Online Appendix:
A Macroeconomic Framework for Quantifying Systemic Risk

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I. Data and Simulation with Alternative Distress Thresholds

Model Simulation and Data

The table presents standard deviations and covariances for intermediary equity growth (Eq), investment growth (I), consumption growth (C), land price growth (PL), and Sharpe ratio (EB). Growth rates are computed as annual changes in log value from t to $t+1$. The Sharpe ratio is the value at $t+1$. The columns labeled data are the statistics for the period 1975 to 2015Q4. The Sharpe ratio is constructed from the excess bond premium, and other variables are standard and defined in the text. The data columns correspond to distress classification of the 10% worst observations and the 20% worst observations. For the model simulation, the distress period is defined as the 10% and 20% worst realizations of the Sharpe ratio.

	Data 10	Model 10	Data 20	Model 20
Panel A: Distress Periods				
vol(Eq)	38.16	32.08	29.63	22.67
vol(I)	8.71	8.14	7.31	7.27
vol(C)	2.39	6.70	1.88	5.72
vol(PL)	21.38	22.45	17.61	18.21
vol(EB)	101.27	113.90	76.66	83.90
cov(Eq, I)	2.71	2.15	1.29	1.27
cov(Eq, C)	0.68	-1.87	0.33	-1.06
cov(Eq, PL)	6.77	6.65	4.05	3.67
cov(Eq, EB)	-27.32	-24.18	-11.39	-11.83
Panel B: Non-distress Periods				
vol(Eq)	20.78	5.99	20.06	5.56
vol(I)	7.31	5.66	6.35	5.56
vol(C)	1.32	3.29	1.34	3.06
vol(PL)	10.73	9.47	10.28	8.84
vol(EB)	25.25	23.30	19.48	17.95
cov(Eq, I)	0.11	0.33	0.03	0.31
cov(Eq, C)	0.01	-0.16	0.01	-0.13
cov(Eq, PL)	0.03	0.56	-0.22	0.49
cov(Eq, EB)	-0.67	-0.56	-0.22	-0.37

II. Derivation of ODE System

Note: equation references with * refer to the main paper text.

A. Asset returns and Intermediary Optimality

We write the evolution of e_t in equilibrium as

$$de_t = \mu_e dt + \sigma_e dZ_t,$$

The functions μ_e and σ_e are state-dependent drift and volatility to be solved in equilibrium.

The terms in equation (13*) can be expressed in terms of the state variables of the model. Consider the risk and return terms on each investment. We can use the rental market clearing condition $C_t^h = H = 1$ to solve for the housing rental rate D_t:

$$D_t = \frac{\phi}{1-\phi} C_t^y = \frac{\phi}{1-\phi} K_t (A - i_t - \frac{\kappa}{2}(i_t - \delta)^2),$$

where we have used the goods market clearing condition in the second equality. Note that i_t, as given in (7*), is only a function of $q(e_t)$. Thus, D_t can be expressed as a function of K_t and e_t.

Given the conjecture $P_t = p(e_t)K_t$, we use Ito's lemma to write the return on housing as,

$$\begin{aligned} (1) \quad dR_t^h &= \frac{dP_t + D_t dt}{P_t} = \frac{K_t dp_t + p_t dK_t + [dp_t, dK_t] + D_t dt}{p_t K_t} \\ &= \left[\frac{p'(e)(\mu_e + \sigma\sigma_e) + \frac{1}{2}p''(e)\sigma_e^2 + \frac{\phi}{1-\phi}\left(A - i_t - \frac{\kappa}{2}(i_t - \delta)^2\right)}{p(e)} + i_t - \delta \right] dt + \sigma_t^h dZ_t, \end{aligned}$$

where the volatility of housing returns is,

$$\sigma_t^h = \sigma + \sigma_e \frac{p'(e)}{p(e)}.$$

The return volatility has two terms: the first term is the exogenous capital quality shock and the second term is the endogenous price volatility due to the dependence of housing prices on the intermediary reputation e (which is equal to equity capital, when the constraint binds). In addition, when e is low, prices are more sensitive to e (i.e. $p'(e)$ is high), which further increases volatility.

