Proof of Proposition 1

The argument is by backward induction. The functions $X_T$, $Q_{T-1}$ and $m_{T-1}$ are uniquely defined. The first step at which multiple equilibria can arise is in the selection of $b_T$ when constructing the bond issuance function $B_T(b_{T-1}, s^T)$. However, when $\delta = 1$, the bond issuance function $B_T$ does not affect the construction of the repayment function $X_{T-1}$ and of the pricing function $Q_{T-2}$, as repayment only depends on the maximum of the function $Q_{T-1}(b_T, s^{T-1}) b_T$ and the term $(1 - \delta) Q_{T-1}(B_T(b_{T-1}, s^{T-1}), s^{T-1})$ in (3) disappears when $\delta = 1$. The same argument applies in all previous periods.

Proof of Proposition 4

In the case considered, the Laffer curve takes the form $[1 - F ((1 + r) b - m)] b$ (omitting time subscripts and dependence on $s^t$ to simplify notation). The slope of the Laffer curve is

$$1 - F ((1 + r) b - m) - (1 + r) f ((1 + r) b - m) b$$

which has the same sign of

$$1 - (1 + r) \frac{f ((1 + r) b - m)}{1 - F ((1 + r) b - m)} b.$$

So if $f/(1 - F)$ is monotone non-decreasing, the derivative can only change sign once.
Proof of Lemma 1

Since $Q_{T-1}(b_T)$ is non-increasing in $b_T$, we need to show that

$$(1 - \delta') b'_{T-1} > (1 - \delta) b_{T-1}. \tag{22}$$

Using (10) we have

$$(1 - \delta') b'_{T-1} = \frac{1 - \delta'}{r + \delta' + (1 - \delta') Q_{T-1}(b_T)} (r + \delta + (1 - \delta) Q_{T-1}(b_T')) b_{T-1}$$

and inequality (22) follows from the fact that the right-hand side is decreasing in $\delta'$.

More on boundary conditions in Section 5.1

Let $v = q b$ denote the value of debt. Multiplying both sides of (12) by $b$, substituting

$$\kappa b = z + q (\dot{b} + \delta \dot{b}),$$

and rearranging, yields

$$(r + \lambda) q b = z + \lambda \Psi(b) b + \dot{q} b + q \dot{b} = z + \lambda \Psi(b) b + \dot{v}.$$

Suppose $b$ is large enough that $z = \bar{z}$ and $\Psi(b) = \phi E[Z]$. Then we can characterize the dynamics of $(q, b)$ for $b$ large enough by studying the following ODEs in $(q, v)$

$$(r + \delta + \lambda) q = \kappa + \lambda \phi E[Z] + \dot{q},$$

$$(r + \lambda) v = \bar{z} + \frac{q}{v} \lambda \phi E[Z] + \dot{v},$$
with terminal conditions
\[
q(T) = 0, \\
v(T) = \phi \bar{z} + \lambda E[Z] / (r + \lambda).
\]

**Proof of Lemma 2**

Using steady-state conditions, the Jacobian can be written as
\[
J = \begin{bmatrix}
\frac{\kappa - b'(b)}{q} - \delta & -\frac{\delta b}{q} \\
-\lambda \Psi'(b) & r + \delta + \lambda
\end{bmatrix}.
\]

A necessary and sufficient condition for a saddle is a negative determinant of \(J\), i.e., \(J_{11}J_{22} < J_{12}J_{21}\). Since \(J_{12} < 0\) and \(J_{22} > 0\), this is equivalent to \(-J_{11}/J_{12} < -J_{21}/J_{22}\), which means that the \(b = 0\) locus is downward sloping and steeper than the \(q = 0\) locus. Condition (16) then follows.

