Monetary Policy, Bounded Rationality, and Incomplete Markets

Online Appendix

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1 Proofs, Details, and Derivations

1.1 Proofs of Propositions 1 and 2

We consider an initial REE $\{R_t, Y_t\}$ which is a steady state with $R_t = R$ and $Y_t = Y$ for all $t \geq 0$. This only requires that $\beta R = 1$. We consider a change $\{\hat{R}_t\}$ in the path for the interest rate ΔR_τ at date τ so that $\hat{R}_\tau = R + \Delta R_\tau$ and $\hat{R}_t = R_t$ for $t \neq \tau$.

We start by computing the new new REE $\{\hat{R}_t, \hat{Y}_t\}$. Because the aggregate model is purely forward looking, we can immediately conclude that for $t > \tau$, $\hat{Y}_t = Y$ and so $\Delta\hat{Y}_t = 0$. And we guess and verify that for $t \leq \tau$, $\hat{Y}_t = Y(1 + \frac{\Delta R}{R})^{-\sigma}$ and so

$$\Delta\hat{Y}_t = Y[(1 + \frac{\Delta R}{R})^{-\sigma} - 1].$$

This immediately implies that

$$\epsilon_\tau = \sigma.$$

We can perform the decomposition into a partial equilibrium effect and a general equilibrium effect. For $t > \tau$, we have $\Delta\hat{Y}_t^{PE} = \Delta\hat{Y}_t^{GE} = 0$, and for $t \le \tau$, we have

$$\Delta\hat{Y}_t^{PE} = Y\frac{\frac{(1+\frac{\Delta R}{R})^{-1}-(1+\frac{\Delta R}{R})^{\sigma-1}}{R^{\tau-t+1}}}{1+\frac{(1+\frac{\Delta R}{R})^{\sigma-1}-1}{R^{\tau-t+1}}},$$
$$\Delta\hat{Y}_t^{GE} = Y[(1+\frac{\Delta R}{R})^{-\sigma}-1] - Y\frac{\frac{(1+\frac{\Delta R}{R})^{-1}-(1+\frac{\Delta R}{R})^{\sigma-1}}{R^{\tau-t+1}}}{1+\frac{(1+\frac{\Delta R}{R})^{\sigma-1}-1}{R^{\tau-t+1}}}.$$

This immediately implies that

$$\epsilon_{t,\tau}^{PE} = \sigma\frac{1}{R^{\tau-t+1}},$$
$$\epsilon_{t,\tau}^{GE} = \sigma(1-\frac{1}{R^{\tau-t+1}}).$$

Next we compute the level-k equilibria $\{\hat{R}_t, \hat{Y}_t^k\}$. We have

$$\hat{Y}_t^k = \frac{\sum_{s=0}^{\tau-t-1}\frac{\hat{Y}_{t+1+s}^{k-1}}{R^{1+s}} + (1+\frac{\Delta R}{R})^{-1}\sum_{s=\tau-t}^{\infty}\frac{\hat{Y}_{t+1+s}^{k-1}}{R^{1+s}}}{\frac{1}{R}\frac{1-\frac{1}{R^{\tau-t}}}{1-\frac{1}{R}} + (1+\frac{\Delta R}{R})^{\sigma-1}\frac{\frac{1}{R^{\tau-t+1}}}{1-\frac{1}{R}}}.$$

This implies that

$$\Delta\hat{Y}_t^k = \frac{\sum_{s=0}^{\tau-t-1}\frac{\Delta\hat{Y}_{t+1+s}^{k-1}}{R^{1+s}} + (1+\frac{\Delta R}{R})^{-1}\sum_{s=\tau-t}^{\infty}\frac{\Delta\hat{Y}_{t+1+s}^{k-1}}{R^{1+s}} + Y\frac{(1+\frac{\Delta R}{R})^{-1}-(1+\frac{\Delta R}{R})^{\sigma-1}}{1-\frac{1}{R}}\frac{1}{R^{\tau-t+1}}}{\frac{1}{R}\frac{1-\frac{1}{R^{\tau-t}}}{1-\frac{1}{R}} + \frac{(1+\frac{\Delta R}{R})^{\sigma-1}}{1-\frac{1}{R}}\frac{1}{R^{\tau-t+1}}}.$$

