mathrm{E}_t \left\{ \widehat{\lambda}_{t+1}^E \right\} + \frac{\left(1-\frac{\beta^E}{\beta^P}\right)}{1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)} \widehat{\mu}_t^E \\ & -\beta^E \left(1-\vartheta^E\right)\left(1-\xi^E\right) \frac{\left(1-\frac{\beta^E}{\beta^P}\right)}{1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)} \mathrm{E}_t \left\{ \widehat{\mu}_{t+1}^E \right\}. \end{aligned} \tag{122}$$

From (60) we get

$$\widehat{\psi}_t^E - \Omega\left(1+\beta^E\right)\widehat{I}_t + \Omega\widehat{I}_{t-1} + \beta^E\Omega \mathrm{E}_t\left\{\widehat{I}_{t+1}\right\} = \widehat{\lambda}_t^E, \tag{123}$$

where we have made use of (79). Equation (61) becomes

$$\begin{aligned}\widehat{\psi}_t^E &= \beta^E r^K \mathrm{E}_t\left\{\widehat{\lambda}_{t+1}^E\right\} + \beta^E \widehat{r}_t^K + \beta^E(1-\delta)\mathrm{E}_t\left\{\widehat{\psi}_{t+1}^E\right\} \\ &+\vartheta^E \frac{\left(\beta^P-\beta^E\right)}{1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)} s^E Q^K\left[\widehat{\mu}_t^E + \widehat{s}_t^E + \mathrm{E}_t\left\{\widehat{Q}_{t+1}^K\right\} - \beta^P\widehat{R}_t\right],\end{aligned} \tag{124}$$

where we have used (71), (76), and (79). Moreover, (64) becomes

$$\widehat{\psi}_t^E = \widehat{\lambda}_t^E + \widehat{Q}_t^K. \tag{125}$$

Finally, (62) is approximated as

$$\begin{aligned}Q\widehat{Q}_t &= \beta^E r^H\left(\mathrm{E}_t\left\{\widehat{\lambda}_{t+1}^E\right\} - \widehat{\lambda}_t^E + \frac{1}{r^H}\widehat{r}_t^H\right) + \beta^E Q\left(\mathrm{E}_t\left\{\widehat{\lambda}_{t+1}^E\right\} + \mathrm{E}_t\left\{\widehat{Q}_{t+1}\right\} - \widehat{\lambda}_t^E\right) \\ &+\vartheta^E s^E \frac{\mu^E}{\lambda^E}\frac{Q}{R}\left[\widehat{s}_t^E + \widehat{\mu}_t^E - \widehat{\lambda}_t^E + \mathrm{E}_t\left\{\widehat{Q}_{t+1}\right\} - \beta^P\widehat{R}_t\right],\end{aligned}$$

which we can rewrite, using (71) and (76), as

$$\begin{aligned}Q\widehat{Q}_t &= \beta^E r^H\left(\mathrm{E}_t\left\{\widehat{\lambda}_{t+1}^E\right\} - \widehat{\lambda}_t^E + \frac{1}{r^H}\widehat{r}_t^H\right) + \beta^E Q\mathrm{E}_t\left(\widehat{\lambda}_{t+1}^E + \widehat{Q}_{t+1} - \widehat{\lambda}_t^E\right) \\ &+\frac{\left(\beta^P-\beta^E\right)}{1-\beta^E\left(1-\vartheta^E\right)\left(1-\xi^E\right)}\vartheta^E s^E Q\left[\widehat{s}_t^E + \widehat{\mu}_t^E - \widehat{\lambda}_t^E + \mathrm{E}_t\left\{\widehat{Q}_{t+1}\right\} - \beta^P\widehat{R}_t\right].\end{aligned} \tag{126}$$

Furthermore, the budget constraint becomes

$$\begin{aligned}&\frac{C^E}{Y}\widehat{C}_t^E + \frac{I}{Y}\widehat{I}_t + \frac{QH^E}{Y}\left(\widehat{H}_t^E - \widehat{H}_{t-1}^E\right) + \frac{B^E}{Y}\widehat{R}_{t-1}^{M,E} + \frac{1}{\beta^P}\widehat{B}_{t-1}^E \\ =\ &\widehat{B}_t^E + \frac{K}{Y}\widehat{r}_{t-1}^K + \frac{H^E}{Y}\widehat{r}_{t-1}^H + (1-\gamma)\phi\widehat{H}_{t-1}^E + (1-\gamma)(1-\phi)\widehat{K}_{t-1},\end{aligned} \tag{127}$$

where we have used (88) and (89). Finally, the borrowing constraint reads as

$$\begin{aligned}Y\widehat{B}_t^E &\leq \vartheta^E s^E \frac{\left(K+QH^E\right)}{R}\left(\widehat{s}_t^E - \beta^P\widehat{R}_t\right) + \vartheta^E s^E\frac{K}{R}\mathrm{E}_t\left\{\widehat{Q}_{t+1}^K + \widehat{K}_t\right\} \\ &+\vartheta^E s^E\frac{QH^E}{R}\mathrm{E}_t\left\{\widehat{Q}_{t+1} + \widehat{H}_t^E\right\} + \left(1-\vartheta^E\right)\left(1-\xi^E\right)Y\widehat{B}_{t-1}^E.\end{aligned} \tag{128}$$

Firms' Optimality Conditions

Firms' first-order conditions, (65), (66), (67) and (68), are log-linearized as

$$\widehat{Y}_t - \widehat{N}_t^P = \widehat{W}_t^P, \tag{129}$$

$$\widehat{Y}_t - \widehat{N}_t^I = \widehat{W}_t^I, \tag{130}$$

$$\mathrm{E}_t\left\{\widehat{Y}_{t+1}\right\} - \widehat{K}_t = \left(r^K\right)^{-1}\widehat{r}_t^K, \tag{131}$$

$$\mathrm{E}_t\left\{\widehat{Y}_{t+1}\right\}-\widehat{H}_t^E=\left(r^H\right)^{-1}\widehat{r}_t^H, \tag{132}$$

respectively.

Market Clearing and Resource Constraints

From the law of motion for capital, (23), we get

$$\widehat{K}_t=(1-\delta)\,\widehat{K}_{t-1}+\delta\widehat{I}_t, \tag{133}$$

where we have used (90). Moreover, from the resource constraint, (30), we have

$$\widehat{Y}_t=\frac{C^P}{Y}\widehat{C}_t^P+\frac{C^I}{Y}\widehat{C}_t^I+\frac{C^E}{Y}\widehat{C}_t^E+\delta\frac{K}{Y}\widehat{I}_t. \tag{134}$$

We also have the log-linearized versions of (26), (28) and (29):

$$\widehat{Y}_t=\widehat{A}_t+\alpha\gamma\widehat{N}_t^P+(1-\alpha)\,\gamma\widehat{N}_t^I+(1-\gamma)\,(1-\phi)\,\widehat{K}_{t-1}+(1-\gamma)\,\phi\widehat{H}_{t-1}^E, \tag{135}$$

$$0=H^P\widehat{H}_t^P+H^I\widehat{H}_t^I+H^E\widehat{H}_t^E, \tag{136}$$

$$0=\widehat{B}_t^P+\widehat{B}_t^I+\widehat{B}_t^E. \tag{137}$$

As for the shocks processes, (27), (14) and (19) imply

$$\widehat{A}_t=\rho_A\widehat{A}_{t-1}+z_t, \tag{138}$$

$$\widehat{\varepsilon}_t=\rho_\varepsilon\widehat{\varepsilon}_{t-1}+u_t, \tag{139}$$

$$\widehat{s}_t=\rho_s\widehat{s}_{t-1}+v_t, \tag{140}$$

respectively. This completes our list of log-linearized equations.

