Aging, Output Per Capita, and Secular Stagnation

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ONLINE APPENDIX

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APPENDIX 1 - TABLES

Table A1: Estimates of the impact of aging on GDP per capita from 1990 to 2014: old > 50 years

	(1) 1990-2014	(2) 1990-2008	(3) 2008-2014	(4) ≈ ZLB	(5) ≠ZLB
Change of the ratio	0.361*	0.784***	-0.528**	-0.429	-0.302
of old to young	(0.208)	(0.282)	(0.205)	(0.320)	(0.310)
Observations	168	168	168	55	113
R-squared	0.019	0.052	0.038	0.038	0.008

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Table A2: Estimates of the impact of aging on GDP per capita: old > 50 years

Panel A: from 1990 to 2014	SAMPLE OF ALL COUNTRIES					OECD COUNTRIES		
	(1 OLS)	(2 OLS)	(3 OLS)	(4 OLS)	(5 IV)	(6 OLS)	(7 OLS)	(8 IV)
Change in the ratio of old to young	0.361*	0.990***	1.067***	0.615**	1.646***	-0.137	0.0909	1.042**
(from 2008 to 2015)	(0.208)	(0.257)	(0.265)	(0.290)	(0.416)	(0.415)	(0.391)	(0.511)
Initial GDP per capita		-0.138***	-0.118***	-0.142***	-0.180***		-0.204***	-0.244***
		(0.0362)	(0.0379)	(0.0447)	(0.0443)		(0.0733)	(0.0856)
First-stage F Statistic					18.96			13.72
Overidentification test p-value					0.65			0.31
Observations	168	168	168	168	168	35	35	35
Differential trends by:								
Population and initial age structure			✓	✓	✓		✓	✓
Regional dummies				✓	✓			
Panel B: from 1990 to 2008	SAMPLE OF ALL COUNTRIES					OECD COUNTRIES		
	(1 OLS)	(2 OLS)	(3 OLS)	(4 OLS)	(5 IV)	(6 OLS)	(7 OLS)	(8 IV)
Change in the ratio of old to young	0.784***	1.150***	1.171***	0.870**	1.947***	-0.255	0.0105	0.924
(from 2008 to 2015)	(0.282)	(0.368)	(0.375)	(0.399)	(0.624)	(0.458)	(0.490)	(0.673)
Initial GDP per capita		-0.0597*	-0.0530	-0.108***	-0.132***		-0.163**	-0.205***
		(0.0326)	(0.0350)	(0.0409)	(0.0414)		(0.0625)	(0.0662)
First-stage F Statistic					13.86			11.96
Overidentification test p-value					0.11			0.32
Observations	168	168	168	168	168	35	35	35
Differential trends by:								
Population and initial age structure			✓	✓	✓		✓	✓
Regional dummies				✓	✓			
Panel C: from 2008 to 2014	SAMPLE OF ALL COUNTRIES					OECD COUNTRIES		
	(1 OLS)	(2 OLS)	(3 OLS)	(4 OLS)	(5 IV)	(6 OLS)	(7 OLS)	(8 IV)
Change in the ratio of old to young	-0.528**	0.138	0.236	-0.247	0.337	-0.350	-0.376*	-0.259
(from 2008 to 2015)	(0.205)	(0.218)	(0.225)	(0.232)	(0.349)	(0.258)	(0.201)	(0.317)
Initial GDP per capita		-0.0615***	-0.0502***	-0.0268	-0.0356**		-0.0395	-0.0391
		(0.0122)	(0.0133)	(0.0171)	(0.0175)		(0.0302)	(0.0279)
First-stage F Statistic					17.38			3.40
Overidentification test p-value					0.58			0.03
Observations	168	168	168	168	168	35	35	35
Differential trends by:								
Population and initial age structure			✓	✓	✓		✓	✓
Regional dummies				✓	✓			

Notes: The table presents long-differences estimates of the impact of aging on GDP per capita in constant dollars from the Penn World Tables for all countries (columns 1 to 5) and OECD countries (columns 6 to 8). Aging is defined as the change in the ratio of the population above 65 to the population between 20 and 64. Columns 5 and 8 present IV estimates in which we instrument aging using the birthrate in 1960, 1965, … , 1980. The bottom rows indicate additional controls included in the models but not reported: The population and age structure controls include the log of the population and the initial value of our aging measure. We report standard errors robust to heteroscedasticity in parentheses.

