Relational Contracts with Private Information on the Future Value of the Relationship: The Upside of Implicit Downsizing Costs

Online Appendix

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1 Proofs for Section 2

Before proving Proposition 2, we establish the following lemma, which details some characteristics of an optimal solution.

Lemma 12 *Assume that the firm's type is publicly observable. Then, there exists a profit-maximizing equilibrium in which the agent never gets a rent, that is,*

- $qb^h(\theta^t) + (1-q)b^l(\theta^t) = n(\theta^t)c$ *and*
- $w(\theta^t) = 0$ *for every history* θ^t.

Furthermore, equilibrium effort only depends on the current state, that is, $n(\theta^t) = n(\theta_t)$.

Proof: We shall first show that there exists an optimal equilibrium such that $U(\theta^t) = 0$ for all histories θ^t. If $U(\theta^1) > 0$, reduce $w(\theta^1)$ by $U(\theta^1)$. For $t > 1$, assume to the contrary that, in an optimal equilibrium, $U^i(\theta^t) > 0$ for some history θ^t and $i \in \{h, l\}$. Now, reduce $w^i(\theta^t)$ by $U^i(\theta^t)$ and increase the respective bonus in the previous period, $b^i(\theta^t)$, by $\delta U^i(\theta^t)$. Since $-b^i(\theta^t) + \delta\Pi^i(\theta^t)$ and $b^i(\theta^t) + \delta U^i(\theta^t)$ remain unchanged, this change leaves the agent's (IC) constraints as well as all of the principal's constraints at history θ^t and all predecessor histories unaffected. Furthermore, the principal's profits at history θ^t as well as in all predecessor histories remain unchanged. We can thus without loss focus on equilibria such that $U(\theta^t) = 0$ for all histories θ^t.

Now, suppose that there exists a history θ^τ after which the (IC) constraint does not bind. Note that a non-binding (IC) constraint implies that either $b^h(\theta^\tau) > 0$ or $b^l(\theta^\tau) > 0$. Thus, there exists an $\varepsilon > 0$ such that, if either $b^h(\theta^\tau)$ is reduced by $\frac{\varepsilon}{q}$ or $b^l(\theta^\tau)$ by $\frac{\varepsilon}{1-q}$, the (IC) constraint is still satisfied. If $w(\theta^\tau)$ is at the same time increased by ε, the (DE) constraint for history θ^τ is relaxed, and all constraints for all other histories θ^t are unaffected by this change. This adjustment potentially increases profits if (DE) for history θ^τ binds, and leaves profits unaffected if (DE) for history θ^τ is slack, hence is optimal. Thus, we have shown that there exists an optimal equilibrium with the property that $w(\theta^t) = 0$, $U(\theta^t) = 0$, and $qb^h(\theta^t) + (1-q)b^l(\theta^t) = n(\theta^t)c$ for all histories θ^t.

To prove the final part of the Lemma, we first rewrite the (DE) constraint:

$$-n(\theta^t)c + \delta\left(q\Pi^h(\theta^t) + (1-q)\Pi^l(\theta^t)\right) \geq 0. \tag{DE}$$

In addition, note that effort levels will never exceed the first best (otherwise, a reduction would increase profits without violating any of the constraints). Now, assume that there are histories $\theta^{\tilde{\tau}}$ and $\theta^{\overline{\tau}}$, with $n^h(\theta^{\tilde{\tau}}) > n^h(\theta^{\overline{\tau}})$. If the profits being produced in the continuation play following $\left(\theta^{\overline{\tau}}, \theta^h\right)$ are higher, it is possible to implement $n^h(\theta^{\tilde{\tau}})$ with the continuation play following $\left(\theta^{\overline{\tau}}, \theta^h\right)$. In this case, the principal can therefore increase her profits following history $\left(\theta^{\overline{\tau}}, \theta^h\right)$ by increasing the current period's effort level to $n^h(\theta^{\tilde{\tau}})$, while leaving the continuation play unchanged. Now, suppose that it is not possible to implement $n^h(\theta^{\tilde{\tau}})$ with the continuation play following $\left(\theta^{\overline{\tau}}, \theta^h\right)$. This implies that the profits created by the continuation play following $\left(\theta^{\overline{\tau}}, \theta^h\right)$ are lower than the continuation play following $\left(\theta^{\tilde{\tau}}, \theta^h\right)$. Furthermore, because $n^h(\theta^{\tilde{\tau}})$ is enforceable, it is possible to replace the continuation play following $\left(\theta^{\overline{\tau}}, \theta^h\right)$ with the continuation play following $\left(\theta^{\tilde{\tau}}, \theta^h\right)$, thereby relaxing the (DE) constraint in $\overline{\tau}$. It thus becomes possible to increase $n^h(\theta^{\overline{\tau}})$ to $n^h(\theta^{\tilde{\tau}})$. This increases both the principal's current and future profits. A similar argument applies to the low state. Hence, equilibrium effort only depends on the current state. ■

Proof of Proposition 2: To ease the notational burden, we write $n^h \equiv n(\theta^h)$ and $n^l \equiv n(\theta^l)$. The Lagrangian for the firm's problem can be written as

$$\mathcal{L} = \left(\theta^h g(n^h) - n^h c\right)\left(1 + \frac{\delta q}{1-\delta}\right) + \left(\theta^l g(n^l) - n^l c\right)\frac{\delta(1-q)}{1-\delta}$$
$$+ \lambda_{DE_h}\left[-n^h c + \frac{\delta}{1-\delta}\left[q(\theta^h g(n^h) - n^h c) + (1-q)(\theta^l g(n^l) - n^l c)\right]\right]$$
$$+ \lambda_{DE_l}\left[-n^l c + \frac{\delta}{1-\delta}\left[q(\theta^h g(n^h) - n^h c) + (1-q)(\theta^l g(n^l) - n^l c)\right]\right],$$

where λ_{DE_i} denotes the Lagrange multiplier associated with the (DE)-constraint, given the current type is $\theta^i \in \{\theta^l, \theta^h\}$.

By strict concavity of g, the first-order conditions are both necessary and sufficient for an optimum. By the Inada Conditions on g, optimal effort levels are interior, and hence characterized by $\frac{\partial \mathcal{L}}{\partial n^i} = 0$, as well as $\lambda_{DE_i}\frac{\partial \mathcal{L}}{\partial \lambda_{DE_i}} = 0$, for both $i \in \{h, l\}$. One computes

$$\frac{\partial \mathcal{L}}{\partial n^h} = \left(\theta^h g'(n^h) - c\right)\left[1 + \frac{\delta}{1-\delta} q(1 + \lambda_{DE_h} + \lambda_{DE_l})\right] - \lambda_{DE_h} c;$$

$$\frac{\partial \mathcal{L}}{\partial n^l} = \left(\theta^l g'(n^l) - c\right)\frac{\delta}{1-\delta}(1-q)(1 + \lambda_{DE_h} + \lambda_{DE_l}) - \lambda_{DE_l} c.$$

As $n^h \geq n^l$ at an optimum, we know that $\lambda_{DE_h} = 0$ implies $\lambda_{DE_l} = 0$. As our system of equations characterizing the solution $\left(n^h, n^l, \lambda_{DE_h}, \lambda_{DE_l}\right)$ is (jointly) continuous in $\left(n^h, n^l, \lambda_{DE_h}, \lambda_{DE_l}, \delta\right)$, the solutions $\left(n^h, n^l, \lambda_{DE_h}, \lambda_{DE_l}\right)$ can be written as continuous functions of δ. Thus, profits Π^h and Π^l are continuous in δ.

The left-hand sides of the (DE_i) constraints are increasing in δ,[1] hence maximum enforceable effort increases in δ as well.

For $\delta \to 1$, (DE_i) are satisfied for first-best effort levels, since $\theta g(n^{FB}(\theta)) - n^{FB}(\theta)c > 0$ for both $\theta \in \{\theta^h, \theta^l\}$. Thus, there exists a $\bar{\delta} \in [0, 1)$ such that $\lambda_{DE_h} = \lambda_{DE_l} = 0$ for all $\delta > \bar{\delta}$. For $\delta = 0$, no positive effort can be enforced. Thus, $\bar{\delta} > 0$. Moreover, by continuity of the (DE_i)-constraints in δ, for every pair of effort levels (n^h, n^l) between zero and the respective first-best effort levels n_l^{FB} and n_h^{FB}, there exists a discount factor $\delta(n^h, n^l)$ such that the constraint (DE_h) holds for $\delta \geq \delta(n^h, n^l)$ and is violated for $\delta < \delta(n^h, n^l)$. Set $\bar{\delta} = \delta(n_h^{FB}, n_l^{FB})$. Since $n_l^{FB} < n_h^{FB}$, (DE_l) holds with slackness at $n^l = n_l^{FB}$

[1] This can be shown formally by an argument analogous to the one underlying the proof of Lemma 8.

for $\delta = \bar{\delta}$. Let $\underline{n}^h(\delta)$ be defined by $\theta^h g'(\underline{n}^h(\delta)) = c\frac{1-\delta(1-q)}{\delta q}$; as g' is continuous, strictly decreasing and takes on all values in $(0, \infty)$, $\underline{n}^h(\delta)$ exists and is unique; furthermore, the Inverse Function Theorem implies that it is a continuous function of δ. As the partial derivative of (DE$_h$) with respect to n^h is always strictly negative at $n^h = n_h^{FB}$, we have that $\underline{n}^h(\delta) < n_h^{FB}$. Clearly, the solution $\hat{n}^h$ to the optimization problem in which only (DE$_h$) is imposed entails $\hat{n}^h \in [\underline{n}^h(\delta), n_h^{FB}]$. Direct computation shows the partial derivative of (DE$_h$) with respect to n^h to be strictly negative on $(\underline{n}^h(\delta), n_h^{FB})$, while its partial derivative with respect to δ is strictly positive and, since $\delta \leq \bar{\delta} < 1$, bounded. Therefore $\hat{n}^h$ is a continuous function of δ, and thus, by continuity of (DE$_l$) in (n^h, δ), there exists a $\underline{\delta} \in (0, \bar{\delta})$ such that (DE$_l$) continues to hold with slackness for all $\delta \in (\underline{\delta}, \bar{\delta}]$. This implies $n^l = n_l^{FB} < n^h < n_h^{FB}$. For $\delta \leq \underline{\delta}$, both (DE) constraints bind, and hence $n^h = n^l$. ■

2 Proofs for Section 3 (Propositions 3-5)

Proof of Proposition 3: The (EC) constraint to enforce first-best effort levels is given by

$$-n^{FB}(\theta_t)c + \delta\left(q\Pi^{h,FB} + (1-q)\Pi_0^{l,FB}\right) - \delta q g(n_l^{FB})\left(\theta^h - \theta^l\right) \geq 0.$$

The left-hand side can be bounded from below by

$$\begin{aligned}
&- n^{FB}(\theta_t)c + \delta q\Pi^{h,FB} - \delta q g(n_l^{FB})\left(\theta^h - \theta^l\right) \\
\geq &- n^{FB}(\theta_t)c + \delta q\left(\theta^h g(n_h^{FB}) - n_h^{FB}c\right)\left(\frac{1-\delta(1-q)}{1-\delta}\right) \\
&- \delta q g(n_l^{FB})\left(\theta^h - \theta^l\right).
\end{aligned}$$

Since $\theta^h g(n_h^{FB}) - n_h^{FB}c > 0$ by assumption and because $g(n_l^{FB})$ is finite, this expression diverges to infinity as $\delta \to 1$. Since, by Lemma 8, (EC) constraints are relaxed by larger values of δ, the claim follows. ■

Proof of Proposition 4: Define $\overline{\delta} \in (0, 1)$ as the smallest discount factor such that (ECh) holds as an equality for first-best effort levels $n^h = n_h^{FB}$

and $n_i^l = n_l^{FB}$, for all $i \in \mathbb{N}$; i.e., $\overline{\delta}$ is the smallest discount factor such that

$$-n_h^{FB}c + \overline{\delta}\left(q\Pi^{h,FB} + (1-q)\Pi^{l,FB}\right) = \overline{\delta}qg(n_l^{FB})\left(\theta^h - \theta^l\right).$$

Note that given first-best effort levels, (ECh) is continuous in δ. Furthermore, $\overline{\delta} > 0$ follows from no effort being enforceable for $\delta = 0$. Because $n_h^{FB} > n_l^{FB}$, all (ECl) constraints are slack at $\overline{\delta}$ for first-best effort levels.