The log-linearized system consists of 30 equations: 18 first-order conditions, 2 budget constraints, 2 credit constraints, 1 production function, 3 market clearing conditions, 1 capital accumulation equation, and 3 shock processes. The 30 variables of the system are given by the vector

$$\left[\begin{array}{c}\widehat{C}_t^P,\widehat{C}_t^I,\widehat{C}_t^E,\widehat{\lambda}_t^P,\widehat{\lambda}_t^I,\widehat{\lambda}_t^E,\widehat{\psi}_t^E,\widehat{\mu}_t^I,\widehat{\mu}_t^E,\widehat{R}_t,\widehat{N}_t^P,\widehat{N}_t^I,\widehat{W}_t^P,\widehat{W}_t^I,\\ \widehat{H}_t^P,\widehat{H}_t^I,\widehat{H}_t^E,\widehat{Q}_t,\widehat{Q}_t^K,\widehat{r}_t^H,\widehat{r}_t^K,\widehat{K}_t,\widehat{I}_t,\widehat{Y}_t,\widehat{B}_t^P,\widehat{B}_t^I,\widehat{B}_t^E,\widehat{A}_t,\widehat{\varepsilon}_t,\widehat{s}_t\end{array}\right],$$

and are determined by equations (111)-(140).

Appendix E. Solution method

We solve the model numerically, as described in the following. When solving the model, we treat the collateral constraints as inequalities, accounting for two complementary slackness conditions (57) and (63). We then adopt the solution method of Holden and Paetz (2012), on which this appendix builds. In turn, Holden and Paetz (2012) expand on previous work by Laséen and Svensson (2011). With first-order perturbations, this solution method is equivalent to the piecewise linear approach discussed by Guerrieri and Iacoviello (2015). We have verified that their proposed solution method does indeed produce identical results. Furthermore, Holden and Paetz (2012) and Guerrieri and Iacoviello (2015) evaluate the accuracy of their respective methods against a global solution based on projection methods. This is done for a very simple model with a borrowing constraint, for which a highly accurate global solution can be obtained and used as a benchmark. They find that the local approximations are very accurate. For the model used in this paper, the large number of state variables (14 endogenous state variables and three shocks) renders the use of global solution methods impractical due to the curse of dimensionality typically associated with such methods.

The collateral constraints put an upper bound on the borrowing of each of the two constrained agents. While the constraints are binding in the steady state, this may not be the case outside the steady state, where the constraints may not bind. Observe that we can reformulate the collateral constraints in terms of restrictions on each agent's shadow value of borrowing; μ_t^j, $j = \{I, E\}$: We know that $\mu_t^j \geq 0$ if and only if the optimal debt level of agent j is exactly at or above the collateral value. In other words, we need to ensure that $\mu_t^j \geq 0$. If this restriction is satisfied with inequality, the constraint is binding, so the slackness condition is satisfied. If it holds with equality, the collateral constraint becomes non-binding, but the slackness condition is still satisfied. If instead $\mu_t^j < 0$, agent j's optimal level of debt is lower than the credit limit, so that treating his collateral constraint as an equality implies that we are forcing him to borrow 'too much'. In this case, the slackness condition is violated. We then need to add shadow price shocks so as to 'push' μ_t^j back up until it exactly equals its lower limit of zero and the slackness condition is satisfied. To ensure compatibility with rational expectations, these shocks are added to the model as 'news shocks'. The idea of adding such shocks to the model derives from Laséen and Svensson (2011), who use such an approach to deal with pre-announced paths for the interest rate setting of a central bank. The contribution of Holden and Paetz (2012) is to develop a numerical method to compute the size of these shocks that are required to obtain the desired level for a given variable in each period, and to make this method applicable to a general class of potentially more complicated problems than the relatively simple experiments conducted by Laséen and Svensson (2011).

We first describe how to compute impulse responses to a single generic shock, e.g., a technology shock. The first step is to add independent sets of shadow price shocks to each of the two log-linearized collateral constraints. To this end, we need to determine the number of periods T for which we conjecture that the collateral constraints may be non-binding. This number may be smaller than or equal to the number of periods for which we compute impulse responses; $T \leq T^{IRF}$. For each period $t \leq T$, we then add shadow price shocks which hit the economy in period t but become known at period 0, that is, at the same time the economy is hit by the technology shock.

Let $\widehat{X_t}$ denote the log-deviation of a generic variable X_t from its steady-state value X, except for the following variables: For the interest rates, $\widehat{R}_t \equiv R_t - R$, $\widehat{r}_t^H \equiv r_t^H - r^H$ and $\widehat{r}_t^K \equiv r_t^K - r^K$, and for debt, $\widehat{B}_t^i \equiv (B_t^i - B^i)/Y$, $i = P, I, E$. We can then write the log-linearized collateral constraints, augmented with the shadow price shocks, as follows:

$$Y\widehat{B}_t^I \leq \frac{\vartheta^I s^I Q H^I}{R}\left(\widehat{s}_t^I + \mathrm{E}_t\left\{\widehat{Q}_{t+1}\right\} + \widehat{H}_t^I - \beta^P \widehat{R}_t\right) + \left(1-\vartheta^I\right)\left(1-\xi^I\right) Y\widehat{B}_{t-1}^I - \sum_{s=0}^{T-1} \varepsilon_{s,t-s}^{SP,I},$$

$$\begin{aligned}
Y\widehat{B}_t^E \;\leq\; & \vartheta^E s^E \frac{\left(K+QH^E\right)}{R}\left(\widehat{s}_t^E - \beta^P \widehat{R}_t\right) + \vartheta^E s^E \frac{K}{R}\mathrm{E}_t\left\{\widehat{Q}_{t+1}^K + \widehat{K}_t\right\} \\
& + \vartheta^E s^E \frac{QH^E}{R}\mathrm{E}_t\left\{\widehat{Q}_{t+1} + \widehat{H}_t^E\right\} + \left(1-\vartheta^E\right)\left(1-\xi^E\right) Y\widehat{B}_{t-1}^E - \sum_{s=0}^{T-1} \varepsilon_{s,t-s}^{SP,E},
\end{aligned}$$