Table A3: Estimates of the impact of Labor Input on GDP per capita: old > 65 years

	(1) 1990-2014	(2) 1990-2008	(3) 2008-2014	(4) ≈ ZLB	(5) ≠ZLB
Change in labor input	-1.827*	-3.297**	3.260***	3.180**	1.620
	(1.084)	(1.419)	(1.005)	(1.501)	(1.680)
Observations	168	168	168	55	113
R-squared	0.019	0.052	0.038	0.067	0.007

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Table A4: Estimates of the impact of Labor Input on GDP per capita: old > 65 years

Panel A: from 1990 to 2014	SAMPLE OF ALL COUNTRIES					OECD COUNTRIES		
	(1 OLS)	(2 OLS)	(3 OLS)	(4 OLS)	(5 IV)	(6 OLS)	(7 OLS)	(8 IV)
Change in labor input	-1.827*	-3.458***	-3.357***	-2.280*	-6.531***	0.720	1.902	-1.197
(from 2008 to 2015)	(1.084)	(1.173)	(1.282)	(1.300)	(2.045)	(2.087)	(1.802)	(2.436)
Initial GDP per capita		-0.103***	-0.0859**	-0.129***	-0.147***		-0.197**	-0.154*
		(0.0323)	(0.0366)	(0.0433)	(0.0423)		(0.0857)	(0.0925)
First-stage F Statistic					12.17			7.88
Overidentification test p-value					0.21			0.18
Observations	168	168	168	168	168	35	35	35
Differential trends by:								
Population and initial age structure			✓	✓	✓		✓	✓
Regional dummies				✓	✓			

Panel B: from 1990 to 2008	SAMPLE OF ALL COUNTRIES					OECD COUNTRIES		
	(1 OLS)	(2 OLS)	(3 OLS)	(4 OLS)	(5 IV)	(6 OLS)	(7 OLS)	(8 IV)
Change in labor input	-3.297**	-3.791**	-3.563**	-3.720**	-9.914***	0.118	1.397	-1.934
(from 2008 to 2015)	(1.419)	(1.535)	(1.661)	(1.706)	(2.789)	(1.966)	(1.674)	(2.064)
Initial GDP per capita		-0.0312	-0.0312	-0.0948**	-0.107***		-0.163**	-0.119
		(0.0278)	(0.0322)	(0.0386)	(0.0372)		(0.0669)	(0.0750)
First-stage F Statistic					10.00			7.98
Overidentification test p-value					0.29			0.49
Observations	168	168	168	168	168	35	35	35
Differential trends by:								
Population and initial age structure			✓	✓	✓		✓	✓
Regional dummies				✓	✓			

Panel C: from 2008 to 2014	SAMPLE OF ALL COUNTRIES					OECD COUNTRIES		
	(1 OLS)	(2 OLS)	(3 OLS)	(4 OLS)	(5 IV)	(6 OLS)	(7 OLS)	(8 IV)
Change in labor input	3.260***	0.539	0.617	1.767	1.186	2.943*	2.322*	3.930
(from 2008 to 2015)	(1.005)	(1.088)	(1.103)	(1.108)	(2.511)	(1.621)	(1.172)	(2.713)
Initial GDP per capita		-0.0552***	-0.0432***	-0.0277	-0.0291*		-0.0280	-0.0291
		(0.0127)	(0.0144)	(0.0168)	(0.0175)		(0.0381)	(0.0348)
First-stage F Statistic					8.43			5.21
Overidentification test p-value					0.52			0.22
Observations	168	168	168	168	168	35	35	35
Differential trends by:								
Population and initial age structure			✓	✓	✓		✓	✓
Regional dummies				✓	✓			

Notes: The table presents long-differences estimates of the impact of aging on GDP per capita in constant dollars from the Penn World Tables for all countries (columns 1 to 5) and OECD countries (columns 6 to 8). Aging is defined as the change in the ratio of the population above 65 to the population between 20 and 64. Columns 5 and 8 present IV estimates in which we instrument aging using the birthrate in 1960, 1965, … , 1980. The bottom rows indicate additional controls included in the models but not reported: The population and age structure controls include the log of the population and the initial value of our aging measure. We report standard errors robust to heteroscedasticity in parentheses.

Table A5: Estimates of the impact of Labor Input Ratio on GDP per adult: old > 65 years

	(1) 1990-2014	(2) 1990-2008	(3) 2008-2014	(4) ≈ ZLB	(5) ≠ZLB
Change in labor input	-2.014*	-3.235**	2.681***	2.425*	1.361
	(1.031)	(1.365)	(1.018)	(1.386)	(1.737)
Observations	168	168	168	55	113
R-squared	0.019	0.052	0.038	0.046	0.005

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Table A6: Estimates of the impact of Labor Input on GDP per adult: old > 65 years

