Fix the realization of sample noise $\varepsilon$. The coefficients $b_0$ and $b_1$ are given by the solution to the first-order conditions of

$$
\min_{b_0,b_1} \sum_{x=0,1} (y_x - b_0 - b_1 x)^2 + c_0 I_{b_1 \neq 0} + c_1 |b_1|
$$

where the dependence of the coefficients $b_0$ and $b_1$ on the noise realization $\varepsilon$ is suppressed for notational ease.

The first-order condition with respect to $b_0$ is

$$(y_0 - b_0) + (y_1 - b_0 - b_1) = 0 \quad (1)$$

while the first-order condition with respect to $b_1$ when $b_1 \neq 0$ gives

$$2(y_1 - b_0 - b_1) = \text{sign}(b_1)c_1 \quad (2)$$

In particular, $2(y_1 - b_0 - b_1) = c_1$ when $b_1 > 0$, and $2(y_1 - b_0 - b_1) = -c_1$ when $b_1 < 0$.

From (1) we obtain

$$b_0 = \frac{1}{2}(y_0 + y_1 - b_1)$$

Plugging this into (2), we obtain the following characterization of $b_1$ cond-
tional on it being non-zero:

\[ b_1 = \begin{cases} 
\beta_1 + \varepsilon_1 - \varepsilon_0 - c_1 & \text{if } b_1 > 0 \\
\beta_1 + \varepsilon_1 - \varepsilon_0 + c_1 & \text{if } b_1 < 0 
\end{cases} \]

This means in particular that when \( \beta_1 + \varepsilon_1 - \varepsilon_0 \in (-c_1, c_1) \), \( b_1 = 0 \).

To complete the characterization of when \( b_1 \neq 0 \), we compute the difference between the Residual Sum of Squares (RSS) when the coefficient \( b_1 \) is admitted and when it is omitted. First,

\[RSS(b_1 \neq 0) = (b_0 - y_0)^2 + (b_0 + b_1 - y_1)^2\]

where \( b_0 \) and \( b_1 \) are given by (1) and (2). In contrast, when \( b_1 \) is omitted, \( b_0 = \frac{1}{2}(y_0 + y_1) \), such that

\[RSS(b_1 = 0) = \left( \frac{1}{2}y_0 + \frac{1}{2}y_1 - y_0 \right)^2 + \left( \frac{1}{2}y_0 + \frac{1}{2}y_1 - y_1 \right)^2 = \frac{1}{2}(y_1 - y_0)^2\]

It follows that

\[RSS(b_1 = 0) - RSS(b_1 \neq 0) = \frac{1}{2}(y_1 - y_0)^2 - (b_0 - y_0)^2 - (b_0 + b_1 - y_1)^2\]

\[= b_1[y_1 - y_0 - \frac{1}{2}b_1]\]

\[= [y_1 - y_0 - \text{sign}(b_1)c_1][y_1 - y_0 - \frac{1}{2}(y_1 - y_0 - \text{sign}(b_1)c_1)]\]

\[= \frac{1}{2}(y_1 - y_0)^2 - \frac{1}{2}(c_1)^2\]

\[= \frac{1}{2}(\beta_1 + \varepsilon_1 - \varepsilon_0)^2 - \frac{1}{2}(c_1)^2\]

The condition for \( b_1 \neq 0 \) is

\[RSS(b_1 = 0) - RSS(b_1 \neq 0) \geq c_0\]
i.e.

$$(\beta_1 + \epsilon_1 - \epsilon_0)^2 \geq (c_1)^2 + 2c_0$$

This concludes the proof.