Similarly, for capital, we can expand (10*):

$$dR_t^k = \left[-\delta + \frac{(\mu_e + \sigma\sigma_e)q'(e) + \frac{1}{2}\sigma_e^2 q''(e) + A}{q(e)} \right] dt + \sigma_t^k dZ_t,$$

with the volatility of capital returns,

$$\sigma_t^k = \sigma + \sigma_e \frac{q'(e)}{q(e)}$$

The volatility of capital has the same terms as that of housing. However, when we solve the model, we will see that $q'(e)$ is far smaller than $p'(e)$ which indicates that the endogenous component of volatility is small for capital.

The supply of housing and capital via the market clearing condition (16*) pins down α_t^k and α_t^h. We substitute these market clearing portfolio shares to find an expression for the equilibrium volatility of the intermediary's portfolio,

$$(2) \qquad \alpha_t^k \sigma_t^k + \alpha_t^h \sigma_t^h = \frac{K_t}{E_t}\left(\sigma_e(q' + p') + \sigma(p+q)\right).$$

From the intermediary optimality condition (13*), we note that:

$$\frac{\pi_t^k}{\sigma_t^k} = \frac{\pi_t^h}{\sigma_t^h} = \gamma \frac{K_t}{E_t}\left[\sigma_e(q' + p') + \sigma(p+q)\right] \equiv \text{Sharpe ratio.} \tag{3}$$

When K_t/E_t is high, which happens when intermediary equity is low, the Sharpe ratio is high. In addition, we have noted earlier that p' is high when E_t is low, which further raises the Sharpe ratio.

We expand (3) to find a pair of second-order ODEs. For capital:

$$(\mu_e + \sigma\sigma_e)\, q' + \frac{1}{2}\sigma_e^2 q'' + A - (\delta + r_t)q = \gamma\left(\sigma q + \sigma_e q'\right)\frac{K_t}{E_t}\left(\sigma_e(q'+p') + \sigma(p+q)\right); \tag{4}$$

and for housing:

$$\begin{aligned} & (\mu_e + \sigma\sigma_e)\, p' + \frac{1}{2}\sigma_e^2 p'' + \frac{\phi}{1-\phi}\left(A - i_t - \frac{\kappa}{2}(i_t - \delta)^2\right) - (\delta + r_t - i_t)\, p \\ = \; & \gamma\left(\sigma p + \sigma_e p'\right)\frac{K_t}{E_t}\left(\sigma_e(q'+p') + \sigma(p+q)\right). \end{aligned} \tag{5}$$

B. *Dynamics of State Variables*

We derive equations for μ_e and σ_e which describe the dynamics of the capital capacity. Applying Ito's lemma to $\mathcal{E}_t = e_t K_t$, and substituting for dK_t from (2*), we find:

$$\frac{d\mathcal{E}_t}{\mathcal{E}_t} = \frac{K_t de_t + e_t dK_t + \sigma_e \sigma K dt}{e_t K_t} = \frac{\mu_e + \sigma_e\sigma + e\left(i_t - \delta\right)}{e} dt + \frac{\sigma_e + e\sigma}{e} dZ_t. \tag{6}$$

We can also write the intermediary reputation dynamics directly in terms of intermediary returns and exit, from (6*). When the economy is not at at a boundary (hence $d\psi = 0$), equity dynamics are given by,

$$\begin{aligned} \frac{d\mathcal{E}_t}{\mathcal{E}_t} &= \alpha_t^k\left(dR_t^k - r_t\right) + \alpha_t^h\left(dR_t^h - r_t\right) + (r_t - \eta)dt \\ &= \alpha_t^k\left(\pi_t^k dt + \sigma_t^k dZ_t\right) + \alpha_t^h\left(\pi_t^h dt + \sigma_t^h dZ_t\right) + (r_t - \eta)dt. \end{aligned}$$

We use (13*) relating equilibrium expected returns and volatilities to rewrite this expression as,

$$\frac{d\mathcal{E}_t}{\mathcal{E}_t} = \gamma\left(\alpha_t^k\sigma_t^k + \alpha_t^h\sigma_t^h\right)^2 dt + \left(\alpha_t^k\sigma_t^k + \alpha_t^h\sigma_t^h\right) dZ_t + (r_t - \eta)dt \tag{7}$$

where the portfolio volatility term is given in (2). We match drift and volatility in both equations (6) and (7), to find expressions for μ_e and σ_e. Matching volatilities, we have,

$$\frac{K_t}{E_t}\left(\sigma_e(q'+p') + \sigma(p+q)\right) = \frac{\sigma_e}{e} + \sigma \tag{8}$$

while matching drifts, we have,

$$\gamma\left(\frac{K_t}{E_t}\left(\sigma_e(q'+p')+\sigma(p+q)\right)\right)^2+r_t-\eta=\frac{\mu_e+\sigma_e\sigma+e\left(i_t-\delta\right)}{e}.$$

Because

$$\frac{E_t}{K_t}=\frac{\min\left(\mathcal{E}_t,(1-\lambda)\,W_t\right)}{K_t}=\min\left(e_t,(1-\lambda)\left(p\left(e\right)+q\left(e\right)\right)\right),$$

these equations can be rewritten to solve for μ_e and σ_e in terms of e, $p\left(e\right)$, $q\left(e\right)$, and their derivatives.