Panel A: from 1990 to 2014	SAMPLE OF ALL COUNTRIES					OECD COUNTRIES		
	(1 OLS)	(2 OLS)	(3 OLS)	(4 OLS)	(5 IV)	(6 OLS)	(7 OLS)	(8 IV)
Change in labor input	-2.014*	-3.487***	-2.979**	-2.144*	-6.082***	0.970	1.995	-1.310
(from 2008 to 2015)	(1.031)	(1.042)	(1.169)	(1.232)	(1.887)	(1.821)	(1.770)	(2.516)
Initial GDP per adult		-0.132***	-0.128***	-0.149***	-0.159***		-0.161*	-0.0827
		(0.0348)	(0.0375)	(0.0442)	(0.0424)		(0.0922)	(0.105)
First-stage F Statistic					12.61			6.61
Overidentification test p-value					0.50			0.13
Observations	168	168	168	168	168	35	35	35
Differential trends by:								
Population and initial age structure			✓	✓	✓		✓	✓
Regional dummies				✓	✓			
Panel B: from 1990 to 2008	SAMPLE OF ALL COUNTRIES					OECD COUNTRIES		
	(1 OLS)	(2 OLS)	(3 OLS)	(4 OLS)	(5 IV)	(6 OLS)	(7 OLS)	(8 IV)
Change in labor input	-3.235**	-3.803***	-3.141**	-3.448**	-8.890***	0.454	1.386	-2.414
(from 2008 to 2015)	(1.365)	(1.406)	(1.532)	(1.623)	(2.598)	(1.622)	(1.541)	(2.095)
Initial GDP per adult		-0.0550*	-0.0685**	-0.117***	-0.120***		-0.124	-0.0396
		(0.0306)	(0.0330)	(0.0395)	(0.0371)		(0.0737)	(0.0878)
First-stage F Statistic					10.31			6.84
Overidentification test p-value					0.71			0.38
Observations	168	168	168	168	168	35	35	35
Differential trends by:								
Population and initial age structure			✓	✓	✓		✓	✓
Regional dummies				✓	✓			
Panel C: from 2008 to 2014	SAMPLE OF ALL COUNTRIES					OECD COUNTRIES		
	(1 OLS)	(2 OLS)	(3 OLS)	(4 OLS)	(5 IV)	(6 OLS)	(7 OLS)	(8 IV)
Change in labor input	2.681***	0.386	0.501	1.355	0.349	2.678*	2.242*	4.194
(from 2008 to 2015)	(1.018)	(1.077)	(1.101)	(1.089)	(2.406)	(1.498)	(1.155)	(2.684)
Initial GDP per adult		-0.0608***	-0.0498***	-0.0265	-0.0284		-0.0235	-0.0262
		(0.0143)	(0.0157)	(0.0181)	(0.0182)		(0.0401)	(0.0364)
First-stage F Statistic					8.85			5.06
Overidentification test p-value					0.59			0.25
Observations	168	168	168	168	168	35	35	35
Differential trends by:								
Population and initial age structure			✓	✓	✓		✓	✓
Regional dummies				✓	✓			

Notes: The table presents long-differences estimates of the impact of aging on GDP per capita in constant dollars from the Penn World Tables for all countries (columns 1 to 5) and OECD countries (columns 6 to 8). Aging is defined as the change in the ratio of the population above 65 to the population between 20 and 64. Columns 5 and 8 present IV estimates in which we instrument aging using the birthrate in 1960, 1965, … , 1980. The bottom rows indicate additional controls included in the models but not reported: The population and age structure controls include the log of the population and the initial value of our aging measure. We report standard errors robust to heteroscedasticity in parentheses.

Table A7: Estimates of the impact of aging on GDP per capita from 1990 to 2014: old > 65 years

Panel A: in non OECD countries

	(1) 1990-2015	(2) 1990-2008	(3) 2008-2015	(4) ≈ ZLB	(5) ≠ZLB
Change of the ratio of old to young	2.365** (1.026)	3.208** (1.422)	-0.854 (0.862)	-0.274 (1.247)	-1.046 (1.242)
Observations	133	133	133	27	106
R-squared	0.045	0.056	0.006	0.001	0.006

Panel B: Excluding countries in the periphery of the European: Portugal, Italy, Ireland, Greece, and Spain

	(1) 1990-2015	(2) 1990-2008	(3) 2008-2015	(4) ≈ ZLB	(5) ≠ZLB
Change of the ratio of old to young	0.989 (0.708)	2.230** (1.024)	-1.795*** (0.533)	-1.649** (0.758)	-1.099 (1.010)
Observations	164	164	164	51	113
R-squared	0.014	0.042	0.049	0.074	0.009

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Table A8: Estimates of the impact of aging on GDP per capita from 2008 to 2014 for different values of ZLB threshold: old > 65 years

ZLB Threshold	0.5%	0.5%	1.0%	1.0%	1.5%	1.5%	2.0%	2.0%
	(1) ≈ ZLB	(2) ≠ZLB	(3) ≈ ZLB	(4) ≠ZLB	(5) ≈ ZLB	(6) ≠ZLB	(7) ≈ ZLB	(8) ≠ZLB
Change of the ratio of old to young	-1.527** (0.723)	-1.451* (0.874)	-1.640* (0.815)	-1.131 (0.969)	-1.843** (0.761)	-1.099 (1.010)	-1.948*** (0.671)	-1.248 (1.090)
Observations	42	126	48	120	55	113	63	105
R-squared	0.075	0.017	0.075	0.010	0.093	0.009	0.107	0.010

Robust standard errors in parentheses
*** p<0.01, ** p<0.05, * p<0.1

Table A9: Estimates of the impact of aging on capital per worker, through real interest rates