We get

$$\epsilon_\tau^1 = \sigma\frac{1}{R^\tau},$$
$$\epsilon_\tau^2 = \sigma\frac{1}{R^\tau}\left[1+(R-1)\tau\right],$$
$$\epsilon_\tau^3 = \sigma\frac{1}{R^\tau}\left[1+(R-1)\tau+(R-1)^2\frac{\tau(\tau-1)}{2}\right],$$

and more generally

$$\epsilon_\tau^k = \sigma\frac{1}{R^\tau}\left[\sum_{n=0}^{k}(R-1)^n\sum_{s_0=0}^{\tau-1}\sum_{s_1=0}^{\tau-1-s_0}\cdots\sum_{s_{n-2}=0}^{\tau-1-s_{n-3}}1\right].$$

1.2 The Perpetual Youth Model of Borrowing Constraints with $\sigma \neq 1$

Individual consumption function. When $\sigma \neq 1$, the individual consumption function is given by

$$c^*(a_t^i; \{r_{t+s}\}, \{Y_{t+s}^e\}) = \frac{a_t^i + \int_0^\infty (1-\delta) Y_{t+s}^e e^{-\int_0^s (r_{t+u}+\lambda) du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)] du} ds}.$$

Aggregate state variable. Exactly as in the case $\sigma = 1$ treated in Section 4.1, the aggregate state variable Ψ_t (the wealth distribution) is not required to characterize the aggregate equilibrium since the reduced-form aggregate consumption function is independent of Ψ_t.

Reduced-form aggregate consumption function. The reduced-form aggregate consumption function is given by

$$C(\{r_{t+s}\}, \{Y_{t+s}^e\}) = \frac{\int_0^\infty \delta Y_{t+s}^e e^{-\int_0^s r_{t+u} du} ds + \int_0^\infty (1-\delta) Y_{t+s}^e e^{-\int_0^s (r_{t+u}+\lambda) du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)] du} ds}.$$

Equilibrium characterization. For concreteness, we briefly characterize the various equilibria in the context of this particular model. Given beliefs $\{Y_t^e\}$, and given the path for interest rates $\{r_t\}$, $\{r_t, Y_t\}$ is a temporary equilibrium if and only if the path for aggregate income $\{Y_t\}$ is given by

$$Y_t = \frac{\int_0^\infty \delta Y_{t+s}^e e^{-\int_0^s r_{t+u} du} ds + \int_0^\infty (1-\delta) Y_{t+s}^e e^{-\int_0^s (r_{t+u}+\lambda) du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)] du} ds} \quad \forall t \geq 0.$$

Similarly, given the path for interest rates $\{r_t\}$, $\{r_t, Y_t\}$ is an REE if and only if the path for aggregate income $\{Y_t\}$ satisfies the fixed point

$$Y_t = \frac{\int_0^\infty \delta Y_{t+s} e^{-\int_0^s r_{t+u} du} ds + \int_0^\infty (1-\delta) Y_{t+s} e^{-\int_0^s (r_{t+u}+\lambda) du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)] du} ds} \quad \forall t \geq 0.$$

Finally given an initial REE $\{r_t, Y_t\}$ and a new interest rate path $\{\hat{r}_t\}$, the level-k equilibria $\{\hat{r}_t, \hat{Y}_t^k\}$ satisfy the following recursion over $k \geq 0$:

$$\hat{Y}_t^k = \frac{\int_0^\infty \delta \hat{Y}_{t+s}^{k-1} e^{-\int_0^s r_{t+u} du} ds + \int_0^\infty (1-\delta) \hat{Y}_{t+s}^{k-1} e^{-\int_0^s (r_{t+u}+\lambda) du} ds}{\int_0^\infty e^{-\int_0^s [(1-\sigma)(r_{t+u}+\lambda)+\sigma(\rho+\lambda)] du} ds} \quad \forall t \geq 0.$$

with the initialization that $\hat{Y}_t^0 = Y_t$ for all $t \geq 0$.

We now turn to the computation of the different interest rate elasticities of output around a steady state REE $\{R_t, Y_t\}$ $Y_t = Y > 0$ and $r_t = r$ for all $t \geq 0$, where the steady-state interest rate r is given by

$$1 = [(1-\sigma)(r+\lambda) + \sigma(\rho+\lambda)][\frac{\delta}{r} + \frac{1-\delta}{r+\lambda}],$$

so that $r = \rho$ in the limit where the frequency of binding borrowing constraints λ goes to 0.