C. *Interest Rate*

Based on the household consumption Euler equation, we can derive the interest rate r_t. Since

$$C_t^y=Y_t-i_tK_t-\frac{\kappa K_t}{2}(i_t-\delta)^2=\left(A-\delta-\frac{q_t-1}{\kappa}-\frac{(q_t-1)^2}{2\kappa}\right)K_t,$$

we can derive $\mathbb{E}_t\left[dC_t^y/C_t^y\right]$ and $Var_t\left[dC_t^y/C_t^y\right]$ in terms of $q\left(e\right)$ (and its derivatives), along with μ_e and σ_e. Then using (8*) it is immediate to derive r_t in these terms as well.

D. *The System of ODEs*

Here we give the expressions of ODEs, expecially write the second-order terms p'' and q'' in terms of lower order terms. For simplicity, we ingore the argument for $p\left(e\right)$, $q\left(e\right)$ and their derivateives. Let

$$\begin{aligned} c^y\left(e\right) &\equiv A-\delta-\widehat{i}\left(e\right)-\frac{\kappa\left[\widehat{i}\left(e\right)\right]^2}{2},w\left(e\right)\equiv p\left(e\right)+q\left(e\right),F\left(e\right)\equiv\frac{w\left(e\right)}{e}-\theta\left(e\right)w'\left(e\right), \\ \text{and } G\left(e\right) &\equiv c^y\left(e\right)\kappa F\left(e\right)+q\left(e\right)q'\left(e\right)\left(1-\theta\left(e\right)\right)w\left(e\right), \end{aligned} \tag{9}$$

and

$$H\left(e\right)\equiv\left(A-\delta\right)\frac{1}{\phi}+\left(p\left(e\right)-\frac{1-\phi}{\phi}\right)\widehat{i}-\frac{1-\phi}{\phi}\frac{\kappa}{2}\widehat{i}_t^2+\delta\left(1-q\left(e\right)\right), \tag{10}$$

where

$$\theta\left(e\right)\equiv\max\left[\frac{w\left(e\right)}{e},\frac{1}{1-\lambda}\right]\text{ and }\widehat{i}\left(e\right)\equiv\frac{q\left(e\right)-1}{\kappa}.$$

We have

$$\sigma_e=\frac{ew\left(e\right)\sigma\left(\theta\left(e\right)-1\right)}{w\left(e\right)-e\theta\left(e\right)w'\left(e\right)}.$$

This, together with (8), implies that the Sharpe ratio is

$$\gamma\frac{K_t}{E_t}\left(\sigma_ew'\left(e\right)+\sigma w\left(e\right)\right)=\gamma\left(\frac{\sigma_e}{e}+\sigma\right)=\gamma\sigma\theta\left(e\right)\frac{w\left(e\right)-ew'\left(e\right)}{w\left(e\right)-e\theta\left(e\right)w'\left(e\right)}. \tag{11}$$