	(1)	(2)	(3)	(4)	(5)
Panel A: Estimates of the impact of aging on Capital per working age adult (age 20-65)					
	1990-2014	1990-2008	2008-2014	≈ ZLB	≠ZLB
Change of the ratio of old to young	2.688***	3.298**	0.598	-1.100	1.742
	(0.920)	(1.260)	(0.753)	(1.075)	(1.709)
Observations	59	59	59	16	43
R-squared	0.066	0.068	0.004	0.036	0.016
Panel B: Estimates of the impact of aging on real interest rate					
	(1)	(2)	(3)	(4)	(5)
	1990-2014	1990-2008	2008-2014	≈ ZLB	≠ZLB
Change of the ratio of old to young	-0.587**	-0.537	-0.489	-0.409	-0.824
	(0.230)	(0.323)	(0.353)	(0.390)	(0.842)
Observations	59	59	59	16	43
R-squared	0.024	0.009	0.024	0.048	0.027
Panel C: Estimates of the impact of real interest rates on Capital per working age adult (age 20-65)					
	(1)	(2)	(3)	(4)	(5)
	1990-2014	1990-2008	2008-2014	≈ ZLB	≠ZLB
Change of real interest rate	-0.846***	-0.706***	0.0929	-0.681	0.218
	(0.253)	(0.211)	(0.365)	(0.648)	(0.403)
Observations	59	59	59	16	43
R-squared	0.095	0.104	0.001	0.048	0.006

Notes: Robust standard errors in parentheses, *** p<0.01, ** p<0.05, * p<0.1. The table presents in Panel A long-differences estimates of the impact of aging on capital per working age adult in constant dollars from the Penn World Tables in countries for which real interest rate data from the World Bank exists between 1990 and 2015. Panel B presents long-differences estimates of the impact of aging on real interest rates data from the World Bank. Real interest rates are defined as lending interest rates adjusted for inflation as measured by the GDP deflator. Panel C presents long-differences estimates of the impact of real interest rates on capital per working age adult .

APPENDIX 2 - MODEL

This appendix summarizes the microfoundations of the simple general equilibrium model of section 4.

A Households

Individuals live for two periods, young and old, and maximize utility from consumption of one aggregate good according to:

$$U_t(c_t^y, c_{t+1}^o) = \max_{c_t^y, c_{t+1}^o} \mathbb{E}_t \left\{ u(c_t^y) + \beta u(c_{t+1}^o) \right\} \tag{1}$$

$$\text{s.t. } c_t^y = w_t l_t - \tau_t - s_t \tag{2}$$

$$c_{t+1}^o = \frac{(1+i_t)}{\Pi_{t+1}} s_t \tag{3}$$

where the $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ is a constant relative risk aversion (CRRA) preference function. c_t^y and c_{t+1}^o are household's consumption respectively when young and old. When young, individuals earn income in period t by renting their labor endowment l_t to firms at wage w_t. After paying taxes τ_t the young use their net income to consume in period t and to save s_t for consumption when old by accumulation of private capital supplied to firms for production during the next period for a gross real rent $\frac{(1+i_t)}{\Pi_{t+1}}$, such that:

$$K_{t+1}^s = N_t^y s_t \tag{4}$$

where N_t^y is the size of young generation at time t. When old, individuals dissave to consume, earning a gross real return $\frac{(1+i_t)}{\Pi_{t+1}}$ on their savings from previous period (3). We derive the first order conditions of this problem by maximizing the Lagrangian[1]:

$$\mathcal{L}_t = u(c_t^y) + \beta u(c_{t+1}^o) - \lambda_t \left(c_t^y - w_t l_t + \tau_t + s_t \right) - \lambda_{t+1} \left(c_{t+1}^o - \frac{(1+i_t)}{\Pi_{t+1}} s_t \right) \tag{5}$$

First-order conditions:

$$\frac{\delta \mathcal{L}_t}{\delta c_t^y} = u_c(c_t^y) - \lambda_t = 0 \tag{6}$$

$$\frac{\delta \mathcal{L}_t}{\delta c_{t+1}^o} = \beta u_c(c_t^o) - \lambda_{t+1} = 0 \tag{7}$$

$$\frac{\delta \mathcal{L}_t}{\delta k_{t+1}^s} = -\lambda_t + \lambda_{t+1} \frac{(1+i_t)}{\Pi_{t+1}} = 0 \tag{8}$$

Perfect foresight young individuals are at an interior solution and their consumption-saving choices satisfy a standard Euler equation given by

$$\lambda_t = \lambda_{t+1} \frac{(1+i_t)}{\Pi_{t+1}} \rightarrow u_c(c_t^y) = \beta R_t u_c(c_t^o) \Leftrightarrow \frac{1}{(c_t^y)^\sigma} = \beta \frac{(1+i_t)}{\Pi_{t+1}} \frac{1}{\left(c_{t+1}^o\right)^\sigma} \tag{9}$$

[1] The expectations operator is ignored since the model is deterministic.