Monetary policy at different horizons under RE. The expressions for the interest rate elasticities of output ϵ_τ and their decompositions $\epsilon_{t,\tau} = \epsilon_\tau^{PE} + \epsilon_\tau^{GE}$ into PE and GE effects can be simplified in three special cases. The first case is when $\sigma = 1$ and is treated in the main body of the paper.

The second case is when the frequency of binding borrowing constraints λ goes to 0, where we get $r = \rho$ and

$$\epsilon_\tau = \sigma, \quad \epsilon_\tau^{PE} = \sigma e^{-r\tau}, \quad \epsilon_\tau^{GE} = \sigma[1 - e^{-r\tau}].$$

The third case is when there is no outside liquidity $\delta = 0$, where we get $r = \rho$ and

$$\epsilon_\tau = \sigma, \quad \epsilon_\tau^{PE} = \sigma e^{-(r+\lambda)\tau}, \quad \epsilon_\tau^{GE} = \sigma[1 - e^{-(r+\lambda)\tau}].$$

1.3 Details for the Model with Sticky Prices and Inflation in Section 5

This section fleshes out the details of the model with sticky prices and inflation in Section 5.

Monetary policy. Monetary policy is described by a path of interest rate rules $R_t(\Pi_t)$ specifying nominal interest rates as a function of the inflation rate $\Pi_t = P_t/P_{t-1}$. In what follows, we often simply write $\{R_t\}$ to denote the path of interest rate rules.

Aggregate variables and beliefs. These modifications change the relevant aggregate variables. In particular, we now need to track not only the path of nominal interest rates $\{R_t\}$ and the paths of output $\{Y_t\}$ and beliefs about output $\{Y_t^e\}$, but also the paths of aggregate real profits $\{X_t\}$ and beliefs about profits $\{X_t^e\}$, the paths of wages $\{W_t\}$ and beliefs about wages $\{W_t^e\}$, the paths of prices of final goods $\{P_t\}$ and beliefs about prices of final goods $\{P_t^e\}$, as well as the paths of prices of upstream goods $\{\hat{P}_t\}$ and beliefs about these prices $\{\hat{P}_t^e\}$. We define $\Omega_t = (Y_t, X_t, W_t, P_t, \hat{P}_t)$, and $\Omega_t^e = (Y_t^e, X_t^e, W_t^e, P_t^e, \hat{P}_t^e)$.

We assume that at every date t, beliefs about future wages, prices of final goods, and prices of upstream goods at date $t+s$ are scaled by P_t/P_t^e so that they are given by $W_{t+s}^e(P_t/P_t^e)$, $P_{t+s}^e(P_t/P_t^e)$, and $\hat{P}_{t+s}^e(P_t/P_t^e)$. This scaling allows the agents to incorporate the accumulated surprise inflation differential P_t/P_t^e that has already been realized but leaves unchanged beliefs about future relative prices $\hat{P}_{t+s}^e/P_{t+s}^e$ and wages W_{t+s}^e/P_{t+s}^e as well as beliefs about future inflation Π_{t+s}^e.

Technology. Final output is produced from intermediates by competitive firms indexed by $h \in [0,1]$ according to $y_t^h = \left(\int y_t^{hj\frac{\theta-1}{\theta}} dj\right)^{\frac{\theta}{\theta-1}}$. The different varieties of intermediates are produced using the upstream good by monopolisticcally competitive firms indexed by $j \in [0,1]$ according to $y_t^j = \hat{y}_t^j$. Finally, the upstream good is produced competitively from effective labor according to $\hat{Y}_t = N_t^{1-\delta}$, where $N_t = \int z_t^i n_t^i di$ is aggregate effective labor and δ is a measure of decreasing returns to scale. Decreasing returns to scale can be thought as arising from an underlying constant returns production function featuring capital and intermediate goods with strong frictions to the adjustment of capital, a standard assumption in the New Keynesian literature.