where $\varepsilon_{s,t-s}^{SP,j}$ is the shadow price shock that hits agent j in period $t=s$, and is anticipated by all agents in period $t=t-s=0$ ensuring consistency with rational expectations. We let all shadow price shocks be of unit magnitude. We then need to compute two sets of weights α_{μ_I} and α_{μ_E} to control the impact of each shock on μ_t^I and μ_t^E. The 'optimal' sets of weights ensure that μ_t^I and μ_t^E are bounded below at exactly zero. The weights are computed by solving the following quadratic programming problem:

$$\begin{aligned}
\alpha^* &\equiv \begin{bmatrix} \alpha_{\mu_I}^{*\prime} & \alpha_{\mu_E}^{*\prime} \end{bmatrix}' \\
&= \arg\min \begin{bmatrix} \alpha'_{\mu_I} & \alpha'_{\mu_E} \end{bmatrix} \left[\begin{bmatrix} \mu^I + \widetilde{\mu}^{I,A} \\ \mu^E + \widetilde{\mu}^{E,A} \end{bmatrix} + \begin{bmatrix} \widetilde{\mu}^{I,\varepsilon^{SP,I}} & \widetilde{\mu}^{I,\varepsilon^{SP,E}} \\ \widetilde{\mu}^{E,\varepsilon^{SP,I}} & \widetilde{\mu}^{E,\varepsilon^{SP,E}} \end{bmatrix} \begin{bmatrix} \alpha_{\mu_I} \\ \alpha_{\mu_E} \end{bmatrix} \right],
\end{aligned}$$

subject to

$$\alpha'_{\mu_j} \geq 0,$$

$$\mu^j + \widetilde{\mu}^{j,A} + \widetilde{\mu}^{j,\varepsilon^{SP,j}} \alpha_{\mu_j} + \widetilde{\mu}^{j,\varepsilon^{SP,k}} \alpha_{\mu_k} \geq 0,$$

$j=\{I,E\}$. Here, μ^j and $\widetilde{\mu}^{j,A}$ denote, respectively, the steady-state value and the unrestricted relative impulse response of μ^j to a technology shock, that is, the impulse-response of μ^j when the collateral constraints are assumed to always bind. In this respect, the vector $\begin{bmatrix} \mu^I + \widetilde{\mu}^{I,A} \\ \mu^E + \widetilde{\mu}^{E,A} \end{bmatrix}$ contains the absolute, unrestricted impulse responses of the two shadow values stacked. Further, each matrix $\widetilde{\mu}^{j,\varepsilon^{SP,k}}$ contains the relative impulse responses of μ^j to shadow price shocks to agent k's constraint for $j,k=\{I,E\}$, in the sense that column s in $\widetilde{\mu}^{j,\varepsilon^{SP,k}}$ represents the response of the shadow value to a shock $\varepsilon_{s,t-s}^{SP,j}$, i.e. to a shadow price shock that hits in period s but is anticipated at time 0, as described above.[44] The off-diagonal elements of the matrix $\begin{bmatrix} \widetilde{\mu}^{I,\varepsilon^{SP,I}} & \widetilde{\mu}^{I,\varepsilon^{SP,E}} \\ \widetilde{\mu}^{E,\varepsilon^{SP,I}} & \widetilde{\mu}^{E,\varepsilon^{SP,E}} \end{bmatrix}$ take into account that the impatient household may be affected if the collateral constraint of the entrepreneur becomes non-binding, and *vice versa*. Following the discussion in Holden and Paetz (2012), a sufficient condition for the existence of a unique solution to the optimization problem is that the matrix $\begin{bmatrix} \widetilde{\mu}^{I,\varepsilon^{SP,I}} & \widetilde{\mu}^{I,\varepsilon^{SP,E}} \\ \widetilde{\mu}^{E,\varepsilon^{SP,I}} & \widetilde{\mu}^{E,\varepsilon^{SP,E}} \end{bmatrix} + \begin{bmatrix} \widetilde{\mu}^{I,\varepsilon^{SP,I}} & \widetilde{\mu}^{I,\varepsilon^{SP,E}} \\ \widetilde{\mu}^{E,\varepsilon^{SP,I}} & \widetilde{\mu}^{E,\varepsilon^{SP,E}} \end{bmatrix}'$ is positive definite. We have checked and verified that this condition is in fact always satisfied.

We can explain the nature of the optimization problem as follows. First, note that $\mu^j + \widetilde{\mu}^{j,A} + \widetilde{\mu}^{j,\varepsilon^{SP,j}} \alpha_{\mu_j} + \widetilde{\mu}^{j,\varepsilon^{SP,k}} \alpha_{\mu_k}$ denotes the combined response of μ_t^j to a given shock (here, a

[44] Each matrix $\widetilde{\mu}^{j,\varepsilon^{SP,k}}$ needs to be a square matrix, so if the number of periods in which we guess the constraints may be non-binding is smaller than the number of periods for which we compute impulse responses, $T < T^{IRF}$, we use only the first T rows of the matrix, i.e., the upper square matrix.

technology shock) *and* a simultaneous announcement of a set of future shadow price shocks for a given set of weights. Given the constraints of the problem, the objective is to find a set of optimal weights so that the impact of the (non-negative) shadow-price shocks is exactly large enough to make sure that the response of μ_t^j is never negative. The minimization ensures that the impact of the shadow price shocks will never be larger than necessary to obtain this. Finally, we only allow for solutions for which the value of the objective function is zero. This ensures that at any given horizon, positive shadow price shocks occur if and only if at least one of the two constrained variables, μ_t^I and μ_t^E, are at their lower bound of zero in that period. As pointed out by Holden and Paetz (2012), this can be thought of as a complementary slackness condition on the two inequality constraints of the optimization problem. Once we have solved the minimization problem, it is straightforward to compute the bounded impulse responses of all endogenous variables by simply adding the optimally weighted shadow price shocks to the unconstrained impulse responses of the model in each period.

We rely on the same method to compute dynamic simulations. In this case, however, we need to allow for more than one type of shock. For each period t, we first generate the shocks hitting the economy. We then compute the unrestricted path of the endogenous variables given those shocks and given the simulated values in $t-1$. The unrestricted paths of the bounded variables (μ_t^I and μ_t^E) then take the place of the impulse responses in the optimization problem. If the unrestricted paths of μ_t^I and μ_t^E never hit the bounds in future periods, our simulation for period t is fine. If the bounds are hit, we follow the method above and add anticipated shadow price shocks for a sufficient number of future periods. We then compute restricted values for all endogenous variables, and use these as our simulation for period t. Note that, unlike the case for impulse responses, in our dynamic simulations not all anticipated future shadow price shocks will eventually hit the economy, as other shocks may occur before the realization of the expected shadow price shocks and push the restricted variables away from their bounds.