Let $R_t \equiv \frac{(1+i_t)}{\Pi_{t+1}} \equiv 1 + r_t$. Then the previous expression can be written as

$$\frac{1}{c_t^y} = \beta_{R_t,\sigma} R_t \frac{1}{c_{t+1}^o} \Leftrightarrow c_{t+1}^o = R_t \left[\beta_{R_t,\sigma} c_t^y\right] \tag{10}$$

where $\beta_{R_t,\sigma} = \beta^{\frac{1}{\sigma}} R_t^{\frac{1-\sigma}{\sigma}} \overset{(\sigma=1)}{=} \beta$. Directly from the budget constraint of the old(3) we have

$$s_t = \beta_{R_t,\sigma} c_t^y \tag{11}$$

Savings of the young s_t can then be derived by replacing the previous expression of c_t^y with respect to s_t in the budget constraint of the young(2):

$$s_t = \frac{\beta_{R_t,\sigma}}{1 + \beta_{R_t,\sigma}} (w_t l_t - \tau_t) \tag{12}$$

Capital supply:

Because aggregate savings in period t is equal to the capital supplied in the following period, we have:

$$N_t^y s_t = K_{t+1}^s \Leftrightarrow s_t = \frac{K_{t+1}^s}{N_t^y} = \frac{K_{t+1}^s}{N_{t+1}^y} \frac{N_{t+1}^y}{N_t^y} = k_{t+1}^s (1 + g_t^y) = \frac{k_{t+1}^s}{A_t} \Rightarrow k_{t+1}^s = A_t s_t \tag{13}$$

where k_t^s is capital supplied per young individual at time t, $1 + g_t^y = N_{t+1}^y / N_t^y$ is the birth rate of the young, and defining an aging parameter as the ratio of old to young at time $t+1$:

$$A_t = \frac{N_{t+1}^o}{N_{t+1}^y} = \frac{N_t^y}{N_{t+1}^y} = \frac{1}{1 + g_t^y} \tag{14}$$

Then,

$$k_{t+1}^s = A_t s_t = A_t \frac{\beta_{R_t,\sigma}}{1 + \beta_{R_t,\sigma}} (w_t l_t - \tau_t) \tag{15}$$

No-arbitrage condition:

The return on savings R_t accounts for the rent R_{t+1}^k on capital firms pay to individuals, and a capital depreciation δ. So, the budget constraint of the old can alternatively be expressed by:

$$c_{t+1}^o = \frac{1}{A_{t+1}} \left[(1-\delta) k_{t+1}^s + R_{t+1}^k k_{t+1}^s\right] = s_t (1 - \delta + r_{t+1}^k) \tag{16}$$

Implying the following no-arbitrage condition:

$$R_{t+1}^k = R_t + \delta - 1 \tag{17}$$

B Firms

We assume that firms produce only one good, are perfectly competitive, and take prices as given. They hire labor at a wage w_t and rent capital at rate r_t^k to maximize period-by-period profits. They operate using a standard Cobb-Douglas production function, and their problem is given by:

$$\max_{L_t,K_t} P_t Y_t - W_t L_t - P_t R_t^k K_t \tag{18}$$

$$\text{s.t. } Y_t = L_t^{1-\alpha} K_t^{\alpha} \tag{19}$$

The firm's capital and labor demand equilibrium conditions are given by:

$$R_t^k = \alpha \frac{Y_t}{K_t} \tag{20}$$

$$w_t = \frac{W_t}{P_t} = (1-\alpha)\frac{Y_t}{L_t} \tag{21}$$

Each individual of the young generation supplies his labor endowment inelastically at $\bar{l}$. Since for now we are assuming wages are flexible, and full-employment, then $L_t = N_t^y \bar{l}$. Let $k_t^d = \frac{K_t}{N_t^y} = \frac{K_t}{L_t}\bar{l}$. Then:

$$w_t = (1-\alpha)\left(\frac{\alpha}{R_t^k}\right)^{\frac{\alpha}{1-\alpha}} \tag{22}$$

$$k_t^d = \bar{l}\left(\frac{\alpha}{R_t^k}\right)^{\frac{1}{1-\alpha}} \tag{23}$$

Defining $\tilde{x} \equiv \ln x$:

$$\tilde{k}_{t+1}^d = \ln\left[\bar{l}\alpha^{\frac{1}{1-\alpha}}\right] - \frac{1}{1-\alpha}\tilde{R}_{t+1}^k \tag{24}$$

C Government

We assume the Government budget is balanced, $G_t = T_t$. And that Government spending is exogenously proportional to full-employment output $G_t = \Omega \bar{Y}_t$.

$$G_t = \mathcal{G}\bar{Y}_t = T_t = N_t^y \tau_t \tag{25}$$

$$\tau_t = \frac{\mathcal{G}}{N_t^y}\bar{Y}_t = \frac{\mathcal{G}}{N_t^y}\frac{w_t \bar{L}_t}{1-\alpha} = w_t \bar{l}\frac{\mathcal{G}}{1-\alpha} = w_t \bar{l}\tau \tag{26}$$

$$\text{where } \tau = \frac{\mathcal{G}}{1-\alpha} \text{ is exogenously determined.} \tag{27}$$

