Positive Long-Run Capital Taxation: Chamley-Judd Revisited Online Appendix

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A Recursive Formulation of (1a)

In our numerical simulations, we use a recursive representation of the Judd (1985) economy. The two constraints in the planning problem feature the variables $C_{t-1}, k_t, C_t, k_{t+1}$ and c_t. This suggests a recursive formulation with (k_t, C_{t-1}) as the state and c_t as a control. The associated Bellman equation is then

$$V(k, C_-) = \max_{c \geq 0, (k', C) \in A} \{u(c) + \gamma U(C) + \beta V(k', C)\} \tag{15}$$
$$c + C + k' + g = f(k) + (1-\delta)k$$
$$\beta U'(C)(C + k') = U'(C_-)k$$
$$c, C, k' \geq 0.$$

Here, A is the *feasible* set, that is, states (k_0, C_{-1}) such that there exists a sequence $\{k_{t+1}, C_t\}$ satisfying all the constraints in (1) including the transversality condition. At $t = 0$, capital k_0 is given, so there is no need to impose $\beta U'(C_0)(C_0 + k_1) = U'(C_{-1})k_0$. Thus, the planner maximizes $V(k_0, C_{-1})$ with respect to C_{-1}. If V is differentiable, the first order condition is

$$V_C(k_0, C_{-1}) = 0.$$

Since one can show that $\mu_t = V_C(k_t, C_{t-1})U''(C_{t-1})k_t$, this is akin to the condition $\mu_0 = 0$ in equation (2a).[41]

[41] Alternatively, we may impose that R_0 is taken as given, with $R_0 = R_0^*$ for example, to exclude an initial capital tax. In that case the planner solves

$$\max_{k_1, c_0, C_0} \{u(c_0) + \gamma U(C_0) + \beta V(k_1, C_0)\}$$

subject to

$$C_0 + k_1 = R_0 k_0$$
$$c_0 + C_0 + k_1 = f(k_0) + (1-\delta)k_0$$
$$c_0, C_0, k_1 \geq 0.$$

This alternative gives rise to similar results.

B Proof of Proposition 3

The proof of Proposition 3 consists of three parts. In the first part, we provide a few definitions that are necessary for the proof. In particular, we define the *feasible* set of states. In the second part, we characterize the feasible set of states geometrically. The proofs for the results in that part are somewhat cumbersome and lengthy, so they are relegated to the end of this section to ensure greater readability. Finally, in the third part, we use our geometric results to prove Proposition 3. Readers interested only in the main steps of the proof are advised to jump straight to the third part.

B.1 Definitions

For the proof of Proposition 3 we make a number of definitions, designed to simplify the exposition. A state (k, C_-) as in the recursive statement (15) of problem (1a) will sometimes be abbreviated by z, and a set of states by Z. The total state space is denoted by $Z_{\text{all}} \subset \mathbb{R}^2_+$ and is defined below. It will prove useful at times to express the set of constraints in (15) as

$$k' = x - C_- \left(\frac{\beta x}{k}\right)^{1/\sigma} \tag{16a}$$

$$C = C_- \left(\frac{\beta x}{k}\right)^{1/\sigma} \tag{16b}$$

$$C_-^{\sigma/(\sigma-1)} \left(\frac{\beta}{k}\right)^{1/(\sigma-1)} \leq x \leq f(k) + (1-\delta)k - g, \tag{16c}$$

where $x = k' + C$ replaces $c = f(k) + (1-\delta)k - g - x$ as control. In the last equation, the first inequality ensures non-negativity of k' while the second inequality is merely the resource constraint. Substituting out x, we can also write the law of motion for capital as $k' = \frac{1}{\beta}\frac{k}{C_-^\sigma}C^\sigma - C$, which we will be using below.

The whole set of future states z' which can follow a given state $z = (k, C_-)$ is denoted by $\Gamma(z)$, which can be the empty set. We will call a path $\{z_t\}$ *feasible* if (a) $z_{t+1} \in \Gamma(z_t)$ for all $t \geq 0$, which precludes $\Gamma(z_t)$ from being empty; and (b) if the transversality condition holds along the path, $\beta^t C_t^{-\sigma} k_{t+1} \to 0$. Similarly, a state z will be called *feasible*, if there exists a feasible (infinite) path $\{z_t\}$ starting at $z_0 = z$. In this case, z is *generated by* $\{z_t\}$. Because $z_1 \in \Gamma(z)$, we also say z is *generated by* z_1. A *steady state* $z = (k, C_-) \in \mathbb{R}^2_+$ is defined to be a state with $C_- = (1-\beta)/\beta k$. For very low and high capital levels k, steady states turn out to be infeasible, but all others are *self-generating*, $z \in \Gamma(z)$, as we argue below. Similarly, a set Z is called *self-generating* if every $z \in Z$ is generated by a sequence in Z. Denote by Z^* ($= A$ in the notation above) the set of all feasible states. An integral part of the proof will be to characterize Z^*.

It will be important to specify between which capital stocks the economy is moving.

For this purpose, define k_g and $k^g > k_g$ to be the two roots to the equation

$$k = \underbrace{f(k) + (1-\delta)k - g}_{\equiv F(k)} - \frac{1-\beta}{\beta}k. \tag{17}$$

Demanding that $k^g > k_g$ is tantamount to specifying $F'(k^g) < 1/\beta < F'(k_g)$. Equation (17) was derived from the resource constraint, demanding that capitalists' consumption is at the steady state level of $C = \frac{1-\beta}{\beta}k$ and workers' consumption is equal to zero. Equation (17) need not have two solutions, not even a single one, in which case government consumption is unsustainably high for *any* capital stock. Such values for g are uninteresting and therefore ruled out. Corresponding to k_g and k^g, we define $C_g \equiv (1-\beta)/\beta\, k_g$ and $C^g \equiv (1-\beta)/\beta\, k^g$ as the respective steady state consumption of capitalists. The steady states (k_g, C_g) and (k^g, C^g) represent the lowest and highest feasible steady states, respectively. The reason for this is that the steady state resource constraint (17) is violated for any $k \notin [k_g, k^g]$.

As in the Neoclassical Growth Model, the set of feasible states of this model is easily seen to allow for arbitrarily large capital stocks. This is why we cap the state space for high values of capital, and we take the total state space to be $Z_{\text{all}} = [0, \bar{k}] \times \mathbb{R}_+$ for states (k, C_-), where $\bar{k} \equiv \max\{k_{\max}, k_0\}$ and $k = k_{\max}$ solves $k = f(k) + (1-\delta)k - g$. This way, the set of capital stocks that are resource feasible given an initial capital stock of k_0 must necessarily lie in the interval $[0, \bar{k}]$, so the restriction for $\bar{k}$ is without loss of generality for any given initial capital stock k_0. Note that with this state space, the set of feasible states Z^* is also capped at $\bar{k}$ in its k-component.

We now characterize the geometry of the set of feasible states Z^*. The results derived there are essential for the actual proof of Proposition 3 in Section B.3.

B.2 Geometry of Z^*

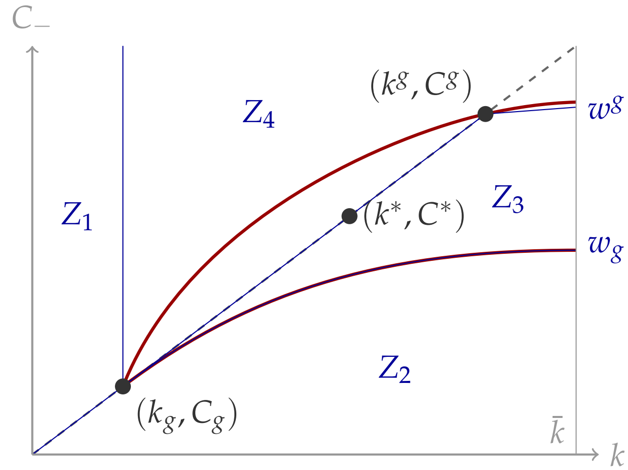
For better guidance through this section, we refer the reader to figure 5, which shows the typical shape of Z^*. The main results in this section are characterizations of the bottom and top boundaries of Z^*. We proceed by splitting up the state space, $Z_{\text{all}} = [0, \bar{k}] \times \mathbb{R}_+$, into four pieces and characterizing the feasible states in each of the four pieces.

Define

$$w_g(k) \equiv \begin{cases} \frac{1-\beta}{\beta}k & \text{for } 0 \le k \le k_g \\ C_g \left(\frac{k}{k_g}\right)^{1/\sigma} & \text{for } k_g \le k \le \bar{k} \end{cases}$$

$$w^g(k) \equiv \begin{cases} \frac{1-\beta}{\beta}k & \text{for } 0 \le k \le k^g \\ C^g \left(\frac{k}{k^g}\right)^{1/\sigma} & \text{for } k^g \le k \le \bar{k}, \end{cases}$$

Figure 5: The state space of the Judd (1985) planning problem.



Note. This figure shows the two-dimensional state space of the Judd (1985) model. The entire state space is denoted by Z_{all}, which includes the feasible set Z^* (between the two red curves), and all sets Z_i (separated by the blue curves). The point (k^*, C^*) is the zero-tax steady state. Showing that this is the qualitative shape of the feasible set Z^* is an integral part of the proof of Proposition 3

and split up the state space as follows (see figure 5)

$$Z_{\text{all}} = \underbrace{\left\{k < k_g, C_- \geq \frac{1-\beta}{\beta}k\right\}}_{Z_1} \cup \underbrace{\{C_- < w_g(k)\}}_{Z_2}$$
$$\cup \underbrace{\{k \geq k_g, w_g(k) \leq C_- \leq w^g(k)\}}_{Z_3} \cup \underbrace{\{k \geq k_g, C_- \geq w^g(k)\}}_{Z_4}.$$

Lemma 1 characterizes the feasible states in sets Z_1 and Z_2.

Lemma 1. *$Z^* \cap Z_1 = Z^* \cap Z_2 = \emptyset$. All states with $k < k_g$ or $C_- < w_g(k)$ are infeasible.*

Proof. See Subsection B.4.1. □

In particular, Lemma 1 shows that all states with $C_- < w_g(k)$ are infeasible. Lemma 2 below complements this result stating that all states with $w_g(k) \leq C_- \leq w^g(k)$ (and $k \geq k_g$) in fact are feasible, that is, lie in Z^*. This means, $\{C_- = w_g(k), k \geq k_g\}$ constitutes the lower boundary of the feasible set Z^*.

Lemma 2. *$Z_3 \subseteq Z^*$, or equivalently, all states with $w_g(k) \leq C_- \leq w^g(k)$ and $k \geq k_g$ are feasible and generated by a feasible steady state. Moreover, states on the boundary $\{C_- = w_g(k), k > k_g\}$ can only be generated by a single feasible state, (k_g, C_g). Thus, there is only a single "feasible" control for those states, $c > 0$.*

Proof. See Subsection B.4.2. □

Lemma 2 finishes the characterization of all feasible states with $C_- \leq w^g(k)$. What remains is a characterization of feasible states with $C_- > w^g(k)$, or in terms of the $k -$

C_- diagram of Figure 5, the characterization of the red top boundary. This boundary is inherently more difficult than the bottom boundary because it involves states that are not merely one step away from a steady state. Rather, paths might not reach a steady state at all in finite time. The goal of the next set of lemmas is an iterative construction to show that the boundary takes the form of an increasing function $\bar{w}(k)$ such that states with $C_- > w^g(k)$ are feasible if and only if $C_- \leq \bar{w}(k)$.

For this purpose, we need to make a number of new definitions: Let $\psi(k, C_-) \equiv (k + C_-)/C_-^\sigma$. Applying the ψ function to the successor (k', C) of a state (k, C_-) and using the IC constraint (1c) gives $\psi(k', C) = \beta^{-1}k/C_-^\sigma$, a number that is *independent of the control* x. Hence, for every state (k, C_-) there exists an iso-ψ curve containing all its potential successor states.

In some situations it will be convenient to abbreviate the laws of motion for capitalists' consumption and capital, equations (16a) and (16b), as $k'(x, k, C_-)$ and $C(x, k, C_-)$.

Finally, define an operator T on the space of continuous, increasing functions $v : [k_g, \bar{k}] \to \mathbb{R}_+$, as,

$$Tv(k) = \sup\{C_- \mid \exists x \in (0, F(k)] : v(k'(x, k, C_-)) \geq C(x, k, C_-)\}, \tag{18}$$

where recall that $F(k) = f(k) + (1 - \delta)k - g$, as in (17). The operator is designed to extend a candidate top boundary of the set of feasible states by one iteration. To make this formal, let $Z^{(i)}$ be the set of states with $C_- \geq w^g(k)$ which are i steps away from reaching $C_- = w^g(k)$. For example, $Z^{(0)} = \{C_- = w^g(k)\}$. Lemma 3 proves some basic properties of the operator T.

Lemma 3. *T maps the space of continuous, strictly increasing functions $v : [k_g, \bar{k}] \to \mathbb{R}_+$ with $\psi(k, v(k))$ strictly decreasing in k and $v(k_g) = C_g$, $v(k^g) = C^g$, into itself.*

Proof. See Subsection B.4.3. □

Lemma 4 uses the operator T to describe the sets $Z^{(i)}$.

Lemma 4. *$Z^{(i)} = \{w^g(k) \leq C_- \leq T^i w^g(k)\}$. In particular $T^i w^g(k) \geq T^j w^g(k) \geq w^g(k)$ for $i \geq j$.*

Proof. See Subsection B.4.4. □

The next two lemmas characterize the limit function $\bar{w}(k)$, whose graph will describe the top boundary of the set of feasible states.

Lemma 5. *There exists a continuous limit function $\bar{w}(k) \equiv \lim_{i\to\infty} T^i w^g(k) = T\bar{w}(k)$, with $\bar{w}(k_g) = C_g$ and $\bar{w}(k^g) = C^g$. All states with $C_- = \bar{w}(k)$ are feasible, but only with policy $c = 0$.*

Proof. See Subsection B.4.5. □

Lemma 6. *No state with $C_- > \bar{w}(k)$ (and $k_g \leq k \leq \bar{k}$) is feasible.*

Proof. See Subsection B.4.6. □

Finally, Lemma 7 shows an auxiliary result which is both used in the proof of Lemma 6 and in Lemma 9 below.

Lemma 7. *Let $\{k_{t+1}, C_t\}$ be a path starting at (k_0, C_{-1}) with controls $c_t = 0$. Let $k_g < k_0 \leq \bar{k}$. Then:*

(a) If $C_{-1} = \bar{w}(k_0)$, $(k_{t+1}, C_t) \to (k^g, C^g)$.

(b) If $C_{-1} > \bar{w}(k_0)$, $(k_{t+1}, C_t) \not\to (k^g, C^g)$.

Proof. See Subsection B.4.7. □

B.3 Proof of Proposition 3

Armed with the results from Section B.2 we now prove Proposition 3 in a series of intermediate results. For all statements in this section, we consider an economy with an initial capital stock of $k_0 \in [k_g, \bar{k}]$. We call a path $\{k_{t+1}, C_t\}$ *optimal path*, if the initial C_{-1} was optimized over given the initial capital stock k_0. Analogously, we call a path $\{k_{t+1}, C_t\}$ *locally optimal path*, if the initial C_{-1} was not optimized over but rather taken as given at a certain level, respecting the constraint that (k_0, C_{-1}) be feasible. If $\{k_{t+1}, C_t\}$ is a locally optimal path, with control c_{t+1} at some point $\{k_{t+1}, C_t\}$ we say this control is *optimal at* $\{k_{t+1}, C_t\}$. Notice that along both optimal and locally optimal paths, first order conditions are *necessary*, as long as paths are interior; they need not be sufficient, in the sense that there could be multiple optima that satisfy our characterization below.

The first lemma proves that the multiplier on the capitalists' IC constraint explodes along an optimal path, and at the same time, workers' consumption drops to zero.

Lemma 8. *Along any optimal path, $c_t \to 0$.*

Proof. Let $\{k_{t+1}, C_t\}$ be the optimal path. Suppose first the optimal path hits the boundary of the feasible set Z^* at some finite time. Given that no path can hit the $k = \bar{k}$ boundary after $t = 0$, and given Lemma 2 this means the path hits the top boundary—the graph of $\bar{w}$—after finite time. Lemma 5 showed that along that boundary, the control is necessarily zero, $c = 0$.

Now suppose the optimal path is interior at all times. In that case, the first order conditions are necessary. Using the notation from problem (1a) the necessary first order conditions are equations (2a)–(2d). In particular, the one for μ_t states

$$\mu_{t+1} = \mu_t \left(\frac{\sigma - 1}{\sigma \kappa_{t+1}} + 1 \right) + \frac{1}{\beta \sigma \kappa_{t+1} v_t}.$$

From Lemma 1 we know that $\kappa_{t+1} = k_{t+1}/C_t$ is bounded away from ∞. Since $\mu_0 = 0$ by (2a) and $\sigma > 1$, it follows that $\mu_t \geq 0$ and $\mu_t \to \infty$. To show that $c_t \to 0$, suppose to the contrary that $c_t \not\to 0$. In this case, there exists $\underline{c} > 0$ and an infinite sequence of indices (t_s) such that $c_{t_s} \geq \underline{c}$ for all s. Along these indices, the FOC for capital (2d) implies

$$\underbrace{u'(c_{t_s})}_{\leq u'(\underline{c})}(f'(k_{t_s}) + (1-\delta)) = \frac{1}{\beta} \underbrace{u'(c_{t_s - 1})}_{\geq 0} + \underbrace{U'(c_{t_s - 1})}_{\text{bounded away from } 0} \cdot \underbrace{(\mu_{t_s} - \mu_{t_s - 1})}_{\geq \text{const} \cdot \mu_{t_s - 1} \to \infty},$$

and so $k_{t_s} \to 0$ for $s \to \infty$, which is impossible within the feasible set Z^* because it violates $k \geq k_g$ (see Lemma 1). This proves that also for interior optimal paths, $c_t \to 0$. $\square$

Lemma 8 is important because it shows that workers' consumption drops to zero. Together with the following lemma, this gives us a crucial geometric restriction of where an optimal path goes in the long run.

Lemma 9. *The set of states where $c = 0$ is an optimal control is the top boundary, the graph of $\bar{w}$. It follows that an optimal path approaches either (k_g, C_g) or (k^g, C^g).*

Proof. First, we show that any state in the interior of Z^* can be generated by a path with positive controls $c > 0$. Any state in the interior of Z^* is element of some $Z^{(i)}$, $i < \infty$, and can thus reach the set $\{C_- \leq w^g(k)\} \setminus \{(k_g, C_g), (k^g, C^g)\}$ in finite time. From there, at most two steps are necessary to reach a interior steady state (k_{ss}, C_{ss}) with $k_g < k_{ss} < k^g$ and hence positive consumption $c_{ss} > 0$. Note that such an interior steady state can be reached without leaving the interior of the feasible set, since by Lemmas 2 and 7, hitting the upper or lower boundary once means convergence to a non-interior steady state.[42] This proves that any state in the interior is generated by such an interior path, with positive controls $c > 0$.

Now take an interior state (k_0, C_{-1}). We prove that any optimal control at that state is positive. Suppose to the contrary, $c_0 = 0$ is an optimal control at (k_0, C_{-1}). This means, (k_0, C_{-1}) is generated by a locally optimal path $\{k_{t+1}, C_t\}$, where (k_1, C_0) is precisely linked to (k_0, C_{-1}) using control $c_0 = 0$, or equivalently, $x_0 = F(k_0)$. Since (k_0, C_{-1}) is interior, any state $(k'(\tilde{x}_0, k_0, C_{-1}), C(\tilde{x}_0, k_0, C_{-1}))$ with slightly positive controls, that is, $\tilde{x}_0 < F(k_0)$, has to be feasible too. Therefore, we find the following first order necessary condition for local optimality of c_0,[43]

$$\frac{u'(c_1)}{u'(c_0)}(f'(k_1) + 1 - \delta) \geq \frac{1}{\beta} + v_0(\mu_1 - \mu_0),$$

where the inequality is there due to the (implicit) boundary condition $c_0 \geq 0$. This condition can only be satisfied if $c_1 = 0$ as well. We can iterate this logic: If (k_1, C_0) is interior, it must be that $c_2 = 0$ is optimal at (k_2, C_1). If (k_1, C_0) is not interior, then it must be on the top boundary of Z^*, that is, on the graph of $\bar{w}$,[44] where it has policy $c = 0$ forever after. This proves, by induction, that if any interior state (k, C_-) has $c = 0$ as an optimal policy, any locally optimal path starting at (k, C_-) with $c = 0$ as initial optimal policy must have $c = 0$ forever, yielding utility $u(0)/(1 - \beta)$. This, however, contradicts local optimality of such a path: We showed above that any interior state (k_0, C_{-1}) is generated by a path with strictly positive controls. Therefore, any optimal control at an interior state (k_0, C_{-1}) is positive.

Finally, notice that states (k, C_-), $k > k_g$, along the bottom boundary of Z^* only admit a single feasible control, which is positive (see Lemma 2). Thus, by Lemma 5, the set

[42] Note that hitting the right boundary at $k = \bar{k}$ (other than with k_0) is of course not feasible due to depreciation.