Now, consider the relaxed problem of maximizing Π^h subject only to (ECh). The Lagrange function for this problem is given by

$$\mathcal{L} = \Pi^h + \lambda_{ECh}\Bigg[-n^hc + \frac{\delta q}{1-\delta(1-q)}\Pi^h + \delta\left(\left(\theta^l - q\theta^h\right)g(n_0^l) - (1-q)n_0^lc\right)$$
$$+\sum_{\tau=1}^{\infty}\left(\delta(1-q)\right)^{\tau+1}\left(\theta^l g(n_\tau^l) - n_\tau^lc\right)\Bigg]$$

where $\Pi^h = \frac{1-\delta(1-q)}{1-\delta}\left(\theta^h g(n^h) - n^hc\right) + \frac{1-\delta(1-q)}{1-\delta}\delta(1-q)\left[\sum_{i=0}^{\infty}\left(\delta(1-q)\right)^i\left(\theta^l g(n_i^l) - n_i^lc\right)\right]$. By our assumptions on g, the objective function and the constraint are twice continuously differentiable in the choice variables $\left(n^h, n_i^l\right)_{i\in\mathbb{N}}$. If $\theta^l \geq q\theta^h$, the Lagrangian is strictly concave in the choice variables, and the first-order conditions are necessary and sufficient for an optimum. If $\theta^l < q\theta^h$, the first-order conditions are necessary for a global optimum.[2]

The first-order conditions for our reduced problem are given by

$$\frac{\partial\mathcal{L}}{\partial n^h} = \left(\theta^h g'(n^h) - c\right)\left(\frac{1-\delta(1-q)}{1-\delta} + \lambda_{ECh}\frac{\delta q}{1-\delta(1-q)}\right) - c\lambda_{ECh} = 0;$$

$$\frac{\partial\mathcal{L}}{\partial n_0^l} = \delta(1-q)\left(\theta^l g'(n_0^l) - c\right)\frac{1-\delta(1-q)}{1-\delta}\left(1+\lambda_{ECh}\right) - \lambda_{ECh}\delta q g'(n_0^l)\left(\theta^h - \theta^l\right) = 0;$$

[2]In this case, one can show that a global optimum exists and that it entails $n^h \in (0, n_h^{FB})$ by substituting the binding (ECh) constraint into the objective. Indeed, considering n_0^l as a function of n^h, one shows that this objective function is strictly concave in n^h, strictly increasing for n^h close to 0, and, given that we can impose without loss that $n_0^l \leq n_l^{FB}$ by Lemma 7, decreasing at $n^h = n_h^{FB}$. Of course, as the global optimum satisfies the first-order conditions, the properties we derive from them apply to the optimum in this case as well.

$$\lambda_{ECh}[-n^h c + \frac{\delta q}{1-\delta(1-q)}\Pi^h$$
$$+\delta\left(\left(\theta^l - q\theta^h\right) g(n_0^l) - (1-q)n_0^l c\right) + \sum_{\tau=1}^{\infty} (\delta(1-q))^{\tau+1}\left(\theta^l g(n_\tau^l) - n_\tau^l c\right)] = 0.$$

Furthermore, optimality requires $\frac{\partial L}{\partial n_i^l} = 0$, implying $\theta^l g'(n_i^l) = c$, for all $i \geq 1$.

Thus, once (ECh) binds and hence $\lambda_{ECh} > 0$, $\theta^h g'(n^h) - c$ must be positive for the respective first-order condition to hold; n^h will thus be below its first-best level. In addition, if $n_0^l > 0$, $\theta^l g'(n_0^l) - c$ must be positive for the first-order condition to hold, so that n_0^l will be below its first-best level as well. Effort levels n_i^l are at their efficient level n_l^{FB} for all $i \geq 1$.

Let $\underline{n}^h(\delta)$ be defined by $\theta^h g'(\underline{n}^h(\delta)) = c\frac{1-\delta(1-q)}{\delta q}$. As g' is continuous, strictly decreasing and takes on all values in $(0, \infty)$, $\underline{n}^h(\delta)$ exists and is unique; furthermore, the Inverse Function Theorem implies that it is a continuous function of δ. Moreover, define $\underline{\tilde{n}}^l(\delta)$ by $g'(\underline{\tilde{n}}^l(\delta)) = c(1-q)\frac{1-\delta(1-q)}{1-\delta}\left[\frac{1-\delta(1-q(1-q))}{1-\delta}\theta^l - q\theta^h\right]^{-1}$ and $\underline{n}^l(\delta)$ by

$$\underline{n}^l(\delta) = \begin{cases} \underline{\tilde{n}}^l(\delta) & \text{if } \frac{1-\delta(1-q(1-q))}{1-\delta}\theta^l - q\theta^h > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Again, as g' is continuous, strictly decreasing and takes on all values in $(0, \infty)$, $\underline{n}^l(\delta)$ exists and is unique; furthermore, the Inverse Function Theorem implies that it is a continuous function of δ. Clearly, the solution $(\hat{n}^h, \hat{n}_0^l)(\delta)$ to the problem in which only (ECh) is imposed entails $(\hat{n}^h, \hat{n}_0^l)(\delta) \in \mathcal{I}$, where $\mathcal{I} := [\underline{n}^h(\delta), n_h^{FB}] \times [\underline{n}^l(\delta), n_l^{FB}]$.[3] Direct computation shows the partial derivatives of (ECh) with respect to n^h and n_0^l respectively to be strictly negative a.e. on $\mathcal{I}$, while, because $\delta \leq \bar{\delta} < 1$, its partial derivative with respect to δ is bounded. Hence, it is feasible to have a policy $(\hat{n}^h, \hat{n}_0^l)$ that is continuous in δ, implying that the optimal profits $\hat{\Pi}^h$ in this problem are a continuous function of δ. As (n^h, n_0^l) impacts the (ECli) constraints only via the profits Π^h, and since these constraints are continuous in Π^h, all (ECli) constraints hold for the solutions of this reduced problem in a neighborhood of $\bar{\delta}$.[4] By the argument underlying

[3]One shows that $\underline{n}^l < n_l^{FB}$ ($\underline{n}^h < n_h^{FB}$) by showing that the partial derivative of (ECh) with respect to n_0^l (n^h) is always strictly negative at $n_0^l = n_l^{FB}$ ($n^h = n_h^{FB}$).

[4]As the only exception, there is a direct impact of $\hat{n}_0^l$ in (ECl0). Yet, as $\hat{n}_0^l \leq n_l^{FB}$,

the proof of Lemma 8, the (ECh) constraint becomes tighter as the discount factor δ decreases. Thus, $\hat{\Pi}^h(\delta)$ is (weakly) increasing. We can thus take $\underline{\delta}$ as low as the discount factor at which the (ECli) constraints, $i \geq 1$, just hold as an equality for $n_i^l = n_l^{FB}$, and $n^h = \hat{n}^h$ and $n_0^l = \hat{n}_0^l$, as characterized by the Kuhn-Tucker system above.

It remains to show that $n^h > n_l^{FB}$. Suppose to the contrary that $n^h \leq n_l^{FB}$. Yet this solution is dominated by $\hat{n}^h = \hat{n}_0^l = n_i^l = n_l^{FB}$, which leads to higher profits and is feasible since all (ECli)-constraints (for $i \geq 1$) hold for $n_i^l = n_l^{FB}$ even for the initial n^h and n_0^l. ■

Proof of Proposition 5: By definition of $\underline{\delta}$, some ECli ($i \geq 1$) will bind in some left-neighborhood of $\underline{\delta}$, while ECl0 remains slack. In this neighborhood, the profit-maximizing n_i^l ($i \geq 1$) are obtained by maximizing Π_1^l.

Thus, we maximize

$$\Pi_1^l = \sum_{\tau=1}^{\infty} (\delta(1-q))^{\tau-1} \left(\theta^l g(n_\tau^l) - n_\tau^l c\right) + \delta q \Pi^h \frac{1}{1-\delta(1-q)}$$

subject to

$$-n_i^l c + \delta q \Pi^h \left(1 + \delta(1-q)\right) + \delta\left(\left(\theta^l - q\theta^h\right) g(n_{i+1}^l) - (1-q) n_{i+1}^l c\right) + \delta^2 (1-q)^2 \Pi_{i+2}^l \geq 0$$

for all $i \geq 1$. We proceed in several steps.

Lemma 13 *For any $i \geq 1$, $\Pi_1^l \geq \Pi_i^l$.*

Proof: Suppose to the contrary that $\Pi_j^l > \Pi_1^l$, for some $j > 1$. For all $i \geq 1$, replace n_i by n_{j+i-1}. (This operation is feasible because all (ECli) were satisfied by assumption.) Thus, our previous Π_1^l cannot solve our maximization problem. ■

Lemma 14 *$n_1^l \geq n_i^l$ for all $i \geq 1$.*

Proof: Suppose to the contrary that there is a $j > 1$ with $n_j^l > n_1^l$. Replace n_j^l with n_1^l and the continuation play following n_j^l with the continuation play following n_1^l. This is clearly feasible and (weakly) profitable (as $\Pi_j^l \leq \Pi_1^l$ by Lemma 13). ■

(ECl0) is slacker than the other (ECli) constraints, and thus continues to hold as well.

Lemma 15 *For all odd* $i \geq 1$, $\Pi_i^l \geq \Pi_{i+2}^l$ *and* $n_i^l \geq n_{i+2}^l$.

Proof: We proceed by induction over i. That $\Pi_3^l \leq \Pi_1^l$ follows from Lemma 13. For the induction step, suppose that $\Pi_j^l \geq \Pi_{j+2}^l$, for some odd integer j. We have to show that $\Pi_{j+2}^l \geq \Pi_{j+4}^l$. Suppose to the contrary that $\Pi_{j+2}^l < \Pi_{j+4}^l$. Now, for all $i \geq j+2$, replace n_i^l by n_{i+2}^l. This is feasible if $n_{j+1}^l \leq n_{j+3}^l$. Therefore, our operation increases Π_{j+2}^l and hence Π_1^l. If $n_{j+1}^l > n_{j+3}^l$, by contrast, we distinguish two cases: (1.) If $\Pi_{j+1}^l \leq \Pi_{j+3}^l$, we can replace n_i^l by n_{i+2}^l for all $i \geq j+1$. This replacement is feasible and weakly increases Π_1^l. (2.) If, however, $\Pi_{j+1}^l > \Pi_{j+3}^l$, we replace n_{i+2}^l by n_i^l for all $i \geq j+1$. This is feasible if $n_j^l \geq n_{j+2}^l$. If, however, $n_j^l < n_{j+2}^l$, we can replace n_{i+2}^l by n_i^l for all $i \geq j$. Because, by the induction hypothesis, $\Pi_j^l \geq \Pi_{j+2}^l$, this increases Π_1^l.

Suppose that $n_j^l < n_{j+2}^l$ for some odd integer j. Replace all n_{i+2}^l by n_i^l for all $i \geq j$. This is clearly feasible and (weakly) profitable (as $\Pi_j^l \geq \Pi_{j+2}^l$).■

Lemma 16 *For all even* $i \geq 2$, $\Pi_i^l \leq \Pi_{i+2}^l$ *and* $n_i^l \leq n_{i+2}^l$.

Proof: Suppose to the contrary that $\Pi_{j+4}^l < \Pi_{j+2}^l$ for some even integer j. Then, we can replace all n_{i+2}^l by n_i^l for all $i \geq j+2$. This is feasible as $n_{j+1}^l \geq n_{j+3}^l$ by Lemma 15. Suppose that $n_j^l > n_{j+2}^l$ for some even integer j. Replace all n_i^l by n_{i+2}^l for all $i \geq j$. This is clearly feasible and (weakly) profitable (as $\Pi_{j+2}^l \geq \Pi_j^l$). ■

Lemma 17 $n_i^l \neq n_{i+2}^l \Rightarrow n_j^l \neq n_{j+2}^l \forall j \leq i$.

Proof: Suppose to the contrary that $n_i^l \neq n_{i+2}^l$ but $n_j^l = n_{j+2}^l$ for some integer $j < i$. Consider the biggest such integer j, i.e., $n_{j+1}^l \neq n_{j+3}^l$. First, assume that j is even, i.e., $j+1$ is odd and, by Lemma 15, $n_{j+1}^l > n_{j+3}^l$. Replace all $n_{\iota+2}^l$ by n_ι^l for all $\iota \geq j+1$. This is feasible as $n_j^l = n_{j+2}^l$ and (weakly) profitable (as $\Pi_{j+1}^l \geq \Pi_{j+3}^l$). Second, assume that j is odd, i.e., $j+1$ is even and, by Lemma 16, $n_{j+1}^l < n_{j+3}^l$. Replace all n_ι^l by $n_{\iota+2}^l$ for all $\iota \geq j+1$. This is feasible as $n_j^l = n_{j+2}^l$ and (weakly) profitable (as $\Pi_{j+1}^l \leq \Pi_{j+3}^l$). ■

Lemma 18 $n_i^l = n_{i+2}^l \Rightarrow n_j^l = n_{j+2}^l \forall j \geq i$.

Proof: Suppose to the contrary that $n_i^l = n_{i+2}^l$ but $n_j^l \neq n_{j+2}^l$ for some integer $j > i$. Consider the smallest such integer j, i.e., $n_{j-1}^l = n_{j+1}^l$. First, assume that $j-1$ is even, i.e., j is odd and, by Lemma 15, $n_j^l > n_{j+2}^l$. Replace all $n_{\iota+2}^l$ by n_ι^l for all $\iota \geq j$. This is feasible as $n_{j-1}^l = n_{j+1}^l$ and (weakly) profitable (as $\Pi_j^l \geq \Pi_{j+2}^l$). Second, assume that $j-1$ is odd, i.e., j is even and, by Lemma 16, $n_j^l < n_{j+2}^l$. Replace all n_ι^l by $n_{\iota+2}^l$ for all $\iota \geq j$. This is feasible as $n_{j-1}^l = n_{j+1}^l$ and (weakly) profitable (as $\Pi_{j1}^l \leq \Pi_{j+2}^l$). ■

Lemma 19 $n_1^l = n_2^l \Rightarrow n_i^l = n_1^l \forall i \geq 1$.