Define

$$
\begin{aligned}
a_{11} &\equiv p'(e)\left(\frac{c^y(e)\kappa}{G(e)}\left(-(1-\theta(e))\xi\left(\frac{\frac{1}{2}q(e)\sigma_e^2}{c^y(e)\kappa}\right)w(e)+\theta(e)\frac{1}{2}\sigma_e^2\right)\right)+\frac{p\phi}{G(e)}\left(q(e)q'(e)\theta(e)\frac{1}{2}\sigma_e^2+\frac{F(e)}{2}q\sigma_e^2\right),\\
a_{12} &\equiv p'(e)\left(\frac{c^y(e)\kappa}{G(e)}\theta(e)\frac{1}{2}\sigma_e^2\right)+\frac{1}{2}\sigma_e^2+\frac{p\xi}{G(e)}\left(q(e)q'(e)\theta(e)\frac{1}{2}\sigma_e^2\right),\\
a_{21} &\equiv q'(e)\left(\frac{c^y(e)\kappa}{G(e)}\left(\left[-(1-\theta(e))\xi\frac{q(e)\sigma_e^2}{2c^y(e)\kappa}\right]w(e)+\frac{1}{2}\theta(e)\sigma_e^2\right)\right)\\
&\quad+\frac{1}{2}\sigma_e^2+\frac{q(e)\xi}{G(e)}\left(q(e)q'(e)\theta(e)\frac{1}{2}\sigma_e^2+\frac{F(e)}{2}q(e)\sigma_e^2\right)\\
a_{22} &\equiv q'(e)\theta(e)\frac{1}{2}\sigma_e^2\left[\frac{c^y(e)\kappa}{G(e)}+\frac{q^2(e)\xi}{G(e)}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
b_1 \equiv\ & \left(p(e)\sigma+p'(e)\sigma_e\right)\sigma\gamma\theta(e)\frac{w(e)-ew'(e)}{eF(e)}\\
&-\frac{p'(e)c^y(e)\kappa}{G(e)}\left(\left[(1-\theta(e))\left(\rho+\xi\left(\hat{i}-\frac{(q'(e))^2\sigma_e^2}{2c^y(e)\kappa}\right)-\frac{\xi(1+\xi)\left[c^y(e)\sigma-\frac{q(e)q'(e)\sigma_e}{\kappa}\right]^2}{2c^y(e)^2}\right)-\hat{i}(e)-\eta\right]w(e)+\theta(e)H(e)\right)\\
&-\frac{1-\phi}{\phi}c^y(e)-p\hat{i}(e)+\frac{p(e)}{G(e)}\left(\begin{array}{c}\left(\rho+\xi\hat{i}(e)\right)c^y(e)\kappa F(e)-\xi\frac{F(e)}{2}(q'(e))^2\sigma_e^2-\frac{\xi(1+\xi)}{2}\frac{F(e)\kappa\left[c^y(e)\sigma-\frac{q(e)q'(e)\sigma_e}{\kappa}\right]^2}{c^y(e)}\\ -\xi q(e)q'(e)\left(-\left(\hat{i}(e)+\eta\right)w(e)+\theta(e)H(e)\right)\end{array}\right),\\
b_2 \equiv\ & \left(\sigma_e q'(e)+q(e)\sigma\right)\sigma\gamma\theta(e)\frac{w(e)-ew'(e)}{eF(e)}\\
&-\frac{q'(e)c^y(e)\kappa}{G(e)}\left(\left[(1-\theta(e))\left(\rho+\xi\left(\hat{i}(e)-\frac{(q'(e))^2\sigma_e^2}{2c^y(e)\kappa}\right)-\frac{\xi(1+\xi)\left[c^y(e)\sigma-\frac{q(e)q'(e)\sigma_e}{\kappa}\right]^2}{2c^y(e)^2}\right)-\hat{i}(e)-\eta\right]w(e)+\theta(e)H(e)\right)\\
&-A+q(e)\delta+\frac{q(e)}{G(e)}\left(\begin{array}{c}\left(\rho+\hat{i}(e)\right)c^y(e)\kappa F(e)-\xi\frac{F(e)}{2}(q'(e))^2\sigma_e^2-\frac{\xi(1+\xi)F(e)\kappa\left[c^y(e)\sigma-\frac{q(e)q'(e)\sigma_e}{\kappa}\right]^2}{2c^y(e)}\\ -\xi q(e)q'(e)\left(-\left(\hat{i}(e)+\eta\right)w(e)+\theta(e)H(e)\right)\end{array}\right).
\end{aligned}
$$

Then the second-order terms can be solved as

$$
\begin{bmatrix} q'' \\ p'' \end{bmatrix}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}=\frac{1}{a_{11}a_{22}-a_{12}a_{21}}\begin{bmatrix} a_{22}b_1-a_{12}b_2 \\ -a_{21}b_1+a_{11}b_2 \end{bmatrix}. \tag{12}
$$