[43] A *locally* optimal path still satisfies the first order conditions (2b)–(2d), just not (2a) which comes from the optimal choice of C_{-1}.

[44] On the lower boundary of Z^* (excluding (k_g, C_g)), a policy of $c = 0$ would not be feasible, see Lemma 2.

where $c = 0$ is an optimal control is precisely the top boundary $\{(k, C_-) \mid k \in [k_g, \bar{k}], C_- = \bar{w}(k)\}$. It follows that an optimal path either hits the boundary of Z^* at some point, in which case it converges either to (k_g, C_g) or (k^g, C^g) (by Lemma 7), or it remains interior forever and thus (by Lemma 8) approaches the set $\{c = 0\}$ of all states where $c = 0$ is an optimal control, that is, the graph of $\bar{w}$.[45] Then it must share the same limiting behavior as states in the set $\{c = 0\}$.[46] By virtue of Lemma 7, it can then either converge to (k_g, C_g) or (k^g, C^g). □

Lemma 10. *If an optimal path $\{k_{t+1}, C_t\}$ converges to (k^g, C^g), then the value function V is locally decreasing in C at each point (k_{t+1}, C_t), for all $t > T$, with T large enough.*

Proof. Let $x_t \equiv F(k_t) - c_t$ and consider the following variation: Suppose that at a point T, (k_{T+1}, C_T) is not at the lower boundary (in which case it cannot converge to (k^g, C^g) anyway) and that $c_t < F(k_t) - F'(k_t)k_t$ for all $t \geq T$.[47] For simplicity, call this $T = -1$. Do the perturbation $\hat{C}_{-1} \equiv C_{-1} - \epsilon$, $\hat{k}_0 = k_0$, but keep the controls c_t at their optimal level for (k_0, C_{-1}), that is $\hat{c}_t = c_t$. Denote the perturbed capital stock and capitalists' consumption by $\hat{k}_{t+1} = k_{t+1} + dk_{t+1}$ and $\hat{C}_t = C_t + dC_t$. Then the control x changes by $dx_t = F'_t dk_t$ to first order. We want to show that $dk_{t+1} > 0$ and $dC_t < 0$ for all $t \geq 0$, knowing that $dC_{-1} = -\epsilon$ and $dk_0 = 0$.

From the constraints we find,

$$dk_{t+1} = \underbrace{F'(k_t)dk_t}_{\geq 0} \underbrace{-\frac{C_t}{C_{t-1}}dC_{t-1}}_{>0} + \underbrace{\frac{1}{\sigma}\frac{C_t}{x_t}\frac{F(k_t) - F'(k_t)k_t - c_t}{k_t}dk_t}_{\geq 0} > 0$$

$$dC_t = \underbrace{\frac{C_t}{C_{t-1}}dC_{t-1}}_{\leq 0} \underbrace{-\frac{1}{\sigma}\frac{C_t}{x_t}\frac{F(k_t) - F'(k_t)k_t - c_t}{k_t}dk_t}_{\leq 0} < 0.$$

Using matrix notation, this local law of motion can be written as

$$\begin{pmatrix} dk_{t+1} \\ dC_t \end{pmatrix} = \begin{pmatrix} a_t + b_t & -d_t \\ -b_t & d_t \end{pmatrix} \begin{pmatrix} dk_t \\ dC_{t-1} \end{pmatrix},$$

with $a_t = F'(k_t)$, $d_t = C_t/C_{t-1}$, $b_t = \frac{1}{\sigma}\frac{C_t}{x_t}\frac{F(k_t)-F'(k_t)k_t-c_t}{k_t}$. Close to (k^g, C^g), this matrix has

[45] By the Maximum Theorem, the control c is upper hemicontinuous in the state, so its graph is closed. Hence, if along a path $\{k_{t+1}, C_t\}$ it holds that $c_t \to 0$, then $\{k_{t+1}, C_t\}$ necessarily approximates the set $\{c = 0\}$, in the sense that the distance between $\{k_{t+1}, C_t\}$ and the set shrinks to zero (or else you could take a subsequence $\{k_{n_t+1}, C_{n_t}, c_{n_t}\}$ in the graph of c whose limit is not in the graph, contradicting the graph being closed).

[46] The formal reason for this is as follows: Suppose the optimal path $\{k_{t+1}, C_t\}$ did not share the limiting behavior of the set $\{c = 0\}$, that is, suppose the path had a convergent subsequence $\{k_{n_t+1}, C_{n_t}\} \to \{k^*, C^*\} \in \{c = 0\} \setminus \{(k_g, C_g), (k^g, C^g)\}$. Suppose $k^* \in (k_g, k^g)$, the case $k^* > k^g$ is analogous. Because $\bar{w}(k^*) > \frac{1-\beta}{\beta}k^*$, $h(k_{n_t+1}, C_{n_t})$ is eventually strictly decreasing in t (see logic around equation (30)) and converges to $h(k^*, C^*)$. But convergence of $h(k_{n_t+1}, C_{n_t})$ implies $C^* = \frac{1-\beta}{\beta}k^*$—a contradiction.

[47] Such a finite $T > 0$ exists for two reasons: (a) because $c_t \to 0$; and (b) because $F(k) - F'(k)k$ which is positive in a neighborhood around $k = k^g$ since k^g was defined by $F(k^g) = k^g/\beta$ and $F'(k^g) < 1/\beta$.

$d \approx 1$. Suppose for one moment that a was zero; the fact that $a > 0$ only works in favor of the following argument. With $a = 0$, the matrix has a single nontrivial eigenvalue of $b + d$, which exceeds 1 strictly in the limit, and the associated eigenspace is spanned by $(1, -1)$. The trivial eigenvalue's eigenspace is spanned by (d, b). Notice that the latter eigenvector is not collinear with the initial perturbation $(0, -1)$, implying that $dk_\infty > 0$ and $dC_\infty < 0$. Hence, $\hat{k}_\infty > k_\infty = k^g$ and $\hat{C}_\infty < C_\infty = C^g$.

But notice that to the bottom right of (k^g, C^g), the new point is interior, which implies a continuation value strictly larger than $u(0)/(1-\beta)$ (see proof of Lemma 9). More formally, this means there must exist a time $T' > 0$ for which the continuation value of $(k_{T'+1}, C_{T'})$ is strictly dominated by the one for $(\hat{k}_{T'+1}, \hat{C}_{T'})$, that is, $V(k_{T'+1}, C_{T'}) < V(\hat{k}_{T'+1}, \hat{C}_{T'})$. Because all controls were equal up until time T', this implies that $V(k_{T+1}, C_T) < V(k_{T+1}, C_T - \epsilon)$ for ϵ small (Recall that we had set $T = -1$ during the proof). Thus, the value function must increase if C_T is lowered, for a path starting at (k_{T+1}, C_T), for large enough T. This proves that the value function is locally decreasing in C at that point. □

And finally, Lemma 11 proves Proposition 3.

Lemma 11. *An optimal path converges to (k_g, C_g).*

Proof. By Lemma 9 it is sufficient to prove that an optimal path does not converge to (k^g, C^g). Suppose the contrary held and there was an optimal path converging to (k^g, C^g). By Lemma 10, this means that the value function is locally decreasing around the optimal path (k_{t+1}, C_t) for $t \geq T$, with $T > 0$ sufficiently large. Consider the following feasible variation for $t = -1, 0, \ldots, T$, $\hat{C}_t = C_t(1 - d\epsilon_t), \hat{k}_{t+1} = k_{t+1}, \hat{x}_t = x_t - C_t d\epsilon_t$ where[48]

$$d\epsilon_t = \left(1 - \frac{1}{\sigma}\frac{C_t}{x_t}\right)^{-1} d\epsilon_{t-1}. \tag{19}$$

Observe that (19) is precisely the relation which ensures that the variation satisfies all the constraints of the system (in particular (16b) of which (19) is the linearized version). Workers' consumption increases with this variation by $dc_t = C_t d\epsilon_t > 0$. Therefore, the value of this path changes by

$$dV = \underbrace{\sum_{t=0}^{T} \beta^t u'(c_t) dc_t}_{>0} + \beta^{T+1} \underbrace{\left(V(k_{T+1}, C_T - C_T d\epsilon_T) - V(k_{T+1}, C_T)\right)}_{>0, \text{ by Lemma } 10} > 0,$$

which is contradicting the optimality of $\{k_{t+1}, C_t\}$. An optimal path converges to (k_g, C_g). □

[48]Notice that $x_t = C_t + k_{t+1} \geq C_t$ by definition of x_t, and $\sigma > 1$. Hence this expression is well defined.

B.4 Proofs of Auxiliary Lemmas

B.4.1 Proof of Lemma 1

Proof. Focus on Z_1 first and consider a state $(k_1, C_0) \in Z_1$, that is, $k_1 < k_g$ and $C_0 \geq \frac{1-\beta}{\beta} k_1$. Suppose (k_1, C_0) was feasible, and as such generated by a path of states $\{(k_{t+1}, C_t)\}_{t\geq 0}$, each of which compatible with (16a)–(16c). We now show by induction the claim that $(k_{t+1}, C_t) \in Z_1$ and $k_{t+1} \leq \beta F(k_t)$ for any $t \geq 0$. This will lead to a contradiction since $\beta F(k)$ is a concave and increasing function with $\beta F(0) < 0$ and smallest fixed point $\beta F(k_g) = k_g$. Thus, any sequence of capital stocks $\{k_{t+1}\}$ satisfying $k_{t+1} \leq \beta F(k_t)$, starting at any $k_1 < k_g$, necessarily drops below zero in finite time, contradicting feasibility.

Pick a point (k_t, C_{t-1}) of the sequence and assume $(k_t, C_{t-1}) \in Z_1$. Then, $x_{t+1} \equiv k_{t+1} + C_t \leq F(k_t)$ by (16c), and so

$$k_{t+1} = x_{t+1} - C_t \underbrace{\left(\frac{\beta x_{t+1}}{k_t}\right)^{1/\sigma}}_{\geq \beta x_{t+1}/k_t} \leq \beta x_{t+1} \left(\frac{1}{\beta} - \frac{C_t}{k_t}\right) \leq \beta x_{t+1} \leq \beta F(k_t), \tag{20}$$

where in the first inequality we used the fact that $\beta x_{t+1}/k_t \leq \beta F(k_t)/k_t < 1$ which holds since $k_t < k_g$; and in the second inequality we used that $C_{t-1} \geq \frac{1-\beta}{\beta} k_t$. Building on (20), the fact that $k_{t+1} \leq \beta x_{t+1}$ proves that

$$C_t = x_{t+1} - k_{t+1} \geq \frac{1-\beta}{\beta} k_{t+1}. \tag{21}$$

To sum up, this implies that $k_{t+1} \leq \beta F(k_t) < k_g$ and that $C_t \geq \frac{1-\beta}{\beta} k_{t+1}$, so $(k_{t+1}, C_t) \in Z_1$. Moreover, $k_{t+1} \leq \beta F(k_t)$. This proves the aforementioned claim and hence the desired contradiction. No state in Z_1 is feasible.

Now consider a state $(k_1, C_0) \in Z_2$. Again, suppose it was generated by a path of feasible states $\{(k_{t+1}, C_t)\}$. Define $h(k, C_-) \equiv k/C_-^\sigma$ for any state (k, C_-). The proof idea is to show the claim that $(k_{t+1}, C_t) \in Z_2$ for all t and that $h(k_{t+1}, C_t)$ is strictly increasing and diverges to $+\infty$. Since k_{t+1} is bounded from above by $\bar{k}$, this will mean that $C_t \to 0$. Moreover, k_{t+1} is bounded away from zero since feasibility requires $\beta F(k) \geq 0$ and $\beta F(k)$ turns negative for k sufficiently close to zero. Lemma 12 below proves that this combination of convergence of C_t to zero and k_{t+1} bounded away from zero violates the transversality condition.

We now prove the aforementioned claim by induction. Take a state $(k_t, C_{t-1}) \in Z_2$ from the sequence. By construction of Z_2, it holds that $C_{t-1} < w_g(k_t)$, or in particular, $(C_{t-1}/C_g)^\sigma < k_t/k_g$.[49] Notice that if the next state in the sequence, (k_{t+1}, C_t), satisfied $C_t \geq \frac{1-\beta}{\beta} k_{t+1}$, we must have $(k_{t+1}, C_t) \in Z_1$ which is infeasible according to the above.[50]

[49]This inequality even holds if $k_t < k_g$ because there, $C_g(k_t/k_g)^{1/\sigma} > (1-\beta)/\beta k_t$. To see this recall that $C_g = (1-\beta)/\beta k_g$ and so $C_g(k_t/k_g)^{1/\sigma}/((1-\beta)/\beta k_t) = (k_t/k_g)^{1/\sigma - 1} > 1$, where we used $\sigma > 1$.

[50]Note that if $C_t \geq (1-\beta)/\beta\, k_{t+1}$, then $k_{t+1} < k_g$. The reason is as follows: The constraints (16a) and (16b) can be rewritten as $k_{t+1} = (C_t/C_{t-1})^\sigma k_t/\beta - C_t$. Because $(C_{t-1}/C_g)^\sigma < k_t/k_g$, this implies that

Therefore, $C_t < \frac{1-\beta}{\beta} k_{t+1}$. Then,

$$h(k_{t+1}, C_t) = \frac{k_{t+1}}{C_t^\sigma} = \frac{k_{t+1}}{C_{t-1}^\sigma \beta x_{t+1}/k_t} = \frac{k_t}{C_{t-1}^\sigma} \underbrace{\frac{k_{t+1}}{\beta(k_{t+1}+C_t)}}_{>1} > h(k_t, C_{t-1}), \tag{22}$$

which, together with $C_t < \frac{1-\beta}{\beta} k_{t+1}$ implies that both $(k_{t+1}, C_t) \in Z_2$ and $h(k_{t+1}, C_t)$ is strictly increasing in t. To show that $h(k_{t+1}, C_t)$ diverges to $+\infty$, suppose it were the case that $h(k_{t+1}, C_t)$ converged to some $H > 0$. Using (22), convergence of $h(k_{t+1}, C_t)$ would imply that $k_{t+1}/\left(\beta(k_{t+1}+C_t)\right) \to 1$, or equivalently that $k_{t+1}/C_t \to \beta/(1-\beta)$. Since k_{t+1} is bounded away from zero (see argument in previous paragraph), this can only be the case if (k_{t+1}, C_t) converges to a feasible steady state,[51] that is some $\left(k, \frac{1-\beta}{\beta} k\right)$ with $k_g \le k \le k^g$. However, as $(k_{t+1}, C_t) \in Z_2$ for any t, it is the case that $(C_t/C_g)^\sigma < k_{t+1}/k_g$, or,

$$h(k_{t+1}, C_t) > h(k_g, C_g) = \sup_{k_g \le k \le k^g} h(k, (1-\beta)/\beta k),$$

where the equality follows because $k/((1-\beta)/\beta k)^\sigma$ is decreasing in k. This shows that $h(k_{t+1}, C_t) \to \infty$ and hence completes the proof by contradiction. No state in Z_2 is feasible. □

Lemma 12. *Suppose that $C_t \to 0$ and k_{t+1} bounded away from zero for a given path of states (k_{t+1}, C_t). Then, this path is not feasible.*

Proof. Suppose the path (k_{t+1}, C_t) is feasible. In particular, this necessitates that the IC condition $\beta U'(C_t)(C_t + k_{t+1}) = U'(C_{t-1})k_t$ and the transversality condition $\beta^t U'(C_t)k_{t+1} \to 0$ hold. We back out (after tax) interest rates from the allocation as $R_t \equiv U'(C_{t-1})/(\beta U'(C_t))$. Thus we can recover the capitalists' per period budget constraint $C_t + k_{t+1} = R_t k_t$, and, using the transversality condition, also present value budget constraints starting at any given time $t_0 \ge 0$,

$$\sum_{t=t_0}^{\infty} \frac{1}{\overline{R}_{t_0,t}} C_t = R_{t_0} k_{t_0}, \tag{23}$$

where we denote $\overline{R}_{t_0,t} \equiv R_{t_0+1} \cdots R_t$. Also, by construction of R_t, consumption can be expressed as

$$C_t = \beta^{(t-t_0)/\sigma} \left(\overline{R}_{t_0,t}\right)^{1/\sigma} C_{t_0}. \tag{24}$$

Define $\underline{K} \equiv \inf_t k_{t+1} > 0$ and $\overline{K} \equiv \sup_t k_{t+1} > 0$. Using the per period budget constraints, we then have

$$R_t = \frac{C_t + k_{t+1}}{k_t} \ge \frac{k_{t+1}}{k_t}$$

$k_{t+1} > \left(C_t/C_g\right)^\sigma k_g/\beta - C_t$. Note that the right hand side of this inequality is increasing in C_t as long as it is positive (which is the only interesting case here). Substituting in $C_t \ge (1-\beta)/\beta\, k_{t+1}$, this gives $k_{t+1} > \left(k_{t+1}/k_g\right)^\sigma k_g/\beta - (1-\beta)/\beta k_{t+1}$. Rearranging, $k_{t+1}/k_g > \left(k_{t+1}/k_g\right)^\sigma$, a condition which can only be satisfied if $k_{t+1}/k_g < 1$ (recall that $\sigma > 1$).

[51] Notice that, if $k_{t+1}/C_t^\sigma \to H > 0$ and $k_{t+1}/C_t \to \beta/(1-\beta)$ then convergence of k_{t+1} and C_{t+1} themselves immediately follows.

and similarly,

$$\overline{R}_{t_0,t} \geq \frac{k_{t+1}}{k_{t_0+1}} \geq \frac{\underline{K}}{\overline{K}}. \tag{25}$$

Combining (23), (24) and (25), we find

$$k_{t_0} = \frac{1}{R_{t_0}} \sum_{t=t_0}^{\infty} \beta^{(t-t_0)/\sigma} \underbrace{\left(\overline{R}_{t_0,t}\right)^{-(1-1/\sigma)}}_{\leq \left(\underline{K}/\overline{K}\right)^{-(1-1/\sigma)}} C_{t_0} \leq C_{t_0} \frac{\left(\underline{K}/\overline{K}\right)^{-(2-1/\sigma)}}{1-\beta^{1/\sigma}}.$$

Since t_0 was arbitrary, this implies that $k_t \to 0$, leading to the desired contradiction. Thus, the path (k_{t+1}, C_t) cannot be feasible. □

B.4.2 Proof of Lemma 2

Proof. Consider a state (k, C_-) with $w_g(k) \leq C_- \leq w^g(k)$ and $k \geq k_g$. In particular, $C_- \leq (1-\beta)/\beta k$, $(C_-/C_g)^\sigma \geq k/k_g$ and $(C_-/C^g)^\sigma \leq k/k^g$.[52] The idea of the proof is to show that in fact such a state can be generated by a steady state (k_{ss}, C_{ss}) (with $C_{ss} = (1-\beta)/\beta k_{ss}$ and $k_g \leq k_{ss} \leq k^g$). By definition of k_g and k^g, such a steady state is always self-generating.

Guess that the right steady state has $k_{ss} = (\beta C_-/(1-\beta))^{\sigma/(\sigma-1)} k^{-1/(\sigma-1)}$ and $C_{ss} = (1-\beta)/\beta k_{ss}$. It is straightforward to check that this steady state can be attained with control $x = (C_{ss}/C_-)^\sigma k/\beta$. This steady state is self-generating because $k_g \leq k_{ss} \leq k^g$, which follows from $(C_-/C_g)^\sigma \geq k/k_g$ and $(C_-/C^g)^\sigma \leq k/k^g$. Finally, the control x is resource-feasible because $C_- \leq (1-\beta)/\beta k$ and thus,

$$x = \frac{1}{\beta}\left[\frac{\left(\frac{\beta}{1-\beta}C_-\right)^\sigma}{k}\right]^{1/(\sigma-1)} \leq \frac{k}{\beta} \leq f(k) + (1-\delta)k - g,$$

where the latter inequality follows from the fact that $k_g \leq k \leq k^g$ and the definition of k_g and k^g. This concludes the proof that all states with $w_g(k) \leq C_- \leq w^g(k)$ and $k \geq k_g$ are feasible.

Now regard a state on the boundary $\{C_- = w_g(k), k > k_g\}$, so we also have that $C_- < (1-\beta)/\beta k$.[53] Such a state is generated by $(k_{ss}, C_{ss}) = (k_g, C_g)$. Moreover, the unique control which moves (k, C_-) to (k_g, C_g) is $x < k/\beta \leq f(k) + (1-\delta)k - g$, or in terms of c, $c > 0$.

To show that (k_g, C_g) is in fact the only feasible state generating (k, C_-), let (k', C) be a state generating (k, C_-). If $k' < k_g$, then (k', C) is not feasible by Lemma 1, and if $k' = k_g$ only (k_g, C_g) generates (k, C_-). Suppose $k' > k_g$. Then, $C < (1-\beta)/\beta k'$,[54] and so we can

[52] These inequalities hold for all $k \geq k_g$. The proofs are analogous to the proofs in footnotes 49 and 53.

[53] This holds because $C_- = w_g(k) = C_g(k/k_g)^{1/\sigma}$ and thus $C_-/((1-\beta)/\beta k) = (k/k_g)^{1/\sigma - 1} < 1$.