Proof: By Lemma 16, $n_1^l = n_2^l \Rightarrow n_j^l = n_1^l$ for all even j. Hence, by Lemma 18, $n_\iota^l = n_{\iota+2}^l$ for all odd $\iota \geq 3$. Suppose to the contrary that $n_1^l > n_3^l$. Replace n_3^l with n_1^l and the continuation play following n_3^l with the continuation play following n_1^l. This is feasible and (weakly) profitable (as $\Pi_j^l \leq \Pi_1^l$ by Lemma 13). ■

Lemma 20 *Assume there is one i for which the (ECli) constraint is slack. Then, the (ECli+1) constraint binds.*

Proof: To the contrary, assume that the (ECli+1) constraint is slack. Increase n_{i+1}^l by a small $\varepsilon > 0$. This is feasible and increases Π_1^l. ■

Lemma 21 *Assume there is one odd $i > 1$ for which the (ECli) constraint is slack. Then, $n_j^l = n_l^{FB} \forall j \geq 1$.*

Proof: Suppose (ECli) is slack for i odd, with $i > 1$. Then, there must exist an optimum with $n_{i+j} = n_j \forall j \geq 1$. This implies that $\Pi_{i+1}^l = \Pi_1^l$, $\Pi_{i+2}^l = \Pi_2^l$, ..., $\Pi_{2i-1}^l = \Pi_{i-1}^l$. By Lemma 16, $\Pi_2^l \leq \Pi_4^l \leq ... \leq \Pi_{i+1}^l \leq \Pi_{i+3}^l \leq ....$ Since $\Pi_{i+1}^l = \Pi_1^l \leq \Pi_{i+3}^l = \Pi_3^l \leq \Pi_{i+5}^l = \Pi_5^l \leq ... \leq \Pi_{2i}^l = \Pi_i^l \leq ... \leq \Pi_1^l$, $\Pi_j^l = \Pi_1^l$ for all even j.

By Lemma 15, we have $\Pi_1^l \geq \Pi_3^l \geq ... \geq \Pi_i^l = \Pi_{2i}^l = \Pi_1^l$, and hence $\Pi_j^l = \Pi_1^l$ for all odd j. Thus, $n_j^l = n_1^l$ for all $j \geq 1$. Therefore, the Lagrange parameters satisfy $\lambda_j = \lambda_{j+1} = 0$ for all j, and $n_1^l = n_l^{FB}$. ■

Lemma 21 implies that, in our left-neighborhood of $\underline{\delta}$, all odd-numbered constraints will bind, i.e. the Lagrange parameters satisfy$\lambda_j > 0$ for all odd integers j.

Lemma 22 *Assume there is one even i for which the (ECli) constraint is slack. Then, the (EClj) constraints are slack for any even j. Moreover, $n_j^l = n_{j+2}^l = ... = n_1^l$ for all odd j, and $n_\iota^l = n_{\iota+2}^l = ... = n_2^l$ for any even ι.*

Proof: Suppose (ECli) is slack for i even. Then, there must exist an optimum with $n_{i+j} = n_j \forall j \geq 1$. This implies that $\Pi^l_{i+1} = \Pi^l_1$, $\Pi^l_{i+2} = \Pi^l_2$, ..., $\Pi^l_{2i-1} = \Pi^l_{i-1}$. By Lemma 16, $\Pi^l_2 \leq \Pi^l_4 \leq ... \leq \Pi^l_i \leq \Pi^l_{i+2} = \Pi^l_2$, and hence $\Pi_\iota = \Pi^l_2$ for all even ι. It follows that (EClι) is slack for all even ι. Thus, $n^l_\iota = n^l_2$ for all even ι.

By Lemma 15, we have $\Pi^l_1 \geq \Pi^l_3 \geq ... \geq \Pi^l_{i+1} = \Pi^l_1$, and hence $\Pi^l_j = \Pi^l_1$ for all odd j. Thus, $n^l_j = n^l_1$ for all odd j. Therefore, the Lagrange parameters $\lambda_\iota = \lambda_{\iota+2} = 0$ for all even ι. ■

The previous lemmata imply that there are two possibilities for an optimum. Either, all even (ECli) constraints are slack, in which case $n^l_j = n^l_1$ for all odd j and $n^l_\iota = n^l_2$ for all even ι. Otherwise, all (ECli) constraints will bind. In the following, we characterize effort levels n^l_i $(i \geq 1)$ for the latter possibility.

Lemma 23 *Assume all (ECli) constraints bind. Then, either* $n^l_1 = n^l_3 = n^l_5 = ...$ *and* $n^l_2 = n^l_4 = n^l_6 = ...$, *or* $n^l_1 > n^l_3 > n^l_5 > ...$ *and* $n^l_2 < n^l_4 < n^l_6 < ...$.

Proof: To the contrary, assume that $n^l_{j+2} > n^l_j$ for j even, but that $n^l_{j+3} = n^l_{j+1}$. By Lemma 18, this implies that $n^l_{j+2} = n^l_{j+4} = ...$ and $n^l_{j+3} = n^l_{j+5} = ...$, and in particular also that $\Pi^l_{j+2} = \Pi^l_{j+4}$. But then, (ECl$j$) can not bind, a contradiction. The same logic can be applied to show that $n^l_{j+2} < n^l_j$ for j odd, but that $n^l_{j+3} = n^l_{j+1}$, is not feasible. ■

Lemma 24 $n^l_i > n^l_j \Rightarrow \Pi^l_i \geq \Pi^l_j$.

Proof: Suppose to the contrary that there exist integers i and j such that $n^l_i > n^l_j$ yet $\Pi^l_i < \Pi^l_j$. Then,

$$\begin{aligned}
\Pi^l_j - \Pi^l_i =& \left(\theta^l g(n^l_j) - cn^l_j\right) + \delta q \Pi^h + \delta(1-q)\Pi^l_{j+1} \\
&- \left[\left(\theta^l g(n^l_i) - cn^l_i\right) + \delta q \Pi^h + \delta(1-q)\Pi^l_{i+1}\right] \\
=& \left[\left(\theta^l g(n^l_j) - cn^l_j\right) - \left(\theta^l g(n^l_i) - cn^l_i\right)\right] + \delta(1-q)\left(\Pi^l_{j+1} - \Pi^l_{i+1}\right) \\
\geq& 0.
\end{aligned}$$

Because $n^l_i > n^l_j$, $\left[\left(\theta^l g(n^l_j) - cn^l_j\right) - \left(\theta^l g(n^l_i) - cn^l_i\right)\right] < 0$. Therefore, $\Pi^l_{j+1} - \Pi^l_{i+1} > 0$. Hence, replacing the history n^l_i by n^l_j and the continuation play after n^l_i by the continuation play after n^l_j is feasible, and also strictly profitable. ■

Lemma 25 *Suppose all (ECli) constraints bind. Then,* $\sup_{j\in\mathbb{N}} n^l_{2j} \leq \inf_{j\in\mathbb{N}} n^l_{2j-1}$.

Proof: Suppose to the contrary that $\sup_{j\in\mathbb{N}} n^l_{2j} > \inf_{j\in\mathbb{N}} n^l_{2j-1}$. Then, by Lemmata 15 and 16, this implies $\limsup_{j\in\mathbb{N}} n^l_{2j} > \liminf_{j\in\mathbb{N}} n^l_{2j-1}$. Therefore, there exists an integer i such that $n^l_{2i} > n^l_{2i-1} \geq n^l_{2i+1}$. By Lemma 24, this implies that $\Pi^l_{2i} > \Pi^l_{2i+1}$. Yet, as all constraints (EClι), and in particular (ECl$2i-2$), are binding, $n^l_{2i} > n^l_{2i-1}$ and $\Pi^l_{2i} > \Pi^l_{2i+1}$ implies that $n^l_{2i-2} > n^l_{2i-1}$, which, by Lemma 24, implies $\Pi^l_{2i-2} > \Pi^l_{2i-1}$. As furthermore $\Pi^l_{2i} \geq \Pi^l_{2i-2}$ by Lemma 16 and all constraints, in particular (ECl$i-2$) and (ECl$i-3$), are binding, we can conclude that $n^l_{2i-2} > n^l_{2i-3}$ and thus, by Lemma 24, $\Pi^l_{2i-2} > \Pi^l_{2i-3}$. Iterating this argument finally yields $n^l_2 > n^l_1$, a contradiction to Lemma 14. ■

Lemma 26 $n^l_1 = n^l_2 \Leftrightarrow q\theta^h = \theta^l$.

Proof: Recall that for $\delta < \underline{\delta}$, the values n^l_i, $i \geq 1$, can be obtained by maximizing

$\Pi^l_1 = \sum_{\tau=1}^{\infty} (\delta(1-q))^{\tau-1} \left(\theta^l g(n^l_\tau) - n^l_\tau c\right) + \delta q \Pi^h \frac{1}{1-\delta(1-q)}$, s.t. (ECli) constraints for $i \geq 1$, and treating Π^h as a constant. The Lagrange function of this problem is

$$
\begin{aligned}
\mathcal{L} = &\sum_{\tau=1}^{\infty} (\delta(1-q))^{\tau-1} \left(\theta^l g(n^l_\tau) - n^l_\tau c\right) + \delta q \Pi^h \frac{1}{1-\delta(1-q)} \\
&+ \lambda_1 \Bigg[- n^l_1 c + \frac{\delta q}{1-\delta(1-q)} \Pi^h + \delta \left(\left(\theta^l - q\theta^h\right) g(n^l_2) - (1-q) n^l_2 c \right) \\
&\qquad + \sum_{\tau=3}^{\infty} (\delta(1-q))^{\tau-1} \left(\theta^l g(n^l_\tau) - n^l_\tau c\right) \Bigg] \\
&+ \lambda_2 \Bigg[- n^l_2 c + \frac{\delta q}{1-\delta(1-q)} \Pi^h + \delta \left(\left(\theta^l - q\theta^h\right) g(n^l_3) - (1-q) n^l_3 c \right) \\
&\qquad + \sum_{\tau=4}^{\infty} (\delta(1-q))^{\tau-2} \left(\theta^l g(n^l_\tau) - n^l_\tau c\right) \Bigg] \\
&\vdots
\end{aligned}
$$

and first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial n_1^l} = \left(\theta^l g'(n_1^l) - c\right) - \lambda_1 c = 0$$

$$\frac{\partial \mathcal{L}}{\partial n_2^l} = \delta(1-q)\left(\theta^l g'(n_2^l) - c\right) + \lambda_1 \delta\left(\left(\theta^l - q\theta^h\right) g'(n_2^l) - (1-q)c\right) - c\lambda_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial n_3^l} = (\delta(1-q))^2 \left(\theta^l g'(n_3^l) - c\right)(1+\lambda_1) + \lambda_2\left[\delta\left(\left(\theta^l - q\theta^h\right) g'(n_3^l) - (1-q)c\right)\right] - c\lambda_3 = 0$$

$$\frac{\partial \mathcal{L}}{\partial n_4^l} = (\delta(1-q))^3 \left(\theta^l g'(n_4^l) - c\right)\left(1+\lambda_1 + \frac{\lambda_2}{\delta(1-q)}\right) + \lambda_3 \delta\left(\left(\theta^l - q\theta^h\right) g'(n_4^l) - (1-q)c\right) - c\lambda_4 = 0$$

$$\vdots$$

$\frac{\partial \mathcal{L}}{\partial n_1^l} = 0$ yields that $n_1^l < n_l^{FB}$ for $\lambda_1 > 0$ which holds for $\delta < \underline{\delta}$. Plugging $\lambda_1 = \frac{\left(\theta^l g'(n_1^l) - c\right)}{c}$ into $\frac{\partial \mathcal{L}}{\partial n_2^l} = 0$ yields

$$\delta(1-q)\theta^l\left(g'(n_2^l) - g'(n_1^l)\right) + \frac{\left(\theta^l g'(n_1^l) - c\right)}{c}\delta\left(\theta^l - q\theta^h\right) g'(n_2^l) - c\lambda_2 = 0.$$

Therefore $q\theta^h > \theta^l$ implies $n_1^l > n_2^l$. To show that $q\theta^h = \theta^l \Rightarrow n_1^l = n_2^l$, we first assume that $\lambda_2 = 0$ and verify later that it holds.