Capital supply per young individual can then be expressed by:

$$k_{t+1}^s = A_t \frac{\beta_{R_t,\sigma}}{1+\beta_{R_t,\sigma}} w_t \bar{l}(\mu_t - \tau) \text{ , where } \mu_t = l_t/\bar{l} \overset{(l_t=\bar{l}_t)}{=} 1 \tag{28}$$

μ_t is the employment ratio of the young, equal to 1 for now. Replacing w_t by (22) and taking logs the previous expression becomes:

$$\tilde{k}^s_{t+1} = \ln\left[\bar{l}(1-\tau)(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right] + \ln\left(\frac{\beta_{R_t,\sigma}}{1+\beta_{R_t,\sigma}}\right) - \frac{\alpha}{1-\alpha}\tilde{R}^k_t + \tilde{A}_t \tag{29}$$

D Comparative statics

Without loss of generality we assume full depreciation of capital in one period $\delta = 1 \Rightarrow R_t = R^k_{t+1}$. Assuming the system is on a steady state equilibrium where $R_t = R$,

$$\tilde{k}^d = \tilde{k}^s \tag{30}$$

where,from (24) and (29)

$$\tilde{k}^d = -\frac{1}{1-\alpha}\tilde{R} + \ln\left[\bar{l}\alpha^{\frac{1}{1-\alpha}}\right] \tag{31}$$

$$\tilde{k}^s = -\frac{\alpha}{1-\alpha}\tilde{R} + \tilde{A} + \ln\left(\frac{\beta_{R,\sigma}}{1+\beta_{R,\sigma}}\right) + \ln\left[\bar{l}(1-\tau)(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\right] \tag{32}$$

(i) If $\sigma = 1$ then $\beta_{R,\sigma} = \beta$ and $\tilde{R}$ and $\tilde{k}$has the following closed form expression

$$\tilde{R} = -\tilde{A} + \ln\left[\left(\frac{1+\beta}{\beta}\right)\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{1-\tau}\right)\right] \tag{33}$$

$$\tilde{k} = \frac{1}{1-\alpha}\tilde{A} + \frac{1}{1-\alpha}\ln\left[\left(\frac{1+\beta}{\beta}\right)\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1}{1-\tau}\right)\right] + \ln\left[\bar{l}\alpha^{\frac{1}{1-\alpha}}\right] \tag{34}$$

(ln)Aging $\tilde{A}$ has a one for one negative impact on $\tilde{R}$

$$\frac{d\tilde{R}}{d\tilde{A}} = -1 \tag{35}$$

(ii) For the general case where $\sigma > 0$ we can use the Theorem of the Implicit Function to express the former derivative

$$\frac{d\tilde{R}}{d\tilde{A}} = -\frac{1+\beta_{R,\sigma}}{\frac{1}{\sigma}+\beta_{R,\sigma}} < 0 \tag{36}$$

which is still negative (and equal to -1 when $\sigma = 1$). Also, aging has a stronger impact on real rates when the Relative Risk Aversion σ is higher. Aging expands the supply of capital which effect has to be offset by a reduction of the real rate in order to sustain a general equilibrium. This real rate change has to be higher if the Elasticity of Intertemporal Substution is lower (or σ higher). This is consistent with the data used.

(iii) Impact of aging on output per capita $\tilde{y}^{pc}$

Let,

$$y_t = \frac{Y_t}{N^y_t} = \left(\frac{K_t}{N^y_t}\right)^\alpha \Rightarrow \tilde{y}_t = \alpha\tilde{k}_t \tag{37}$$

Since we are assuming full-employment $L_t = N_t^y$. Then,

$$y_t^{pc} = \frac{Y_t}{N_t^y + N_t^o} = \frac{Y_t}{N_t^y}\frac{N_t^y}{N_t^y + N_t^o} = \frac{Y_t}{N_t^y}\frac{1}{1+\frac{N_t^o}{N_t^y}} = y_t\frac{1}{1+A_{t-1}} \tag{38}$$

using logs,

$$\tilde{y}_t^{pc} = \tilde{y}_t - \ln\left(1+A_{t-1}\right) \tag{39}$$

replacing $\tilde{y}_t = \alpha\tilde{k}_t$

$$\tilde{y}_t^{pc} = \alpha\tilde{k}_t - \ln\left(1+A_{t-1}\right) \tag{40}$$

now replacing $\tilde{k}_t^d = \ln\left[\bar{l}\alpha^{\frac{1}{1-\alpha}}\right] - \frac{1}{1-\alpha}\tilde{R}_t$

$$\tilde{y}_t^{pc} = -\frac{\alpha}{1-\alpha}\tilde{R}_t - \ln\left(1+A_{t-1}\right) + \alpha\ln\left[\bar{l}\alpha^{\frac{1}{1-\alpha}}\right] \tag{41}$$

Finally by replacing $\tilde{R}$ by its steady state expression and taking the derivative of $\tilde{y}_t^{pc}$ with respect to $\tilde{A}$

$$\frac{d\tilde{y^{pc}}}{d\tilde{A}} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1+\beta_{R,\sigma}}{\frac{1}{\sigma}+\beta_{R,\sigma}}\right) - \left(\frac{A}{1+A}\right) \tag{42}$$