[54] This holds because by the IC constraint (1c), $\beta(k' + C)/C^\sigma = k_g/C_g^\sigma$ or equivalently $(k' + C)/C = 1/(1-\beta)\,(C/C_g)^\sigma$. Thus, letting $\kappa = k'/C$, $(\kappa+1)\kappa^\sigma = (1-\beta)^{-1} \cdot (\beta/(1-\beta))^\sigma \cdot (k'/k_g)^\sigma$. Since the right

recycle equation (22) to see $h(k', C) > h(k, C_-)$. Because $h(k, C_-) = h(k_g, C_g)$ however, this implies that $h(k', C) > h(k_g, C_g)$, or put differently, $C < w_g(k')$. Again by Lemma 1 such a (k', C) is not feasible. Therefore, the only state that can generate a state on the boundary $\{C_- = w_g(k), k > k_g\}$ is (k_g, C_g), and the associated unique control involves positive c. $\square$

B.4.3 Proof of Lemma 3

Proof. Let $\mathcal{V}$ ($\tilde{\mathcal{V}}$) be the space of all continuous, weakly (strictly) increasing functions $v : [k_g, \bar{k}] \to \mathbb{R}_+$ with $\psi(k, v(k))$ weakly (strictly) decreasing in k, and $v(k_g) = C_g$, $v(k^g) = C^g$. For these functions, T is well-defined since for small values of C_-, $k'(F(k), k, C_-)$ tends to $F(k) \in (k_g \bar{k}]$. Moreover, the supremum in (18) is attained for all $k \in [k_g, \bar{k}]$ since the set of C_- in (18) is closed and bounded. We next show that (a) instead of considering all possible controls x, it is sufficient to consider $x = F(k)$; and (b) instead of looking for C_- that satisfy the inequality in (18), it suffices to look for solutions to the corresponding relation with equality. This will allow us to write

$$Tv(k) = \max\{C_- \,|\, v(k'(F(k), k, C_-)) = C(F(k), k, C_-)\}, \tag{26}$$

The formal arguments behind these two steps are:

(a) Fix $k \in [k_g, \bar{k}]$ and $v \in \mathcal{V}$. Suppose the supremum in (18) is attained by C_-, with control $x_0 < F(k)$. Define $\Phi_{v,k,C_-} : [0, F(k)] \to \mathbb{R}$ by

$$\Phi_{v,k,C_-}(x) = \underbrace{\psi\left(k'(x,k,C_-), C(x,k,C_-)\right)}_{\text{constant in } x} - \underbrace{\psi\left(k'(x,k,C_-), v(k'(x,k,C_-))\right)}_{\text{decreasing in } x} \tag{27}$$

and notice that $v(k'(x_0, k, C_-)) \geq C(x_0, k, C_-)$ is equivalent to $\Phi_{v,k,C_-}(x_0) \geq 0$. Since $\Phi_{v,k,C_-}(x)$ is weakly increasing in x due to $v \in \mathcal{V}$, $\Phi_{v,k,C_-}(F(k)) \geq \Phi_{v,k,C_-}(x_0)$ and so $v(k'(F(k), k, C_-)) \geq C(F(k), k, C_-)$. Therefore, focusing on controls $x = F(k)$ is without loss in (18).

(b) Now argue that equality (rather than inequality) is without loss in (18). Suppose the supremum were attained by C_- with control $x = F(k)$ and *strict* inequality, $v(k'(F(k), k, C_-)) > C(F(k), k, C_-)$. Since both sides of this inequality are continuous in C_-, it follows that slightly increasing C_- still satisfies the inequality and hence C_- could not have attained the supremum in the first place.
Notice also that the equation $v(k'(F(k), k, C_-)) = C(F(k), k, C_-)$ can never have more than one solution since raising C_- weakly decreases the left hand side and strictly increases the right hand side.

hand side is increasing in κ, the fact that $k' > k_g$ tells us that $\kappa > \beta/(1-\beta)$, which is what we set out to show.

Now we argue that T maps $\mathcal{V}$ into $\tilde{\mathcal{V}}$. Take $v \in \mathcal{V}$. To show Tv is continuous and strictly increasing, define first the auxiliary function $\Psi_v : [k_g, \bar{k}] \times \mathbb{R}_{++} \to \mathbb{R}$ by

$$\Psi_v : (k, C_-) = \underbrace{\psi\left(k'(F(k), k, C_-), C(F(k), k, C_-)\right)}_{\nearrow \text{ in } k \text{ and } \searrow \text{ in } C_-} - \underbrace{\psi\left(k'(F(k), k, C_-), v(k'(F(k), k, C_-))\right)}_{\searrow \text{ in } k \text{ and } \nearrow \text{ in } C_-}.$$

The function Ψ_v is continuous and consists of two terms: The first term is equal to $\beta^{-1}k/C_-^\sigma$, using the definition of ψ, and hence strictly increasing in k and strictly decreasing in C_-. For the second term, recall that

$$k'(F(k), k, C_-) = F(k)\left(1 - C_-\left(\frac{\beta}{kF(k)^{\sigma-1}}\right)^{1/\sigma}\right)$$

is strictly increasing in k and strictly decreasing in C_-, and v is such that $\psi(k, v(k))$ is weakly decreasing in k. Thus, the second term is weakly decreasing in k and weakly increasing in C_-. Putting both terms together gives us that $\Psi_v(k, C_-)$ is continuous, strictly increasing in k, and strictly decreasing in C_-. We can rewrite Tv as

$$Tv(k) = C_- \quad \text{where } C_- \text{ is the unique number with } \Psi_v(k, C_-) = 0.$$

Since Ψ_v is continuous, strictly increasing in k, strictly decreasing in C_- and admits a unique solution $C_- = Tv(k)$ to the equation $\Psi_v(k, C_-) = 0$, it follows that $Tv(k)$ is continuous and strictly increasing.[55]

To prove that $k \mapsto \psi(k, Tv(k))$ is strictly decreasing, pick $k_1 < k_2$ in $[k_g, \bar{k}]$. Suppose $\psi(k_1, Tv(k_1)) \leq \psi(k_2, Tv(k_2))$. Since $Tv(k)$ is strictly increasing, it follows that

$$\frac{k_1}{Tv(k_1)^\sigma} - \frac{k_2}{Tv(k_2)^\sigma} < \underbrace{\frac{k_1}{Tv(k_1)^\sigma} + Tv(k_1)^{1-\sigma}}_{\psi(k_1, Tv(k_1))} \underbrace{- \frac{k_2}{Tv(k_2)^\sigma} - Tv(k_2)^{1-\sigma}}_{-\psi(k_2, Tv(k_2))} \leq 0.$$

Defining $k_i' \equiv k'(F(k_i), k_i, Tv(k_i))$ and $C_i \equiv C(F(k_i), k_i, Tv(k_i))$, we find

$$\psi(k_1', C_1) = \beta^{-1}\frac{k_1}{Tv(k_1)^\sigma} < \beta^{-1}\frac{k_2}{Tv(k_2)^\sigma} = \psi(k_2', C_2). \tag{28}$$

This, however, implies that $Tv(k_2)$ cannot have been optimal: Pick an alternative consumption level $C_{2,-}$ as $C_{2,-} = Tv(k_1)(k_2/k_1)^{1/\sigma}$, which exceeds $Tv(k_2)$ by (28). Moreover, pick the policy $x_2 \equiv F(k_1)$, which is feasible, $x_2 \leq F(k_2)$. Since $k_1/Tv(k_1)^\sigma = k_2/C_{2,-}^\sigma$ by construction, it follows that $(k'(x_2, k_2, C_{2,-}), C(x_2, k_2, C_{2,-})) = (k_1', C_1)$, which lies on the

[55] This is a fact that holds more generally: If $I_1, I_2 \subset \mathbb{R}$ are intervals and $f : I_1 \times I_2 \to \mathbb{R}$ is continuous, strictly increasing in x, and strictly decreasing in y with the property that for each x there exists a unique $y^*(x)$ s.t. $f(x, y^*(x)) = 0$, then $y^*(x)$ must be continuous and strictly increasing in x.

graph of v. Hence $Tv(k_2)$ cannot have been optimal and so $\psi(k, Tv(k))$ is decreasing in k.

Finally, we prove that $Tv(k_g) = C_g$. Note that $k'(F(k_g), k_g, C_g) = k_g$ and $C(F(k_g), k_g, C_g) = C_g$. Because $k'(F(k_g), k_g, C_-)$ is strictly decreasing in C_- and so $k'(F(k_g), k_g, C_-) < k_g$ for $C_- > C_g$ (for $k < k_g$, $v(k)$ is not even defined), this implies that $Tv(k_g) = C_g$, concluding the proof that $T(\mathcal{V}) \subset \tilde{\mathcal{V}}$. □

B.4.4 Proof of Lemma 4

Proof. Note that any state (k, C_-) reaches the space $\{C_- \leq v(k)\}$ in one step if and only if $C_- \leq Tv(k)$ (provided that v satisfies the regularity properties in Lemma 3). Thus, by iteration, $Z^{(i)} = \{w^g(k) \leq C_- \leq T^i w^g(k)\}$. Because $Z^{(i)} \supseteq Z^{(j)}$ for $i \geq j$, it holds that $T^i w^g(k) \geq T^j w^g(k)$.[56] □

B.4.5 Proof of Lemma 5

Proof. The existence of the limit $\lim_{i\to\infty} T^i w^g(k)$ is straightforward for every k (monotone sequence, bounded above because for large values of C_-, $k'(F(k), k, C_-) < k_g$ for any k). It can easily be verified that $w^g \in \mathcal{V}$. Thus, using Lemma 3, $\bar{w}$ must be weakly increasing, $\bar{w}(k_g) = C_g$, $\bar{w}(k^g) = C^g$, and $\psi(k, \bar{w}(k))$ must be weakly decreasing. To show $\bar{w} \in \mathcal{V}$, suppose now that $\bar{w}$ were not continuous. Then, there would have to be two arbitrarily close values of k, $k_1 < k_2$ with a significant gap between $T^N w^g(k_1)$ and $T^N w^g(k_2) > T^N w^g(k_1)$ for some large N. Since $k'(...)$ and $C(...)$ are both continuous, k_1 and k_2 can be chosen sufficiently close so that

$$k'_1 \equiv k'(F(k_1), k_1, T^N w^g(k_1)) > k'(F(k_2), k_2, T^N w^g(k_2)) \equiv k'_2,$$

yet the inequality is reversed for $C(...)$, $C_1 \equiv C(F(k_1), k_1, T^N w^g(k_1)) < C(F(k_2), k_2, T^N w^g(k_2)) \equiv C_2$. However, this contradicts the definition of $T^N w^g$ since both pairs (k'_1, C_1) and (k'_2, C_2) have to lie on the graph of the same increasing function $T^{N-1} w^g$ but the latter is to the top left of the former. Therefore, $\bar{w}$ is continuous and $\bar{w} \in \mathcal{V}$.

Applying Dini's Theorem, the convergence of $T^n w^g$ to $\bar{w}$ is also uniform, and by interchanging limits we find that

$$\begin{aligned}\bar{w}(k'(F(k), k, \bar{w}(k)) &= \lim_{n\to\infty} T^n w^g(k'(F(k), k, T^{n+1} w^g(k))) \\ &= \lim_{n\to\infty} C(F(k), k, T^{n+1} w^g(k)) = C(F(k), k, \bar{w}(k)), \quad (29)\end{aligned}$$

and thus, by the representation of T in (26), $\bar{w} = T\bar{w}$. This also means that $\bar{w} \in \tilde{\mathcal{V}}$, so $\bar{w}$ is strictly increasing and $\psi(k, \bar{w}(k))$ strictly decreasing. Hence, for any given k, the only feasible policy at point $(k, \bar{w}(k))$ is $x = F(k)$ (or equivalently $c = 0$) since for any feasible policy x, $\Phi_{\bar{w},k,\bar{w}(k)}(x)$ from (27) needs to be non-negative; but by $\bar{w} \in \tilde{\mathcal{V}}$ and (29), $\Phi_{\bar{w},k,\bar{w}(k)}$ is strictly increasing with $\Phi_{\bar{w},k,\bar{w}(k)}(F(k)) = 0$, so $x = F(k)$ is the only feasible policy. □

[56] A subtlety here is that $Z^{(i)} \supseteq Z^{(j)}$ only holds because states in the set $\{C_- = w^g(k)\}$ is "self-generating", that is, if a path hits the set $\{C_- = w^g(k)\}$ after j steps, it can stay in that set forever. In particular, it can hit the set after $i \geq j$ steps as well. This explains why $Z^{(i)} \supseteq Z^{(j)}$.

B.4.6 Proof of Lemma 6

Proof. Define h as before, $h(k', C) \equiv k'/C^\sigma$. Fix a state (k_1, C_0) with $C_0 > \bar{w}(k_1)$. First, consider the case $C_0 \geq (1-\beta)/\beta k_1$ and suppose it were generated by a feasible path $\{(k_{t+1}, C_t)\}$. As an intermediate result we now establish that $C_t > (1-\beta)/\beta k_{t+1}$ along such a path. We do this by distinguishing the following two cases:

(a) If $k_{t+1} \leq k^g$, this follows directly from $C_t > \bar{w}(k_{t+1}) \geq (1-\beta)/\beta k_{t+1}$. The former inequality holds by construction of $\bar{w}$,[57] the latter by Lemma 4.

(b) If instead $k_{t+1} > k^g$, it must be the case that $k_{s+1} > k^g$ for all $s < t$ as well.[58] But then, using that $x_t \leq F(k_t) < k_t/\beta$ for $k_t > k^g$,

$$\frac{k_{t+1}}{C_t} = \frac{x_t}{C_t} - 1 < \frac{k_t/\beta}{C_{t-1}} - 1 = \frac{\beta}{1-\beta}.$$

We use our intermediate result as follows (still for the case $C_0 \geq (1-\beta)/\beta k_1$). Consider

$$h(k_{t+1}, C_t) = \frac{k_{t+1}}{C_t^\sigma} = \frac{k_t}{C_{t-1}^\sigma} \underbrace{\frac{k_{t+1}}{\beta(k_{t+1} + C_t)}}_{<1} < h(k_t, C_{t-1}). \tag{30}$$

If $h(k_{t+1}, C_t)$ converges to zero, then either $k_{t+1} \to 0$ or $C_t \to \infty$ (in which case $k_{t+1} \to 0$ by the law of motion for capital and the fact that $k_t \leq \bar{k}$). Such a path is not feasible because then $F(k_{t+1})$ drops below zero in finite time (see the proof of Lemma 1 for a similar argument). Hence, suppose $h(k_{t+1}, C_t) \to \underline{h} > 0$. Then, $k_{t+1}/(\beta(k_{t+1} + C_t)) \to 1$, so the path must approximate the steady state line described by $\{(k, C_-) \,|\, C_- = (1-\beta)/\beta k\}$. Because $C_t > \bar{w}(k_{t+1})$ along the path, (k_{t+1}, C_t) must be converging to (k^g, C^g).

Next we show that this convergence is still true if we take c_t to be zero. Suppose there were times with $c_t > 0$. Then, define a new path $\{(\hat{k}_{t+1}, \hat{C}_t)\}$, starting at the same initial state (k_1, C_0) but with controls $c_t = 0$. Observe that

$$h(\hat{k}_{t+1}, \hat{C}_t) = \psi(\hat{k}_{t+1}, \hat{C}_t) - \hat{C}_t^{1-\sigma} = \beta^{-1} h(\hat{k}_t, \hat{C}_{t-1}) - h(\hat{k}_t, \hat{C}_{t-1})^{(\sigma-1)/\sigma} (\beta F(\hat{k}_t))^{-(\sigma-1)/\sigma}$$

$$\hat{k}_{t+1} = F(\hat{k}_t) - \left(\frac{\beta F(\hat{k}_t)}{h(\hat{k}_t, \hat{C}_{t-1})} \right)^{1/\sigma},$$

[57] If it were violated, $C_0 \leq T^t \bar{w}(k_1) = \bar{w}(k_1)$ by construction of $\bar{w}$. This would contradict our assumption that $C_0 > \bar{w}(k_1)$.

[58] The reason for this is that for any state (k, C_-) with $k \leq k^g$ and $C_- > \bar{w}(k)$ we have that $k' \equiv k'(x, k, C_-) \leq k^g$ for any control $x \leq F(k)$. First, if $\psi(k', C) \geq \psi(k^g, C^g)$, then the curve $\{(k'(x, k, C_-), C(x, k, C_-)), x > 0\}$ and the graph of $\bar{w}$ necessarily intersect at a state $\tilde{k}$ with capital less than k^g. The intersection is unique since $\psi(k, \bar{w}(k))$ is strictly increasing. Since $C_- > \bar{w}(k)$ it cannot be that $\tilde{k} = k'(x, k, C_-)$ for a feasible $x \leq F(k)$ and therefore, any $k'(x, k, C_-)$ with a feasible $x \leq F(k)$ is necessarily less than $\tilde{k} \leq k^g$. Second, if $\psi(k', C) < \psi(k^g, C^g)$, that is, $k/C_-^\sigma < k^g/(C^g)^\sigma$, then $k' \leq F(k) - C_- \left(\frac{\beta F(k)}{k}\right)^{1/\sigma} < F(k^g) - C^g \left(\frac{\beta F(k^g)}{k^g}\right)^{1/\sigma} = k^g$.

where the first equation is increasing in $h(\hat{k}_t, \hat{C}_{t-1})$ for the relevant parameters for which $h(\hat{k}_{t+1}, \hat{C}_t) \geq 0$, and similarly the second equation is increasing in $F(\hat{k}_t)$ if $\hat{k}_{t+1} \geq 0$. By induction over t, if $h(\hat{k}_t, \hat{C}_{t-1}) \geq h(k_t, C_{t-1})$ and $\hat{k}_t \geq k_t$ (induction hypothesis), then, because $F(\hat{k}_t) \geq x_t$,

$$
\begin{aligned}
h(\hat{k}_{t+1}, \hat{C}_t) &\geq \beta^{-1} h(k_t, C_{t-1}) - h(k_t, C_{t-1})^{(\sigma-1)/\sigma} (\beta x_t)^{-(\sigma-1)/\sigma} = h(k_{t+1}, C_t) \\
\hat{k}_{t+1} &\geq F(k_t) - \left(\frac{\beta F(k_t)}{h(k_t, C_{t-1})} \right)^{1/\sigma},
\end{aligned}
$$

confirming that $\hat{k}_t \geq k_t$ and $h(\hat{k}_t, \hat{C}_{t-1}) \geq h(k_t, C_{t-1})$ for all t. Given that $h(k_{t+1}, C_t) \to \underline{h} > 0$, either $\left(\hat{k}_{t+1}, \hat{C}_t\right) \to (k^g, C^g)$ as well, or $\{(\hat{k}_{t+1}, \hat{C}_t)\}$ converges to some steady state between k_g and k^g. The latter cannot be because of $\hat{C}_t > \bar{w}(\hat{k}_{t+1})$ along the path. But the former is precluded by Lemma 7 below. This provides a contradiction, proving that a state (k_1, C_0) with $C_0 > \bar{w}(k_1)$ and $C_0 > (1-\beta)/\beta k_1$ cannot be feasible.

Now, consider the case $C_0 < (1-\beta)/\beta k_1$. Due to $C_0 > \bar{w}(k_1)$, this can only be the case if $k_1 > k^g$. Again, suppose (k_1, C_0) were generated by a feasible path $\{(k_{t+1}, C_t)\}$. Given the first half of this proof, if at any point (k_{t+1}, C_t) lies above the steady state line, we have the desired contradiction. Therefore, suppose $C_t < (1-\beta)/\beta k_{t+1}$ for all t. In that case,

$$
h(k_{t+1}, C_t) = \frac{k_{t+1}}{C_t^\sigma} = \frac{k_t}{C_{t-1}^\sigma} \underbrace{\frac{k_{t+1}}{\beta(k_{t+1} + C_t)}}_{>1} > h(k_t, C_{t-1}).
$$

Note that $h(k_{t+1}, C_t)$ is bounded from above, for example by $h(k_g, C_g)$ (because all states below the steady state line with h equal to $h(k_g, C_g)$ are below the graph of w^g and thus below $\bar{w}$ as well). So, $h(k_{t+1}, C_t)$ converges and $k_{t+1}/(\beta(k_{t+1} + C_t)) \to 1$. The state approximates the steady state line. Because the only feasible steady state with below the steady state line but above the graph of $\bar{w}$ is (k^g, C^g) it follows that $(k_{t+1}, C_t) \to (k^g, C^g)$. Following the same steps as before, it can be shown that without loss of generality, controls c_t can be taken to be zero along the path. By Lemma 7 below this is a contradiction, concluding our proof that no state (k_1, C_0) with $C_0 > \bar{w}(k_1)$ is feasible. □

B.4.7 Proof of Lemma 7

Proof. We prove each of the results in turn.