If $\lambda_2 = 0$, the condition gives $n_1^l = n_2^l$. Furthermore, if $\lambda_2 = 0$, Lemma 22 implies that $n_1^l = n_3^l = ...$ and $n_2^l = n_4^l = ...$ Then, $\Pi_1^l = \frac{\left(\theta^l g(n_1^l) - n_1^l c\right) + \delta(1-q)\left(\theta^l g(n_2^l) - n_2^l c\right)}{1-(\delta(1-q))^2} + \delta q \Pi^h \frac{1}{1-\delta(1-q)}$, and the (binding) (ECl1) constraint equals

$$-n_1^l c + \frac{\delta q}{1-\delta(1-q)}\Pi^h + \delta\left(\left(\theta^l - q\theta^h\right) g(n_2^l) - (1-q)n_2^l c\right) + \frac{(\delta(1-q))^2}{1-(\delta(1-q))^2}\left[\left(\theta^l g(n_1^l) - n_1^l c\right) + \delta(1-q)\left(\theta^l g(n_2^l) - n_2^l c\right)\right] = 0$$

Plugging

$$
\begin{aligned}
&\frac{\delta q}{1-\delta(1-q)}\Pi^h \\
=&n_1^l c - \delta\left(\left(\theta^l - q\theta^h\right) g(n_2^l) - (1-q) n_2^l c\right) \\
&- \frac{(\delta(1-q))^2}{1-(\delta(1-q))^2}\left[\left(\theta^l g(n_1^l) - n_1^l c\right) + \delta(1-q)\left(\theta^l g(n_2^l) - n_2^l c\right)\right]
\end{aligned}
$$

into (ECl2) gives

$$
\left(g(n_1^l) - g(n_2^l)\right)\delta\left(\theta^l - q\theta^h - q\delta(1-q)\left(\theta^h - \theta^l\right)\right) + c\left(n_1^l - n_2^l\right) \geq 0.
$$

For $q\theta^h = \theta^l$ and $n_1^l = n_2^l$, the left hand side equals zero, hence (ECl2) is satisfied. ■

This concludes the proof of Proposition 5. ■

3 Timing – Details

3.1 θ_t Revealed at Beginning of Period t

First, we analyze the case of public information. There, we consider a quasi-stationary equilibrium in the sense that bonus and effort are only a function of today's type. The wage might be a function of today's and yesterday's type, if it is used to provide incentives for yesterday's effort. We use left and right superscripts to describe wages (and profits) as functions of θ_{t-1} (left) and θ_t (right). For example, if the type in both periods is high, profit is ${}^h\Pi^h$ and wages are ${}^h w^h$. Profits can thus be written as

$$
\begin{aligned}
{}^h\Pi^h &= \theta^h g(n^h) - b^h - {}^h w^h + \delta\overline{\Pi}^h \\
{}^l\Pi^h &= \theta^h g(n^h) - b^h - {}^l w^h + \delta\overline{\Pi}^h \\
{}^h\Pi^l &= \theta^l g(n^l) - b^l - {}^h w^l + \delta\overline{\Pi}^l \\
{}^l\Pi^l &= \theta^l g(n^l) - b^l - {}^l w^l + \delta\overline{\Pi}^l,
\end{aligned}
$$

with $\overline{\Pi}^h = q\,^h\Pi^h + (1-q)\,^h\Pi^l$ and $\overline{\Pi}^l = q\,^l\Pi^h + (1-q)\,^l\Pi^l$. The agent's utilities are described accordingly.

We maximize $\overline{\Pi}^h$, subject to the following constraints:[5]

$$-n^h c + b^h + \delta\overline{U}^h \geq 0 \tag{ICh}$$

$$-n^l c + b^l + \delta\overline{U}^l \geq 0 \tag{ICl}$$

$$-b^h + \delta\overline{\Pi}^h \geq 0 \tag{DEh}$$

$$-b^l + \delta\overline{\Pi}^l \geq 0. \tag{DEl}$$

First, we show that it is weakly optimal only to use the bonus to provide incentives, while setting wages equal to zero: If any fixed wages were strictly positive, a reduction accompanied by a corresponding increase of the respective bonus would leave all constraints unaffected (for example, if $^hw^h > 0$, reducing $^hw^h$ by a small $\varepsilon > 0$ and increasing b^h by $\delta q\varepsilon$ has no effect on ICh and DEh) and not decrease profits. Furthermore, as in Lemma 12, we can show that it is feasible and optimal to set $b^h = n^h c$ and $b^l = n^l c$. Then, the two remaining constraints are

$$-n^h c + \delta\frac{q\left(\theta^h g(n^h) - n^h c\right) + (1-q)\left(\theta^l g(n^l) - n^l c\right)}{(1-\delta)} \geq 0$$

$$-n^l c + \delta\frac{q\left(\theta^h g(n^h) - n^h c\right) + (1-q)\left(\theta^l g(n^l) - n^l c\right)}{(1-\delta)} \geq 0,$$

which are the same as in our main setting with public information. Therefore, profit-maximizing effort levels are also characterized by Proposition 2, with levels of the discount factor, $\overline{\delta}$ and $\underline{\delta}$ ($0 < \underline{\delta} < \overline{\delta} < 1$), such that $n^h = n_h^{FB}$ and $n^l = n_l^{FB}$ for $\delta \geq \overline{\delta}$; $n^l = n_l^{FB} < n^h < n_h^{FB}$ for $\underline{\delta} < \delta < \overline{\delta}$; and $n^h = n^l \leq n_l^{FB}$ for $\delta \leq \underline{\delta}$.

[5]Note that maximizing any other of the above profit streams would yield identical outcomes because the equilibrium – as we will see below – is now sequentially efficient.

3.2 θ_{t+1} Revealed at Beginning of Period t

3.2.1 Public Types

We use superscripts to indicate equilibrium values as functions of this and next period's types. For example, n^{hh} is equilibrium effort in case today's and tomorrow's types are high, n^{hl} is equilibrium effort if today's type is high and tomorrow's type is low, and so on. By standard arguments, with public information it is without loss to analyze equilibria where, after all histories, actions depend only on today's and tomorrow's types. In the following, we call these equilibria *quasi-stationary*.

Then, on-path profit streams can take one of the four values

$$\begin{aligned}
\Pi^{hh} &= \theta^h g(n^{hh}) - w^{hh} - b^{hh} + \delta\overline{\Pi}^h \\
\Pi^{hl} &= \theta^h g(n^{hl}) - w^{hl} - b^{hl} + \delta\overline{\Pi}^l \\
\Pi^{lh} &= \theta^l g(n^{lh}) - w^{lh} - b^{lh} + \delta\overline{\Pi}^h \\
\Pi^{ll} &= \theta^l g(n^{ll}) - w^{ll} - b^{ll} + \delta\overline{\Pi}^l,
\end{aligned}$$

where $\overline{\Pi}^h \equiv q\Pi^{hh} + (1-q)\Pi^{hl}$ and $\overline{\Pi}^l \equiv q\Pi^{lh} + (1-q)\Pi^{ll}$. The agent's utilities are defined equivalently. Bonus payments are bounded by dynamic enforcement constraints,

$$-b^{hh} + \delta\overline{\Pi}^h \geq 0 \tag{DEhh}$$

$$-b^{hl} + \delta\overline{\Pi}^l \geq 0 \tag{DEhl}$$

$$-b^{lh} + \delta\overline{\Pi}^h \geq 0 \tag{DElh}$$

$$-b^{ll} + \delta\overline{\Pi}^l \geq 0, \tag{DEll}$$

whereas effort levels are bounded by incentive compatibility constraints,

$$-n^{hh}c + b^{hh} + \delta\overline{U}^h \geq 0 \tag{IChh}$$

$$-n^{hl}c + b^{hl} + \delta\overline{U}^l \geq 0 \tag{IChl}$$

$$-n^{lh}c + b^{lh} + \delta\overline{U}^h \geq 0 \tag{IClh}$$

$$-n^{ll}c + b^{ll} + \delta\overline{U}^l \geq 0. \tag{ICll}$$

Now, although the bonus is a function of next period's type, it is certain at the time of the agent's effort choice. This is different from the main part of our paper, where next period's type is revealed immediately before today's bonus is paid, and therefore *uncertain* at the time of the agent's effort choice. Furthermore, for reasons similar to above (Lemma 12), it is feasible and weakly optimal to set $w^{hh} = w^{hl} = w^{lh} = w^{ll} = 0$ and let (IC) constraints hold as equalities. Therefore, $b^{hh} = n^{hh}c$, $b^{hl} = n^{hl}c$, $b^{lh} = n^{lh}c$ and $b^{ll} = n^{ll}c$.

Again, our objective is to maximize $\overline{\Pi}^h$, now subject to

$$-n^{hh}c + \delta\overline{\Pi}^h \geq 0 \tag{DEhh}$$

$$-n^{hl}c + \delta\overline{\Pi}^l \geq 0 \tag{DEhl}$$

$$-n^{lh}c + \delta\overline{\Pi}^h \geq 0 \tag{DElh}$$

$$-n^{ll}c + \delta\overline{\Pi}^l \geq 0. \tag{DEll}$$

3.2.2 Private Types

With private types, we keep the notation from our analysis with public types (proof of Lemma 1). Though this restriction is not without loss of generality here, we continue to focus on the same kind of quasi-stationary equilibria as with public types (that is, actions depend only on today's and tomorrow's types), where fixed wages equal zero and (IC) constraints bind. We will show below that, in contrast to before, the relevant truth-telling constraints can now either be satisfied by a reduction of effort levels, or by an ex-ante payment made to the agent. If these payments can be extracted by the principal at the beginning of the game, such an agreement would indeed maximize the principal's profits.

Now, two types of truth-telling constraints arise. First, the principal might misreport her type and then proceed with play as prescribed by equilibrium (like in our main case). This yields the constraints

$$\Pi^{hh} \geq \tilde{\Pi}^{hl} \tag{TThh}$$

$$\Pi^{hl} \geq \tilde{\Pi}^{hh} \tag{TThl}$$

$$\Pi^{lh} \geq \tilde{\Pi}^{ll} \tag{TTlh}$$

$$\Pi^{ll} \geq \tilde{\Pi}^{lh}, \tag{TTll}$$

where

$$\tilde{\Pi}^{hh} = \theta^h g(n^{hh}) - n^{hh}c + \delta\left[q\left(\theta^l g(n^{hh}) - n^{hh}c + \delta\overline{\Pi}^h\right) + (1-q)\left(\theta^l g(n^{hl}) - n^{hl}c + \delta\overline{\Pi}^l\right)\right]$$
$$= \Pi^{hh} - \delta\left(\theta^h - \theta^l\right)\left[qg(n^{hh}) + (1-q)g(n^{hl})\right]$$

are the principal's profits in case today's type is high and tomorrow's type is low, but where she falsely reports tomorrow's type to be high.

The respective values $\tilde{\Pi}^{hl}$, $\tilde{\Pi}^{lh}$ and $\tilde{\Pi}^{ll}$ are obtained in similar fashion. The second kind of truth-telling constraints prevent the principal from misreporting her type and subsequently shutting down.

These constraints are

$$\theta^h g(n^{hh}) - n^{hh}c + \delta\overline{\Pi}^h \geq \theta^h g(n^{hl}) \tag{TThh2}$$
$$\theta^h g(n^{hl}) - n^{hl}c + \delta\overline{\Pi}^l \geq \theta^h g(n^{hh}) \tag{TThl2}$$
$$\theta^l g(n^{lh}) - n^{lh}c + \delta\overline{\Pi}^h \geq \theta^l g(n^{ll}) \tag{TTlh2}$$
$$\theta^l g(n^{ll}) - n^{ll}c + \delta\overline{\Pi}^l \geq \theta^l g(n^{lh}) \tag{TTll2}$$

Note that these kinds of constraints are not needed in our main case. There, next period's type is revealed after today's effort and output have been realized. They are thus sunk when the principal's announces next period's type. Therefore, these constraints coincide with the respective dynamic enforcement constraints.

Finally, (DE) constraints as specified in the proof to Lemma 1, the case with public information, must hold.

4 Impermanent Shocks – Details

Besides the q^h/q^l notation introduced in the main text, we shall also write $q(\theta_t)$ for the probability of next period's type being high given the current-period type θ_t. We focus on a subset of the parameter space for which our solution is qualitatively similar to our previous results, with overshooting and gradual recovery.

Here, the truth-telling and dynamic enforcement constraints amount to

$$-b^h(\theta^t) + \delta\Pi^h(\theta^t) \geq -b^l(\theta^t) + \delta\tilde{\Pi}^l(\theta^t) \tag{TTh}$$

$$-b^l(\theta^t) + \delta\Pi^l(\theta^t) \geq -b^h(\theta^t) + \delta\tilde{\Pi}^h(\theta^t) \tag{TTl}$$

$$-b^h(\theta^t) + \delta\Pi^h(\theta^t) \geq 0 \tag{DEh}$$

$$-b^l(\theta^t) + \delta\Pi^l(\theta^t) \geq 0, \tag{DEl}$$

with

$$\begin{aligned}\Pi^h(\theta^t) =& \theta^h g(n^h(\theta^t)) - w^h(\theta^t) \\ &+ q^h\left(-b^{hh}(\theta^t) + \delta\Pi^{hh}(\theta^t)\right) + (1-q^h)\left(-b^{hl}(\theta^t) + \delta\Pi^{hl}(\theta^t)\right),\end{aligned}$$

$$\begin{aligned}\Pi^l(\theta^t) =& \theta^l g(n^l(\theta^t)) - w^l(\theta^t) \\ &+ q^l\left(-b^{lh}(\theta^t) + \delta\Pi^{lh}(\theta^t)\right) + (1-q^l)\left(-b^{ll}(\theta^t) + \delta\Pi^{ll}(\theta^t)\right),\end{aligned}$$

$$\begin{aligned}\tilde{\Pi}^l(\theta^t) =& \theta^h g(n^l(\theta^t)) - w^l(\theta^t) \\ &+ q^h\left(-b^{lh}(\theta^t) + \delta\Pi^{lh}(\theta^t)\right) + (1-q^h)\left(-b^{ll}(\theta^t) + \delta\Pi^{ll}(\theta^t)\right)\end{aligned}$$

and

$$\begin{aligned}\tilde{\Pi}^h(\theta^t) =& \theta^l g(n^h(\theta^t)) - w^h(\theta^t) \\ &+ q^l\left(-b^{hh}(\theta^t) + \delta\Pi^{hh}(\theta^t)\right) + (1-q^l)\left(-b^{hl}(\theta^t) + \delta\Pi^{hl}(\theta^t)\right).\end{aligned}$$

Note that our formulations of $\tilde{\Pi}^h(\theta^t)$ and $\tilde{\Pi}^l(\theta^t)$ again make use of the One-deviation principle (see Hendon, Jacobsen, and Sloth (1996)).