E. *Boundary Conditions and Numerical Methods*

When $e\to\infty$ without capital constraint. — When $e\to\infty$, we have q and p as constants. Let $\hat{i}=\frac{q-1}{\kappa}$, and since

$$
C_t^y=\left(A-\delta-\hat{i}-\frac{\kappa\hat{i}^2}{2}\right)K_t,
$$

we have $dC_t^y/C_t^y = dK_t/K_t = \hat{i}dt + \sigma dZ_t$. As a result, both assets have the same return volatility $\sigma_R^k = \sigma_R^h = \sigma$, and the interest rate is

$$r = \rho + \xi\hat{i} - \frac{\xi(1+\xi)}{2}\sigma^2.$$

Because the intermediary's portfolio weight $\theta = \frac{1}{1-\lambda}$, the banker's pricing kernel is $\sigma\gamma\theta = \frac{\gamma\sigma}{1-\lambda}$. Therefore

$$\frac{\mu_R^k - r}{\sigma_R^k} = \frac{\gamma\sigma}{1-\lambda} \Rightarrow \mu_R^k = \frac{\gamma\sigma^2}{1-\lambda} + \rho + \xi\hat{i} - \frac{\xi(1+\xi)}{2}\sigma^2 = \rho + \phi\hat{i} + \frac{2\gamma - \xi(1+\xi)(1-\lambda)}{2(1-\xi)}\sigma^2.$$

Because $\mu_R^k = -\delta + \frac{A}{q}$ by definition, we can solve for

$$q = \frac{A}{\rho + \delta + \xi\hat{i} + \frac{2\gamma - \xi(1+\xi)(1-\lambda)}{2(1-\xi)}\sigma^2}.$$

Because $\hat{i} = \frac{q-1}{\kappa}$, plugging in the above equation we can solve for

$$q = \frac{-\left(\rho + \delta + \frac{2\gamma-\xi(1+\xi)(1-\lambda)}{2(1-\xi)}\sigma^2 - \frac{\xi}{\kappa}\right) + \sqrt{\left(\rho + \delta + \frac{2\gamma-\xi(1+\xi)(1-\lambda)}{2(1-\xi)}\sigma^2 - \frac{\phi}{\kappa}\right)^2 + \frac{4A\xi}{\kappa}}}{\frac{2\xi}{\kappa}} \tag{13}$$

which gives the value of q and $\hat{i}$ when $e = \infty$.

Now we solve for p. Using $\frac{\mu_R^h - r}{\sigma_R^h} = \frac{\gamma\sigma}{1-\lambda}$ we know that $\mu_R^h = \rho + \xi\hat{i} + \frac{2\gamma-\xi(1+\xi)(1-\lambda)}{2(1-\xi)}\sigma^2$. Since $\frac{\frac{\phi}{1-\phi}\left(A - \delta - \hat{i} - \frac{\kappa\hat{i}^2}{2}\right)}{p} + \hat{i} = \mu_R^h$ by definition, we have

$$p = \frac{\frac{(1-\phi)}{\phi}\left(A - \delta - \hat{i} - \frac{\kappa\hat{i}^2}{2}\right)}{\rho + (\xi - 1)\hat{i} + \frac{2\gamma-\xi(1+\xi)(1-\lambda)}{2(1-\xi)}\sigma^2}. \tag{14}$$

Numerically, instead of (13) and (13) we impose the slope conditions $p'(\infty) = q'(\infty) = 0$ which gives more stable solutions.

Lower entry barrier. — Consider the boundary condition at $\underline{e}$ which is a reflecting barrier due to linear technology of entry. More specifically, at the entry boundary $\underline{e}$, we have

$$d\mathcal{E}_t = \theta(e_t)\left[dR_t^{agg} - r_t dt\right]\mathcal{E}_t dt + dU_t$$

where dU_t reflects $\mathcal{E}_t$ at $\underline{e}K$. Heuristically, suppose that at $\underline{\mathcal{E}} = \underline{e}K$, a negative shock ϵ sends $\mathcal{E}$ to $\underline{e}K - \epsilon$ which is below $\underline{e}K$. Then immediately there will be βx unit of physical capital to be converted into x units of E, so that the new level $\widehat{\mathcal{E}} = \underline{e}K - \epsilon + x = \underline{e}\widehat{K} = \underline{e}(K - \beta x)$. This implies that the amount of capital to be converted to $\mathcal{E}$ is $x = \frac{\epsilon}{1+\underline{e}\beta} > 0$, and the new capital is $\widehat{K} = K - \beta x = K - \beta\frac{\epsilon}{1+\underline{e}\beta}$.