Appendix F. Data description and estimation strategy

As described in the main text, we use data for the following five macroeconomic variables of the U.S. economy spanning the period 1952:I–1984:II: The year-on-year growth rates (in log-differences) of real GDP, real private consumption, real non-residential investment, and real house prices, and the cyclical component of the LTA series in Figure 2, with the trend being computed as in Müller and Watson (2018). Since the cyclical components of the two LTA series are strongly correlated, we use the one obtained for the households.[45] All data series are taken from the Federal Reserve's FRED database, with the exception of the house price, which is provided by the US Census Bureau. The series are the following:

- Growth rate of *Real Gross Domestic Product*, billions of chained 2009 dollars, seasonally adjusted, annual rate (FRED series name: GDPC1).
- Growth rate of *Real Personal Consumption Expenditures*, billions of chained 2009 dollars, seasonally adjusted, annual rate (FRED series name: PCECC96).
- Growth rate of *Real private fixed investment: Nonresidential* (chain-type quantity index), index 2009=100, seasonally adjusted (FRED series name: B008RA3Q086SBEA).
- Growth rate of *Price Index of New Single-Family Houses Sold Including Lot Value*, index 2005=100, not seasonally adjusted. This series is available only from 1963:Q1 onwards.

[45] All results are robust to using the corporate one.

– To obtain the house price in real terms, this series is deflated using the GDP deflator (*Gross Domestic Product: Implicit Price Deflator*, index 2009=100, seasonally adjusted, FRED series name: GDPDEF).

- LTA data: We employ the series in the right panel of Figure 2 for the period up until 1984:II. As described in Appendix A1, we extract the trend from these series using the method of Müller and Watson (2018). We then use the cyclical component in the estimation of the model. Since the cyclical components of the two series are strongly correlated, we use the series for households, but all results are robust to using the series for firms instead.

Estimation

We use 16 empirical moments in the SMM estimation: The standard deviations and first-order autoregressive parameters of each of the five variables described above, the correlation of consumption, investment, and house prices with output, and the skewness of output, consumption, and investment. These moments are matched to their simulated counterparts from the theoretical model. Our estimation procedure seeks to minimize the sum of squared deviations between empirical and simulated moments. As some of the moments are measured in different units (e.g., standard deviations vs. correlations), we use the percentage deviation from the empirical moment in each case. In order for the minimization procedure to converge, it is crucial to use the same set of shocks repeatedly, making sure that the only change in the simulated moments from one iteration to the next is that arising from updating the parameter values. In practice, since the list of parameter values to be estimated includes the variance of the shocks in the model, we draw from the standard normal distribution with zero mean and unit variance, and then scale the shocks by the variance of each of the three shock processes, allowing us to estimate the latter. We use a draw of 2000 realizations of each of the three shocks in the model, thus obtaining simulated moments for 2000 periods.[46] To make sure that the draw of shocks used is representative of the underlying distribution, we make 501 draws of potential shock matrices, rank these in terms of the standard deviations of each of the three shocks, and select the shock matrix closest to the median along all three dimensions. This matrix of shocks is then used in the estimation. In the estimation, we impose only very general bounds on parameter values: All parameters are bounded below at zero, and the habit formation parameters along with all AR(1)-coefficients are bounded above at 0.99—a bound that is never attained.

To initiate the estimation procedure a set of initial values for the estimated parameters are needed. These are chosen based on values reported in the existing literature. The estimation results proved robust to changes in the set of initial values, as long as these remain within the range of available estimates. In line with the existing literature, we set the initial values of the investment adjustment cost parameter (Ω) and the parameters governing habit formation in consumption for the three agents to 4 and 0.5, respectively.[47] For the technology shock, we choose values similar to those used in most of the real business cycle literature, $\rho_A = 0.97$ and $\sigma_A = 0.005$ (see, e.g., Mandelman *et al.*, 2011). For the land-demand shock, we set $\rho_\varepsilon = 0.99$

[46] Our simulated sample is thus more than 15 times longer than the actual dataset (which spans 130 quarters). Ruge-Murcia (2012) finds that SMM is already quite accurate when the simulated sample is five or ten times longer than the actual data.

[47] Unlike the other estimated parameters, θ^P and θ^I also affect the steady state of the model. To account for this, we rely on the following iterative procedure: We first calibrate the model based on the starting value for θ^P and θ^I. Upon estimation, but before simulating the model, we recalibrate it for the estimated values of the habit parameters. This leads only to a very small change in the values of ε, ϕ, and s^I, while the remaining parameters are unaffected.

and $\sigma_\varepsilon = 0.03$, in line with Liu *et al.* (2013). Finally, for the credit limit shock, we set the persistence parameter $\rho_s = 0.95$, while the standard deviation is set to $\sigma_s = 0.04$.

We abstain from using an optimal weighting matrix in the estimation. This choice is based on the findings of Altonji and Segal (1996), who show that when GMM is used to estimate covariance structures and, potentially, higher-order moments such as variances, as in our case, the use of an optimal weighting matrix causes a severe downward bias in estimated parameter values. Similar concerns apply to SMM as to GMM. The bias arises because the moments used to fit the model itself are correlated with the weighting matrix, and may thus be avoided by the use of fixed weights in the minimization. Altonji and Segal (1996) demonstrate that minimization schemes with fixed weights clearly dominate optimally weighted ones in such circumstances. Ruge-Murcia (2012) points out that parameter estimates remain consistent for any positive-definite weighting matrix, and finds that the accuracy and efficiency gains associated with an optimal weighting matrix are not overwhelming. The empirical moments and their model counterparts upon estimation are reported in Table F1.

When computing standard errors, we rely on a version of the delta method, as described, e.g., in Hamilton (1994). We approximate the numerical derivative of the moments with respect to the estimated parameters using the secant that can be computed by adding and subtracting ϵ to/from the estimates, where ϵ is a very small number. The covariance (or spectral density) matrix is estimated using the Newey-West estimator.