Now, we maximize $\Pi(\theta^1)$ subject to the (TTh), (DEl) and (IC) constraints (and omit the (DEh) and (TTl) constraints).

Combining (TTh) and (DEl) yields the following (EC) constraints, which are necessary (but may not be sufficient) for equilibrium:

$$-q^h b^h(\theta^t) - \left(1-q^h\right) b^l(\theta^t) + \delta q^h \left(\Pi^h(\theta^t) - \tilde{\Pi}^l(\theta^t)\right) + \delta \Pi^l(\theta^t) \geq 0, \quad \text{(ECh)}$$

$$-q^l b^h(\theta^t) - \left(1-q^l\right) b^l(\theta^t) + \delta q^l \left(\Pi^h(\theta^t) - \tilde{\Pi}^l(\theta^t)\right) + \delta \Pi^l(\theta^t) \geq 0. \quad \text{(ECl)}$$

It is straightforward to verify that, if δ is large enough, first-best effort levels satisfy these constraints. In contrast to the case of iid shocks, (ECl) might bind for higher discount factors than (ECh) constraints. This is because autocorrelated shocks make not only first-best, but also implementable, effort a function of today's state of the world. We shall, however, focus on the case that (ECh) binds before (ECl) does, as we did for permanent shocks.

While (ECl) constraints can thus be omitted, (TTh) constraints (which constitute one part of (ECl) constraints) will bind for all subsequent histories. Indeed, suppose to the contrary that there exists a subsequent history, $\hat{\theta}^{t+\tau}$, such that (TTh) at $\hat{\theta}^{t+\tau}$ is slack, and (EC) binds. Increase $b^h(\hat{\theta}^{t+\tau})$ by some $\varepsilon > 0$ and reduce $w(\hat{\theta}^{t+\tau})$ by $q(\hat{\theta}_{t+\tau})\varepsilon$. This relaxes the (EC) constraint at history θ^t and leaves all other (EC) constraints unaffected.

From this observation, it follows that we can plug binding (TTh) constraints into (EC), and rewrite the latter as

$$\begin{aligned}&-q^h b^h(\theta^t) - \left(1-q^h\right) b^l(\theta^t) + \delta\left(q^h \Pi^h(\theta^t) + (1-q^h)\Pi^l(\theta^t)\right)\\ &\geq \delta q^h\left(\theta^h - \theta^l\right)\left\{g(n^l(\theta^t)) + \delta\left(q^h - q^l\right)\left[g(n^{ll}(\theta^t)) + \delta\left(q^h-q^l\right)\left(g(n^{lll}(\theta^t)) + \ldots\right)\right]\right\}.\end{aligned}$$

By the same argument as in the proof of Lemma 6, it follows that $n^h(\theta^t)$ will be the same for all θ^t. By the same token, low-type effort can be written as n_i^l, where the i indicates the number of consecutive low periods immediately preceding period t along a given history θ^t.

Furthermore having (IC) constraints hold as equalities and using $U(\theta^t) = 0$ for all θ^t, we solve

$$\begin{aligned}&\max_{n^h, n_i^l} \Pi^h\\ &= \left(1-\delta(1-q^l)\right) \frac{\theta^h g(n^h) - n^h c + \delta(1-q^h)\sum_{i=0}^{\infty}\left(\delta(1-q^l)\right)^i \left(\theta^l g(n_i^l) - n_i^l c\right)}{(1-\delta)\left(1-\delta\left(q^h-q^l\right)\right)},\end{aligned} \tag{1}$$

subject to

$$-n^h c + \delta q^h \Pi^h + \delta\left(1-q^h\right)\Pi_0^l \geq \delta q^h\left(\theta^h-\theta^l\right)\sum_{i=0}^{\infty}\left[\delta\left(q^h-q^l\right)\right]^i g(n_i^l) \quad \text{(ECh)}$$

Proposition 9 *The solution to the constrained maximization problem* (1) *has the following features: There exists a* $\delta^h < 1$ *such that*

- *for* $\delta \geq \delta^h$, $n^h = n_h^{FB}$ *and* $n_i^l = n_l^{FB}$ *for all* i*;*
- *for discount factors in some left neighborhood of* δ^h, $n^h < n_h^{FB}$*. Furthermore, for all* $i \in \mathbb{N}$, $n_i^l < n_{i+1}^l < n_l^{FB}$*, with* $\lim_{i\to\infty} n_i^l = n_l^{FB}$*.*

Proof: Note that

$$\Pi_0^l = \sum_{i=0}^{\infty}\left(\delta(1-q^l)\right)^i\left(\theta^l g(n_i^l) - n_i^l c\right) + \delta q^l \frac{\left(\theta^h g(n^h) - n^h c\right) + \delta(1-q^h)\sum_{i=0}^{\infty}\left(\delta(1-q^l)\right)^i\left(\theta^l g(n_i^l) - n_i^l c\right)}{(1-\delta)\left(1-\delta\left(q^h-q^l\right)\right)}$$

and that the (ECh) constraint can be rewritten to

$$-n^h c + \delta\Pi^h \frac{q^h - \delta\left(q^h-q^l\right)}{1-\delta(1-q^l)} + \delta\left(1-q^h\right)\sum_{i=0}^{\infty}\left(\delta(1-q^l)\right)^i\left(\theta^l g(n_i^l) - n_i^l c\right)$$
$$\geq \delta q^h\left(\theta^h-\theta^l\right)\sum_{i=0}^{\infty}\left[\delta\left(q^h-q^l\right)\right]^i g(n_i^l) \quad \text{(ECh)}$$

Denoting by λ the Lagrange parameter associated with the (ECh) constraint, the Lagrange function equals

$$
\begin{aligned}
\mathcal{L} = & \left(1-\delta(1-q^l)\right) \\
& \times \frac{\left(\theta^h g(n^h) - n^h c\right) + \delta(1-q^h)\sum_{i=0}^{\infty}\left(\delta(1-q^l)\right)^i\left(\theta^l g(n_i^l) - n_i^l c\right)}{(1-\delta)\left(1-\delta\left(q^h-q^l\right)\right)} \\
& \qquad \times \left(1 + \delta\frac{q^h - \delta\left(q^h-q^l\right)}{1-\delta(1-q^l)}\lambda\right) \\
& \quad + \lambda\Bigg[- n^h c + \delta\left(1-q^h\right)\sum_{i=0}^{\infty}\left(\delta(1-q^l)\right)^i\left(\theta^l g(n_i^l) - n_i^l c\right) \\
& \qquad - \delta q^h\left(\theta^h-\theta^l\right)\sum_{i=0}^{\infty}\left[\delta\left(q^h-q^l\right)\right]^i g(n_i^l)\Bigg],
\end{aligned}
$$

yielding first-order conditions

$$
\frac{\partial \mathcal{L}}{\partial n^h} = \left(\theta^h g'(n^h) - c\right)\frac{\left(1-\delta(1-q^l)\right)}{(1-\delta)\left(1-\delta\left(q^h-q^l\right)\right)}\left(1+\delta\frac{q^h-\delta\left(q^h-q^l\right)}{1-\delta(1-q^l)}\lambda\right) - \lambda c = 0 \tag{2}
$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial n_i^l} = & \delta^{i+1}\Bigg\{\left(\theta^l g'(n_i^l) - c\right)\left(\frac{\left(1-\delta(1-q^l)\right)\left(1-q^h\right)}{(1-\delta)\left(1-\delta\left(q^h-q^l\right)\right)}\left(1-q^l\right)^i(1+\lambda)\right) \\
& - \lambda q^h\left(q^h-q^l\right)^i\left(\theta^h-\theta^l\right)g'(n_i^l)\Bigg\} = 0
\end{aligned} \tag{3}
$$

The existence of a δ^h, with $\lambda = 0$ for $\delta \geq \delta^h$, follows from the enforceability of first-best effort levels for $\delta \to 1$.

Now, consider a left neighborhood of δ^h where n_h^{FB} and n_l^{FB} do not satisfy (ECh), and thus $\lambda > 0$. Condition (2) gives $n^h < n_h^{FB}$, whereas (3) gives $n_i^l < n_l^{FB}$. Condition (3) also implies that $\lim_{i\to\infty} n_i^l = n_l^{FB}$: Since $\left(q^h-q^l\right)^i < 1$, $\lim_{i\to\infty}\left(q^h-q^l\right)^i = 0$, hence $\lim_{i\to\infty}\left(\theta^l g'(n_i^l) - c\right) = 0$.

To show that $n_i^l < n_{i+1}^l$, rewrite conditions (3) for n_i^l and for n_{i+1}^l as

$$
\begin{aligned}
& \left(\theta^l g'(n_i^l) - c\right)\frac{\left(1-\delta(1-q^l)\right)\left(1-q^h\right)\left(1-q^l\right)^i}{(1-\delta)\left(1-\delta\left(q^h-q^l\right)\right)} \\
= & -\lambda\left[\left(\theta^l g'(n_i^l) - c\right)\frac{\left(1-\delta(1-q^l)\right)\left(1-q^h\right)}{(1-\delta)\left(1-\delta\left(q^h-q^l\right)\right)}\left(1-q^l\right)^i - q^h\left(q^h-q^l\right)^i\left(\theta^h-\theta^l\right)g'(n_i^l)\right]
\end{aligned}
$$

$$
\left(\theta^l g'(n^l_{i+1}) - c\right) \frac{\left(1-\delta(1-q^l)\right)\left(1-q^h\right)\left(1-q^l\right)^{i+1}}{(1-\delta)\left(1-\delta\left(q^h-q^l\right)\right)}
$$

$$
= -\lambda \left[\left(\theta^l g'(n^l_{i+1}) - c\right) \frac{\left(1-\delta(1-q^l)\right)\left(1-q^h\right)}{(1-\delta)\left(1-\delta\left(q^h-q^l\right)\right)} \left(1-q^l\right)^{i+1} - q^h \left(q^h-q^l\right)^{i+1} \left(\theta^h-\theta^l\right) g'(n^l_{i+1}) \right]
$$

Dividing the first by the second equality and reformulating yields the necessary condition

$$
\frac{\left(\theta^l g'(n^l_{i+1}) - c\right)}{\left(\theta^l g'(n^l_i) - c\right)} = \frac{\left(q^h - q^l\right)}{(1-q^l)} \frac{g'(n^l_{i+1})}{g'(n^l_i)}.
$$

As $\frac{\left(q^h-q^l\right)}{\left(1-q^l\right)} < 1$, this implies $\frac{\left(\theta^l g'(n^l_{i+1})-c\right)}{\left(\theta^l g'(n^l_i)-c\right)} < \frac{g'(n^l_{i+1})}{g'(n^l_i)}$. This is equivalent to $g'(n^l_i) > g'(n^l_{i+1})$, which yields $n^l_{i+1} > n^l_i$ due to the strict concavity of $g(\cdot)$. ■

Proposition 9 suggests that recovery may be gradual and never complete, as in the case of permanent shocks. The solution to the maximization underlying this proposition constitutes an equilibrium for parameter values such that the (ECl) and (TTl) constraints hold at the solution. While we can show that this is the case for an open, non-empty, subset of the parameter space, we leave a complete characterization of this subset outside the scope of this paper.

Concerning the intuition of this result, recall that with persistent shocks, falsely claiming that the type is low forces the principal to stick to announcing the low state forever thereafter. This is not the case with persistent, impermanent, shocks. Indeed, by the One-deviation-principle (Hendon, Jacobsen, and Sloth (1996)), the costs of a deviation today are increasing in the size of tomorrow's high-type bonus b^h – because the likelihood of having to pay b^h is larger off the equilibrium path. Therefore, tomorrow's high-type bonus b^{lh} is set as high as feasible, bounded as it is by the respective truth-telling constraint. This truth-telling constraint is again relaxed by a large high-type bonus the day after tomorrow, b^{llh}, and so on. In contrast to the iid case, these consecutively binding truth-telling constraints make it optimal to distort later n^l_i as well. Because of discounting and the decreasing difference between on-path and off-path likelihoods of having to pay high-type bonuses, these distortions decrease with i, and eventually vanish, as for permanent shocks.