The first term of the expression is the capital deepening effect of aging which is positive, and the second one is the negative demographic effect of aging. Aging has a positive impact on output per capita when the capital deepening effect prevail over the demographic effect:

$$\frac{d\tilde{y^{pc}}}{d\tilde{A}} > 0 \Leftrightarrow \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{1+\beta_{R,\sigma}}{\frac{1}{\sigma}+\beta_{R,\sigma}}\right) > \left(\frac{A}{1+A}\right) \tag{43}$$

We see directly from this expression that for greater values of σ the capital deepening effect is stronger, such that we would expect a stronger positive impact of aging on output per capita in those countries. Note also that the demographic effect $\frac{A}{1+A} = \frac{N^o}{N^y+N^o}$, so in countries where people live longer we would expect a weaker positive relation between aging and output per capita. This is suggested by the data where the significance of the results for OECD countries is much weaker.

E Transition dynamics

Define

$$\tilde{x}^* \equiv \textit{ steady state of } \ln(x) \tag{44}$$
$$\hat{x}_t \equiv \tilde{x} - \tilde{x}^* \tag{45}$$

then from (24) and (29), and having $R_t = R_{t+1}^k$,

$$\hat{k}_{t+1}^d = -\frac{1}{1-\alpha}\hat{R}_{t+1}^k \tag{46}$$

$$\hat{k}_{t+1}^s = -\frac{\alpha}{1-\alpha}\hat{R}_t^k + \hat{A}_t + \left[\ln\left(\frac{\beta_{R_{t+1}^k,\sigma}}{1+\beta_{R_{t+1}^k,\sigma}}\right) - \ln\left(\frac{\beta_{R^*,\sigma}}{1+\beta_{R^*,\sigma}}\right)\right] \tag{47}$$

Equilibrium

$$\hat{k}_t^d = \hat{k}_t^s \tag{48}$$

$$\hat{R}_{t+1}^k = \alpha \hat{R}_t^k - (1-\alpha)\hat{A}_t - (1-\alpha)\left[\ln\left(\frac{\beta_{R_{t+1}^k,\sigma}}{1+\beta_{R_{t+1}^k,\sigma}}\right) - \ln\left(\frac{\beta_{R^*,\sigma}}{1+\beta_{R^*,\sigma}}\right)\right] \tag{49}$$

Transition from one steady state to another. Initial steady state: at $t = t_o - 1$ aging $A_{t_o-1} = A_1^*$ and $R_{t_0-1} = R_1^* = R_{t_0}$. At $t = t_0$ aging changes for a change in g from A_1^* to A_2^*. Define $\hat{A}^* \equiv \tilde{A}_1^* - \tilde{A}_2^*$, $\hat{R}^{k*} \equiv \tilde{R}_1^{k*} - \tilde{R}_2^{k*}$, and $\hat{R}_t^k \equiv \tilde{R}_t^k - \tilde{R}_2^{k*}$.

(i) $\sigma = 1$ and $\delta = 1$:

$$\hat{R}_{t+1}^k = \alpha \hat{R}_t^k \text{ for } t \geq t_0 \tag{50}$$

$$\hat{R}_t^k = \alpha^{t-t_0}\hat{R}^{k*} \tag{51}$$

$$\tilde{R}_t^k = \alpha^{t-t_0}\left(\tilde{R}_1^{k*} - \tilde{R}_2^{k*}\right) + \tilde{R}_2^{k*} \tag{52}$$

$\alpha \in]0;1[$, the series converges monotonically to the new steady state. The sign of the convergence process is opposite to aging change. Note that if $\sigma = 1$ then $\hat{R}^* = -\hat{A}^*$

$$\tilde{R}_t^k = \tilde{R}_1^{k*} - (1-\alpha^{t-t_0})\left(\tilde{A}_2^* - \tilde{A}_1^*\right) \tag{53}$$

(ii) General case for σ and $\delta \in]0,1]$: log linearizing (49),

$$\hat{R}_{t+1}^k = (\alpha_{R^{k*},\sigma})\hat{R}_t^k \text{ for } t \geq t_0 \tag{54}$$

$$\hat{R}_t^k = (\alpha_{R^{k*},\sigma})^{t-t_0}\hat{R}^{k*} \tag{55}$$

$$\tilde{R}_t^k = (\alpha_{R^{k*},\sigma})^{t-t_0}\left(\tilde{R}_1^{k*} - \tilde{R}_2^{k*}\right) + \tilde{R}_2^{k*} \tag{56}$$

$$\text{where } \alpha_{R^{k*},\sigma} = \alpha\frac{1+\beta_{R^{k*}}}{1+\beta_{R^{k*}} + (1-\alpha)\left(\frac{1}{\sigma}-1\right)\frac{R^{k*}}{R^{k*}+(1-\delta)}} \in]0;1[\tag{57}$$

the series always converges monotonically to the new steady state. The sign of the convergence process is opposite to aging change. The convergence process takes longer for higher level of σ and lower levels of δ.