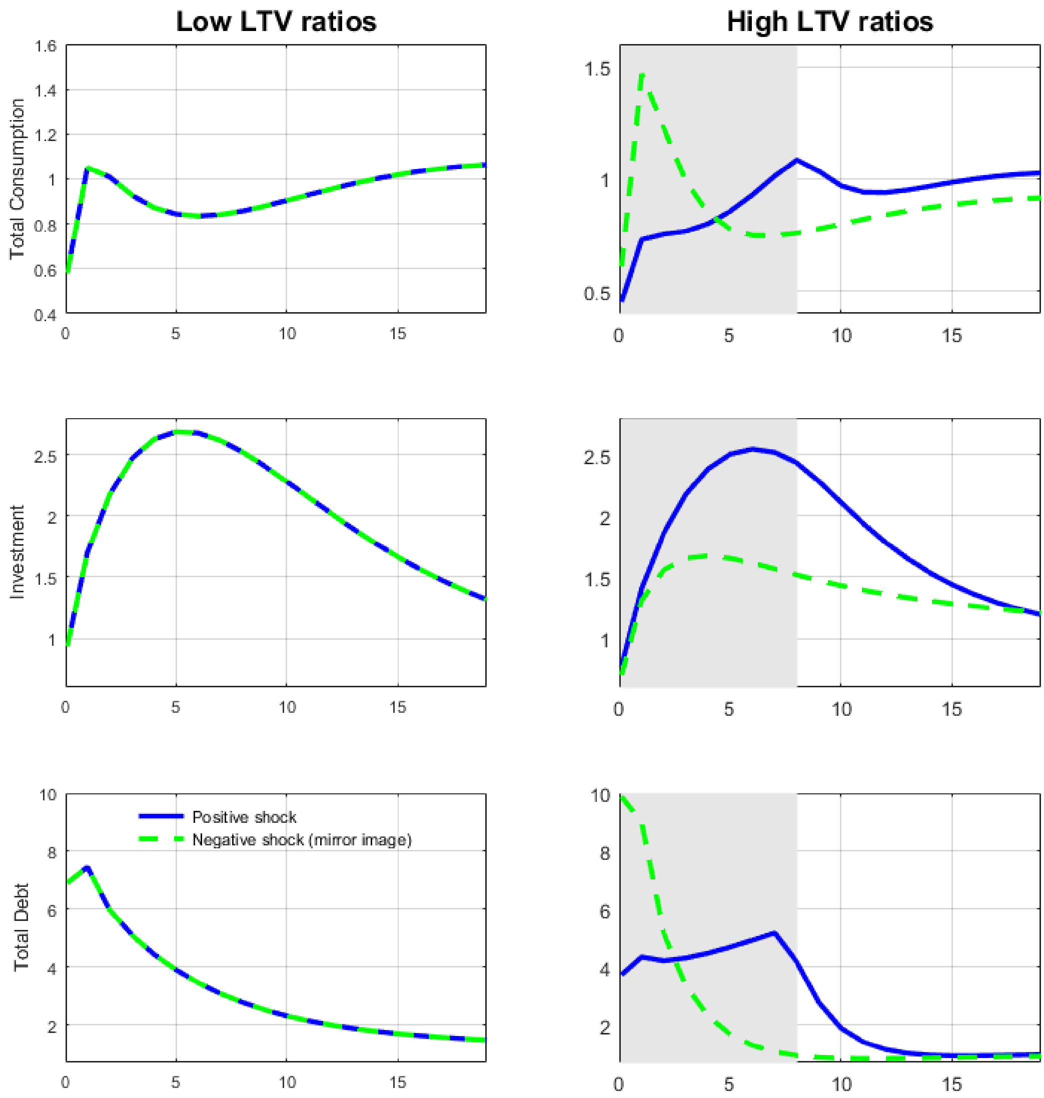
Table F1. Empirical and simulated moments

	Model simulations	U.S. data (1952:I–1984:II)
	Standard deviations (percent)	
Output	2.72	2.84
Consumption	1.87	2.23
Investment	6.52	6.91
House price	4.26	3.05
LTV ratio	6.32	5.68
	Skewness	
Output	−0.14	−0.38
Consumption	−0.21	−0.31
Investment	−0.02	−0.41
	Autocorrelations	
Output	0.90	0.82
Consumption	0.85	0.81
Investment	0.94	0.84
House price	0.63	0.79
LTV ratio	0.85	0.94
	Correlations with output	
Consumption	0.92	0.85
Investment	0.93	0.75
House price	0.73	0.38