5 Proofs for Section 4

Proof of Proposition 6: If the type is the principal's private information, additional truth-telling constraints, now imposed at the beginning of a period, must hold:

$$\theta^h g(n^h) - b^h - {}^h w^h + \delta\overline{\Pi}^h \geq \theta^h g(n^l) - b^l - {}^h w^l + \delta\overline{\Pi}^l \tag{TThh}$$
$$\theta^h g(n^h) - b^h - {}^l w^h + \delta\overline{\Pi}^h \geq \theta^h g(n^l) - b^l - {}^l w^l + \delta\overline{\Pi}^l \tag{TTlh}$$
$$\theta^l g(n^l) - b^l - {}^h w^l + \delta\overline{\Pi}^l \geq \theta^l g(n^h) - b^h - {}^h w^h + \delta\overline{\Pi}^h \tag{TThl}$$
$$\theta^l g(n^l) - b^l - {}^l w^l + \delta\overline{\Pi}^l \geq \theta^l g(n^h) - b^h - {}^l w^h + \delta\overline{\Pi}^h \tag{TTll}$$

To show that these constraints can be omitted, we plug the results from the case with public information, ${}^h w^h = {}^h w^l = {}^l w^h = {}^l w^l = 0$ and $b^h = n^h c$ and $b^l = n^l c$, into the conditions. Then, $\overline{\Pi}^h = \overline{\Pi}^l = \frac{q\left(\theta^h g(n^h) - n^h c\right) + (1-q)\left(\theta^l g(n^l) - n^l c\right)}{(1-\delta)}$, and the constraints become

$$\theta^h g(n^h) - n^h c \geq \theta^h g(n^l) - n^l c \tag{TThh}$$
$$\theta^h g(n^h) - n^h c \geq \theta^h g(n^l) - n^l c \tag{TTlh}$$
$$\theta^l g(n^l) - n^l c \geq \theta^l g(n^h) - n^h c \tag{TThl}$$
$$\theta^l g(n^l) - n^l c \geq \theta^l g(n^h) - n^h c \tag{TTll}$$

which are satisfied for the respective effort levels. ■

Proof of Lemma 1: It is immediate that $\overline{\Pi}^h \geq \overline{\Pi}^l$, i.e. that a high type is associated with higher profits. Therefore, $\theta_{t+1} = \theta^h$ allows for a credible promise of a higher bonus, and therefore for the implementation of a higher effort level, in period t. The *desired* effort levels if today's type is high (n^{hh} and n^{hl}) are also larger than if today's type is low. If the discount factor is sufficiently close to 1, none of the constraints bind and first-best levels $n^{hh} = n^{hl} = n_h^{FB}$ and $n^{ll} = n^{lh} = n_l^{FB}$ can be implemented. For a lower discount factor, (DEhl) will eventually bind, and $n^{hl} < n^{hh} = n_h^{FB}$. For even lower discount factors, (DEhh) and/or (DEll) will at some point bind as well. This yields the result. ■

Proof of Proposition 7: First, we show that $n^{hh} = n^{hl} \equiv n^h$ *and* $n^{lh} = n^{ll} \equiv n^l$. To do so, we omit (TT) constraints and solve the problem only subject to (TT2) and (DE) constraints. Then, we show that the solution to this relaxed problem also satisfies (TT) constraints.

The reduced problem maximizes $\overline{\Pi}^h$, subject to

$$-n^{hh}c + \delta\overline{\Pi}^h \geq 0 \quad \text{(DEhh)}$$
$$-n^{hl}c + \delta\overline{\Pi}^l \geq 0 \quad \text{(DEhl)}$$
$$-n^{lh}c + \delta\overline{\Pi}^h \geq 0 \quad \text{(DElh)}$$
$$-n^{ll}c + \delta\overline{\Pi}^l \geq 0. \quad \text{(DEll)}$$

$$\theta^h g(n^{hh}) - n^{hh}c + \delta\overline{\Pi}^h \geq \theta^h g(n^{hl}) \quad \text{(TThh2)}$$
$$\theta^h g(n^{hl}) - n^{hl}c + \delta\overline{\Pi}^l \geq \theta^h g(n^{hh}) \quad \text{(TThl2)}$$
$$\theta^l g(n^{lh}) - n^{lh}c + \delta\overline{\Pi}^h \geq \theta^l g(n^{ll}) \quad \text{(TTlh2)}$$
$$\theta^l g(n^{ll}) - n^{ll}c + \delta\overline{\Pi}^l \geq \theta^l g(n^{lh}) \quad \text{(TTll2)}$$

Note that effort is never above the respective first-best effort level. Now, assume to the contrary that $n^{hh} > n^{hl}$. If (DEhl) binds, plugging $-n^{hl}c+\delta\overline{\Pi}^l = 0$ into (TThl2) yields $\theta^h g(n^{hl}) \geq \theta^h g(n^{hh})$ which is violated for $n^{hh} > n^{hl}$. If (DEhl) does not bind, increase n^{hl} by a small $\varepsilon > 0$. This operation increases $\overline{\Pi}^h$, relaxes (TThl2), and does not violate (TThh2), (DEhl) or any other constraint. Continue until either $n^{hh} = n^{hl}$ or (DEhl) binds. In the latter case, recall that (TThl2) is violated for a binding (DEhl) constraint and $n^{hh} > n^{hl}$.

Next, assume $n^{hh} < n^{hl}$. If (DEhh) binds, plugging $-n^{hh}c + \delta\overline{\Pi}^h = 0$ into (TThh2) yields $\theta^h g(n^{hh}) \geq \theta^h g(n^{hl})$ which is violated for $n^{hh} < n^{hl}$. If (DEhh) does not bind, increase n^{hh} by a small $\varepsilon > 0$. This operation increases $\overline{\Pi}^h$, relaxes (TThh2), and does not violate (TThl2) and (DEhh) or any other constraint. Continue until either $n^{hh} = n^{hl}$ or (DEhh) binds. In the latter case, recall that (TThh2) is violated for a binding (DEhh) constraint and $n^{hh} > n^{hl}$.

Thus, we have shown that $n^{hh} = n^{hl} \equiv n^h$ in this reduced problem. Accordingly, it can be shown that $n^{lh} = n^{ll} \equiv n^l$. Taking this into account, the remaining constraints in the reduced problem are

$$-n^h c + \delta\overline{\Pi}^h \geq 0 \tag{DEhh}$$

$$-n^h c + \delta\overline{\Pi}^l \geq 0 \tag{DEhl}$$

$$-n^l c + \delta\overline{\Pi}^h \geq 0 \tag{DElh}$$

$$-n^l c + \delta\overline{\Pi}^l \geq 0. \tag{DEll}$$

Together with $\overline{\Pi}^h = \left(\theta^h g(n^h) - n^h c\right) + \delta q \overline{\Pi}^h + \delta(1-q)\overline{\Pi}^l$ and $\overline{\Pi}^l = \left(\theta^l g(n^l) - n^l c\right) + \delta q \overline{\Pi}^h + \delta(1-q)\overline{\Pi}^l$, this implies that $n^h \geq n^l$ and $\overline{\Pi}^h > \overline{\Pi}^l$, which allows us to omit (DEhh) and (DElh), and leaves the remaining constraints

$$-n^h c + \delta\overline{\Pi}^l \geq 0 \tag{DEh}$$

$$-n^l c + \delta\overline{\Pi}^l \geq 0. \tag{DEl}$$

Therefore, effort levels to this constrained maximization problem are given by discount factors, $\overline{\delta}$ and $\underline{\delta}$, with $0 < \underline{\delta} < \overline{\delta} < 1$, with

$n^h = n_h^{FB}$ and $n^l = n_l^{FB}$ for $\delta \geq \overline{\delta}$

$n^l = n_l^{FB} < n^h < n_h^{FB}$ for $\underline{\delta} < \delta < \overline{\delta}$

$n^l = n^h \leq n_l^{FB}$ for $\delta \leq \underline{\delta}$

To complete the proof, we have to show that these effort levels do not violate any of the (TT) constraints. These amount to

$$\left(\theta^h g(n^{hh}) - n^{hh}c + \delta\overline{\Pi}^h\right) - \left(\theta^h g(n^{hl}) - n^{hl}c + \delta\overline{\Pi}^l\right) - \delta\left(\theta^h - \theta^l\right)\left[q g(n^{lh}) + (1-q) g(n^{ll})\right] \geq 0 \tag{TThh}$$

$$-\left[\left(\theta^h g(n^{hh}) - n^{hh}c + \delta\overline{\Pi}^h\right) - \left(\theta^h g(n^{hl}) - n^{hl}c + \delta\overline{\Pi}^l\right)\right] + \delta\left(\theta^h - \theta^l\right)\left[q g(n^{hh}) + (1-q) g(n^{hl})\right] \geq 0 \tag{TThl}$$

$$\left(\theta^l g(n^{lh}) - n^{lh}c + \delta\overline{\Pi}^h\right) - \left(\theta^l g(n^{ll}) - n^{ll}c + \delta\overline{\Pi}^l\right) - \delta\left(\theta^h - \theta^l\right)\left[q g(n^{lh}) + (1-q) g(n^{ll})\right] \geq 0 \tag{TTlh}$$

$$- \left[\left(\theta^l g(n^{lh}) - n^{lh}c + \delta\overline{\Pi}^h\right) - \left(\theta^l g(n^{ll}) - n^{ll}c + \delta\overline{\Pi}^l\right)\right]$$
$$+ \delta\left(\theta^h - \theta^l\right)\left[qg(n^{hh}) + (1-q)g(n^{hl})\right] \geq 0. \quad \text{(TTll)}$$

Plugging $n^{hh} = n^{hl} = n^h$ and $n^{lh} = n^{ll} = n^l$ into the (TT) constraints, rearranging and making use of $\overline{\Pi}^h - \overline{\Pi}^l = \left(\theta^h g(n^h) - n^h c\right) - \left(\theta^l g(n^l) - n^l c\right)$, yields an equivalence of (TThh) and (TTlh), as well as of (TThl) and (TTll). Therefore, the remaining (TT) constraints are

$$\delta\left[\left(\theta^h g(n^h) - n^h c\right) - \left(\theta^h g(n^l) - n^l c\right)\right] \geq 0 \qquad \text{(TThh)}$$
$$\delta\left[\left(\theta^l g(n^h) - n^h c\right) - \left(\theta^l g(n^l) - n^l c\right)\right] \leq 0 \qquad \text{(TThl)}$$

For $\delta > \underline{\delta}$, these conditions hold strictly (the latter because n_l^{FB} maximizes $\theta^l g(n^l) - n^l c$), for $\delta \leq \underline{\delta}$ and hence $n^l = n^h$ they hold as equalities. ■

Proof of Proposition 8: We first omit (DEl) constraints and show ex post that they hold at the solutions of the relaxed problem. Denoting by λ the Lagrange parameter associated with the (ECh) constraint, the Lagrange function equals

$$\mathcal{L} = \frac{\theta^h g(n^h) - n^h c + \delta(1-q)\sum_{i=0}^{\infty}\delta^i\left(\theta^l g(n_i^l) - n_i^l c\right)}{1-\delta q}(1+\delta q\lambda)$$
$$+ \lambda\left[-n^h c + \sum_{i=0}^{\infty}\delta^{i+1}\left[\left((1-q)\,\theta^l - \left(\theta^h - \theta^l\right)q^{i+1}\right)g(n_i^l) - (1-q)\,n_i^l c\right]\right],$$

yielding first-order conditions

$$\frac{\partial\mathcal{L}}{\partial n^h} = \frac{\theta^h g'(n^h) - c}{1-\delta q}(1+\delta q\lambda) - \lambda c = 0 \qquad (4)$$

$$\frac{\partial\mathcal{L}}{\partial n_i^l} = \delta^{i+1}\left\{\left(\theta^l g'(n_i^l) - c\right)\left(\frac{(1-q)}{1-\delta q}(1+\delta q\lambda) + \lambda(1-q)\right)\right.$$
$$\left. -\lambda q^{i+1}\left(\theta^h - \theta^l\right)g'(n_i^l)\right\} = 0 \qquad (5)$$

$\delta < \delta^h$ implies $\lambda > 0$. Hence, condition (4) gives $n^h < n_h^{FB}$, whereas (5) gives $n_i^l < n_l^{FB}$. Condition (5) also implies that $\lim_{i\to\infty} n_i^l = n_l^{FB}$: Since $q < 1$,

$\lim_{i\to\infty} q^{i+1} = 0$, hence $\theta^l g'(n_i^l) - c = 0$.

To show that $n_i^l < n_{i+1}^l$, rewrite conditions (5) for n_i^l and for n_{i+1}^l as
$\left(\theta^l g'(n_i^l) - c\right) \frac{(1-q)}{1-\delta q} = -\lambda \left[\frac{(1-q)}{1-\delta q} \left(\theta^l g'(n_i^l) - c\right) - q^{i+1} \left(\theta^h - \theta^l\right) g'(n_i^l)\right]$
$\left(\theta^l g'(n_{i+1}^l) - c\right) \frac{(1-q)}{1-\delta q} = -\lambda \left[\frac{(1-q)}{1-\delta q} \left(\theta^l g'(n_{i+1}^l) - c\right) - q^{i+2} \left(\theta^h - \theta^l\right) g'(n_{i+1}^l)\right]$.
Dividing the first by the second equality yields the necessary condition

$$\frac{\theta^l g'(n_i^l) - c}{\theta^l g'(n_{i+1}^l) - c} = \frac{\frac{(1-q)}{1-\delta q} \left(\theta^l g'(n_i^l) - c\right) - q^{i+1} \left(\theta^h - \theta^l\right) g'(n_i^l)}{\frac{(1-q)}{1-\delta q} \left(\theta^l g'(n_{i+1}^l) - c\right) - q^{i+2} \left(\theta^h - \theta^l\right) g'(n_{i+1}^l)},$$

which becomes

$$q^{i+1} \left(\theta^h - \theta^l\right) \frac{g'(n_i^l) \left(\theta^l g'(n_{i+1}^l) - c\right) - \left(\theta^l g'(n_i^l) - c\right) q g'(n_{i+1}^l)}{\left(\theta^l g'(n_{i+1}^l) - c\right) \left[\frac{(1-q)}{1-\delta q} \left(\theta^l g'(n_{i+1}^l) - c\right) - q^{i+2} \left(\theta^h - \theta^l\right) g'(n_{i+1}^l)\right]} = 0$$

The denominator of this expression must be different from zero:

$\left(\theta^l g'(n_{i+1}^l) - c\right) > 0$ because $n_{i+1}^l < n_l^{FB}$. The term in squared brackets must be strictly negative: It captures the partial derivative of the left hand side of the (ECh) constraint with respect to n_{i+1}^l. If it were positive, a larger value of n_{i+1}^l (which is feasible) would relax the (ECh) constraint, contradicting that it binds. Therefore, the term is zero if and only if its numerator is zero, yielding

$$\frac{\left(\theta^l g'(n_{i+1}^l) - c\right)}{\left(\theta^l g'(n_i^l) - c\right)} = q \frac{g'(n_{i+1}^l)}{g'(n_i^l)}.$$

As $q < 1$, this implies $\frac{\left(\theta^l g'(n_{i+1}^l) - c\right)}{\left(\theta^l g'(n_i^l) - c\right)} < \frac{g'(n_{i+1}^l)}{g'(n_i^l)}$. This is equivalent to $g'(n_i^l) > g'(n_{i+1}^l)$, which yields $n_{i+1}^l > n_i^l$ due to the strict concavity of $g(\cdot)$.