(a) Notice that $c = 0$ takes any state on the graph of $\bar{w}$ to another state on the graph of $\bar{w}$ (because $T\bar{w} = \bar{w}$). Suppose $k_1 < k^g$ (the case $k_1 > k^g$ is analogous). Then, no future capital stock k_{t+1} can exceed k^g. Because if it did, there would have to be a capital stock $k \in (k_g, k^g)$ with $k'(F(k), k, \bar{w}(k)) = k^g$, by continuity of $k \mapsto k'(F(k), k, \bar{w}(k))$. But this is impossible by definition of k^g.[59] Thus, along the path, $C_t > (1-\beta)/\beta k_{t+1}$

[59] By definition of k^g, $F(k^g) = k^g + C^g$, and so, $F(k) < k^g + C^g$ for $k < k^g$.

and so $h(k_{t+1}, C_t)$ is decreasing. As $h(k_g, C_g) > h(k, \varpi(k))$ for all $k > k_g$,[60] this means $(k_{t+1}, C_t) \to (k^g, C^g)$.

(b) For simplicity, focus on the case $k_0 < k^g$. Again, the case $k_0 > k^g$ is completely analogous. Suppose (k_{t+1}, C_t) were converging to (k^g, C^g). Note that at k^g, $F(k)/k$ is decreasing.[61] Thus, there exists a time $T > 0$ for which the capital stock k_T is sufficiently close to k^g that $F(k)/k$ is decreasing for all k in a neighborhood of k^g which includes $\{k_t\}_{t \geq T}$. Let $\{\hat{k}_{t+1}, \hat{C}_t\}$ denote the path with $c_t = 0$, starting from $(k_T, \varpi(k_T))$. We already know that $\{\hat{k}_{t+1}, \hat{C}_t\}$ does indeed converge to (k^g, C^g) from the first part of this proof. Also, observe that both (k_{t+1}, C_t) and $(\hat{k}_{t+1}, \hat{C}_t)$ have controls $c_t = 0$ here, unlike in the proof of Lemma 6.
In the remainder of this proof, we denote the "zero control $c = 0$" laws of motion for capital and capitalists' consumption by $L_k(k, C_-) \equiv k'(F(k), k, C_-)$ and $L_C(k, C_-) \equiv C(F(k), k, C_-)$ (only for this proof). Since $F(k)/k$ is locally decreasing, it follows that $dL_k/dk > 0$, $dL_k/dC_- < 0$ and $dL_C/dk < 0$, $dL_C/dC_- > 0$. This implies that because $C_{T-1} > \varpi(k_T)$ (which must hold or else $C_0 \leq \varpi(k_1)$ by construction of ϖ), $C_t > \hat{C}_t$ and $k_{t+1} > \hat{k}_{t+1}$ for all $t \geq T$. Moreover, borrowing from equation (22), we know that

$$h(k_{t+1}, C_t) = h(k_t, C_{t-1}) \left(\frac{1}{\beta} - \left(\frac{1}{h(k_t, C_{t-1})} \right)^{1/\sigma} \frac{1}{(\beta F(k_t))^{1-1/\sigma}} \right),$$

which implies that by induction $h(k_{t+1}, C_t) \leq h(\hat{k}_{t+1}, \hat{C}_t)$, that is,

$$\begin{aligned}
&\log h(k_{t+T}, C_{t+T-1}) \\
&\quad = \log h(k_T, C_{T-1}) + \sum_{s=0}^{t-1} \log \left(\frac{1}{\beta} - \left(\frac{1}{h(k_{T+s}, C_{T+s-1})} \right)^{1/\sigma} \frac{1}{(\beta F(k_{T+s}))^{1-1/\sigma}} \right) \\
&\quad \leq \log h(k_T, C_{T-1}) + \sum_{s=0}^{t-1} \log \left(\frac{1}{\beta} - \left(\frac{1}{h(\hat{k}_{T+s}, \hat{C}_{T+s-1})} \right)^{1/\sigma} \frac{1}{(\beta F(\hat{k}_{T+s}))^{1-1/\sigma}} \right) \\
&\qquad\qquad = \log h(\hat{k}_{t+T}, \hat{C}_{t+T-1}) + \log h(k_T, C_{T-1}) - \log h(\hat{k}_T, \hat{C}_{T-1}).
\end{aligned}$$

As $t \to \infty$, this equation yields

$$\log h(k^g, C^g) \leq \log h(k^g, C^g) + \underbrace{\log h(k_T, C_{T-1}) - \log h(\hat{k}_T, \hat{C}_{T-1})}_{= -k_T(\hat{C}_{T-1}^{-\sigma} - C_{T-1}^{-\sigma}) < 0},$$

which is a contradiction. Therefore, $(k_{t+1}, C_t) \not\to (k^g, C^g)$. $\square$

[60] Note that $\varpi(k) > w_g(k)$ and $h(k, w_g(k)) = \text{const}$, see Lemmas 1 and 2 above.
[61] This holds because $F'(k^g) < 1/\beta$ and $F(k^g) = 1/\beta k^g$, and so $\frac{d}{dk} F(k)/k < 0$.

C Numerical Method

To solve the Bellman equation (15) we must first compute the feasible set Z^*. We restrict the range of capital to a closed interval $[\underline{k}, \bar{k}]$ with $\underline{k} \geq k_g$. This leads us to seek a subset $Z^{*\underline{k}} \subset Z^*$ of the feasible set Z^*. We compute this set numerically as follows.

Start with the set $Z^*_{(0)}$ defined by $C_- = \frac{1-\beta}{\beta} k$ and $k \in [\underline{k}, \bar{k}]$. This set is self generating and thus $Z^*_{(0)} \subset Z^{*\underline{k}}$. We define an operator that finds all points (k, C_-) for which one can find c, K', C satisfying the constraints of the Bellman equation and $(k', C) \in Z^*_{(0)}$. This gives a set $Z^*_{(1)}$ with $Z^*_{(0)} \subset Z^*_{(1)}$. Iterating on this procedure we obtain $Z^*_{(0)}, Z^*_{(1)}, Z^*_{(2)}, \ldots$ and we stop when the sets do not grow much. We then solve the Bellman equation by value function iteration. We start with a guess for V_0 that uses a feasible policy to evaluate utility. This ensures that our guess is below the true value function. Iterating on the Bellman equation then leads to a monotone sequence $V_0, V_1, \ldots$ and we stop when iteration n yields a V_n that is sufficiently close to V_{n-1}. Our procedure uses a grid that is defined on a transformation of (k, C_-) that maps Z^* into a rectangle. We linearly interpolate between grid points.

The code was programmed in Matlab and executed with parallel 'parfor' commands, to improve speed and allow denser grids, on a cluster of 64-128 workers. Grid density was adjusted until no noticeable difference in the optimal paths were observed.

D Proof of Proposition 4

As in Appendix B we use the notation that $F(k) = f(k) + (1-\delta)k$. The derivatives of S evaluated at some time τ are denoted by $S_{I,\tau} \equiv \frac{\partial S_\tau}{\partial I_\tau}$ and $S_{\tau,R_t} \equiv \frac{\partial S_\tau}{\partial R_t}$, for $t > \tau$.

Define the following object,

$$\omega_\tau = \frac{dW_\tau}{dk_{\tau+1}} = \sum_{\tau' \geq \tau+1} \beta^{\tau'-\tau} u'(c_{\tau'})(F'(k_{\tau'}) - R_{\tau'}) \left(\prod_{s=\tau+1}^{\tau'-1} S_{I,s} R_s \right), \tag{31}$$

which corresponds to the response in welfare W_τ, measured in units of period τ utility, of a change in savings by an infinitesimal unit between periods τ and $\tau+1$. Now consider the effect of a one-time change in the capital tax, effectively changing R_t to $R_t + dR$ in period t. This has three types of effects on total welfare: It changes savings behavior in all periods $\tau < t$ through the effect of R_t on S_τ. It changes capitalists' income in period t through the effect of R_t on $R_t k_t$. And finally it changes workers' income in period t directly through the effect of R_t on $F(k_t) - R_t k_t$. Summing up these three effects, one obtains a total effect of

$$dW = \sum_{\tau=0}^{t-1} \beta^{\tau-t}\omega_\tau \underbrace{S_{\tau,R_t} dR}_{\text{change in savings in period } \tau<t} + \omega_t \underbrace{S_{I,t} k_t dR}_{\text{change in savings in period } t} - u'(c_t) \underbrace{k_t dR}_{\text{change in workers' income in period } t}.$$

The total effect needs to net out to zero along the optimal path, that is,

$$\omega_t S_{I,t} - u'(c_t) = -\frac{1}{k_t}\sum_{\tau=0}^{t-1} \beta^{\tau-t}\omega_\tau S_{\tau,R_t}. \tag{32}$$

By optimization over the initial interest rate R_0, we find the condition

$$\omega_0 S_{I,0} k_0 - u'(c_0) k_0 = 0. \tag{33}$$

This shows that $S_{I,0} > 0$ and so $\omega_0 \in (0,\infty)$. By their definition (31), the ω_τ satisfy the recursion

$$\omega_\tau = u'(c_{\tau+1})(F'(k_{\tau+1}) - R_{\tau+1}) + \beta S_{I,\tau+1} R_{\tau+1}\omega_{\tau+1}.$$

Since it is easy to see that $R_{\tau+1} > 0$ for all τ,[62] it follows that ω_τ is finite for all τ. Then, due to the recursive nature of (32), if $\omega_\tau > 0$ for $\tau < t$,

$$\omega_t S_{I,t} - u'(c_t) = -\frac{1}{k_t}\sum_{\tau=0}^{t-1} \beta^{\tau-t} \underbrace{\omega_\tau}_{>0} \underbrace{S_{\tau,R_t}}_{\leq 0} \geq 0.$$

In particular, using the initial condition (33), this proves by induction that

$$\omega_t S_{I,t} - u'(c_t) \geq 0 \quad \text{for all } t > 0. \tag{34}$$

Now suppose the economy were converging to an interior steady state with non-positive limit tax (either zero or negative), that is, $\Delta_t \equiv F'(k_t) - R_t$ converges to a non-positive number, $c_t \to c > 0$ and $S_{I,t}R_t \to S_I R > 0$. It is immediate by (31) that if Δ_t converges to a negative number, then ω_t must eventually become negative—contradicting (34). Hence suppose $\Delta_t \to 0$. Distinguish two cases.

Case I: Suppose first that $\beta S_I R > 1$. Thus, $\prod_{s=1}^{\tau}(\beta S_{I,s} R_s)$ is unbounded and diverges to ∞. Then, because ω_0 is finite, we have that the partial sums in the expression for ω_0

[62]Otherwise capital would be zero forever after due to $S(0,\ldots) = 0$, a contradiction to the allocation converging to an interior steady state.

coming from (31) have to converge to zero,

$$\bar{\omega}_\tau \equiv \sum_{\tau' \geq \tau+1} \beta u'(c_{\tau'})(F'(k_{\tau'}) - R_{\tau'}) \prod_{s=1}^{\tau'-1} (\beta S_{I,s} R_s) \to 0, \quad \text{as } \tau \to \infty.$$

Hence,

$$\omega_\tau = \left(\prod_{s=1}^{\tau} (\beta S_{I,s} R_s) \right)^{-1} \bar{\omega}_\tau \to 0,$$

contradicting the fact that ω_t is bounded away from zero by $u'(c)/S_I$. Therefore, $\beta S_I R > 1$ is not compatible with any interior steady state. (This argument does not use the fact that we focus on $\Delta_t \to 0$.)

Case II: Suppose $\beta S_I R < 1$. In this case, we show convergence of ω_τ to zero directly. Fix $\epsilon > 0$. Let τ be large enough such that $\beta S_{I,s} R_s < b$ for some $b < 1$ and that $|u'(c_{\tau'})\Delta_{\tau'}| < \epsilon(1-b)$. Then,

$$|\omega_\tau| \leq \sum_{\tau'=\tau+1} \epsilon(1-b) b^{\tau'-1-\tau} = \epsilon.$$

Again, this contradicts the fact that ω_t is bounded away from zero by $u'(c)/S_I$.

This concludes our proof, establishing that the capital tax $\mathcal{T}_t = \Delta_t / F'(k_t)$ must converge to a positive number at the interior steady state.

E Derivation of the Inverse Elasticity Rule (4) and Proof of the Corollary

Derivation of the Inverse Elasticity Rule. In this section, we continue using the notation and results of Section D. Consider equation (32). Because $\beta S_I R < 1$, ω_τ converges to

$$\omega = \frac{\beta}{1 - \beta S_I R} (F'(k) - R) u'(c).$$

We make the additional convergence assumption

$$\sum_{\tau=1}^{t} \beta^{-\tau} \frac{\omega_{t-\tau} k_{t-\tau}}{\omega_t k_t} \epsilon_{S_{t-\tau}, R_t} \to \sum_{\tau=1}^{\infty} \beta^{-\tau} \epsilon_{S,\tau} \in [-\infty, \infty], \quad \text{as } t \to \infty, \tag{35}$$

which amounts to first taking the limit of the summands as $t \to \infty$, and then taking the limit of the series, instead of considering both limits simultaneously. Under this order of limits assumption, we can characterize the limit of equation (32) as $t \to \infty$,

$$\underbrace{S_{I,t} - \frac{u'(c_t)}{\omega_t}}_{\to S_I - \frac{u'(c)}{\omega}} = - \underbrace{\sum_{\tau=1}^{t} \beta^{-\tau} \frac{\omega_{t-\tau} k_{t-\tau}}{\omega_t k_t} \epsilon_{S_{t-\tau}, R_t}}_{\to \sum_{\tau=1}^{\infty} \beta^{-\tau} \epsilon_{S,\tau}}. \tag{36}$$

Distinguish two cases according to whether $\omega = 0$ or $\omega \neq 0$. First, if $\omega = 0$, or equivalently the limit tax $\mathcal{T}$ is zero, then (36) reveals that $\sum_{\tau=1}^{\infty} \beta^{-\tau} \epsilon_{S,\tau}$ is either plus or minus infinity. Therefore, the inverse elasticity formula holds in this case as both sides of (4) converge to zero.

Second, if $\omega \neq 0$, then by taking the limit of (32) as $t \to \infty$ and using the condition (35), we find

$$S_I - \frac{u'(c)}{\omega} = -\sum_{\tau=1}^{\infty} \beta^{-\tau} \epsilon_{S,\tau},$$

which can be rewritten as

$$\frac{\beta S_I R}{1-\beta S_I R}(F'(k) - R) - R = -\frac{1}{1-\beta S_I R}(F'(k) - R) \sum_{\tau=1}^{\infty} \beta^{-\tau+1} \epsilon_{S,\tau}.$$

Note that $F'(k) - R = \frac{\mathcal{T}}{1-\mathcal{T}} R$. Therefore, we can rearrange the condition to

$$\begin{aligned} \frac{\beta S_I R}{1-\beta S_I R} - \frac{1-\mathcal{T}}{\mathcal{T}} &= -\frac{1}{1-\beta S_I R} \sum_{\tau=1}^{\infty} \beta^{-\tau+1} \epsilon_{S,\tau} \\ \Rightarrow \quad \mathcal{T} &= \frac{1-\beta R S_I}{1+\sum_{t=1}^{\infty} \beta^{-t+1} \epsilon_{S,t}}. \end{aligned}$$

This is precisely the inverse elasticity formula (4).

Proof of the Corollary. Notice that by Proposition 4 the limit tax rate is positive, $\mathcal{T} > 0$, conditional on convergence to an interior steady state. If now the inverse elasticity formula implies a negative tax rate, then either the regularity condition for the inverse elasticity rule is not satisfied or the allocation does not converge to an interior steady state.

F Infinite Sum of Elasticities with Recursive Utility

In this section, we prove the result that the infinite sum $\sum_{\tau=1}^{\infty} \beta^{-\tau} \epsilon_{S,\tau}$ does not converge for any recursive utility function that is locally non-additive.

More specifically, we consider the capitalist's optimization problem as in Section 2.3, just with recursive preferences as in Section 3.1, with $U = c$. In particular, the capitalist's utility is characterized by the recursion $V_t = W(C_t, V_{t+1})$, assuming W is twice continuously differentiable and strictly increasing in both arguments. Analogous to our analysis in Section 3.1, we define $\overline{\beta}(c) \equiv W_V(c, V(c))$ as the steady state discount factor along a constant consumption stream yielding steady state utility $V(c) = W(c, V(c))$.

Any such recursive utility function naturally yields an optimal savings function $a_{t+1} = S(R_t a_t, R_{t+1}, \ldots)$. Fix now constant interest rates R and a steady state of the capitalist's optimization problem (a, c, V). Let $\beta = W_V(c, V(c)) = \overline{\beta}(c)$ the discount factor in that specific steady state. Define $\epsilon_{S,\tau} = \frac{1}{a} R \frac{\partial \log S}{\partial \log R_\tau}$. The following proposition characterizes the behavior of the infinite sum $\sum_{\tau=1}^{\infty} \beta^{-\tau} \epsilon_{S,\tau}$.

Proposition 12. *Suppose capitalists have recursive preferences represented by* (5a) *(see Section 3.1, with $U = c$). Then, if the discount factor is locally non-constant, $\overline{\beta}'(c) \neq 0$, the series $\sum_{\tau=1}^{T} \beta^{-\tau}\epsilon_{S,\tau}$ does not have a finite limit as $T \to \infty$.*

Proof. We first compute the elasticities $\epsilon_{S,\tau}$ and then prove that the infinite sum does not have a finite limit. To compute $\epsilon_{S,\tau}$, we consider an agent with the recursive preferences introduced above, who is at a steady state (a, c, V) given a constant interest rates R. Note that because utility is strictly increasing in a permanent increase in consumption at the steady state, we have $\beta = W_V \in (0, 1)$.

The conditions for optimality are then,

$$V_t = W\left(R_t a_t - a_{t+1}, V_{t+1}\right)$$
$$W_C\left(R_t a_t - a_{t+1}, V_{t+1}\right) = R_{t+1} W_V\left(R_t a_t - a_{t+1}, V_{t+1}\right) W_C\left(R_{t+1} a_{t+1} - a_{t+2}, V_{t+2}\right).$$

The first equation is the recursion for utility V_t and the second equation is the Euler equation. In particular, note that the latter implies that $\beta R = 1$ at the steady state. Linearizing these equations around the steady state (denoted without time subscripts) yields,

$$W_V\, dV_{t+1} = -W_C R\, da_t + W_C\, da_{t+1} + dV_t - W_C a\, dR_t \tag{37}$$

and

$$\begin{aligned}\left(R W_C W_{VC} - R W_{CC} - W_{CC}\right) da_{t+1} + W_{CC}\, da_{t+2} - \left(W_V W_C + W_{CC} a\right) dR_{t+1} \\ + \left(W_{CV} - R W_C W_{VV}\right) dV_{t+1} - W_{CV}\, dV_{t+2} \\ = \left(R^2 W_C W_{VC} - W_{CC} R\right) da_t + \left(R W_C W_{VC} a - W_{CC} a\right) dR_t,\end{aligned} \tag{38}$$

where all derivatives are evaluated at the steady state $\left((R-1)\, a, V\right)$. To save on notation, we define $\omega \equiv W_{VC} - \beta W_{CC}/W_C \in \mathbb{R}$, which is a term that will appear multiple times below. We solve (37) and (38) by the method of undetermined coefficients, guessing

$$da_{t+1} = \omega\lambda\, da_t + a \sum_{\tau=0}^{\infty} \beta^{\tau} \theta_{\tau}\, dR_{t+\tau} \tag{39a}$$

$$dV_t = W_C R\, da_t + (W_C a) \sum_{\tau=0}^{\infty} \beta^{\tau}\, dR_{t+\tau}. \tag{39b}$$

The form of equation (39b) is what is required by the Envelope condition. We are left to find λ and the sequence $\{\theta_\tau\}$, where $\theta_\tau = \beta^{-\tau}\epsilon_{S,\tau}$, for $\tau \geq 1$, is exactly the sequence of elasticities were are looking for. Substituting the guesses (39a) and (39b) into (38), we obtain an expression featuring da_t, da_{t+1}, da_{t+2} and $dR_{t+\tau}$ for $\tau = 0, 1, \ldots$. Setting the coefficient on da_t to zero gives a quadratic for λ,

$$\omega^2\lambda^2 + \left(-(1+R)\omega + (R-1)\overline{\beta}'(c)\right)\lambda + R = 0. \tag{40}$$

Note that the solution of this equation can never be zero, i.e. $\lambda \neq 0$. Also, if $\overline{\beta}'(c) = 0$, the term in parentheses simplifies to $-(1+R)\omega$ and the solutions are just $\lambda = \omega^{-1}$ and $\lambda = \omega^{-1}R$.