Appendix G. Additional numerical evidence

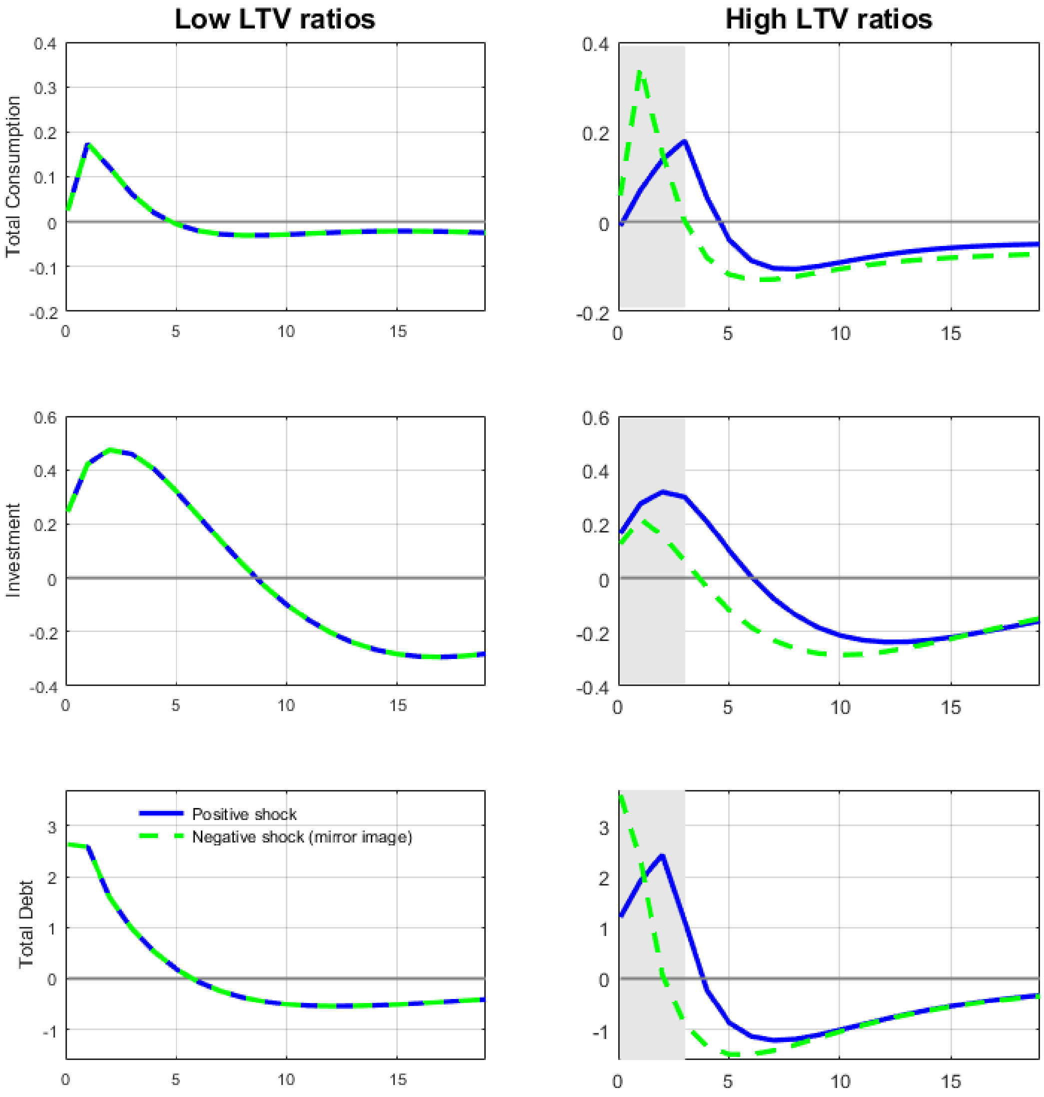
G1. Impulse responses

Figure G1. Impulse responses to a technology shock



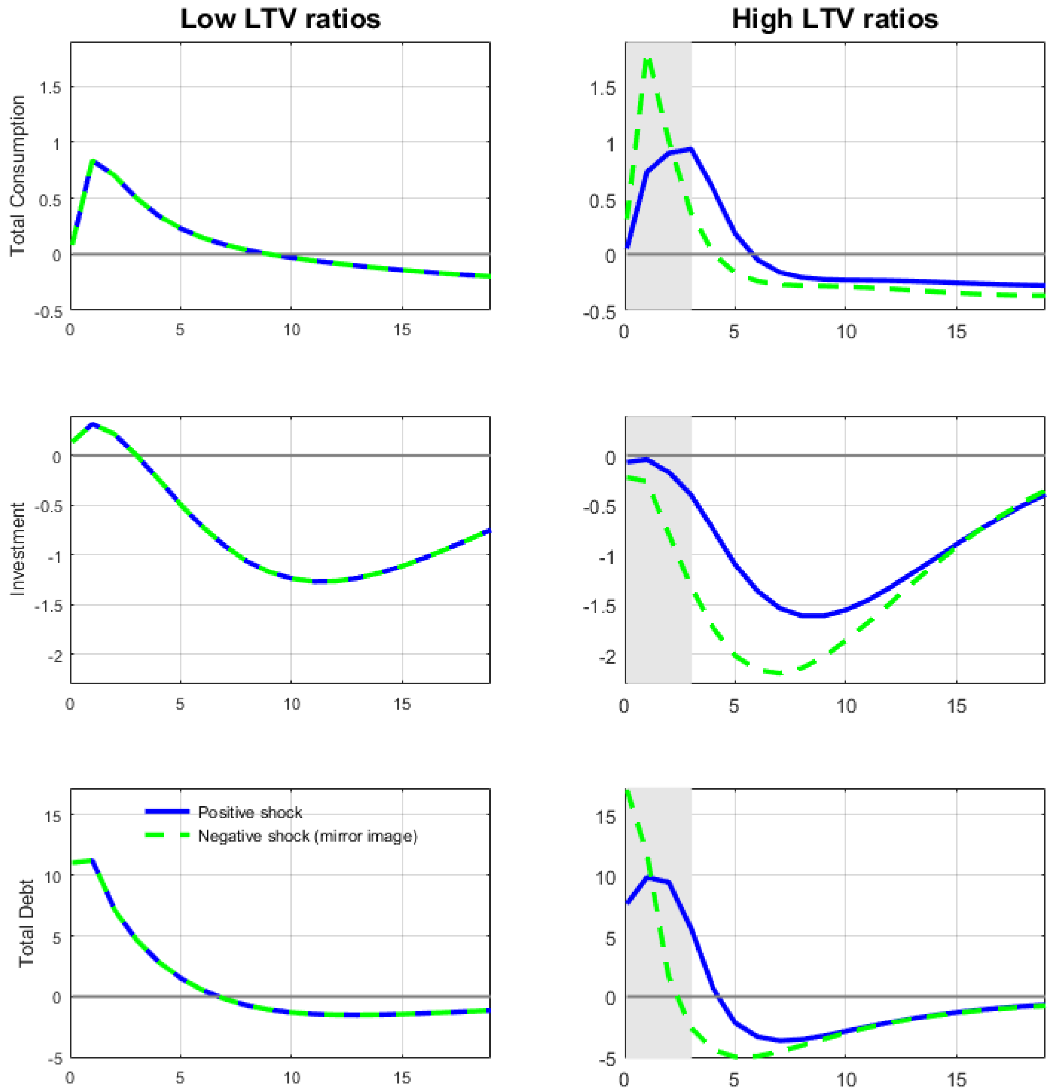
Notes: Impulse responses of key macroeconomic variables (in percentage deviation from the steady state) to a one-standard deviation shock to technology. Left column: $s^I = 0.67$, $s^E = 0.76$; right column: $s^I = 0.85$, $s^E = 0.94$. The shadowed bands indicate the periods in which the entrepreneurs are financially unconstrained.

Figure G2. Impulse responses to a land demand shock



Notes: Impulse responses of key macroeconomic variables (in percentage deviation from the steady state) to a two-standard deviations shock to land demand. Left column: $s^I = 0.67$, $s^E = 0.76$; right column: $s^I = 0.85$, $s^E = 0.94$. The shadowed bands indicate the periods in which the entrepreneurs are financially unconstrained.

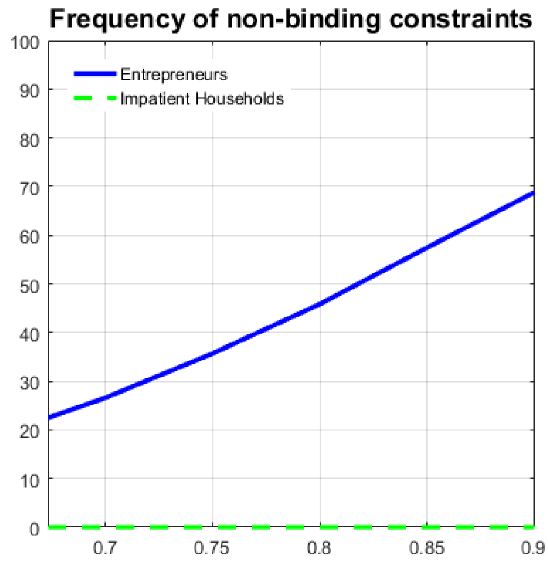
Figure G3. Impulse responses to a credit limit shock



Notes: Impulse responses of key macroeconomic variables (in percentage deviation from the steady state) to a one-standard deviation shock to credit limits. Left column: $s^I = 0.67$, $s^E = 0.76$; right column: $s^I = 0.85$, $s^E = 0.94$. The shadowed bands indicate the periods in which the entrepreneurs are financially unconstrained.