Now we give the boundary conditions for $p(\cdot)$ and $q(\cdot)$. First, although entry reduces physical capital K, since q is measured as per unit of K, the price should not change during entry. Therefore we must have $q'(\underline{e}) = 0$. For scaled housing price $p(\cdot)$, there will be a non-zero slope. Intuitively, entry lowers the aggregate physical capital K, hence future equilibrium consumption as well as future equilibrium housing rents are lower, translating to a lower P directly. Formally, right after the negative shock described above, the housing price is $p\left(\frac{\mathcal{E}}{K}\right)K$ can be rewritten as $p\left(\frac{\mathcal{E}}{K}\right)K = p\left(\underline{e} - \frac{\epsilon}{K}\right)K$, which must equal the housing price $p\left(\underline{e}\right)\widehat{K} = p\left(\underline{e}\right)\left(K - \beta\frac{\epsilon}{1+\underline{e}\beta}\right)$ right after the adjustment (otherwise there will be an arbitrage). Hence,

$$p\left(\underline{e}\right)\left(K - \beta\frac{\epsilon}{1+\underline{e}\beta}\right) = p\left(\underline{e} - \frac{\epsilon}{K}\right)K = p\left(\underline{e}\right)K - p'\left(\underline{e}\right)\epsilon \Rightarrow p'\left(\underline{e}\right) = \frac{p\left(\underline{e}\right)\beta}{1+\underline{e}\beta} > 0$$

where we have used the fact that ϵ can be arbitrarily small in the continuous-time limit.

Define $\Delta \equiv \frac{p(\underline{e})\beta}{1+\underline{e}\beta}$. In numerical solution instead of imposing β, we directly impose the following boundary conditions for equilibrium pricing functions

$$p'\left(\underline{e}\right) = \Delta \text{ and } q'\left(\underline{e}\right) = 0. \tag{15}$$

We will treat ξ as our primitive parameter, calibrated to match land price volatility.

Numerical method. — Given (15), the following results is useful. From (11), we know that at $\underline{e}$ the Sharpe ratio is (recall $w\left(e\right) = p\left(e\right) + q\left(e\right)$)

$$B = \sigma\gamma\theta\left(\underline{e}\right)\frac{w\left(\underline{e}\right) - \underline{e}w'\left(\underline{e}\right)}{w\left(\underline{e}\right) - \underline{e}\theta\left(\underline{e}\right)w'\left(\underline{e}\right)} = \sigma\gamma\frac{w\left(\underline{e}\right)}{\underline{e}}\frac{w\left(\underline{e}\right) - \underline{e}w'\left(\underline{e}\right)\Delta}{w\left(\underline{e}\right) - \underline{e}\frac{w(\underline{e})}{\underline{e}}\Delta} = \sigma\gamma\frac{w\left(\underline{e}\right) - \underline{e}\Delta}{\underline{e}\left(1-\Delta\right)}$$

which implies that

$$p\left(\underline{e}\right) + q\left(\underline{e}\right) = w\left(\underline{e}\right) = \frac{B\underline{e}\left(1-\Delta\right)}{\sigma\gamma} + \underline{e}\Delta. \tag{16}$$

Based on (16) numerically we use the following 2-layer loops to solve the ODE system in (II.D) with endogenous entry boundary $\underline{e}$.

1) In the inner loop, we fix $\underline{e}$. Consider different trials of $q\left(\underline{e}\right)$; given $q\left(\underline{e}\right)$, we can get $p\left(\underline{e}\right) = \frac{B\underline{e}(1-\Delta)}{\sigma\gamma} + \underline{e}\Delta - q\left(\underline{e}\right)$. Then based on the four boundary conditions

$$p\left(\underline{e}\right), q\left(\underline{e}\right), p'\left(\infty\right) = q'\left(\infty\right) = 0,$$

we can solve this 2-equation ODE system with boundary conditions using the *Matlab* builtin ODE solver bvp4c. We then search for the right $q\left(\underline{e}\right)$ so that $p'\left(\underline{e}\right) - q'\left(\underline{e}\right) = \Delta$ holds.

2) In the outer loop, we search for appropriate $\underline{e}$. For each trial of $\underline{e}$, we take the inner loop, and keep searching until $q'\left(\underline{e}\right) = 0$.

III. Households with Direct Asset Holdings

Consider the modification in which the household sector consists of three groups of household members: equity households, debt households, and asset households. For the newly introduced asset households, we assume that these households directly invest $\left(1-\zeta\chi\right)P_tH (= \left(1-\zeta\chi\right)p_tK_t)$ dollars into housing and

$(1-\chi)\,q_t K_t$ dollars into capital, where $\chi \in (0,1)$ and $\zeta \in (0,1)$ are constants. These household investors are constrained in allocating their wealth into H and K.