Individual firm price setting. The monopolistic firms producing the different varieties of intermediate goods are subject to a price setting friction à la Calvo. They only get a chance to change their price with probability $1-\lambda$ at every date, and these opportunities are independent across firms. A firm that gets a chance to change its price at date $t-1$ can change its price from date t onwards, and then chooses so set it to the following reset price

$$p_t^*(\{R_{t-1+s}\}, \Omega_{t-1}, \{\Omega_{t-1+s}^e\}) = \frac{\theta}{\theta-1} \frac{\sum_{s=0}^{\infty} \frac{\lambda^s}{\prod_{u=0}^{s-1}\left[R_{t+u}(\Pi_{t+u}^e)\right]} Y_{t+s}^e (P_{t+s}^e)^{\theta} \hat{P}_{t+s}^e}{\sum_{s=0}^{\infty} \frac{\lambda^s}{\prod_{u=0}^{s-1}\left[R_{t+u}(\Pi_{t+u}^e)\right]} Y_{t+s}^e (P_{t+s}^e)^{\theta}},$$

where $\theta/(\theta-1) > 1$ is the desired markup, $P_t = [\int (p_t^j)^{1-\theta} dj]^{1/(1-\theta)}$ is the aggregate price index and $\hat{P}_t$ is the price index for the upstream good.

Profits and Lucas trees. Real aggregate profits from the monopolistic intermediate good sector are given by $X_t = Y_t - \frac{\hat{P}_t}{P_t}\hat{Y}_t$. A fraction δX_t are capitalized by Lucas trees, and the remainder $(1-\delta)X_t$ is directly distributed to households in every period. The real aggregate profits $\delta \frac{\hat{P}_t}{P_t} \hat{Y}_t$ of the competitive upstream sector are also capitalized by Lucas trees and can be thought of as the rental income of capital. The trees real fruit for each period is thus given by $\delta X_t + \delta \frac{\hat{P}_t}{P_t} \hat{Y}_t = \delta Y_t$. The value of the trees can be calculated by no arbitrage

$$V_t = \delta Y_t + \frac{\Pi^e_{t+1}}{R_t(\Pi_t)} V^e_{t+1} \quad \forall t \geq 0,$$
$$V^e_t = \delta Y^e_t + \sum_{s=0}^{\infty} \prod_{u=0}^{s} \left[\frac{\Pi^e_{t+1+u}}{R_{t+u}(\Pi^e_{t+u})} \right] \delta Y^e_{t+1+s} \quad \forall t \geq 0. \tag{1}$$

Individual agent problem. We first describe the individual problem. The problem at date t with real financial wealth a^i_t

$$\max_{\{\tilde{c}^i_{t+s}, \tilde{n}^i_{t+s}, \tilde{a}^i_{t+1+s}\}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s [\log(\tilde{c}^i_{t+s}) - \frac{(\tilde{n}^i_t)^{1+\gamma}}{1+\gamma}],$$

subject to the current actual budget constraint

$$\tilde{c}^i_t = \frac{W_t}{P_t} z^i_t \tilde{n}^i_t + \alpha^i_t X_t + a^i_t - \frac{\Pi^e_{t+1}}{R_t(\Pi_t)} \tilde{a}^i_{t+1},$$

the future expected budget constraints

$$\tilde{c}^i_{t+1+s} = \frac{W^e_{t+1+s}}{P^e_{t+1+s}} z^i_{t+1+s} \tilde{n}^i_{t+1+s} + \alpha^i_{t+1+s} X^e_{t+1+s} + \tilde{a}^i_{t+1+s} - \frac{\Pi^e_{t+2+s}}{R_{t+1+s}(\Pi^e_{t+1+s})} \tilde{a}^i_{t+2+s} \quad \forall s \geq 0,$$

and the borrowing constraints

$$\tilde{a}_{t+1+s} \geq 0 \quad \forall s \geq 0,$$

where z^i_t is an idiosyncratic productivity shock and $z^i_t \tilde{n}^i_t$ is effective labor. We assume that this shock follows the process $\log(z^i_t) = \rho_\epsilon \log(z^i_{t-1}) + \epsilon^i_t$ where ϵ^i_t is i.i.d. over time, independent across agents, and follows a normal distribution with variance σ^2_ϵ and mean $\mathbb{E}[\epsilon^i_t] = -\sigma^2_\epsilon (1-\rho^2_\epsilon)^{-1}/2$.