G2 On the occurrence of non-binding collateral constraints

Figure G4. Leverage and non-binding collateral constraints



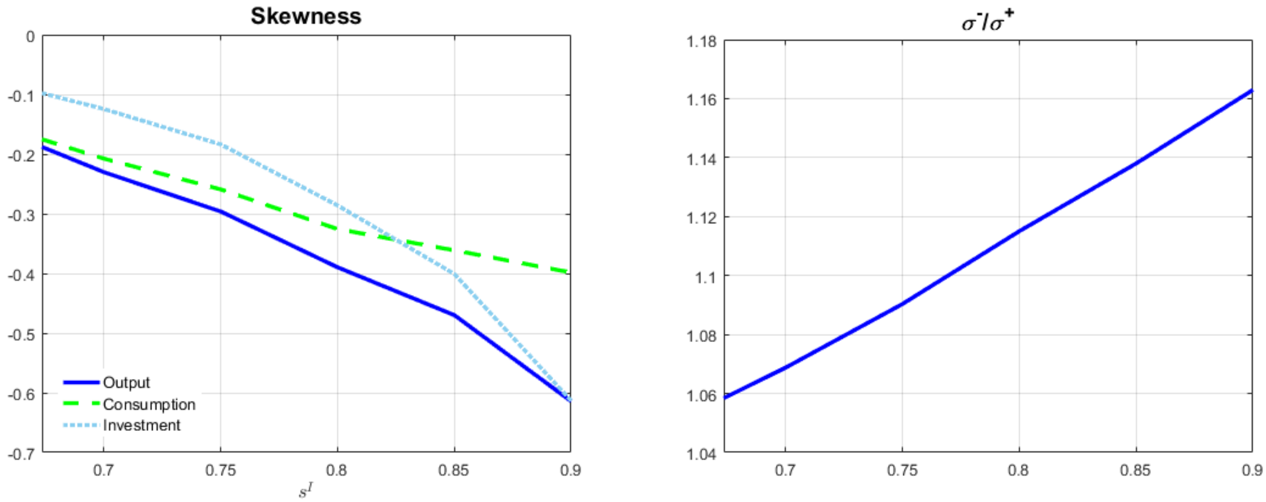
Notes: Frequency of non-binding constraints for the entrepreneurs (solid-blue line) and the impatient households (dashed-green line). Both statistics are graphed for different average LTV ratios faced by the impatient household. Across all the simulations the entrepreneurial average LTV ratio is adjusted so as to be 9 basis points greater than any value we consider for impatient households' credit limits, in line with the baseline calibration of the model.

G3. The model with no household debt

In this appendix we report numerical evidence from an alternative model with no role for collateralized household debt. We effectively exclude impatient households from the model by setting their income share to a very low number (i.e., $1 - \alpha = 0.01$). All other parameters are as described in Section 5.1. We then perform the same simulation exercise as that reported in Section 6.2. The results are reported below.[48] As displayed by Figure G5, the alternative model generates an amount of skewness similar to that of the baseline framework. However, as illustrated in the left panel of Figure G6, the model's ability to reproduce the increase in the duration of expansions observed in the data is impaired substantially. This can be explained based on the fact that impatient households contract long-term debt, which induces a certain smoothness in the consumption/investment profiles of all agents in the model. In addition, the left panel of Figure G7 indicates that the alternative model implies a much larger increase in output volatility when leverage increases, and a much smaller reversal. This pattern represents a further challenge to a model with no household borrowing, as it makes our findings harder to reconcile with the Great Moderation in output volatility. In fact, attaining such a fall in volatility would entail a rather large scaling of the structural shocks (recall the analysis in Section 6.3.1).

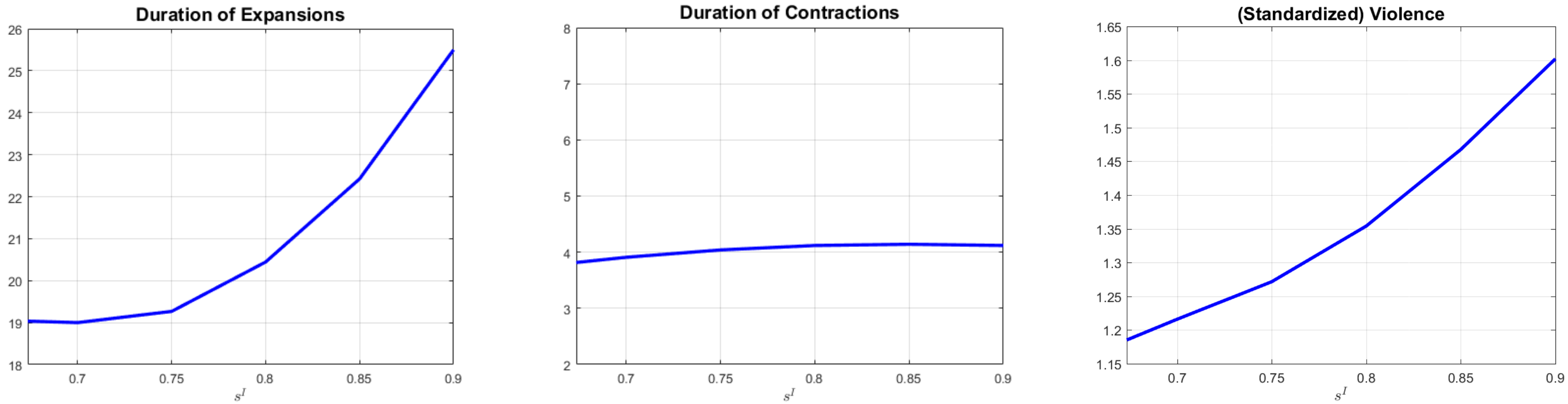
[48] Note that the impatient household is still present in the model, albeit playing a very small role. Thus, when reporting the results from this model, we choose to keep s^I on the horizontal axis, so as to facilitate comparison with the results in the main text.

Figure G5. Business cycle asymmetry



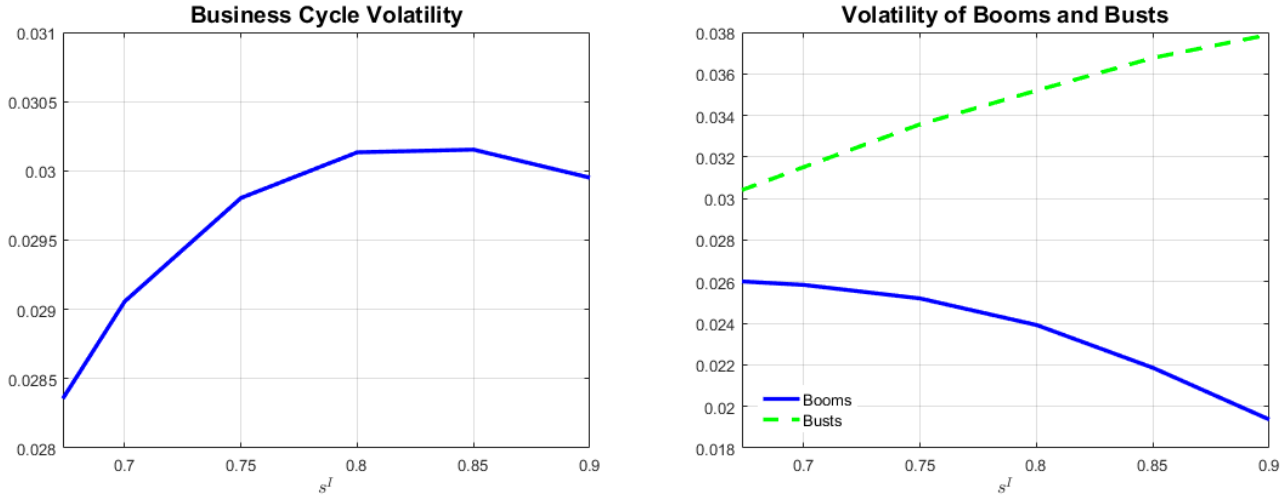
Notes: The left panel of the figure reports the skewness of the year-on-year growth rate of output, consumption and investment, while the right panel displays the ratio between the downside and the upside semivolatility of year-on-year output growth, for different average LTV ratios faced by the financially constrained agents. To identify the recessionary episodes in the simulated series, we use the Harding and Pagan (2002) algorithm. Across all the simulations the entrepreneurial average LTV ratio is adjusted to be 9 basis points greater than any value we consider for impatient households' credit limits, in line with the baseline calibration of the model.