In this modification, debt and equity households invest their wealth in the intermediary sector, as in our baseline model. Their wealth, and hence the size of the inermediary balance sheet, is:

$$\underbrace{\chi\left(\zeta p_t + q_t\right) K_t}_{\text{Intermediary Asset Holdings}} = \underbrace{W_t}_{\text{Aggregate Wealth}} - \underbrace{\left[\left(1-\zeta\chi\right) p_t K_t + \zeta q_t K_t\right]}_{\text{Direct Holding of Asset Households}} \tag{17}$$

(with $W_t = K_t\left(p_t + q_t\right)$). We maintain the same assumption on equity and debt households as in the baseline model, i.e., equity households invest $1-\lambda$ fraction of $\chi\left(\zeta p_t + q_t\right) K_t$ into intermediary equity, but constrained by $\mathcal{E}_t$ which is the aggregate intermediary equity capacity.

In (17), χ captures the household's direct holdings of real assets relative to the indirectly, through intermediaries, holdings of these assets. Intermediaries' aggregate balance sheets consists of $\zeta\chi p_t K_t$ housing assets and $\chi q_t K_t$ physical capital sitting. Note that the ratio of housing to capital assets is $\zeta p_t / q_t$. But, from the perspective of the entire economy, the ratio between these two assets is p_t/q_t. Therefore the introduction of asset households offers a degree of flexibility in calibrating the parameter ζ, which allows relative ratio between housing to capital assets held by intermediaries to differ from the relative ratio in the economy. But as we discuss when calibrating the model, the data suggest that these relative ratios are similar, so that we set $\zeta = 1$ in the calibration.

The modified model can be solved in a similar way as the baseline, after redefining the state variable to be the intermediary capital capacity $\mathcal{E}_t$ divided by χK_t:

$$e_t \equiv \frac{\mathcal{E}_t}{\chi K_t}.$$

We highlight that the resulting ODE system for $p\left(e\right)$ and $q\left(e\right)$ depends on ζ, but not χ. More specifically, when $\zeta = 1$, as in our calibration, so that the relative shares of housing and capital held by intermediaries and that held directly by households are the same, the modified model admits the same solutions (same pricing functions, e.g. $p\left(e\right)$, $p\left(e\right)$ and same policy functions e.g. $i\left(e\right)$) and dynamics as the baseline model without asset households. Potentially the parameter χ matters in determining an initial condition for a simulation, but it is irrelevant in our paper because our simulation exercises focus on the steady state distribution.

Intuitively, the pricing equations are still determined by the banker's optimization behavior, which depend on the relative supplies of housing and capital held by bankers, and hence not on χ. Given this result, investment policy I_t and the aggregate household consumption $C_t = AK_t - I_t$ do not change, implying the same equilibrium interest rate.

When $\zeta \neq 1$, one can show that we have the same boundary conditions as in Section (II.E), and the form of ODE system is the same as in (12) but with some modifications. More specifically, keep the same definition as in (9), a_{11}, a_{21}, b_1 and b_2; but modify the definition of $H\left(e\right)$ in (10) to be

$$H\left(e\right) \equiv \left(A-\delta\right)\frac{\zeta+\phi-\zeta\phi}{\phi} + \zeta\left(p_t - \frac{\left(1-\phi\right)}{\phi}\right)\widehat{i}\left(e_t\right) - \zeta\frac{1-\phi}{\phi}\frac{\kappa}{2}\widehat{i}_t^2 + \delta\left(1-q\right).$$

and a_{12} and a_{22} to be

$$
\begin{aligned}
a_{12} &\equiv p'(e)\left(\frac{c^y\kappa}{G(e)}\theta(e)\frac{\zeta}{2}\sigma_e^2\right)+\frac{1}{2}\sigma_e^2+\frac{p(e)\gamma}{G(e)}\left(q(e)q'(e)\theta(e)\frac{\zeta}{2}\sigma_e^2\right),\\
a_{22} &\equiv q'(e)\frac{c^y\kappa}{G(e)}\theta(e)\frac{\zeta}{2}\sigma_e^2+\frac{q^2(e)\gamma}{G(e)}q'(e)\theta(e)\frac{\zeta}{2}\sigma_e^2.
\end{aligned}
$$