Setting the coefficient on dR_t to zero gives

$$\theta_0 = \beta\omega\lambda.$$

Similarly for dR_{t+1} we find after various simplifications,

$$\theta_1 = \omega\lambda\left(\theta_0 - 1\right) + \lambda\left(\beta^2 a^{-1} + (1-\beta)\overline{\beta}'(c)\right)$$

and for $dR_{t+\tau}$ after some more simplifications

$$\theta_\tau = \omega\lambda\theta_{\tau-1} + \lambda\left(1-\beta\right)\overline{\beta}'(c), \tag{41}$$

for $\tau = 2, 3, \ldots$. The result then follows from this expression: If $\overline{\beta}'(c) \neq 0$, the sum $\sum_{\tau=1}^{T} \beta^{-\tau}\epsilon_{S,\tau} = \sum_{\tau=1}^{T} \theta_\tau$ cannot converge. To see this, consider

$$\sum_{\tau=1}^{T} \theta_\tau = \theta_1 + \sum_{\tau=2}^{T} \theta_\tau = \theta_1 + \sum_{\tau=1}^{T-1} \omega\lambda\theta_\tau + \sum_{\tau=2}^{T} \lambda(1-\beta)\overline{\beta}'(c).$$

If the left hand side of this equation converged to some limit $\Theta \in \mathbb{R}$, the right hand side of this equation would diverge since the last sum diverges (while all other terms would remain finite). Therefore, $\sum_{\tau=1}^{T} \beta^{-\tau}\epsilon_{S,\tau}$ cannot converge to a finite limit. $\square$

G Linearized Dynamics and Proof of Proposition 5

A natural way to prove Proposition 5 would be to linearize our first order conditions in (2), and to solve forward for the multipliers μ_t and λ_t using transversality conditions, arriving at an approximate law of motion of the form

$$\begin{pmatrix} k_{t+1} \\ C_t \end{pmatrix} - \begin{pmatrix} k_t \\ C_{t-1} \end{pmatrix} = \hat{J} \begin{pmatrix} k_t - k^* \\ C_{t-1} - C^* \end{pmatrix}.$$

To maximize similarity with Kemp et al. (1993), however, we do not take that route; rather we start with the continuous time problem, derive its first order conditions and linearize them around the zero tax steady state. The problem in continuous time is

$$\begin{aligned} &\max \int_0^\infty e^{-\rho t}\left(u(c_t) + \gamma U(C_t)\right) dt \\ &\text{s.t. } c_t + C_t + g + \dot{k}_t = f(k_t) - \delta k_t \\ &\qquad \dot{C}_t = \frac{C_t}{\sigma}\left(\frac{f(k_t)}{k_t} - \delta - \frac{c_t}{k_t} - \rho\right). \end{aligned}$$

Let p_t and q_t denote the costates corresponding respectively to the states k_t and C_t. The FOCs are,

$$\begin{aligned}
u_t'(c_t) &= p_t c_t + q_t \frac{1}{\sigma}\frac{C_t}{k_t} \\
\dot{p}_t &= \rho p_t - p_t(f'(k_t) - \delta) + q_t \frac{\dot{C}_t}{k_t} - q_t \frac{C_t}{k_t}(f'(k_t) - \delta) \\
\dot{q}_t &= \rho q_t - \gamma U'(C_t) - q_t \frac{1}{\sigma}\left(\frac{f(k_t)}{k_t} - \delta - \frac{c_t}{k_t} - \rho\right).
\end{aligned}$$

Just like Kemp et al. (1993), we require the two transversality conditions to hold,

$$\lim_{t\to\infty} e^{-\rho t} q_t C_t = 0 \quad \text{and} \quad \lim_{t\to\infty} e^{-\rho t} p_t k_t = 0. \tag{42}$$

Denote the 4-dimensional state of this dynamic system by $x_t = (k_t, C_t, p_t, q_t)$ and its unique positive steady state (the zero-tax steady state) by $x^* = (k^*, C^*, p^*, q^*)$. The linearized system is,

$$\dot{x}_t = J(x_t - x^*), \tag{43}$$

where J is a 4×4 matrix with determinant

$$\det J = (1-\sigma)\underbrace{\frac{f''(k^*)u'(c^*)}{u''(c^*)}}_{>0}\frac{\rho^2}{\sigma^2}. \tag{44}$$

Its eigenvalues can be written as,

$$\lambda_{1-4} = \frac{\rho}{2} \pm \left[\left(\frac{\rho}{2}\right)^2 - \frac{\chi}{2} \pm \frac{1}{2}\left(\chi^2 - 4\det J\right)^{1/2}\right]^{1/2}, \tag{45}$$

with

$$\chi = \frac{\rho}{\sigma}\frac{u'(c^*) - \gamma U'(C^*)}{u''(c^*)k^*} - \frac{f''(k^*)u'(c^*)}{u''(c^*)}. \tag{46}$$

There are two "$\pm$" signs in (45). In the remainder, we number eigenvalues according to those two signs in (45): λ_1 has $++$, λ_2 has $+-$, λ_3 has $-+$, and λ_4 has $--$. For convenience, define γ^* by $\gamma^* = u'(c^*)/U'(C^*)$.

In general, a solution x_t to the linearized FOCs (43) can load on all four eigenvalues. However, taking the two transversality conditions into account restricts the system to only load on eigenvalues with $\text{Re}(\lambda_i) \leq \rho/2$. In Lemma 13 below, we show that this means the solution loads on eigenvalues λ_3 and λ_4.

Lemma 13. *The eigenvalues in (45) can be shown to satisfy the following properties.*

(a) It is always the case that

$$Re\lambda_1 \geq Re\lambda_2 \geq \rho/2 \geq Re\lambda_4 \geq Re\lambda_3.$$

(b) *If $\sigma > 1$, then* $\det J < 0$*, implying that*

$$Re\lambda_1 = \lambda_1 > \rho > Re\lambda_2 \geq \rho/2 \geq Re\lambda_4 > 0 > \lambda_3 = Re\lambda_3. \tag{47}$$

In particular, there is a exactly one negative eigenvalue. The system is saddle-path stable.

(c) *If $\sigma < 1$ and $\gamma \leq \gamma^*$, then* $\det J > 0$ *and $\delta < 0$, implying that*

$$Re\lambda_1, Re\lambda_2 > \rho > 0 > Re\lambda_4, Re\lambda_3. \tag{48}$$

In particular, there exist exactly two eigenvalues with negative real part. The system is locally stable.

(d) *If $\sigma < 1$ and $\gamma > \gamma^*$, the system may either be locally stable, or locally unstable (all eigenvalues having positive real parts).*

Proof. We follow the convention that the square root of a complex number a is defined as the *unique* number b that satisfies $b^2 = a$ *and has nonnegative real part* (if $\mathrm{Re}(b) = 0$ we also require $Im(b) \geq 0$). Hence, the set of all square roots of a is given by $\{\pm b\}$. We prove the results in turn.

(a) First, observe the following fact: Given a real number x and a complex number b with nonnegative real part, it holds that $\mathrm{Re}\left(\sqrt{x+b}\right) \geq \mathrm{Re}\left(\sqrt{x-b}\right)$.[63] From there, it is straightforward to see that $\mathrm{Re}\lambda_1 \geq \mathrm{Re}\lambda_2$ and $\mathrm{Re}\lambda_4 \geq \mathrm{Re}\lambda_3$. Finally $\mathrm{Re}\lambda_2 \geq \rho/2 \geq \mathrm{Re}\lambda_4$ holds according to our convention of square roots having nonnegative real parts.

(b) The negativity of $\det J$ follows immediately from (44). This implies

$$-\frac{\delta}{2} + \frac{1}{2}\left(\delta^2 - 4\det J\right)^{1/2} > 0 > -\frac{\delta}{2} - \frac{1}{2}\left(\delta^2 - 4\det J\right)^{1/2},$$

and so (47) holds, using monotonicity of $\mathrm{Re}\sqrt{x}$ for real numbers x.

(c) The signs of $\det J$ and δ follow immediately from (44) and (46). In this case, $-\delta/2 \pm 1/2\mathrm{Re}\left(\delta^2 - 4\det J\right)^{1/2} > 0$ proving (48).

(d) This is a simple consequence of the fact that if $\det J > 0$, then either $-\delta/2 \pm 1/2\mathrm{Re}\left(\delta^2 - 4\det J\right)^{1/2} > 0$, or $-\delta/2 \pm 1/2\mathrm{Re}\left(\delta^2 - 4\det J\right)^{1/2} < 0$, where under the latter condition the system is locally unstable. □

[63] To prove this, let $\bar{b}$ denote the complex conjugate of b and note that $\mathrm{Re}\left(\sqrt{x+b}\right)$ is monotonic in the real number x. Then, $\mathrm{Re}\left(\sqrt{x+b}\right) = \mathrm{Re}\left(\sqrt{x+\bar{b}}\right) = \mathrm{Re}\left(\sqrt{x-b+(\bar{b}+b)}\right) \geq \mathrm{Re}\left(\sqrt{x-b}\right)$ where $\bar{b}+b = 2\mathrm{Re}(b) \geq 0$ and monotonicity are used.

H Proof of Proposition 6

In this proof, we first exploit the recursiveness of the utility $\mathcal{V}$ to recast the IC constraint (7) entirely in terms of V_t and $W(U, V')$. Then, using the first order conditions, we are able to characterize the long-run steady state. Throughout this section, we denote by X_{zt} the derivative of quantity X with respect to z, evaluated at time t. To save on notation, we define $f(k, n) \equiv F(k, n) + (1 - \delta)k$.

Let $\beta_t \equiv \prod_{s=0}^{t-1} W_{Vs}$. Using the definition of the aggregator in (3) this implies that $\mathcal{V}_{ct} = \beta_t W_{Ut} U_{ct}$ and $\mathcal{V}_{nt} = \beta_t W_{Ut} U_{nt}$. Thus the IC constraint (7) can be rewritten as

$$\sum_{t=0}^{\infty} \beta_t W_{Ut}(U_{ct}c_t + U_{nt}n_t) = W_{U0}U_{c0}\left(R_0 k_0 + R_0^b b_0\right), \tag{49}$$

and the planning problem becomes

$$\begin{aligned} \max_{\{V_t, c_t, n_t, R_0\}} \quad & V_0 \\ \text{s.t. } & V_t = W(U(c_t, n_t), V_{t+1}) \\ & \text{RC (6), IC (49), } R_t \geq 1. \end{aligned} \tag{50}$$

To state the first order conditions, define for each $t \geq 0$, $A_{t+1} \equiv \frac{1}{\beta_{t+1}} \frac{\partial}{\partial V_{t+1}} \sum_{s=0}^{\infty} \beta_s W_{Us}(U_{cs}c_s + U_{ns}n_s)$ and $B_t \equiv \frac{1}{\beta_t} \sum_{s=0}^{\infty} \frac{\partial(\beta_s W_{Us})}{\partial U_t}(U_{cs}c_s + U_{ns}n_s)$. Let $\beta_t \nu_t$ be the present value multiplier on the Koopmans constraint (50), $\beta_t \lambda_t$ the present value multiplier on the resource constraint (6), and μ the multiplier on the IC constraint (49). As stated in the proposition, we assume that the capital tax bound $R_t \geq 1$ is not binding eventually, say from period T onwards. The first order conditions for V_{t+1}, c_t, n_t, and k_{t+1} (in that order) are for each $t \geq T$ given by

$$\begin{aligned} -\nu_t + \nu_{t+1} + \mu A_{t+1} &= 0 \\ -\nu_t W_{Ut} U_{ct} + \mu W_{Ut}\left(U_{ct} + U_{cc,t}c_t + U_{nc,t}n_t\right) + \mu B_t U_{ct} &= \lambda_t \\ \nu_t W_{Ut} U_{nt} - \mu W_{Ut}\left(U_{nt} + U_{cn,t}c_t + U_{nn,t}n_t\right) - \mu B_t U_{nt} &= \lambda_t f_{nt} \\ -\lambda_t + \lambda_{t+1} W_{Vt} f_{kt+1} &= 0. \end{aligned}$$

Suppose the allocation converges to an interior steady state in c, k, and n. Then U_t and V_t converge, as well as their first and second derivatives (when evaluated at c_t, k_t, and n_t). Similarly, the representative agent's assets a_t converge to a value a, which can be characterized using a time $t+1$ version of the IC constraint,

$$\begin{aligned} a &= \lim_{t \to \infty} a_{t+1} = \lim_{t \to \infty} \left(W_{Ut+1} U_{ct+1} \beta_{t+1} R_{t+1}\right)^{-1} \sum_{s=t+1}^{\infty} \beta_s W_{Us}(U_{cs}c_s + U_{ns}n_s) \\ &= \left((1 - \beta) U_c R\right)^{-1} \left(U_c c + U_n n\right), \end{aligned}$$

where $\beta \equiv \bar{\beta}(V) = W_V \in (0, 1)$ (see footnote 30). Using this expression, we see that A_{t+1}, which can be written as,

$$A_{t+1} = \frac{W_{UV,t}}{W_{Vt}}(U_{ct}c_t + U_{nt}n_t) + \frac{W_{VV,t}}{W_{Vt}} \underbrace{\beta_{t+1}^{-1} \sum_{s=t+1}^{\infty} \beta_s W_{Us}(U_{cs}c_s + U_{ns}n_s)}_{W_{Ut+1}U_{ct+1}R_{t+1}a_{t+1} \to W_U U_c Ra},$$

converges as well, to some limit A,

$$\begin{aligned} A_{t+1} &\to \frac{\beta_U}{\beta}(U_c c + U_n n) + \frac{\beta_V}{\beta} W_U U_c Ra \\ &= \left(\frac{1-\beta}{W_U}\beta_U + \beta_V\right)\frac{1}{\beta} W_U U_c Ra = \frac{\bar{\beta}'(V)}{\beta} W_U U_c Ra \equiv A. \end{aligned} \tag{51}$$

where we defined $\beta_X \equiv W_{VX}$ and $X = U, V$. Similarly, we can show that B_t converges to some finite value B. Taking the limits of quantities in the first order conditions above, we thus find a system of equations for multipliers ν_t, μ, λ_t,

$$-\nu_t + \nu_{t+1} + \mu A = 0 \tag{52a}$$

$$-\nu_t + \mu\left(1 + \frac{U_{cc}c}{U_c} + \frac{U_{nc}n}{U_c}\right) + \mu\frac{B}{W_U} = \lambda_t \frac{1}{W_U U_c} \tag{52b}$$

$$-\nu_t + \mu\left(1 + \frac{U_{cn}c}{U_n} + \frac{U_{nn}n}{U_n}\right) + \mu\frac{B}{W_U} = -\lambda_t \frac{f_n}{W_U U_n} \tag{52c}$$

$$-\lambda_t + \lambda_{t+1}\beta f_k = 0. \tag{52d}$$

Substituting out λ_t from (52d) using (52a) and (52b), we find

$$\beta f_k - 1 = \frac{\lambda_t}{\lambda_{t+1}} - 1 = -\frac{W_U U_c}{\lambda_{t+1}}\mu A. \tag{53}$$

We now move to the two main results of this section. First, we show that steady state capital taxes are zero. Second, we show that steady state labor taxes are also zero, unless $\bar{\beta}'(V) = 0$, when preferences are locally additive separable.

Lemma 14. *At an interior steady state, capital taxes are zero, i.e.* $\beta f_k = 1$.

Proof. If $A = 0$ or $\mu = 0$ the result is immediate from (53). Suppose instead that $A \neq 0$ and $\mu \neq 0$. Then, (52a) implies that ν_t and hence λ_t diverges to $+\infty$ or $-\infty$. Then again, $\beta f_k = 1$ follows from (53).[64] □

We move to our second result.

Lemma 15. *At an interior steady state, labor taxes are zero, i.e.* $\tau^n \equiv 1 + \frac{U_n}{U_c f_n} = 0$ *if* $\bar{\beta}'(V) \neq 0$ *and* $a > 0$.

[64]Notice that $\lambda_t \to 0$ requires $\mu = 0$ by (54), so the optimal allocation is first best to begin with, implying $\beta f_k = 1$.

Proof. By combining equations (52b) and (52c) we find an expression for τ^n,

$$\lambda_t \tau^n = \mu \frac{W_U U_n}{f_n} \left(\frac{U_{cc} c}{U_c} + \frac{U_{nc} n}{U_c} - \frac{U_{cn} c}{U_n} - \frac{U_{nn} n}{U_n} \right), \tag{54}$$

Note that by normality of consumption and labor the term in brackets is negative, $\frac{U_{cc} c}{U_c} + \frac{U_{nc} n}{U_c} - \frac{U_{cn} c}{U_n} - \frac{U_{nn} n}{U_n} < 0$. It is immediate from (54) that $\tau^n = 0$ if λ_t diverges to either $+\infty$ or $-\infty$.[65] Suppose $\lambda_t \to \lambda \in \mathbb{R}$. We distinguish whether $\mu = 0$ or $\mu \neq 0$. If $\mu = 0$, the economy was first best to start with, and the labor tax must be zero at any date, including at the steady state. If $\mu \neq 0$, convergence of λ_t (equivalent to convergence of ν_t) necessitates that $A = 0$, using (52a). But then (51) contradicts our assumptions that preferences are not locally additively separable, $\bar{\beta}'(V) \neq 0$, and steady state assets are positive $a > 0$. □

I Proof of Proposition 7

In this section, we prove Proposition 7. The proof is organized as follows. In Section I.1 we introduce the planning problem, derive and discuss the first order conditions, and define the largest feasible level of initial government debt $\bar{b}$. Section I.2 then focuses on parts A and B (i) of Proposition 7. Finally, Section I.3 proves the bang-bang property and parts B (ii) and C of Proposition 7.

I.1 Planning problem and first order conditions

As in the statement of the proposition, we fix some positive initial level of capital $k_0 > 0$. The problem under scrutiny is

$$V(b_0) \equiv \max_{\{c_t, n_t, k_t, r_t\}} \int_0^\infty e^{-\rho t} \left(u(c_t) - v(n_t) \right) dt \tag{55a}$$

$$\dot{c}_t = c_t \frac{1}{\sigma} (r_t - \rho) \tag{55b}$$

$$c_t + g + \dot{k}_t = f(k_t, n_t) - \delta k_t \tag{55c}$$

$$\int_0^\infty e^{-\rho t} \left(u'(c_t) c_t - v'(n_t) n_t \right) dt \geq u'(c_0)(k_0 + b_0) \tag{55d}$$

$$c_t > 0, n_t \geq 0, k_t \geq 0, r_t \geq 0$$

where recall that $u(c) = c^{1-\sigma}/(1-\sigma)$ and $v(n) = n^{1+\zeta}/(1+\zeta)$, $\zeta > 0$. In the entire analysis in this section, we write value functions such as $V(b_0)$ without explicit reference to k_0 since we treat k_0 as fixed. The current-value Hamiltonian of this optimal control problem

[65] Since $A_t \to A \neq 0$ and μ is constant over time, ν_t and thus also λ_t have a well-defined limit in $[-\infty, \infty]$.

with subsidiary condition (55d) (see, e.g., Gelfand and Fomin, 2000) can be written as

$$H(c,k;n,r;\lambda,\eta,\mu) = \Phi_u u(c) - \Phi_v v(n) + \eta c\frac{1}{\sigma}(r-\rho) + \lambda\left(f(k,n) - \delta k - c - g\right), \tag{56}$$

where we defined $\Phi_v \equiv 1 + \mu(1+\zeta)$ and $\Phi_u \equiv 1 + \mu(1-\sigma)$ with μ being the multiplier on the IC constraint; and where we denoted the costates of consumption and capital by η_t and λ_t, respectively. Notice that $\eta_t \leq 0$ or else $r_t = \infty$ were optimal, violating the resource constraint. Problem (55a) implies the following first order conditions for the controls $\{n_t, r_t\}$,

$$\Phi_v v'(n_t) = \lambda_t f_n(k_t, n_t) \tag{57a}$$

$$r_t \begin{cases} = 0 & \text{if } \eta_t < 0 \\ \in [0,\infty) & \text{if } \eta_t = 0, \end{cases} \tag{57b}$$

the following laws of motion for the costates,

$$\dot{\eta}_t - \rho\eta_t = \eta_t\frac{\rho}{\sigma} + \lambda_t - \Phi_u u'(c_t) \tag{57c}$$

$$\dot{\lambda}_t = (\rho - r_t^*)\lambda_t \tag{57d}$$

and the following optimality condition for the initial state of consumption c_0,

$$\eta_0 = -\mu\sigma c_0^{-\sigma-1}(k_0 + b_0). \tag{57e}$$

In equation (57d) we defined the before-tax return on capital as $r_t^* = f_k(k_t, n_t) - \delta$. The conditions (57a)–(57e), together with the constraints (55b)–(55d) and the two transversality conditions

$$\lim_{t\to 0} e^{-\rho t}\lambda_t k_t = 0 \tag{57f}$$

$$\lim_{t\to 0} e^{-\rho t}\eta_t c_t = 0 \tag{57g}$$

are sufficient for an optimum *if* we are able to establish that the planning problem (55a) is a concave maximization problem, or can be transformed into one using variable transformations.

The first order conditions (57a)–(57e) (though not the transversality conditions (57f) and (57g)) are necessary at an optimum since interiority is ensured by the imposition of Inada conditions; that is, with the exception when that optimum is also maximizing the subsidiary constraint, which is the IC constraint in our case (see Gelfand and Fomin, 2000). More specifically, the above first order conditions are not necessary when the optimum to (55a) achieves the supremum in

$$\overline{b} \equiv \sup_{\{c_t,n_t,k_t\}} \frac{1}{u'(c_0)}\int_0^\infty e^{-\rho t}\left(u'(c_t)c_t - v'(n_t)n_t\right)dt - k_0 \tag{58}$$

subject to the two other constraints (55b) and (55c). We deliberately formulated (58) in a way to define $\bar{b}$ as the highest level of b_0 for which there can possibly exist a feasible allocation. Notice that $\bar{b} \in [-\infty, \infty]$, allowing for $\bar{b} = -\infty$ if no feasible allocation exists at all (which might happen if g is very large), and $\bar{b} = \infty$ if there exists a feasible allocation for any value of b_0.