Finally, note that the derived n_i^l satisfy all (DEli) constraints, $-n_i^l c + \delta \Pi_{i+1}^l \geq 0$. Since $n_{i+1}^l > n_i^l \forall i$, $\Pi_{i+1}^l > \frac{\theta^l g(n_i^l) - n_i^l c}{1-\delta}$, hence it is sufficient to show that

$-n_i^l c + \delta \frac{\theta^l g(n_i^l) - n_i^l c}{1-\delta} \geq 0$, that is $-n_i^l c + \delta \theta^l g(n_i^l) \geq 0$, holds. Because $\delta \geq \delta^l$, this condition would hold for $n_i^l = n_l^{FB}$. Because $g(\cdot)$ is strictly increasing and concave, and because $g(0) = 0$, $-n_l^{FB} c + \delta \theta^l g(n_l^{FB}) \geq 0$ implies that this also holds for all $n_i^l < n_l^{FB}$. ■

6 Proofs for Appendix A

Proof of Lemma 2: Adding (DEl) and (TTh) gives $-b^h(\theta^t) + \delta\Pi^h(\theta^t) \geq \delta g(n^l(\theta^t)) \left(\theta^h - \theta^l\right)$. Since the right hand side is positive, this implies (DEh).■

Proof of Lemma 3: Assume there is a history θ^τ where both constraints bind simultaneously even though $n^h(\theta^\tau) \neq n^l(\theta^\tau)$. Then, (TTh) implies $b^h(\theta^\tau) = b^l(\theta^\tau) + \delta\Pi^h(\theta^\tau) - \delta\tilde{\Pi}^l(\theta^\tau)$. Plugging this into the binding (TTl) constraint yields $g(n^l(\theta^\tau)) \left(\theta^h - \theta^l\right) = g(n^h(\theta^\tau)) \left(\theta^h - \theta^l\right)$. Since $\theta^h - \theta^l > 0$ and g is strictly increasing, this contradicts the claim that both constraints bind for $n^h(\theta^\tau) \neq n^l(\theta^\tau)$. ■

Proof of Lemma 4: We start with proving the first two parts. Suppose to the contrary that there exists a history θ^t of length $t \geq 1$ and an equilibrium such that, following history θ^t, the principal is strictly better off in this equilibrium than in any equilibrium satisfying points 1.-2. We show by construction that this cannot be the case.

1. Assume that, in an optimal equilibrium, $U^i(\theta^t) > 0,\ i \in \{h, l\}$ for some history θ^t of length t. Reduce $w^i(\theta^t)$ by $U^i(\theta^t)$ and increase the respective bonus in the previous period, $b^i(\theta^t)$, by $\delta U^i(\theta^t)$. Since $-b^i(\theta^t) + \delta\Pi^i(\theta^t)$ and $b^i(\theta^t) + \delta U^i(\theta^t)$ remain unchanged, this change leaves the agent's (IC) and (IR) constraints as well as all of the principal's constraints at history θ^t and all predecessor histories unaffected. Furthermore, the principal's profits at history θ^t as well as in all predecessor histories remain unchanged. Repeat this step for all histories of length t and of length $t+1$.

2. Assume that $\Pi^h(\theta^t) < \tilde{\Pi}^l(\theta^t)$. Replace play after (θ^t, θ^h) by play after (θ^t, θ^l). This leads to on-path profits of $\hat{\Pi}^h(\theta^t) = \tilde{\Pi}^l(\theta^t)$. Set $b^h_{new}(\theta^t) = b^l_{new}(\theta^t) = n(\theta^t)c$, while increasing $w(\theta^t)$ by $\delta q \left(\hat{\Pi}^h(\theta^t) - \Pi^h(\theta^t)\right) + q\left(b^h_{old}(\theta^t) - b^h_{new}(\theta^t)\right) + (1-q)\left(b^l_{old}(\theta^t) - b^l_{new}(\theta^t)\right)$.(By Step 1. and the fact that (IC) at history θ^t holds, this increase is weakly larger than $q\delta\left(\hat{\Pi}^h(\theta^t) - \Pi^h(\theta^t)\right)$.) (TTh), (TTl) and (IC) at history θ^t now hold with equality. Previous constraints remain unchanged, with the exception of previous (IC)-constraints, which are relaxed. It remains to be

shown that the (DEl)-constraint at history θ^t continues to hold. As the proof of Lemma 5 shows, the fact that (DEl) and (TTh) previously held at history θ^t, together with Step 1, implies

$$-n(\theta^t)c + \delta\left\{q\left[\Pi^h(\theta^t) - \tilde{\Pi}^l(\theta^t)\right]\Pi^l(\theta^t)\right\} \geq 0.$$

As $\Pi^h(\theta^t) < \tilde{\Pi}^l(\theta^t)$, this implies $-n(\theta^t)c + \delta\Pi^l(\theta^t) \geq 0$, which was to be shown.

Furthermore, we can show (for later use) that, for histories θ^t such that $n^h(\theta^t) \leq n^l(\theta^t)$, $\Pi^l(\theta^t) \geq \tilde{\Pi}^h(\theta^t)$. To the contrary, assume that $\Pi^l(\theta^t) < \tilde{\Pi}^h(\theta^t)$. Replace play after (θ^t, θ^l) by play after (θ^t, θ^h). This leads to on-path profits of $\hat{\Pi}^l(\theta^t) = \tilde{\Pi}^h(\theta^t)$. Set $b^h_{new}(\theta^t) = b^l_{new}(\theta^t) = n(\theta^t)c$, while increasing $w(\theta^t)$ by $\delta(1-q)\left(\hat{\Pi}^l(\theta^t) - \Pi^l(\theta^t)\right) + q\left(b^h_{old}(\theta^t) - b^h_{new}(\theta^t)\right) + (1-q)\left(b^l_{old}(\theta^t) - b^l_{new}(\theta^t)\right)$. (TTh), (TTl) and (IC) at history θ^t now hold with equality. Previous constraints remain unchanged, with the exception of previous (IC) and (IR) constraints, which are relaxed. It remains to be shown that (DEl)-constraint at history θ^t continues to hold. As the proof of Lemma 2 shows, the fact that (DEl) and (TTh) previously held at history θ^t, together with Step 1, implies

$$-n(\theta^t)c + \delta\left\{q\left[\Pi^h(\theta^t) - (\theta^h - \theta^l)g(n^l(\theta^t))\right] + (1-q)\Pi^l(\theta^t)\right\} \geq 0.$$

As $\Pi^l(\theta^t) < \Pi^h(\theta^t) - (\theta^h - \theta^l)g(n^h(\theta^t)) = \tilde{\Pi}^h(\theta^t)$, this implies $-n(\theta^t)c + \delta\Pi^h(\theta^t) \geq \delta(\theta^h - \theta^l)\left(qg(n^l(\theta^t) + (1-q)g(n^h(\theta^t))\right)$. As $n^h(\theta^t) \leq n^l(\theta^t)$, this implies $-n(\theta^t)c + \delta\Pi^h(\theta^t) \geq \delta(\theta^h - \theta^l)g(n^h(\theta^t))$, or $-n(\theta^t)c + \delta\hat{\Pi}^l(\theta^t) \geq 0$, which was to be shown.

After Operation 2., we have to repeat Operations 1. As Operations 1. leave profits and effort levels unchanged, there is no need to repeat Operation 2. after that. Furthermore, we can repeat these operations for all histories of length t and after that for all histories of length $t-1$, $t-2$, $\cdots$. Finally, assume $U(\theta^1) > 0$. Reducing $w(\theta^1)$ by $U(\theta^1)$ increases $\Pi(\theta^1)$ and only affects the agent's first-period (IR) constraint, which continues to hold.

To show that $b^h(\theta^t) \geq b^l(\theta^t)$ for all histories θ^t, assume to the contrary that there exists a history θ^t such that $b^h(\theta^t) < b^l(\theta^t)$. Because of part 2, this implies that (TTh) is slack. Increase $b^h(\theta^t)$ by a small $\varepsilon > 0$ and reduce $b^l(\theta^t)$ by $\frac{q}{1-q}\varepsilon$. This leaves all (IC) constraints unaffected and relaxes the (DEl) and (TTl) constraints at history θ^t. (TTh) is tightened, while nonetheless remaining slack as long as $b^h(\theta^t) < b^l(\theta^t)$. Finally, all constraints and profits at predecessor histories remain unchanged.

We now show that the (TTl) constraint can be omitted and the (IC) constraint will bind. If $n^h(\theta^t) \leq n^l(\theta^t)$, it immediately follows from the fact that $b^h(\theta^t) \geq b^l(\theta^t)$ and $\Pi^l(\theta^t) \geq \tilde{\Pi}^h(\theta^t)$ that (TTl) can be omitted. So suppose that $n^h(\theta^t) > n^l(\theta^t)$, and suppose that the (TTl) constraint binds. By Lemma 3, this implies that the (TTh) constraint is slack. We can therefore increase $b^h(\theta^t)$ by a small $\varepsilon > 0$ while decreasing $w(\theta^t)$ by $q\varepsilon$. This leaves all previous constraints and profits unaffected yet relaxes the current (IC) and (TTl) constraints (while tightening the current (TTh) constraint and leaving the current (DEl) constraint unaffected). Now suppose that the (IC) constraint is slack. If $b^l(\theta^t) > 0$, we can decrease $b^h(\theta^t) > 0$ and $b^l(\theta^t) > 0$ by some $\varepsilon > 0$, while increasing $w(\theta^t)$ by ε. This leaves all previous profits as well as all previous and current constraints unaffected, with the exception of the current (DEl)-constraint, which is relaxed. If now $b^l(\theta^t) = 0$ and the (IC) and (TTl) constraints are slack, we can decrease $b^h(\theta^t)$ by some $\varepsilon > 0$, while increasing $w(\theta^t)$ by $\frac{\varepsilon}{q}$. This leaves all previous constraints and profits unaffected, yet relaxes the current (TTh) constraint (while tightening the current (TTl) and (IC) constraints and leaving the current (DEl) constraint unaffected). If $b^l(\theta^t) = 0$ and the (TTl) constraint binds, we can replace play after (θ^t, θ^l) by play after (θ^t, θ^h) while setting $b^h_{new}(\theta^t) = b^l_{new}(\theta^t) = n(\theta^t)c$ and increasing $w(\theta^t)$ by $(1-q)(\Pi^l_{new}(\theta^t) - \Pi^l_{old}(\theta^t)) + qb^h_{old}(\theta^t) - n(\theta^t)c$. As $\Pi^l_{new}(\theta^t) = \tilde{\Pi}^h(\theta^t) \geq \tilde{\Pi}^h(\theta^t) - \frac{b^h_{old}(\theta^t)}{\delta} = \Pi^l_{old}(\theta^t)$, and $b^h_{old}(\theta^t) \geq n(\theta^t)c$ by the (IC) constraint, the increase in $w(\theta^t)$ is positive. Therefore, previous (IC) and (IR) constraints are relaxed while all other previous constraints remain unaffected by our change. Furthermore, the current (TTh), (TTl) and (IC) constraints all hold with equality by construction. It remains to show that the current (DEl) constraint continues to hold, i.e. that $-n(\theta^t)c + \delta\Pi^l_{new}(\theta^t) = -n(\theta^t)c + \delta\tilde{\Pi}^h(\theta^t) \geq 0$. Yet, the binding (TTl) implies that $\delta\Pi^l_{old}(\theta^t) = -b^h_{old}(\theta^t) + \delta\tilde{\Pi}^h(\theta^t) \geq 0$, which implies that the current (DEl) constraint will hold after our change, as $b^h_{old}(\theta^t) \geq \frac{n(\theta^t)c}{q} \geq n(\theta^t)c$ by the (IC) constraint.