F Aggregate Demand

(i) Consumption function

From the Euler equation (10) and budget constraint of the old (16), and assuming full depreciation of capital in each period, $\delta = 1$

$$\begin{aligned}
C_t &= C_t^y + C_t^o && (58)\\
&= N_t^y \frac{s_t}{\beta_{R_t,\sigma}} + R_{t-1} N_t^o s_{t-1} && (59)\\
&= \frac{1}{1+\beta_{R_t,\sigma}} \left(w_t L_t - G_t \right) + R_t^k K_t^s && (60)\\
&= \frac{1}{1+\beta_{R_t,\sigma}} \left[(1-\alpha) Y_t - G_t \right] + \alpha Y_t && (61)\\
&= \left[\frac{(1-\alpha)}{1+\beta_{R_t,\sigma}} + \alpha \right] Y_t - \frac{1}{1+\beta_{R_t,\sigma}} G_t && (62)
\end{aligned}$$

(ii) Investment function

$$I_t = K_{t+1} = \alpha \frac{Y_{t+1}}{R_{t+1}^k} = \alpha \frac{Y_{t+1}}{R_{t+1}} \tag{63}$$

(iii) Aggregate Demand

$$\begin{aligned}
Y_t &= C_t + I_t + G_t && (64)\\
&= \left[\frac{(1-\alpha)}{1+\beta_{R_t,\sigma}} + \alpha \right] Y_t + \alpha \frac{Y_{t+1}}{R_{t+1}} + \frac{\beta_{R_t,\sigma}}{1+\beta_{R_t,\sigma}} G_t && (65)
\end{aligned}$$

(iv) Aggregate Demand per capita

$$y_t^{pc} = \left[\frac{(1-\alpha)}{1+\beta_{R_t,\sigma}} + \alpha \right] y_t^{pc} + \left(\frac{\alpha}{R_{t+1}} \right) \left[\frac{1}{A_t} \left(\frac{1+A_t}{1+A_{t-1}} \right) \right] y_{t+1}^{pc} + \frac{\beta_{R_t,\sigma}}{1+\beta_{R_t,\sigma}} G_t^{pc} \tag{66}$$

(v) Aggregate Demand per capita in steady state

$$y^{pc} = \left[\frac{1-\alpha}{1+\beta_{R,\sigma}} + \alpha + \frac{\alpha}{A} \frac{1}{R} \right] y^{pc} + \frac{\beta_{R,\sigma}}{1+\beta_{R,\sigma}} G^{pc} \tag{67}$$

Assuming that the system is determined, and taking logs, $\tilde{y}^{pc}$ is expressed in terms of R and A, when Government spending per capita is constant, and when it is proportional to full employment output:

$$\text{If } G^{pc} = \bar{G}^{pc} \quad \text{then } \tilde{y}^{pc} = -\ln \left[(1-\alpha) \frac{\beta_{R,\sigma}}{1+\beta_{R,\sigma}} - \frac{\alpha}{A} \frac{1}{R} \right] + \ln \left(\frac{\beta_{R,\sigma}}{1+\beta_{R,\sigma}} \bar{G}^{pc} \right) \tag{68}$$

$$\text{If } G = \mathcal{G} \bar{Y} \quad \text{then } \tilde{y}^{pc} = -\ln \left[(1-\alpha) \frac{\beta_{R,\sigma}}{1+\beta_{R,\sigma}} - \frac{\alpha}{A} \frac{1}{R} \right] + \ln \left(\frac{\beta_{R,\sigma}}{1+\beta_{R,\sigma}} \frac{1}{1+A} \left(\frac{\alpha}{R} \right)^{\frac{\alpha}{1-\alpha}} \right) \tag{69}$$

Where $\bar{y}^{pc} = \bar{y} \frac{1}{1+A} = \frac{1}{1+A} \left(\frac{\alpha}{R} \right)^{\frac{\alpha}{1-\alpha}}$ is full-employment output per capita.

G Impact of aging on output per capita at the ZLB

We now assume that $i = 0$, $\Pi = R = 1$,and also that $\sigma = 1$ without loss of generality. Then an increase in aging leads unambiguously to a decrease of output per capita, , when Government spending per capita is constant, and when it is proportional to full employment output:

$$\text{If } G^{pc} = \bar{G}^{pc} \qquad \text{then } \frac{d\tilde{y^{pc}}}{dA} = -\left[(1-\alpha)\frac{\beta}{1+\beta} - \frac{\alpha}{A}\right]^{-1}\frac{\alpha}{A^2} < 0 \tag{70}$$

$$\text{If } G = \mathcal{G}\bar{Y} \qquad \text{then } \frac{d\tilde{y^{pc}}}{dA} = -\left[(1-\alpha)\frac{\beta}{1+\beta} - \frac{\alpha}{A}\right]^{-1}\frac{\alpha}{A^2} - \frac{1}{1+A} < 0 \tag{71}$$