Since in the case that $b_0 = \bar{b}$ the supremum in (58) is attained, there are still necessary first order conditions the allocation satisfies, namely the ones corresponding to (58). These are exactly the same as (57a)–(57e) after substituting $\mu\eta_t$ for η_t and $\mu\lambda_t$ for λ_t, and then dividing by μ and setting $\mu = \infty$. This replaces Φ_u by $(1-\sigma)$ and Φ_v by $(1+\zeta)$ in (57a)–(57c), leaves (57d) unchanged and alters (57e) to $\eta_0 = -\sigma c_0^{-\sigma-1}(k_0 + b_0)$.

One additional remark about the setup in (55a) is in place. We stated an inequality IC constraint (55d), corresponding to a non-negative multiplier μ. This is without loss of generality in our setup, since at any optimum, μ will indeed be non-negative: From the first order condition (57e), we see that our assumption of positive initial private wealth, $k_0 + b_0 > 0$, together with the non-positivity of η_0 means that $\mu \geq 0$.

I.2 Proof of parts A and B (i)

Our proof in this subsection proceeds in three steps. First, we characterize the space of solutions to a *restricted* planning problem, in which the length T of capital taxation is restricted to be infinity. Then we use these insights to prove that $T = \infty$ is in fact optimal in the *unrestricted* planning problem for levels of initial debt $b_0 \in [\underline{b}, \bar{b}]$ (with non-empty interior if $\sigma > 1$). Finally, we show that for all $b_0 < \bar{b}$ there are feasible policies with $T < \infty$. Throughout, we assume that $\sigma \geq 1$, as is assumed in parts A and B (i) of Proposition 7.

1^{st} **step: The restricted problem.** We start by studying a restricted planning problem, where we restrict ourselves to the case of indefinite capital taxation (at its upper bound). Effectively, this implies that $r_t = 0$ for all t and the path of c_t is entirely characterized by c_0 and (55b). To characterize this restricted problem, it will prove useful to define the minimum discounted sum of labor disutilities, henceforth *effective disutility from labor*, needed to sustain this path $\{c_t\}$ as

$$\tilde{v}(c_0) \equiv \min_{\{n_t, k_t\}} \int_0^\infty e^{-\rho t} v(n_t) dt \tag{59a}$$

$$\text{s.t. } c_0 e^{-\rho/\sigma t} + g + \dot{k}_t = f(k_t, n_t) - \delta k_t. \tag{59b}$$

We prove important properties of the effective disutility $\tilde{v}$ and the optimal control problem (59a) in Lemma 16.

Lemma 16. *The function $\tilde{v} : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly convex, strictly increasing and continuously differentiable at any $c_0 \in \mathbb{R}_{++}$. It satisfies $\tilde{v}(0) > 0$. Moreover, for any $c_0 \geq 0$, there exists a unique solution $\{n_t^\infty, k_t^\infty\}$ and a costate of capital $\{\lambda_t^\infty\}$. Upon defining $c_t^\infty = c_0 e^{-\rho/\sigma t}$ it holds* *that,* $\{c_t^\infty, n_t^\infty, k_t^\infty, \lambda_t^\infty\} \to (c^\infty, n^\infty, k^\infty, \lambda^\infty)$*, where*

$$
\begin{aligned}
c^\infty &= 0 \\
f(k^\infty, n^\infty) &= \delta k^\infty + g && \text{(60a)} \\
f_k(k^\infty, n^\infty) &= \rho + \delta && \text{(60b)} \\
v'(n^\infty) &= \lambda^\infty f_n(k^\infty, n^\infty). && \text{(60c)}
\end{aligned}
$$

In particular, the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda_t^\infty k_t^\infty = 0$ *holds and*

$$\tilde{v}'(c_0) = \int_0^\infty e^{-(\rho+\rho/\sigma)t}\lambda_t dt. \tag{60d}$$

Proof. The proof has 4 steps: First, we prove existence and uniqueness of the solution to a "bounded" version of the optimal control problem (59a) with bounds on n_t and k_t. Second, we characterize the optimal paths (k_t^∞, n_t^∞) of this problem. Third, we show that increasing the bounds on k_t and n_t makes the bounded problem equivalent to (59a). Finally, we establish that the claimed properties of $\tilde{v}$.

First step. For this step, relax the constraint (59b) to be an inequality "$\leq$" and introduce upper bounds $\overline{k} > 0$ and $\overline{n} > 0$ on k and n. Using the definition of k^∞ and n^∞ in (60a)–(60b), pick $\overline{k} > \max\{k_0, k^\infty\}$ and pick $\overline{n}$ large enough so that $\overline{n} > n^\infty$ and so that $\dot{k}_0 > 0$ is feasible at time $t = 0$.[66] This means the problem is given by

$$
\begin{aligned}
\tilde{v}(c_0) \equiv \min_{\{n_t,k_t\}} &\int_0^\infty e^{-\rho t} v(n_t) dt && \text{(61a)} \\
\text{s.t. } & c_0 e^{-\rho/\sigma t} + g + \dot{k}_t \leq f(k_t, n_t) - \delta k_t \\
& k_t \in [0, \overline{k}],\ n_t \in [0, \overline{n}].
\end{aligned}
$$

This problem is clearly a strictly convex minimization problem (strictly convex objective and a convex constraint), even without bounds on k and n, and therefore at most admits a single solution. A straightforward application of Seierstad and Sydsaeter (1987, Section 3.7, Theorem 15) to the optimal control problem (61a) reveals that there always exist paths $\{n_t^\infty, k_t^\infty\}$ that attain the minimum in (61a).[67]

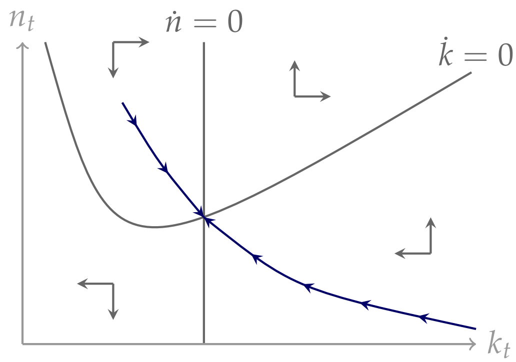
Second step. We now study the long-run properties of the solution to the problem (61a). Before we dive into the details, we note that $k^\infty > 0$ and $n^\infty > 0$ are uniquely determined by (60a) and (60b) due to the Inada properties of $f_k(\cdot, n)$ and the fact that $f/k \geq f_k$. λ^∞ follows from (60c). At each point where $k_t < \overline{k}$ and $n_t < \overline{n}$, the necessary first order conditions corresponding to (61a) are given by

$$
\begin{aligned}
v'(n_t) &= \lambda_t f_n(k_t, n_t) && \text{(62a)} \\
\dot{\lambda}_t &= \lambda_t(\rho - r_t^*), && \text{(62b)}
\end{aligned}
$$

[66] $\dot{k}_0 > 0$ iff $f(k_0, \overline{n}) - \delta k_0 - g - c_0 > 0$.

[67] This relies on our choice of $\overline{n}$ which ensures that $\dot{k}_0 > 0$, so even for low values of k_0 there exist admissible paths $\{n_t, k_t\}$.

Figure 6: Phase diagram characterizing the solution to the restricted problem (59a).



where λ_t denotes the costate of k_t. Notice that n_t is continuous, as an immediate consequence of (62a) and of the fact that both k_t and λ_t are continuous. Also note that (62a) implies $\lambda_t \geq 0$, meaning our relaxation of the resource constraint (59b) to an inequality was without loss of generality. Using the resource constraint (59b) and (62a)–(62b), we can derive an ODE system entirely in terms of n_t and k_t, consisting of the resource constraint (59b) itself and of

$$(\zeta + \alpha_t)\frac{\dot{n}_t}{n_t} = \rho + (1-\alpha_t)\delta - \alpha_t \frac{g + c_t}{k_t},$$

where $\alpha_t = \alpha(k_t/n_t) \equiv \frac{\partial \log f_n}{\partial \log(k_t/n_t)}$. We can also abbreviate the ODEs as $\dot{k} = \dot{k}(k, n, c_t)$ and $\dot{n} = \dot{n}(k, n, c_t)$. Define the two sets

$$\mathcal{A}_t \equiv \{(k,n) \,|\, \dot{n}(k,n,c_t) > 0,\, \dot{k}(k,n,c_t) > 0\}$$
$$\mathcal{B}_t \equiv \{(k,n) \,|\, \dot{n}(k,n,c_t) < 0,\, \dot{k}(k,n,c_t) < 0\}.$$

To illustrate these sets, note that for large t, $c_t \approx 0$, we can draw the phase diagram that corresponds to the ODE system. This is done in Figure 6 for the Cobb-Douglas case where $\alpha_t = const$. In that figure, $\mathcal{A}_t$ is the top right area, while $\mathcal{B}_t$ is the bottom left area. We now argue that the state (k_t, n_t) can never be in $\mathcal{A}_t$ for any t, and never be in $\mathcal{B}_t$ for large t. If for any t, $(k_t, n_t) \in \mathcal{A}_t$, n_t can be lowered to achieve $\dot{k}_t = 0$ at all times, clearly improving the objective. If there does not exist a time s such that $(k_t, n_t) \notin \mathcal{B}_t$ for $t > s$, then it must be that asymptotically $(k_t, n_t) \in \mathcal{B}_t$ for all sufficiently large t. But in that case, $k_t \to 0$, contradicting feasibility (since government spending is positive, $g > 0$). Therefore, it must be that $(k_t^\infty, n_t^\infty) \to (k^\infty, n^\infty)$.

Note that the optimal costate λ_t^∞ can be computed using the first order condition for labor, (62a). Due to the steady state convergence of the system, the transversality condition $\lim_{t\to\infty} e^{-\rho t}\lambda_t^\infty k_t^\infty = 0$ naturally holds.

Third step. We now show that there exists a sufficiently large bound $\overline{n}$ such that the solutions of the problem without bounds, (59a) and the problem with bounds (61a) coincide. This is the case if there exists a $\overline{n}$ such that $n_t^\infty < \overline{n}$ at the optimum at all times t. Assume the contrary held, that is, no matter how large $\overline{n}$ is, at the corresponding optimal path, which we denote by $(k_t^\infty(\overline{n}), n_t^\infty(\overline{n}))$ to emphasize the dependence on $\overline{n}$, there exist times t where $n_t^\infty(\overline{n}) = \overline{n}$. Since $n_t^\infty(\overline{n})$ can never approach $\overline{n}$ from below (this would require $(k_t, n_t) \in \mathcal{A}_t$), it must be that there exists a time $s > 0$ such that $n_t^\infty(\overline{n}) = \overline{n}$ for any $t \in [0, s]$ and any arbitrarily large $\overline{n}$. It is straightforward to see that this lets $k_s^\infty(\overline{n})$ grow unboundedly large, in particular leading to $(k_s^\infty(\overline{n}), n_s^\infty(\overline{n})) \in \mathcal{A}_s$—a contradiction. This completes our proof that problem (59a) admits a unique solution, which approaches the steady state (k^∞, n^∞) asymptotically.

Fourth step. In our final step, we derive the claimed properties of $\tilde{v}$. First, since the objective is strictly convex, $\tilde{v}$ is strictly convex. It is also strictly increasing since the constraint tightens with larger c_0. $\tilde{v}(0) > 0$ follows directly from $g > 0$. For differentiability, pick any $\hat{c}_0 \in \mathbb{R}_{++}$ and denote the associated optimal path for capital by $\{\hat{k}_t^\infty\}$. Following the logic in Benveniste and Scheinkman (1979) we can define a strictly convex and differentiable function $w(c_0) = \int_0^\infty e^{-\rho t} \frac{1}{1+\zeta} N\left(\hat{k}_t^\infty, c_0 e^{-\rho/\sigma t} + g + \dot{\hat{k}}_t^\infty + \delta \hat{k}_t^\infty\right)^{1+\zeta} dt$ where $N(k, y) \equiv f(k, \cdot)^{-1}(y)$ is the level of labor needed to fund output $y \geq 0$ given capital $k > 0$. By contsruction, $w(\hat{c}_0) = v(\hat{c}_0)$ and $w(c_0) \geq v(c_0)$ locally around $\hat{c}_0$.[68] This implies that $\tilde{v}$ is differentiable at any $c_0 \in \mathbb{R}_{++}$ with derivative[69]

$$\tilde{v}'(c_0) = \int_0^\infty e^{-(\rho+\rho/\sigma)t} \frac{v'(n_t^\infty)}{f_n(k_t^\infty, n_t^\infty)} dt. \tag{63}$$

This formula for the derivative of $\tilde{v}$ is equivalent to (60d) after expressing the former in terms of λ_t using the first order condition for labor (62a). This concludes our proof of Lemma 16. □

The effective disutility $\tilde{v}(c_0)$ is convenient since in the original planning problem (55a), labor disutility appears in present value terms both in the objective as well as in the IC constraint (55d). Moreover, due to the assumption of power disutility, both present values are essentially $\tilde{v}(c_0)$ up to a constant factor. The restricted version of the original planning problem (55a) can now be simply written as restricted problem

$$V_\infty(b_0) \equiv \max_{c_0 > 0} u(c_0)\frac{\sigma}{\rho} - \tilde{v}(c_0) \tag{64a}$$

$$c_0^{1-\sigma}\frac{\sigma}{\rho} - (1+\zeta)\tilde{v}(c_0) \geq c_0^{-\sigma}(k_0 + b_0). \tag{64b}$$

We obtained (64a) from the original problem (55a) by requiring that $T = \infty$ and using the definition of $\tilde{v}$. We characterize the restricted problem (64a) in the following lemma.

Lemma 17. *There exists a level of initial debt $\overline{b}^r \in \mathbb{R}$ such that a solution to the restricted planner's problem (64a) exists if and only if $b_0 \leq \overline{b}^r$. For each $b_0 \leq \overline{b}^r$, there is a unique optimum $c_0^\infty(b_0) \in \mathbb{R}_{++}$ and for each $b_0 < \overline{b}^r$ there is a unique multiplier $\mu^\infty(b_0) \in [0, \infty)$ on the IC constraint (64b) such that*

$$\Phi_u u'(c_0)\frac{\sigma}{\rho} - \Phi_v \tilde{v}'(c_0) = -\sigma \mu c_0^{-\sigma-1}(k_0 + b_0), \tag{65}$$

[68]The expression for w is obtained by substituting the resource constraint (59b) into the objective (59a).

[69]Notice that the derivative must be finite since $\tilde{v}$ is strictly convex and finite-valued for any $c_0 \in \mathbb{R}_+$.

for $c_0 = c_0^\infty(b_0)$*,* $\mu = \mu^\infty(b_0)$*. Finally, there exists some* $b^* < \overline{b}^r$ *such that* $\mu^\infty : [b^*, \overline{b}^r) \to [0, \infty)$ *is a continuous and strictly increasing bijection.*

Proof. First, notice that the IC constraint of the restricted planning problem, (64b), can be rewritten as

$$c_0 \frac{\sigma}{\rho} - (1+\zeta) c_0^\sigma \tilde{v}(c_0) \geq k_0 + b_0. \tag{66}$$

Observe that this is a convex constraint, as its left hand side is strictly concave. It is also strictly increasing at $c_0 = 0$ and diverges to $-\infty$ for large c_0.[70] Therefore, there exists an interior maximum at some $\bar{c} > 0$. By definition, $c_0 = \bar{c}$ is the only value that is compatible with the IC constraint if $b_0 = \overline{b}^r$, where we defined

$$\overline{b}^r \equiv \max_{c_0 > 0} c_0 \frac{\sigma}{\rho} - (1+\zeta) c_0^\sigma \tilde{v}(c_0) - k_0. \tag{67}$$

The maximizer $\bar{c}$ is then characterized by the first order conditions

$$\frac{\sigma}{\rho} \bar{c}^{-\sigma} = (1+\zeta)\sigma \bar{c}^{-1} \tilde{v}(\bar{c}) + (1+\zeta) \tilde{v}'(\bar{c}). \tag{68}$$

For any $b_0 > \overline{b}^r$ the set of feasible c_0 compatible with the IC constraint (64b) is empty, so the restricted planning problem (64a) has a solution precisely when $b_0 \leq \overline{b}^r$.

An advantage of writing the IC constraint as in (66) is that it allows us to see that the restricted problem (64a) has a strictly concave objective with a convex and bounded constraint set. The objective attains its unconstrained maximum at some $c^* \in (0, \infty)$ satisfying $u'(c^*) \frac{\sigma}{\rho} = \tilde{v}'(c^*)$. We can show that $c^* > \bar{c}$ since the objective is increasing at $\bar{c}$,

$$u'(\bar{c}) \frac{\sigma}{\rho} - \tilde{v}'(\bar{c}) = (1+\zeta)\sigma \bar{c}^{-1} \tilde{v}(\bar{c}) + \zeta \tilde{v}'(\bar{c}) > 0,$$

where we used the first order condition for $\bar{c}$, (68). Define $b^* \equiv c^* \frac{\sigma}{\rho} - (1+\zeta) c^* \tilde{v}(c^*) - k_0$, so that c^* lies in the constraint set (64b) if and only if $b_0 \leq b^*$—or in other words, the constraint holds with equality for any $b_0 \geq b^*$. We next show that there exists (a) a strictly decreasing (and hence continuous) bijection $c^\infty : [b^*, \overline{b}^r) \to (\bar{c}, c^*]$ and (b) a strictly increasing (and hence continuous) bijection $\mu^\infty : [b^*, \overline{b}^r) \to [0, \infty)$ such that $c^\infty(b_0)$ is the unique solution to the strictly concave problem (64a), and constraint (64b) has Lagrange multiplier $\mu^\infty(b_0)$, for any $b_0 \in [b^*, \overline{b}^r)$.

Take any $c_0 \in (\bar{c}, c^*]$. Clearly, c_0 is optimal with Lagrange multiplier μ when initial debt is b_0 if the three objects c_0, μ, b_0 satisfy the first order condition of the problem—which can easily be seen to be given by (65)—and the constraint (64b). By substituting out b_0 from (65) using the constraint, the first order condition can be expressed

[70] Note that for $\sigma = 1$, (66) reads $c_0(\frac{\sigma}{\rho} - (1+\zeta)\tilde{v}(c_0)) \geq k_0 + b_0$ and by positivity of $k_0 + b_0$ and monotonicity of $\tilde{v}$, this means that $\frac{\sigma}{\rho} - (1+\zeta)\tilde{v}(0) > 0$ (which is exactly equal to the derivative of the left hand side of (66) at $c_0 = 0$).

as function of μ,

$$\mu = \frac{\frac{\sigma}{\rho} - c_0^{\sigma}\tilde{v}'(c_0)}{(1+\zeta)\sigma c_0^{\sigma-1}\tilde{v} + (1+\zeta)c_0^{\sigma}\tilde{v}'(c_0) - \sigma/\rho} \equiv M(c_0).$$

For $c_0 \in (\bar{c}, c^*]$, the denominator is positive and strictly increasing in c_0, approaching 0 for $c_0 \searrow \bar{c}$; while the numerator is strictly decreasing and non-negative, with a zero at $c_0 = c^*$. This defines a strictly decreasing bijection $M : (\bar{c}, c^*] \to [0, \infty)$. From the constraint (64b), we see that

$$b_0 = c_0\frac{\sigma}{\rho} - (1+\zeta)c_0^{\sigma}\tilde{v}(c_0) - k_0 \equiv B(c_0)$$

which, by definition of $\overline{b}^r$ and $\bar{c}$, defines a strictly decreasing bijection $B : (\bar{c}, c^*] \to [\tilde{b}, \overline{b}^r)$. It follows that for any $b_0 \in [\tilde{b}, \overline{b}^r)$, the unique solution to (64a) is given by $c^{\infty}(b_0) = B^{-1}(b_0)$, with associated multiplier $\mu^{\infty}(b_0) = M(B^{-1}(b_0))$. This concludes the proof. □

We finished our characterization of the restricted planning problem and are now ready for the second and main part of the proof of Proposition 7.

2^{nd} **step: Optimality of $T = \infty$ in the unrestricted problem.** Before we proceed to prove the optimality of $T = \infty$ in the unrestricted problem, we establish that $\overline{b}^r$ is not just the upper bound of possible initial debt in the restricted planning problem, but equal to $\overline{b}$, the one in the unrestricted planning problem (55a).

Lemma 18. *Let $b_0 \in \mathbb{R}$ and $\sigma \geq 1$. The constraints (55b), (55c), (55d) define a non-empty set for $\{c_t, n_t, k_t, r_t\}$ if and only if $b_0 \leq \overline{b}^r$. In particular, $\overline{b} = \overline{b}^r$. Moreover, if $b_0 = \overline{b}^r$ then capital is necessarily taxed at the maximum, $T = \infty$.*

Proof. It suffices to show that the constraint set in the original problem is empty for $b_0 > \overline{b}^r$, and that $T = \infty$ is necessary for $b_0 = \overline{b}^r$. We show both by proving that any $b_0 \geq \overline{b}^r$ is infeasible with if capital is not taxed at its upper bound in all periods.