We assume that the share α^i_t of aggregate real profits X_t from the monopolistic intermediate goods sector received by any given agent is proportional to its equilibrium labor income $z^i_t n^i_t$. This means that profits are rebated lump sum so that agents take the profits accruing to them as given when they make their labor supply decisions, since deviations from equilibrium leave α^i_t unchanged. As in Section 4.2, we assume that the borrowing contracts have the same form as the Lucas trees, and that agents cannot borrow. Taken together, these choices ensure that under rational expectations, the incomplete-markets irrelevance result holds, and the interest rate elasticity of output and inflation coincide with those of a complete-markets or representative-agent model.

We denote the policy function for consumption by $c^*(a_t^i, z_t^i; \{R_{t+s}\}, \Omega_t, \{\Omega^e_{t+s}\})$ and the policy function for labor by $n^*(a_t^i, z_t^i; \{R_{t+s}\}, \Omega_t, \{\Omega^e_{t+s}\})$. By analogy with Section 4.2, the law of motion for a_t^i is given

$$a^i_{t+1} = [\frac{R_t(\Pi_t)}{\Pi^e_{t+1}} + \frac{\delta(Y_t - Y^e_t)}{V_t - \delta Y_t}][\frac{W_t}{P_t} z^i_t n^*(a^i_t, z^i_t; \{R_{t+s}\}, \Omega_t, \{\Omega^e_{t+s}\}) + \alpha^i_t X_t + a^i_t \\ - c^*(a^i_t, z^i_t; \{R_{t+s}\}, \Omega_t, \{\Omega^e_{t+s}\})].$$

Temporary, RE, and level-k equilibria. We denote by $\Psi_t = \{a_t^i, z_t^i\}$ the joint distribution of wealth and productivity shocks. The law of motion for Ψ_t is entirely determined by the laws of motion for individual financial wealth and income shocks given an initial condition Ψ_0 with $\int z_0 d\Psi(a_0, z_0) = 1$ and $\int a_0 d\Psi(a_0, z_0) = V_0$, where V_0 is given by the no-arbitrage conditions (1).

Temporary equilibria, RE equilibria, and level-k equilibria are defined in a similar way as in the the general reduced form model described in Section 2. The main differences are that in each of these constructions, we must ensure not only that the goods market clears

$$Y_t = \int_0^1 c^*(a_t, z_t; \{R_{t+s}\}, \Omega_t, \{\Omega^e_{t+s}\}) d\Psi_t(a_t, z_t),$$

but also that the labor market clears

$$N_t = \int_0^1 z_t n^*(a_t, z_t; \{R_{t+s}\}, \Omega_t, \{\Omega^e_{t+s}\}) d\Psi_t(a_t, z_t).$$

We must solve not only for aggregate output $Y_t = \int y_t^h dh$ but also for aggregate effective labor $N_t = \int z_t^i n_t^i di$, the wage W_t, the price of final goods P_t, and the price of upstream goods $\hat{P}_t$. Because it aggregates the prices of intermediate goods producers, the aggreagte price index must follow the difference equation

$$P_t = [(1-\lambda)(p_t^*(\{R_{t-1+s}\}, \Omega_{t-1}, \{\Omega^e_{t-1+s}\}))^{1-\theta} + \lambda (P_{t-1})^{1-\theta}]^{\frac{1}{1-\theta}},$$

with initial condition $P_0 = P$. In addition, because of the optimality condition of the upstream goods producers, we must have

$$\hat{P}_t = W_t \frac{N_t^{\delta}}{1-\delta},$$

and

$$\Delta_t Y_t, = N_t^{1-\delta}$$

where Δ_t is an index of price dispersion which satisfies the difference equation

$$\Delta_t = \lambda \Pi_t^{\theta} \Delta_{t-1} + (1-\lambda) \left[\frac{1-\lambda \Pi_t^{\theta-1}}{1-\lambda} \right]^{\frac{\theta}{\theta-1}},$$

with initial condition $\Delta_0 = 0$, which encapsulates the efficiency costs of misallocation arising from inflation.

The changes required to handle these differences involve the definition of a reduced-form aggregate consumption function and of a reduced-form aggregate effective labor supply function along the lines of the above equations. They also involve the definition of a reduced-form price of upstream goods function, of a reduced-form aggregate price of final goods function, and of a reduced-form aggregate wage function, along the lines of the above equations. The necessary steps are somewhat tedious but conceptually straightforward and so we omit them in the interest of space.