Because $U(\theta^t) = w(\theta^t) - n(\theta^t)c + qb^h(\theta^t) + (1-q)b^l(\theta^t) = 0$, a binding (IC) constraint implies that $w(\theta^t) = 0$ for all histories θ^t. ■

Proof of Lemma 5: By Lemma 4, we can without loss focus on equilibria in which

$$n(\theta^t)c = qb^h(\theta^t) + (1-q)b^l(\theta^t) \tag{6}$$

at every history θ^t. Using (6) and multiplying (TTh) with q and adding it to (DEl) yields (EC).

To prove that (EC) implies (TTh) and (DEl) given (6), assume that we are at an optimum satisfying the properties of Lemma 4 and that (EC) holds. We shall now show that it is always possible to find non-negative bonus payments $b^h(\theta^t)$ and $b^l(\theta^t)$ such that (6) holds, and that (DEl) and (TTh) are both satisfied. Toward this purpose, we set $b^l(\theta^t) = \min\left\{\delta\Pi^l(\theta^t), n(\theta^t)c\right\} \geq 0$. First suppose that $n(\theta^t)c \leq \delta\Pi^l(\theta^t)$. In this case, we set $b^h(\theta^t) = n(\theta^t)c$. Now, (DEl) will trivially hold (with slackness if $n(\theta^t)c < \delta\Pi^l(\theta^t)$). Using $b^h(\theta^t) = n(\theta^t)c$ in (TTh) yields $\delta\Pi^h(\theta^t) \geq \delta g(n^l(\theta^t))\left(\theta^h - \theta^l\right) + \delta\Pi^l(\theta^t)$, which is implied by the second part of Lemma 4. Now suppose that $n(\theta^t)c > \delta\Pi^l(\theta^t)$. In this case, we set $b^h(\theta^t) = \frac{1}{q}\left[n(\theta^t)c - \delta(1-q)\Pi^l(\theta^t)\right] > 0$. Clearly, (DEl) will trivially hold with equality (because $b^l(\theta^t) = \delta\Pi^l(\theta^t)$). Substituting $b^h(\theta^t)$ into (TTh) yields $\frac{1}{q}$ times (EC). ■

Proof of Lemma 6: Consider an optimum satisfying the properties of Lemmas 4 and 5. Suppose that there exists a history θ^t such that $\Pi^h(\theta^t) < \max_{\hat{\theta}^\tau}\Pi^h(\hat{\theta}^\tau)$. Replace the continuation play following $\left(\theta^t, \theta^h\right)$ by the continuation play following $\left(\tilde{\theta}, \theta^h\right)$, where $\tilde{\theta} \in \operatorname{argmax}_{\hat{\theta}^\tau}\Pi^h(\hat{\theta}^\tau)$. By virtue of our iid assumption, this is feasible. This increases profits and relaxes some (EC) constraints without tightening any previous ones. This establishes that $\Pi^h(\theta^t) = \overline{\Pi}^h$ for all θ^t (if two different continuation plays lead to $\operatorname{argmax}_{\hat{\theta}^\tau}\Pi^h(\hat{\theta}^\tau)$, we select one to be played after all histories $\left(\theta^t, \theta^h\right)$). Therefore, there exists an optimum in which for any history θ^t, $n^h(\theta^t) = \overline{n}^h$ and $n^l(\theta^t) = n^l_{i(\theta^t)}$. ■

Proof of Lemma 7:

Consider an optimum satisfying the properties of Lemmas 4, 5 and 6. Suppose there exists a history θ^t such that $n(\theta^t) > n^{FB}(\theta_t)$. Reduce $n(\theta^t)$ by a small $\varepsilon > 0$. This increases profits and relaxes the (EC) constraints at all predecessor histories. ■

Proof of Lemma 8: Consider a given discount factor $\hat{\delta}$ and the associated sequence of optimal actions $\left(n^h(\hat{\delta}), n^l_i(\hat{\delta})\right)_{i\in\mathbb{N}}$. We first show that a higher δ relaxes (EC) constraints; i.e., for any discount factor $\tilde{\delta} > \hat{\delta}$, previously optimal actions $n^h(\hat{\delta})$ and $n_i^l(\hat{\delta})$ continue to satisfy the (EC) constraints. We show this by induction over the number of periods, starting from the first period, in which the discount factor rises from $\hat{\delta}$ to $\tilde{\delta}$. First, suppose only the discount factor between the first and the second period rises. The (EC) constraint in the first period can be written as $-n^h c + \delta q\left[\Pi^h - g(n_0^l)\left(\theta^h - \theta^l\right)\right] + \delta(1-q)\Pi_0^l \geq 0$. In Lemma 4 we showed that, at our optimum, $\Pi^h(\theta^t) \geq \Pi^l(\theta^t) + g(n^l(\theta^t))\left(\theta^h - \theta^l\right)$ for all histories θ^t. Since $\Pi^l(\theta^t) \geq 0$, the term in square brackets is non-negative. Hence, (EC) in period 1 becomes slacker, and the actions that were optimal at the discount factor $\hat{\delta}$ can still be enforced at the higher discount factor $\tilde{\delta}$. By Lemma 7, these actions lead to (weakly) higher profits. The argument for the induction step is analogous. ■

7 Proofs for Appendix B

Proof of Lemma 9: Suppose to the contrary that a policy $\sigma = \left(n^h, n_i^l\right)_{i\in\mathbb{N}}$ such that $n^h < \sup_{i\in\mathbb{N}} n_i^l =: \bar{n}^l$ is optimal. Then, as $n(\theta^t) \leq n^{FB}(\theta_t)$, the policy $\hat{\sigma} = \left(\hat{n}^h, \hat{n}_i^l\right)_{i\in\mathbb{N}}$ given by $\hat{n}^h = \hat{n}_i^l = \bar{n}^l$, $\hat{w}^h = \hat{w}_i^l = 0$, and $\hat{b}^h = \hat{b}_i^l = \bar{n}c$, for all $i \in \mathbb{N}$ leads to higher profits $\hat{\Pi}^h > \Pi^h$ and $\hat{\Pi}^l \geq \Pi_i^l$ ($i \in \mathbb{N}$), where $\hat{\Pi}^h$ (Π^h) and $\hat{\Pi}^l$ (Π_i^l) are the profits associated with policy $\hat{\sigma}$ (σ), respectively. As policy σ satisfies all (DEli)-constraints, we have that $-n_i^l c + \delta\hat{\Pi}^l \geq -n_i^l c + \delta\Pi_i^l \geq 0$. This implies $-\bar{n}^l c + \delta\hat{\Pi}^l \geq 0$, i.e., the policy $\hat{\sigma}$ satisfies all (DEli)-constraints. Moreover, (TTh) and (TTl) hold with equality. This is a contradiction to policy σ being optimal. ■

Proof of Lemma 10: Lemma 9 implies that, if both (TTh) and (TTl) bind, $n^h = n_\tau^l = \bar{n}$ for all $\tau \in \mathbb{N}$. In this case, $b^h = b_0^l \geq \bar{n}c$.

Now, if there exists a $\tau \in \mathbb{N}$ such that $n^h > n_\tau^l$, Lemma 9 implies that $\frac{g(n^h)}{1-\delta q} > \sum_{\tau=0}^{\infty}(\delta q)^\tau g(n_\tau^l)$. Suppose that (TTl) binds. As $\frac{g(n^h)}{1-\delta q} > \sum_{\tau=0}^{\infty}(\delta q)^\tau g(n_\tau^l)$, (TTh) is slack. We can therefore increase b^h by a small $\varepsilon > 0$ while decreasing w^h by $q\varepsilon$. This leaves all constraints and profits unaffected yet relaxes the (IC) and (TTl) constraints (while tightening the (TTh) constraint and leaving the (DEli) constraints unaffected). Now suppose that the (IC) constraint is slack. If $b_0^l > 0$, we can decrease $b^h > 0$ and $b_0^l > 0$ by some $\varepsilon > 0$, while increasing w^h by ε. This leaves profits as well as all constraints unaffected, with the exception of the (DEl0)-constraint, which is relaxed. If now $b_0^l = 0$ and the (IC) and (TTl) constraints are slack, we can decrease b^h by some $\varepsilon > 0$, while increasing w^h by $\frac{\varepsilon}{q}$. This leaves all constraints and profits unaffected, yet relaxes the (TTh) constraint (while tightening the (TTl) and (IC) constraints and leaving the (DEl0) constraint unaffected). If $b_0^l = 0$ and the (TTl) constraint binds, we can replace n_τ^l by n^h for all $\tau \in \mathbb{N}$ while setting $b_{new}^h = b_{\tau,new}^l = n^h c$. The (TTh), (TTl) and (IC) constraints all hold with equality by construction. It remains to show that the (DEli) constraints continue to hold, i.e. that $-\bar{n}c + \delta\Pi_{i,new}^l = -\bar{n}c + \delta\tilde{\Pi}^h \geq 0$. Yet, the binding (TTl) implies that $\delta\Pi_{0,old}^l = -b_{old}^h(\theta^t) + \delta\tilde{\Pi}^h(\theta^t) \geq 0$, which implies that the (DEli) constraints will hold after our change, as $b_{old}^h \geq \frac{n^h c}{q} \geq n^h c$ by the (IC) constraint.

Because $U^h = w^h - n^h c + qb^h + (1-q)b_0^l = 0$, a binding (IC) constraint implies that $w^h = 0$. By the same token, $U_\tau^l = w_\tau^l - n_\tau^l c + b_{\tau+1}^l = 0$, a binding (IC) constraint implies that $w_\tau^l = 0$ for all $\tau \in \mathbb{N}$. ■

Proof of Lemma 11: Suppose the discount factor rises from $\hat{\delta}$ to $\tilde{\delta} > \hat{\delta}$. The actions that were optimal at $\hat{\delta}$ continue to satisfy all (DEli) for $\tilde{\delta}$. By Lemma 7, these actions lead to weakly higher profits. It thus only remains to show that (ECh) is relaxed as δ increases. For this, we compute the derivative $\mathcal{D}$ of (ECh) with respect to δ, which works out as

$$\mathcal{D} = q\left[\Pi^h + \delta\Pi^{h'} - (\theta^h - \theta^l)\sum_{i=0}^{\infty}(1+i)(\delta q)^i g(n_i^l)\right] + (1-q)\left[\Pi_0^l + \delta\Pi_0^{l'}\right].$$

As

$$\Pi^h = \frac{1}{1-\delta q}\left[\theta^h g(n^h) - n^h c + \delta(1-q)\Pi_0^l\right],$$

we have

$$\Pi^{h'} = \frac{1-q}{1-\delta q}[\Pi_0^l + \delta\Pi_0^{l'}] + \frac{q}{(1-\delta q)^2}\left[\theta^h g(n^h) - n^h c + \delta(1-q)\Pi_0^l\right].$$

Furthermore, as

$$\Pi_0^l = \sum_{i=0}^{\infty}\delta^i\left(\theta^l g(n_i^l) - n_i^l c\right),$$

we have

$$\Pi_0^l + \delta(1-\delta q)\Pi_0^{l'} = \sum_{i=0}^{\infty}(1+(1-\delta q)i)\delta^i \left(\theta^l g(n_i^l) - n_i^l c\right).$$

Inserting this gives us

$$\begin{aligned}(1-\delta q)^2\mathcal{D} = q(\theta^h g(n^h) - n^h c) + (1-q)\sum_{i=0}^{\infty}(1+(1-\delta q)i)\,\delta^i \left(\theta^l g(n_i^l) - n_i^l c\right) \\ - q(\theta^h - \theta^l)(1-\delta q)^2 \sum_{i=0}^{\infty}(1+i)(\delta q)^i g(n_i^l).\end{aligned}$$

To show that $\mathcal{D} \geq 0$, it is sufficient to show that

$$q(\theta^h g(n^h) - n^h c) + (1-q)\sum_{i=0}^{\infty}(1+(1-\delta q)i)\,\delta^i \left(\theta^l g(n_i^l) - n_i^l c\right) - q(\theta^h - \theta^l)g(\bar{n}^l) \geq 0,$$

where we have used that $\sum_{i=0}^{\infty}(1+i)(\delta q)^i = \frac{1}{(1-\delta q)^2}$ and $\sup_{i\in\mathbb{N}} n_i^l =: \bar{n}^l$. We can rewrite this as

$$\begin{aligned}q\left[\theta^h(g(n^h) - g(\bar{n}^l)) - \left(n^h - \sum_{i=0}^{\infty}(1+(1-\delta q)i)\delta^i n_i^l\right)c\right. \\ \left. +\theta^l\left(g(\bar{n}^l) - \sum_{i=0}^{\infty}(1+(1-\delta q)i)\delta^i g(n_i^l)\right)\right] \\ + \sum_{i=0}^{\infty}(1+(1-\delta q)i)\delta^i(\theta^l g(n_i^l) - n_i^l c) \geq 0.\end{aligned}$$

By Lemma 9, we know that $n^h \geq \bar{n}^l$; by Lemma 7, this implies that $\theta^h g(n^h) - n^h c \geq \theta^h g(\bar{n}^l) - \bar{n}^l c$. Thus, it is sufficient for $\mathcal{D} \geq 0$ that

$$q\left[\theta^l g(\bar{n}^l) - \bar{n}^l c - \sum_{i=0}^{\infty}(1+(1-\delta q)i)\delta^i(\theta^l g(n_i^l) - n_i^l c)\right] + \sum_{i=0}^{\infty}(1+(1-\delta q)i)\delta^i(\theta^l g(n_i^l) - n_i^l c) \geq 0,$$

which was to be shown. ■