Hence fix some $b_0 \geq \overline{b}^r$ and assume it was achievable without $T = \infty$ by $\{c_t, n_t, k_t, r_t\}$. Then, it must be that $r_t > 0$ on some non-trivial interval, and the path of consumption is described by the Euler equation (55b), as always. Let the initial consumption value be c_0 and denote by $\hat{c}_t$ the path which starts at the same initial consumption $\hat{c}_0 = c_0$ but keeps falling at the fastest possible rate $-\rho/\sigma$ forever, corresponding to $T = \infty$. Similarly, define by $\hat{n}_t$ the path for labor which keeps k_t fixed but satisfies the resource constraint with consumption equal to $\hat{c}_t$. Clearly, $\hat{n}_t \leq n_t$ for all t and $\hat{n}_t < n_t$ on a positive-measure set of times t. Because the left hand side of (55d) is weakly decreasing in c_t and strictly decreasing in n_t, this strictly relaxes the IC constraint. Hence,

$$\int_0^{\infty} e^{-\rho t}\hat{c}_t^{1-\sigma}dt - \int_0^{\infty} e^{-\rho t}v(\hat{n}_t)dt > \hat{c}_0^{-\sigma}(k_0 + b_0).$$

Notice, however, that for $T = \infty$, we can do even better by optimizing over labor (not necessarily keeping capital constant, see (59a)), leading to

$$\hat{c}_0^{1-\sigma}\frac{\sigma}{\rho} - (1+\zeta)\tilde{v}(\hat{c}_0) > \hat{c}_0^{-\sigma}(k_0 + b_0).$$

By definition of $\overline{b}^r$ in (67) this is a contradiction to $b_0 \geq \overline{b}^r$. Therefore, $\overline{b}^r$ is equal to the highest sustainable debt level in the original problem, $\overline{b}$, and can only be achieved with $T = \infty$. □

Our next lemma establishes that the unrestricted problem (55a) is a strictly concave maximization problem with convex constraints. This will be helpful when proving uniqueness in Lemma 20 below.

Lemma 19. *Suppose $\sigma \geq 1$. The unrestricted problem (55a) can be transformed into a strictly concave maximization problem with convex constraints, using variable substitution. Therefore, any optimum of (55a) is unique when $\sigma \geq 1$.*

Proof. We rewrite (55a) in terms of the two variables $u_t \equiv u(c_t) \in (-\infty, 0)$ and $v_t \equiv v(n_t) \in [0, \infty)$ instead of c_t and n_t. We only consider the case $\sigma > 1$; the case $\sigma = 1$ is analogous. For $\sigma > 1$, the substitution yields

$$V(b_0) \equiv \max_{\{u_t, v_t, k_t\}} \int_0^\infty e^{-\rho t}(u_t - v_t)\,dt \tag{69}$$
$$\dot{u}_t \geq (\sigma - 1)\frac{\rho}{\sigma}u_t$$
$$((1-\sigma)u_t)^{-1/(\sigma-1)} + g + \dot{k}_t \leq f\left(k_t, ((1+\zeta)v_t)^{1/(1+\zeta)}\right) - \delta k_t$$
$$\int_0^\infty e^{-\rho t}((1-\sigma)u_t - (1+\zeta)v_t)\,dt \geq ((1-\sigma)u_0)^{\sigma/(\sigma-1)}(k_0 + b_0)$$
$$u_t < 0, v_t \geq 0, k_t > 0.$$

We made two additional simplifications in (69): We incorporated the inequality for the control $r_t \geq 0$ in the Euler equation constraint (55b); and the (strictly convex) resource constraint was relaxed to be an inequality, which is without loss of generality since by (57a) we know that its Lagrange multiplier, the costate of capital λ_t, is necessarily positive at any optimum. Since the resource constraint binds and is strictly convex, all other constraints in (69) are also convex and the objective is linear, this planning problem can at most have a single solution. And, (57a)–(57e), (57f), (57g), (55b)–(55d) are sufficient conditions to find this solution. □

Our next lemma finally establishes the optimality of $T = \infty$ in the unrestricted problem (55a).

Lemma 20. *Suppose $\sigma > 1$ and define $\underline{b} \equiv (\mu^\infty)^{-1}\left(\frac{1}{\sigma-1}\right)$ with μ^∞ as in Lemma 17. Indefinite capital taxation is optimal in the Chamley problem (55a) if and only if $b_0 \in [\underline{b}, \overline{b}]$.*

Proof. As a consequence of Lemma 19, the unrestricted planning problem (55a) can be transformed into a strictly concave maximization problem with convex constraints. This implies that the first order conditions (57a)–(57e), together with transversality conditions (57f), (57g), and constraints (55b)–(55d) are in fact sufficient to characterize the unique optimum of the unrestricted planning problem (55a). In this proof we guess a solution and verify the sufficient conditions in a first step. In a second step, we prove that any $b_0 < \underline{b}$ does not imply positive long run capital taxation, where $T < \infty$. Throughout the proof, we focus on $b_0 < \bar{b}$ since we know from Lemma 18 that initial debt of $\bar{b}$ requires indefinite capital taxation.

First step: Let $b_0 \in [\underline{b}, \bar{b})$. We now construct an allocation $\{c_t, n_t, k_t, r_t\}$ and multipliers $\{\lambda_t, \eta_t\}, \mu$ that satisfy all the sufficient conditions. We define $c_0 \equiv c^\infty(b_0)$ as in Lemma 17; given c_0, $\{c_t, n_t, k_t\} \equiv \{c_t^\infty, n_t^\infty, k_t^\infty\}$ and $\lambda_t \equiv \Phi_v \cdot \lambda_t^\infty$ with notation as in Lemma 16; $\mu \equiv \mu^\infty(b_0)$ as in Lemma 17; $\eta_0 \equiv \Phi_u u'(c_0)\frac{\sigma}{\rho} - \Phi_v \tilde{v}'(c_0)$ (which is negative since $\Phi_u \leq 0$ by construction of μ) and η_t as solution to the ODE (57c) with initial condition η_0. The first order conditions (57a)–(57d) are satisfied by construction and by the fact that the allocation $\{n_t^\infty, k_t^\infty, \lambda_t^\infty\}$ satisfies (62a) and (62b). The first order condition for initial consumption (57e) is equivalent to (65) in Lemma 17. The Euler equation constraint (55b) is trivially satisfied by construction of $\{c_t\}$. The resource constraint holds for $\{c_t^\infty, n_t^\infty, k_t^\infty\}$ (see (59b) and Lemma 16) and therefore also for $\{c_t, n_t, k_t\}$. Due to the fact that $\{n_t^\infty, k_t^\infty\}$ solves (59a) and $c_t = c_0 e^{-\rho/\sigma t}$, the IC constraint (55d) can be seen to be equivalent to (64b) and hence is satisfied since c_0 was chosen to be $c^\infty(b_0)$. Finally, Lemma 16 implies that the transversality condition for capital, (57f), holds. And, concluding the second step, the transversality condition for consumption, (57g), holds since

$$e^{-\rho t}\eta_t c_t = c_0 e^{-(\rho+\rho/\sigma)t}\eta_t = -c_0 \int_t^\infty e^{-(\rho+\frac{\rho}{\sigma})s}\lambda_t dt + c_0 \Phi_u u'(c_0)\frac{\sigma}{\rho}e^{-\frac{\rho}{\sigma}t} \to 0. \tag{70}$$

and by this expression it also follows that $\eta_t < 0$ at all times t. The second equality in (70) builds on an integral version of the law of motion of η_t, which we obtained by combining (57c) with our definition of η_0 as $\Phi_u u'(c_0)\frac{\sigma}{\rho} - \Phi_v \tilde{v}'(c_0)$ and the expression for $\tilde{v}'(c_0)$ in (60d) from Lemma 16. It will become important in the second step below that (70) also reveals the limiting behavior of η_t itself: $\lim_{t\to\infty}\eta_t = -\infty$ but $\lim_{t\to\infty} e^{-\rho t}\eta_t = \Phi_u u'(c_0)\frac{\sigma}{\rho}$.

Second step: We proceed by contradiction. Suppose $b_0 < \underline{b}$ gave rise to indefinite capital taxation (at the maximum rate). Then, reversing the logic of the first step, it must be the case that the allocation $\{c_t, n_t, k_t\}$ is also optimal in the labor disutility minimization problem (59a) with multipliers $\lambda_t^\infty = \frac{1}{\Phi_v}\lambda_t$, given c_0; and c_0 and μ must be optimal given b_0 in the restricted planning problem (64a), that is, $c_0 = c^\infty(b_0)$ and $\mu = \mu^\infty(b_0) < \frac{1}{\sigma-1}$. Since the first order condition (57e) is necessary, it must then be the case that $\eta_0 = \Phi_u u'(c_0)\frac{\sigma}{\rho} - \Phi_v \tilde{v}'(c_0)$ by comparing it to (65). Equation (70) thus holds as in the second step, implying $\lim_{t\to\infty} e^{-\rho t}\eta_t = \Phi_u u'(c_0)\frac{\sigma}{\rho}$ which now is positive since $\Phi_u > 0$, a contradiction to the optimality of capital taxes. □

3^{rd} **step: Feasibility of finite capital taxation for all $b_0 < \bar{b}$.** We now move to the third and last part of this section. Here, we establish:

Lemma 21. *For any initial government debt level $b_0 < \overline{b}$, there are implementable allocations with nonzero capital taxation for only a finite time, $T < \infty$.*

Proof. Fix $b_0 \leq \overline{b}$ and fix the allocation $\{c_t^\infty, n_t^\infty, k_t^\infty\}$ that is optimal among all allocations with indefinite capital tax. By construction, this allocation satisfies the restricted problem (64a). We now explicitly construct an allocation $\{\tilde{c}_t, \tilde{n}_t, \tilde{k}_t\}$ for which there is no capital tax, $\dot{\tilde{c}} = \frac{1}{\sigma}(r_t^* - \rho)\tilde{c}_t$, after time some time $T < \infty$ but that is feasible—satisfying constraints (55b)–(55d)—with initial debt $b_0 - \epsilon$, for $\epsilon > 0$ arbitrarily small. First, we describe the allocation for all times $t \geq T$. Consider

$$
\begin{aligned}
V^{\text{zero tax}}(\hat{k}) \equiv \max_{\{c_t, n_t, k_t\}_{t\geq T}} & \int_T^\infty e^{-\rho(t-T)} \left(u(c_t) - v(n_t)\right) dt \\
\text{s.t. } & c_t + g + \dot{k}_t = f(k_t, n_t) - \delta k_t \\
& k_T = \hat{k} \\
& c_t > 0, n_t \geq 0, k_t \geq 0
\end{aligned}
$$

which is the social planning problem of a standard neoclassical growth model with power utilities in consumption and labor, and a Cobb-Douglas technology (i.e. zero labor and zero capital taxes). It is known that such a model has optimal paths $\{c_t^*, n_t^*, k_t^*\}$ that monotonically converge to a unique positive steady state (c^*, n^*, k^*). This implies that $\{n_t^*\}$ is bounded from above by $\overline{n}(\hat{k}) = \max\{n^*, n(\hat{k})\}$ where $n(\cdot)$ denotes the (continuous) policy function for labor supply. Moreover, the undistorted Euler condition holds along the path for consumption $\{c_t^*\}$. Also, it is well known that the consumption policy function $c(k)$ of this problem is continuous and strictly increasing, with $c(k) > 0$ for any $k > 0$. Fix $\hat{k} \equiv k_T^\infty$. Since k_t^∞ converges to a positive limit $k^\infty > 0$ but $c_t^\infty \to 0$ (see Lemma 16), it is the case for sufficiently large T that $c(\hat{k}) > c_T^\infty$. Focus on such T. Also let $\overline{n} \equiv \sup_t \overline{n}(k_t^\infty) < \infty$ be an upper bound for labor (which by construction is uniform in T). Notice that $\overline{n} < \infty$ since k_t^∞ converges to some $k^\infty > 0$.

Now construct the paths $\{\tilde{c}_t, \tilde{n}_t, \tilde{k}_t\}$ by piecing together $\{c_t^\infty, n_t^\infty, k_t^\infty\}$ for $t < T$ and a zero-tax path $\{c_t^*, n_t^*, k_t^*\}$, starting with $k_T^* = k_T^\infty$, for $t \geq T$. By design, the capital stock is continuous at $t = T$ and consumption jumps upwards at $t = T$.[71] Using this construction, the allocation satisfies the resource constraint at all periods, and the Euler equation with equality for $t > T$. Also,

$$
\begin{aligned}
&\int_0^\infty e^{-\rho t} \left(u'(c_t^\infty)c_t^\infty - v'(n_t^\infty)n_t^\infty\right) dt - \int_0^\infty e^{-\rho t} \left(u'(\tilde{c}_t)\tilde{c}_t - v'(\tilde{n}_t)\tilde{n}_t\right) dt = \\
&\underbrace{\int_T^\infty e^{-\rho t} \left(u'(c_t^\infty)c_t^\infty - u'(c_t^*)c_t^*\right) dt}_{\leq e^{-\rho T} u'(c_T^\infty) c_T^\infty \frac{\sigma}{\rho}} + \underbrace{\int_T^\infty e^{-\rho t} \left(v'(n_t^*)n_t^* - v'(n_t^\infty)n_t^\infty\right) dt}_{\leq e^{-\rho T} \frac{1}{\rho} \overline{n}^{1+\zeta}}. \qquad (71)
\end{aligned}
$$

[71]We think of this as a very high capital subsidy for a very short amount of time (which would definitely not be violating any capital tax constraints). If one prefers to avoid this simple limit case, one could easily smooth out this jump over some very small interval. This makes no difference whatsoever for the argument that follows.

As $e^{-\rho T} u'(c_T^\infty) c_T^\infty \to 0$ both terms in (71) approach zero. This is why for T sufficiently large, $\int_0^\infty e^{-\rho t}\left(u'(\tilde{c}_t)\tilde{c}_t - v'(\tilde{n}_t)\tilde{n}_t\right) dt$ approaches $u'(\tilde{c}_0)(k_0 + b_0)$. Thus, for any $\epsilon > 0$, there exists a T such that the allocation $\{\tilde{c}_t, \tilde{n}_t, \tilde{k}_t\}$ is implementable without capital taxes after time T, for initial debt $b_0 - \epsilon$,

$$\int_0^\infty e^{-\rho t}\left(u'(\tilde{c}_t)\tilde{c}_t - v'(\tilde{n}_t)\tilde{n}_t\right) dt \geq u'(\tilde{c}_0)(k_0 + b_0 - \epsilon)$$

which is what we set out to show. This proves that for any $b_0 < \bar{b}$, there exists a feasible (but not necessarily optimal) path with only a finite period of positive capital taxation. □

Summary This concludes the proof of parts A and B (i) of Proposition 7. For part A, we proved (i) in Lemma 18 , (ii) in Lemma 20 and (iii) in Lemma 21. Part B (i) was shown in Lemma 18.

I.3 Proof of the bang-bang property and parts B (ii) and C

We proceed in three steps. We first establish a transversality condition that is necessary at any optimum (in general, transversality conditions are not necessary). Then, using this transversality condition, we derive the "bang-bang" property of capital taxes. Notice that previous proofs of this property relied on the assumption that indefinite capital taxation is not optimal, which we showed is not the case. The bang-bang property lets us summarize an optimal capital tax plan by the date $T \in [0, \infty]$ at which capital taxes jump from the upper bound $\bar{\tau}$ to zero. In the final step, we prove parts B (ii) and C, that is, $T < \infty$ if either $\sigma < 1$ or $\sigma = 1$ and $b_0 = \bar{b}$.

1^{st} **step: A necessary transversality condition.**

Lemma 22. *Let $\{c_t, n_t, k_t, r_t\}$ be a solution to problem (55a), with multipliers $\{\lambda_t, \eta_t, \mu\}$. If $\exists\, s \geq 0$ such that $c_t = c_s e^{-\rho(t-s)/\sigma}$ for all $t \geq s$, then the transversality condition for consumption (57g) holds.*

Proof. We first establish that under the conditions of the lemma, $\{k_t, n_t\}$ converges to a positive steady state. If $c_t = c_s e^{-\rho(t-s)/\sigma}$, then $\{k_t, n_t\}_{t \geq s}$ must be minimizing the stream of labor disutilities (61a) given initial capital k_s and initial consumption c_s. Therefore, $\{k_t, n_t\} \to \{k^\infty, n^\infty\}$, using the notation from Lemma 16.

Thus, there exists some large enough $\bar{n} > 0$ such that

$$f(k_t, \bar{n}) - \delta k_t - c_t - g > 0 \tag{72}$$

for all t. Since the time t controls maximize the time t Hamiltonian H_t (see (56)), we then have for any $\bar{n}$

$$e^{-\rho t}(\Phi_u u(\hat{c}_t) - \Phi_v v(\bar{n})) + e^{-\rho t}\eta_t \hat{c}_t \frac{1}{\sigma}(-\rho) + e^{-\rho t}\lambda(f(\hat{k}_t, \bar{n}) - \delta\hat{k}_t - \hat{c}_t - g) \leq e^{-\rho t} H_t \to 0 \tag{73}$$

where the left hand side is the present value Hamiltonian with controls $r_t = 0$ and $n_t = \overline{n}$, and the right hand side is the present value Hamiltonian with optimal controls r_t, n_t (both along the optimal path for c_t, k_t). The right hand side converges to zero following Michel (1982). Notice that in (73), $e^{-\rho t}(\Phi_u u(c_t) - \Phi_v v(\overline{n})) \to 0$. Suppose $\liminf_{t\to\infty} e^{-\rho t}\eta_t c_t$ were negative. Then, according to (73) it would have to be that

$$\limsup_{t\to\infty} e^{-\rho t}\lambda(f(k_t,\overline{n}) - \delta k_t - c_t - g) \leq \liminf_{t\to\infty} e^{-\rho t}\eta_t c_t \frac{1}{\sigma}\rho < 0$$

contradicting (72). This means the transversality condition for consumption (57g) holds. □

2^{nd} **step: The bang bang property.** We move to the first main result of this subsection.

Lemma 23. *A solution to problem (55a) is of the form that the capital tax τ_t binds at the upper bound for some time $T \in [0, \infty]$ and is equal to zero thereafter.*

Proof. Let $\{c_t, n_t, k_t, r_t\}$ be an optimal allocation solving (55a), for some initial debt $b_0 \in \mathbb{R}$. Let $\{\lambda_t, \eta_t, \mu\}$ be a set of multipliers such that allocation and multipliers satisfy the necessary first order conditions for the case $b < \overline{b}$, (57a)–(57e) and constraints (55b)–(55d). Our proof is analogous if $b = \overline{b}$. We first show that if $\tau_t < \bar{\tau}$ on some non-trivial interval, then $\tau_t = 0$ on that interval. Second, we prove that $\tau_t = 0$ at all times after that interval as well. The proof utilizes the fact that once $\tau_t = 0$, it must not only be that $r_t^* \geq 0$ at that time (or else the $r_t \geq 0$ constraint would be binding); but also that $r_t^* > 0$ for all future times. We formally prove this fact in Lemma 24 below.

First, suppose $\tau_t < \bar{\tau}$ for some non-trivial interval $t \in [s_0, s_1]$. Then, by (57b), $r_t > 0$ and $\eta_t = 0$ on that interval. Hence, by (57c), $\lambda_t = \Phi_u u'(c_t)$. Taking logs and differentiating implies an undistorted Euler equation of the agent. Therefore, $\tau_t = 0$ for $t \in [s_0, s_1]$. Second, suppose there is a later time where capital taxes are positive, that is $s' \equiv \inf\{t > s_1 \,|\, \eta_t < 0\} < \infty$. Observe that, between $t = s_1$ and $t = s'$, both $u'(c_t)$ and λ_t grow at the common rate $\rho - r_t^*$, so $\lambda_{s'} = \Phi_u u'(c_{s'})$. For any $t > s'$, $u'(c_t)$ still grows at least as fast as λ_t, and, by definition of s', for a positive-measure set of times t after s', $u'(c_t)$ grows at the faster rate $\rho > \rho - r_t^*$ since $\eta_t < 0$ and $\tau_t = \bar{\tau}$ for those t. Therefore, for any $t > s'$, $\Phi_u u'(c_t) > \lambda_t$. By (57c), this means $\dot{\eta}_t < \eta_t\left(\rho + \frac{\rho}{\sigma}\right)$, or in other words, $\eta_t < 0$ and $c_t = c_{s'} e^{-\rho(t-s')/\sigma}$ for $t > s'$. Moreover, $\limsup_{t\to\infty} e^{-\rho t}\eta_t c_t < e^{-\rho s'}\eta_{s'} c_{s'} < 0$, contradicting Lemma 22. This concludes our proof of Lemma 23. □

Lemma 24. *If $\tau_s = 0$ for $s \geq 0$, then $r_s^* \geq 0$ and $r_{s'}^* > 0$ for all $s' > s$.*

Proof. For convenience we introduce $R_t \equiv f_k(k_t, n_t)$. R_t has the following law of motion,

$$(\zeta + \alpha_t)\beta_t^{-1}\frac{\dot{R}_t}{R_t} = \rho + (1+\zeta)\delta + \zeta\frac{g + c_t}{k_t} - \zeta f(1, h(R_t)) - R_t,$$

which was obtained by log-differentiating the first order condition of labor (57a) and combining it with the resource constraint (55c). Here, $\alpha_t \equiv \frac{\partial \log f_n}{\partial(k_t/n_t)}$ as before, $\beta_t \equiv \frac{\partial \log f_k}{\partial(n_t/k_t)}$, and

$h(x) \equiv f_k(1,\cdot)^{-1}(x)$. Observe that $h : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and bijective. Since $\dot{R}$ depends implicitly (through α and β) and explicitly on k_t, R_t, and c_t, we also write $\dot{R}_t = \dot{R}(k, R, c)$.