IV. Derivation for Hidden Leverage Case

The dynamics of the state variable are, $de_t=\mu_e dt+\sigma_e dZ_t$. We recompute μ_e and σ_e based on the higher leverage. The reputation dynamics are:

$$
\frac{d\mathcal{E}_t}{\mathcal{E}_t}=\alpha_t^k\left(\pi_t^k dt+\sigma_t^k dZ_t\right)+\alpha_t^h\left(\pi_t^h dt+\sigma_t^h dZ_t\right)+(r_t-\eta)dt,
$$

where $\alpha_t^k=\frac{1}{1-\widehat{\lambda}}\frac{q_tK_t}{W_t}$ and $\alpha_t^k=\frac{1}{1-\widehat{\lambda}}\frac{P_t}{W_t}$ are larger than the baseline equilibrium portfolio shares to reflect the higher leverage based on $\widehat{\lambda}$. For illustration here we focus on the case where capital constraint is not binding and the leverage is simply the intermeidary leverage is simply $\frac{1}{1-\widehat{\lambda}}$. When the capital constraint is binding, the leverage is determined by $W_t/\mathcal{E}_t$ as the baseline model.

We assume that the interest rate (r_t) and ex-ante risk premia (π_t^k,π_t^h) are the functions of e_t that solve the model based on λ rather than $\widehat{\lambda}$. That is we hold expected returns and interest rates fixed in the experiment. Recall that,

$$
\sigma_t^h=\sigma+\sigma_e\frac{p'(e)}{p(e)}\quad\text{and,}\quad\sigma_t^k=\sigma+\sigma_e\frac{q'(e)}{q(e)}.
$$

We also assume that the price functions, $p(e)$ and $q(e)$, solve the model based on λ rather than $\widehat{\lambda}$. We account for the fact that higher leverage implies a more volatile σ_e which in turn means that σ_t^h and σ_t^k rises. That is, a given shock dZ_t causes e_t to fall which feeds back into a further fall in asset prices and a larger fall in e_t. It is essential to account for this amplification since it is the non-linearity of the model. Thus,

$$
\begin{aligned}
\frac{d\mathcal{E}_t}{\mathcal{E}_t} &= \alpha_t^k\left(\pi_t^k dt+\left(\sigma+\sigma_e\frac{q'(e)}{q(e)}\right)dZ_t\right)+\alpha_t^h\left(\pi_t^h dt+\left(\sigma+\sigma_e\frac{p'(e)}{p(e)}\right)dZ_t\right)+(r_t-\eta)dt\\
&= \left(\alpha_t^k\pi_t^k+\alpha_t^h\pi_t^h+r_t-\eta\right)dt+\frac{1}{1-\widehat{\lambda}}\left[\sigma+\frac{w'(e)}{w(e)}\sigma_e\right]dZ_t \qquad (18)
\end{aligned}
$$

where the second equality uses the fact that $\alpha_t^k=\frac{1}{1-\widehat{\lambda}}\frac{q_tK_t}{W_t}$, $\alpha_t^k=\frac{1}{1-\widehat{\lambda}}\frac{P_t}{W_t}$, and $W_t=K_t(p(e)+q(e))=K_tw(e)$. From (6), we can also write,

$$
\frac{d\mathcal{E}_t}{\mathcal{E}_t}=\frac{\mu_e+\sigma_e\sigma+e(i_t-\delta)}{e}dt+\frac{\sigma_e+e\sigma}{e}dZ_t. \qquad (19)
$$

By matching (18) and (19), we can solve for μ_e and σ_e with hidden leverage. For instance, for σ_e, we have

$$
\frac{1}{1-\widehat{\lambda}}\left[\sigma+\frac{w'(e)}{w(e)}\sigma_e\right]=\frac{\sigma_e}{e}+\sigma\Rightarrow\sigma_e=\sigma\frac{\frac{1}{1-\widehat{\lambda}}-1}{\frac{1}{e}-\frac{1}{1-\widehat{\lambda}}\frac{w'(e)}{w(e)}}
$$

Increasing λ to $\widehat{\lambda}$ increases the numerator and decreases the denominator. In particular, σ_e rises more than one for one with the increase in the leverage $\frac{1}{1-\hat{\lambda}}$, which is 1.5 times of the leverage in the base model. Moreover, this amplification effect is stronger when the economy is closer to crisis. We find that the σ_e rises by around 1.5 times relative to the baseline in the first quarter of the simulation, but rises by about 15 times at the point in the simulation when the capital constraint binds.