Figure G6. Expansions, contractions and leverage



Notes: The first two panels report the duration (in quarters) of expansions and contractions, respectively. The last panel displays the standardized violence of recessionary episodes. To identify expansions and contractions in the simulated gross output series, we use the Harding and Pagan (2002) algorithm. We compute the violence as the average fall of output over a given recession, divided by the length of the recession itself. Finally, we standardize the violence by means of the volatility of year-on-year output growth over the five years prior to the recession. All statistics are conditional on different average LTV ratios faced by the financially constrained agents. Across all the simulations the entrepreneurial average LTV ratio is adjusted to be 9 basis points greater than any value we consider for impatient households' credit limits, in line with the baseline calibration of the model.

Figure G7. Leverage and volatility

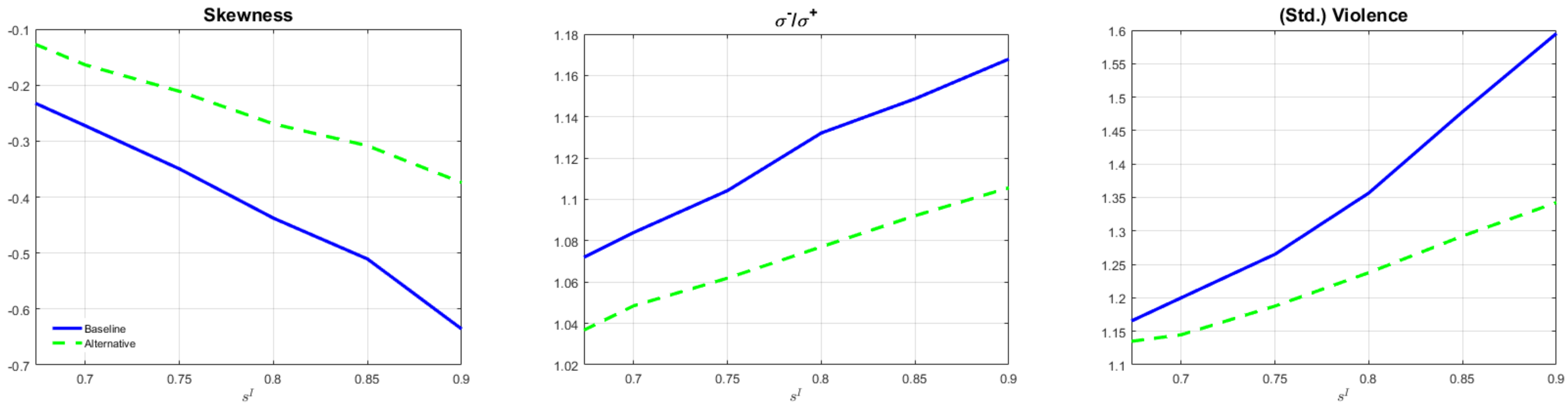


Notes: The left panel reports the standard deviation of year-on-year output growth, while the right panel reports the standard deviation of expansions (solid-blue line) and contractions (dashed-green line) in economic activity. These are determined based on whether output is above or below its steady-state level. Across all the simulations the entrepreneurial average LTV ratio is adjusted so as to be 9 basis points greater than any value we consider for impatient households' credit limits, in line with the baseline calibration of the model.

G4. Asymmetry and collateral prices

In this appendix we report results obtained by simulating an alternative version of the model where the collateral assets are pledged at their steady-state prices. We also report results from our baseline model for comparison.

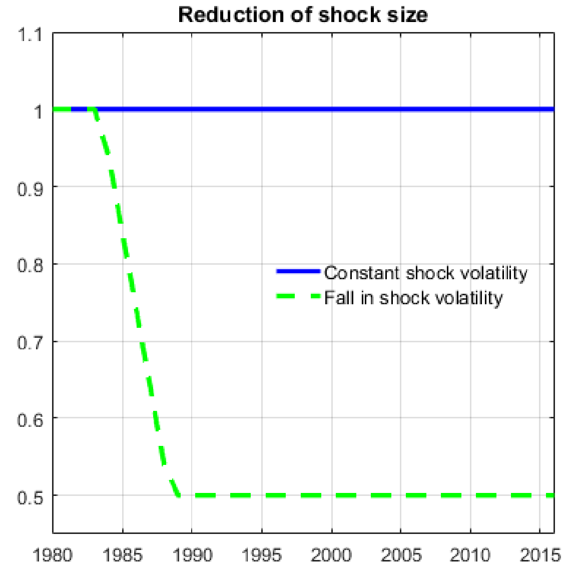
Figure G8. Asymmetry and collateral prices



Notes: Solid-blue lines: baseline model. Dashed-green lines: alternative model with asset prices fixed at their steady-state values in the collateral constraints.

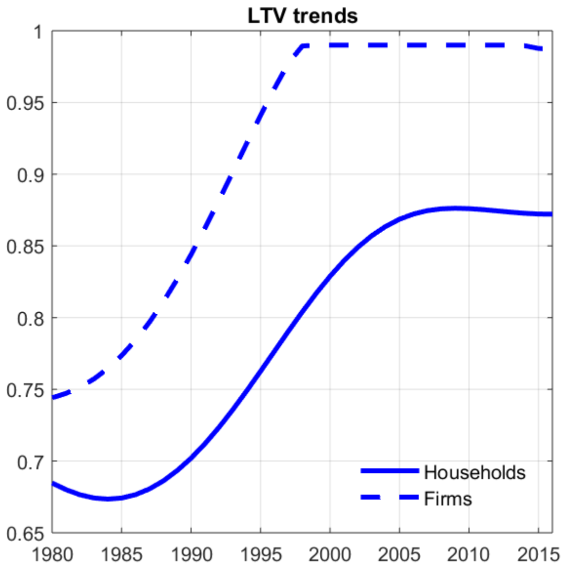
G5. Counterfactual exercise

Figure G9. Scaling factor



Notes: Scaling factor applied to the shocks to attain a 40% reduction in the volatility of output growth over the 1984-1989 time window.

Figure G10. Sequences of LTV ratios



Notes: Sequence of LTV ratios used in each of the two counterfactual scenarios reported in Figure 10. We use the long-term components reported in Figure A1, to which we add a constant in order to match the calibrated LTV ratios from Section 5.1.1 in 1984. If the resulting LTV ratio exceeds 0.99, we cap it at this value.

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