Our proof of this lemma proceeds in two steps. First, we show an auxiliary result, namely that whenever

$$(k_t, R_t, c_t) \in \mathcal{A} \equiv \{(k, R, c) \,|\, R \le \delta,\ \dot{R}(k, R, c) \le 0\},$$

for some time $t = t_0$, then $(k_t, R_t, c_t) \in \mathcal{A}$ for all later times $t > t_0$ too. Second, we establish the result stated in the lemma.

First step. To prove the auxiliary result, it suffices to consider points (k_t, R_t, c_t) at the boundary of $\mathcal{A}$ and study whether the flows induced by the differential equation point to the inside of $\mathcal{A}$. There are two kinds of boundary points. If $R_t = \delta$, it trivially holds that $\frac{d}{dt}R_t \le \frac{d}{dt}\delta = 0$. Suppose now that $\dot{R}(k_t, R_t, c_t) = 0$ and ask whether $\frac{d}{dt}\dot{R}(k_t, R_t, c_t) \le 0$. Generally, whenever $(k_t, R_t, c_t) \in \mathcal{A}$, it is straightforward to see that

$$\frac{\dot{k}}{k} = \frac{f(k,n)}{k} - \delta - \frac{g + c_t}{k_t} \ge \frac{\rho}{\zeta} > 0. \tag{74}$$

Moreover, $\dot{c}_t = -\frac{\rho}{\sigma}c_t$ since $r_t^* = R_t - \delta \le 0$. The fact that k_t is increasing and c_t is decreasing mean that

$$(\zeta + \alpha_t)\beta_t^{-1}\frac{\frac{d}{dt}\dot{R}_t}{R_t} = \frac{d}{dt}(\zeta + \alpha_t)\beta_t^{-1}\frac{\dot{R}}{R} = \underbrace{\frac{d}{dt}\frac{g + c_t}{k_t}}_{<0} - \underbrace{\frac{d}{dt}\left(\zeta f(1, h(R)) + R\right)}_{=0} < 0$$

establishing the auxiliary result.

Second step. Suppose $\tau_s = 0$ for some $s \ge 0$. The fact that $r_s^* \ge 0$ follows directly from the constraint $(1 - \tau_s)r_s^* = r_s \ge 0$. Let $s' \equiv \inf\{t > s | r_t^* \le 0\}$ and suppose $s' < \infty$. Since r_t^* is continuous and differentiable, this means that $r_{s'}^* = 0$ and $\frac{d}{dt}r_t^*|_{t=s'} \le 0$, or in terms of R_t, $R_{s'} = \delta$ and $\dot{R}_{s'} \le 0$. Applying the auxiliary result, $(k_t, R_t, c_t) \in \mathcal{A}$ for any $t > s'$. Moreover, $k_t \to \infty$ due to (74) at all times $t > s'$. This is in sharp contradiction to Lemma 16 (which applies here using $k_{s'}$ as initial capital stock since $\dot{c}_t = -\frac{\rho}{\sigma}c_t$ for all $t \ge s'$. Therefore $r_t^* > 0$ for all $t > s$. □

3^{rd} step: Finite capital taxation $T < \infty$ in parts B (ii) and C.

Lemma 25. *If either $\sigma < 1$ or $\sigma = 1$ and $b_0 < \bar{b}$, then $T < \infty$.*

Proof. If either $\sigma < 1$ or $\sigma = 1$ and $b_0 < \bar{b}$, then $\Phi_u > 0$ for any $\mu \ge 0$.[72] In the following, we prove that this is incompatible with $T = \infty$. By contradiction, suppose it were the case that there exists an optimal allocation $\{c_t, n_t, k_t, r_t\}$ with $T = \infty$, i.e. $c_t = c_0 e^{-\rho t/\sigma}$. Applying Lemma 22, $(k_t, n_t) \to (k^\infty, n^\infty)$. In particular, $r_t^* \to \rho > 0$ following the definition of (k^∞, n^∞) in Lemma 16. Now, $\Phi_u u'(c_t)$ grows at rate ρ while λ_t only grows at

[72] If $b_0 = \bar{b}$ and $\sigma < 1$, then as we explain in Section I, Φ_u can be taken to be $(1 - \sigma)$, and thus is positive here.

rate $\rho - r_t^* < \rho$. Therefore, there exists some finite time s such that $\lambda_t < \Phi_u u'(c_t)$ for all $t > s$. Using law of motion of η_t, (57c), this means $\dot{\eta}_t < \eta_t \left(\rho + \frac{\rho}{\sigma}\right)$ for all $t > s$ and so $\limsup_{t\to\infty} e^{-\rho t}\eta_t c_t < e^{-\rho s}\eta_s c_s < 0$, contradicting Lemma 22. $\square$

Summary. This concludes our proofs of the bang bang property (Lemma 23) and parts B (ii) and C (Lemma 25).

J Proof of Proposition 8

We proceed by providing an explicit solution to the first order and transversality conditions to problem (55a) with zero government spending and certain combinations of k_0, b_0. We do so in two steps. First, taking k_0 as given, we find paths $\{c_t, n_t, k_t, r_t\}$, $\{\lambda_t, \eta_t\}$, μ and a level of initial debt b_0 which together satisfy all first order conditions, transversality conditions and constraints, with the one exception that η_t need not necessarily be negative. In a second step, we choose k_0 such that $\mu \geq 1/(\sigma - 1)$ which will ensure that $\eta_t < 0$ at all times t.

The reason this construction is analytically tractable is that along the optimum, c_t, n_t, k_t will all fall to zero at the exact same growth rate, which needs to equal $\frac{\rho}{\sigma}$ by the Euler equation (55b). At the same time, $r_t = 0$ (since $T = \infty$). Taken together, to find the solution for a given k_0, it is necessary to find $c_0, n_0, \{\lambda_t, \eta_t\}, \mu, b_0$. Again, we use the previous notation $\Phi_u = 1 + \mu(1-\sigma)$ and $\Phi_v = 1 + \mu(1+\zeta)$.

First step. We conjecture that $c_t = c_0 e^{-\rho/\sigma t}$, $n_t = n_0 e^{-\rho/\sigma t}$, $k_t = k_0 e^{-\rho/\sigma t}$, $r_t = 0$, $\lambda_t = \lambda_0 e^{-\zeta\rho/\sigma t}$. The Euler equation (55b) obviously holds. The resource constraint (55c) is satisfied iff

$$c_0 = f(k_0, n_0) - \delta k_0 + \frac{\rho}{\sigma} k_0. \tag{75}$$

The IC constraint (55d) is satisfied iff

$$b_0 = c_0 \frac{\sigma}{\rho} - \frac{1}{\rho + (1+\zeta)\rho/\sigma} c_0^{\sigma} n_0^{1+\zeta} - k_0. \tag{76}$$

The first order condition for labor (57a) and the costate λ_t (57d) hold iff

$$f_k(k_0, n_0) = \zeta \frac{\rho}{\sigma} + \rho + \delta \tag{77}$$

and

$$\Phi_v n_0^{\zeta} = \lambda_0 f_n(k_0, n_0). \tag{78}$$

Given k_0, (77) pins down n_0, (75) c_0, and (76) b_0. The law of motion of η_t (57c) and the associated transversality condition (57g) are satisfied iff

$$\eta_t = -\frac{\lambda_0}{\rho + (1+\zeta)\rho/\sigma} e^{-\zeta\rho/\sigma t} + \frac{\sigma}{\rho} \Phi_u c_0^{-\sigma} e^{\rho t}. \tag{79}$$

Notice that (57b) holds, i.e.$\eta_t < 0$, as long as $\Phi_u \leq 0$, requiring $\mu \geq \frac{1}{\sigma-1}$. The transversality condition for capital (57f) obviously holds.

It remains to determine λ_0, η_0, and μ subject to (79) (at $t = 0$), $\mu \geq \frac{1}{\sigma-1}$, (78), and the first order condition for c_0 (57e). For expositional reasons, define the initial labor tax as $\tau_0^\ell \equiv 1 - n_0^\zeta c_0^\sigma / w^*$. Then, we can solve for μ as

$$\mu = \frac{\tau_0^\ell + \sigma + \zeta}{\sigma\left((1-\tau_0^\ell)\frac{n_0}{c_0}w^* - 1\right) - \tau_0^\ell(1+\zeta)}. \tag{80}$$

Notice that whenever $\mu \in [\frac{1}{\sigma-1}, \infty)$, $\lambda_0 > 0$ is given by (78) and $\eta_0 < 0$ is given by (79) (at $t = 0$). So the last step in our construction is to determine whether there are levels for k_0 for which $\mu \in [\frac{1}{\sigma-1}, \infty)$.

Second step. The only object on the right hand side of (80) that depends on k_0 is τ_0^ℓ, and τ_0^ℓ is a strictly decreasing function of $k_0 \in [0, \infty)$, with $\tau_0^\ell \to 1$ as $k_0 \to 0$ and $\tau_0^\ell \to -\infty$ as $k_0 \to \infty$. Moreover, μ is increasing in $\tau_0^\ell \in (-\infty, 1]$ and has a pole at $\tau_{0,\text{pole}}^\ell = \frac{\sigma w^* n_0/c_0 - \sigma}{\sigma w^* n_0/c_0 + 1 + \zeta} < 1$, where it rises to $+\infty$ from the left. For $\tau_0^\ell = 1$, $\mu = -1 < 0$. We define $\underline{k}$ to be the value of k_0 corresponding to $\tau_{0,\text{pole}}^\ell$. Putting the mapping from k_0 to τ_0^ℓ and the one from τ_0^ℓ to μ together, we find a function $\mu(k_0)$ with the properties that

$$\begin{aligned} \mu(k_0) &< 0 && \text{for} \quad k_0 < \underline{k} \\ \mu(k_0) &\geq 1/(\sigma-1) && \text{for} \quad k_0 \in (\underline{k}, \bar{k}] \\ \mu(k_0) &< 1/(\sigma-1) && \text{for} \quad k_0 > \bar{k}, \end{aligned}$$

where $\bar{k} \equiv \inf_{k_0 \geq \underline{k}} \left\{ k_0 \geq \underline{k} \,|\, \mu(k_0) < \frac{1}{\sigma-1} \right\} \in (\underline{k}, \infty]$. This proves that for $k_0 \in (\underline{k}, \bar{k}]$, there exists a debt level $b_0(k_0)$ for which the quantities c_t, n_t, k_t all fall to zero at equal rate $-\rho/\sigma$ and the sufficient optimality conditions of the problem are satisfied.

K Proof of Proposition 9

First, we show that the planner's problem is equivalent to (13). Then we show that the functions $\psi(T)$ and $\tau(T)$ are increasing, have $\psi(0) = \tau(0) = 0$ and bounded derivatives.

The planner's problem in this linear economy can be written using a present value resource constraint, that is,

$$\begin{aligned}
&\max \int_0^\infty e^{-\rho t}\left(u(c_t) - v(n_t)\right) dt \\
&\text{s.t. } \dot{c} \geq c\frac{1}{\sigma}((1-\bar{\tau})r^* - \rho) \\
&\int_0^\infty e^{-r^* t}(c_t - w^* n_t)dt + G = k_0 \\
&\int_0^\infty e^{-\rho t}\left[(1-\sigma)u(c_t) - (1+\zeta)v(n_t)\right] dt \geq u'(c_0)a_0,
\end{aligned} \tag{81}$$

where $G = \int_0^\infty e^{-r^* t} g dt$ is the present value of government expenses, k_0 is the initial capital stock, a_0 is the representative agent's initial asset position, and per-period utility from consumption and disutility from work are given by $u(c_t) = c_t^{1-\sigma}/(1-\sigma)$ and $v(n_t) = n_t^{1+\zeta}/(1+\zeta)$. Note that we assumed $\sigma > 1$. The FOCs for labor imply that given n_0,

$$n_t = n_0 e^{-(r^*-\rho)t/\zeta}. \tag{82}$$

An analogous argument to the bang-bang result in Appendix **??** implies the existence of $T \in [0, \infty]$ such that $\tau_t = \bar{\tau}$ for $t \leq T$ and zero thereafter. In particular, the after-tax (net) interest rate will be $r_t = (1-\bar{\tau})r^* \equiv \bar{r}$ for $t \leq T$ and $r_t = r^*$ for $t > T$. Then, by the representative agent's Euler equation, the path for consumption is determined by

$$c_t = c_0 e^{-\frac{\rho-\bar{r}}{\sigma}t + \frac{r^*-\bar{r}}{\sigma}(t-T)^+}. \tag{83}$$

Substituting equations (82) and (83) into (81), the planner's problem simplifies to,

$$\begin{aligned}
&\max_{T, c_0, \bar{n}} \quad \psi_1(T)u(c_0) - \psi_3 v(n_0) \\
&\text{s.t. } \psi_2(T)(\chi^*)^{-1}c_0 + G = k_0 + \psi_3 w^* n_0 \\
&\psi_1(T)u'(c_0)c_0 - \psi_3 v'(n_0)n_0 = \chi^* u'(c_0)a_0,
\end{aligned} \tag{84}$$

where $\psi_1(T) = \frac{\chi^*}{\chi}\left(1 - e^{-\chi T}\right) + e^{-\chi T}$, $\psi_2(T) = \frac{\chi^*}{\hat{\chi}}\left(1 - e^{-\hat{\chi} T}\right) + e^{-\hat{\chi} T}$, $\psi_3 = \chi^*\left(r^* + \frac{r^*-\rho}{\zeta}\right)^{-1}$ and $\chi = \frac{\sigma-1}{\sigma}\bar{r} + \frac{\rho}{\sigma}$, $\chi^* = \frac{\sigma-1}{\sigma}r^* + \frac{\rho}{\sigma}$, $\hat{\chi} = r^* + \frac{\rho-\bar{r}}{\sigma}$. Notice that $\hat{\chi} > \chi^* > \chi$.

Now normalize consumption and labor

$$c \equiv \psi_1(T)^{1/(1-\sigma)} c_0 / \chi^* \qquad\qquad n \equiv \psi_3^{1/(1+\zeta)} n_0 / \left(\chi^*\right)^{(1-\sigma)/(1+\zeta)}$$

and define an efficiency cost $\psi(T) \equiv \psi_2(T)\psi_1(T)^{1/(\sigma-1)} - 1$, a capital levy $\tau(T) \equiv 1 - \psi_1(T)^{-\sigma/(\sigma-1)}$, and the present value of wage income $\omega n \equiv w^* \psi_3^{\zeta/(1+\zeta)} n$. Here, we note that by definition, ψ is bounded away from infinity and τ is bounded away from 1. Then, we can rewrite problem (84) as

$$\begin{aligned}\max_{T,c,n} \quad & u(c) - v(n) \\ \text{s.t.} \; & (1+\psi(T))c + G = k_0 + \omega n \\ & u'(c)c - v'(n)n = (1-\tau(T))u'(c)a_0,\end{aligned}$$

which is what we set out to show. Notice that $\psi_1(0) = \psi_2(0) = 1$ and so $\psi(0) = \tau(0) = 0$. Further, given our assumption that $\sigma > 1$, $\psi_1(T)$ and $\tau(T)$ are increasing in T. To show that $\psi'(T) \geq 0$, notice that, after some algebra,

$$\frac{d}{dT}\left(\psi_2 \psi_1^{1/(\sigma-1)}\right) \geq 0 \quad \Leftrightarrow \quad \hat{\chi}\left(e^{\chi T} - 1\right) \leq \chi\left(e^{\hat{\chi} T} - 1\right),$$

which is true for any $T \geq 0$ because $\hat{\chi} > \chi$. Therefore, $\psi'(T) \geq 0$, with strict inequality for $T > 0$, implying that $\psi(T)$ is strictly increasing in T.

Now consider the ratio of derivatives,

$$\frac{\psi'(T)}{\tau'(T)} = \frac{1}{\sigma}\psi_2 \psi_1^{(1+\sigma)/(\sigma-1)}\left((\sigma-1)\frac{\psi_2'}{\psi_2}\frac{\psi_1}{\psi_1'} + 1\right).$$

Notice that $\psi_1(T) \in [1, \chi^*/\chi]$ and $\psi_2(T) \in [\chi^*/\hat{\chi}, 1]$, so both are bounded away from infinity and zero. Further, the ratio ψ_2'/ψ_1' is also bounded away from infinity, $\psi_2'/\psi_1' = -\frac{1}{\sigma-1}e^{-(\hat{\chi}-\chi)T} \in [-1/(\sigma-1), 0]$, implying that $\psi'(T)/\tau'(T)$ is bounded away from ∞.

L Proof of Proposition 10

The planning problem is given by

$$\sup_{\{c_t, C_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{85}$$

$$c_t + C_t + k_{t+1} = f(k_t) + (1-\delta)k_t \tag{86}$$

$$\sum_{t=0}^{\infty} \beta^t C_t^{1-\sigma} = C_0^{-\sigma} a_0. \tag{87}$$

First, note that a_0 must be positive or else the IC constraint (87) cannot be satisfied (recall that $\sigma > 1$). Second, note that there exists a unique solution $C_0(\varphi)$ to the equation

$$C_0^{\sigma} \varphi_0 \varphi^{1-\sigma} + C_0 = a_0$$

for any $\varphi_0, \varphi > 0$, and that $C_0(\varphi) \to 0$ as $\varphi \to 0$. We now use this to construct a sequence of feasible paths $\{C_t^{(n)}\}_{t=0}^{\infty}$, $n = 0, 1, \ldots$, with $C_t^{(n)}$ uniformly converging to 0 as $n \to \infty$.

Take any sequence $\{C_t^{(0)}\}_{t=0}^{\infty}$ that satisfies (87). Define

$$C_t^{(n)} = \begin{cases} C_0(\varphi^n) & t = 0 \\ \varphi^n C_t^{(0)} & t > 0 \end{cases}$$

for some $\varphi \in (0,1)$, $\varphi_0 \equiv C_0^{-\sigma}(a_0 - C_0)$. By construction, $C_t^{(n)} \to 0$ uniformly and the supremum in (85) approaches the maximum of the planning problem of a standard neoclassical growth model,

$$\max_{\{c_t, C_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$c_t + k_{t+1} = f(k_t) + (1-\delta)k_t.$$

The way $\{C_t^{(n)}\}$ was constructed in this proof, it suggests an implementation via a wealth tax $\mathcal{T}_1 = R_1/R_1^* \to 100\%$. Analogous to this construction, a wealth tax approaching 100% in any period would implement the same allocation. This also shows that only a single period of unconstrained taxation is necessary to implement the supremum.

M Proof of Proposition 11

As in Section 2, labor supply is inelastic at $n_t = 1$. Denote the capitalist's initial wealth by $a_0 \equiv R_0 k_0 + R_0^b b_0$. The planning problem is then

$$\max_{\{c_t, C_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{88a}$$
$$C_{t+1} \geq C_t \beta^{1/\sigma} \tag{88b}$$
$$c_t + C_t + k_{t+1} = f(k_t) + (1-\delta)k_t \tag{88c}$$
$$\sum_{t=0}^{\infty} \beta^t U'(C_t) C_t = U'(C_0) a_0. \tag{88d}$$

The necessary first order conditions for C_t and c_t in problem (88a) are

$$\beta^{1/\sigma}\eta_t - \beta^{-1}\eta_{t-1} = \lambda_t - \Phi_u U'(C_t) \tag{89}$$
$$u'(c_t) = \lambda_t \tag{90}$$
$$\beta^{1/\sigma}\eta_0 = \lambda_0 - \Phi_u U'(C_0) - \mu\sigma C_0^{-\sigma-1} a_0 \tag{91}$$

where we defined $\Phi_u \equiv \mu(1-\sigma)$. Here, μ is the multiplier on the IC constraint (88d), λ_t is the multiplier of the resource constraint (88c)—which is positive by (90)—and η_t denotes the costate of capitalists' consumption C_t. If $\eta_t < 0$, constraint (88b) is binding. Also, it follows from (88d) that

$$\sigma C_0^{-\sigma-1} a_0 = \sigma C_0^{-1} \cdot U'(C_0) a_0 = \sigma C_0^{-1} \cdot \sum_{t=0}^{\infty} \beta^t U'(C_t) C_t > (\sigma - 1) U'(C_0),$$

where the inequality is obtained by dropping all terms with $t > 0$ from the infinite sum and observing that $\sigma > \sigma - 1$. Using this inequality, (91) implies that μ must be positive and $\Phi_u < 0$.

Suppose now there existed a period $T \geq 0$ where constraint (88b) is slack. In that case, $\eta_T = 0$ and (89) becomes for $t = T + 1$

$$\Phi_u U'(C_{T+1}) = \lambda_{T+1} - \beta^{1/\sigma} \eta_{T+1} > 0$$

contradicting $\Phi_u < 0$. Therefore, (88b) binds in all periods, or equivalently, $R_t = 1$ for all t.