**Proof of Proposition 5**

Consider the functions on the right-hand sides of (13) and (14), which are both continuous for \(b > 0\). If there is a saddle-path stable steady state at \(b'\), the second function is steeper, from Lemma 2, and so is below the first function at \(b' + \epsilon\) for some \(\epsilon > 0\). Taking limits for \(b \to \infty\) the the second function yields \(q \to \kappa/\delta\) and the first yields
\[
q \to \frac{\kappa + \lambda \Psi(S)}{r + \delta + \lambda} < \frac{\kappa}{\delta},
\]
where the inequality can be proved using \(\Psi(S) < 1\) and \(\kappa = r + \delta\). Therefore, the second function is above the first for some \(b''\) large enough. The intermediate value theorem implies that a second steady state exists in \((b' + \epsilon, b'')\).
Proof of Proposition 6

Consider the path that solves our ODE system going backwards in time, starting on the saddle path converging to the low-debt steady state, at some value of $b = b' + \epsilon$. Given a small enough $\epsilon > 0$ the saddle path must lie above the $\dot{q} = 0$ locus. Moreover, between $b'$ and $b''$ the $\dot{q} = 0$ locus lies strictly above the $\dot{\bar{b}} = 0$ locus. Therefore, the path can never cross the $\dot{q} = 0$ locus because along the path $\dot{b} < 0$ and $\dot{q} > 0$. Therefore, it is possible to solve the ODE backwards until $b$ approaches $b''$ from below. This implies that for all $b(0) < b''$ we can select a path with $\dot{b} < 0$ and $b \to b'$. Consider next the path that solves the ODE going backwards starting at $(\hat{b}, \hat{q})$. By construction the point $(\hat{b}, \hat{q})$ must lie in the region of the phase diagram below both the $\dot{b} = 0$ locus and the $\dot{q} = 0$ locus (to see this notice that at the definition of $\hat{b}$ implies that $\dot{b} > 0$ at $(\hat{b}, \hat{q})$ and the constancy of $qb$ implies $\dot{q} < 0$). If $\hat{b} < b''$ the path with $qb = \hat{v}$ is an equilibrium for all initial conditions in $[\hat{b}, \infty)$, so the interesting case is $\hat{b} > b''$. In this case, we can solve backward the ODE. As long as $b > b''$ the $\dot{b} = 0$ locus lies strictly above the $\dot{q} = 0$ locus. Therefore, the path can never cross the $\dot{q} = 0$ locus, because along the path $\dot{b} > 0$ and $\dot{q} < 0$. Therefore, it is possible to solve the ODE backwards until $b$ approaches $b''$ from above. This implies that for all $b(0) > b''$ we can select a path with $\dot{b} > 0$ and $b \to \infty$.

Turning to multiplicity, consider the first path constructed above. As we approach $b''$ two possibilities arise. Either $q$ remains bounded away from its steady state value $q''$ or $q$ converges to $q''$. In the first case, $\dot{b}$ is bounded above by a negative value, so we must cross $b''$ and can extend the solution in some interval $[b'', b'' + \epsilon)$. In this case, we have multiple equilibria because for some $b > b''$ we can select both an equilibrium path with $\dot{b} < 0$ and an equilibrium path with $\dot{b} > 0$. In the second case, the path converges to the steady state $(b'', q'')$ along a monotone path with $\dot{b} < 0$. However, if the local dynamics near $(b'', q'')$ are characterized by a spiral, we reach a contradiction (since the path must cross the arms of the spiral and then convergence can no longer be monotone).
Proof of Proposition 9

To prove the proposition, we construct an equilibrium which implements the desired outcome. The equilibrium pricing function satisfies \( Q(d^i, q^{i-1}) = q^* \) for any history \((d^i, q^{i-1})\) with \( q^{i-1} = \{q^*, ..., q^*\} \). The strategy of the government is to issue \( b^* - (1 - \delta) b_\bot - \sum_{j=0}^i d_j \) and consume after any history with \( q^{i-1} = \{q^*, ..., q^*\} \). The government strategy is optimal following any history with \( q^{i-1} = \{q^*, ..., q^*\} \) because the maximum utility the government can reach following any future deviation is

\[
\max_b u(\bar{y} + q^*(b - (1 - \delta) b_\bot - \kappa b_\bot) + \beta W(b)
\]

and issuing \( b^* \) reaches the maximum by construction. The pricing function satisfy rational expectations because the government will reach a total stock of debt \( b^* \) independently of the past history. It is not difficult to complete the description of the equilibrium constructing continuation strategies after histories with \( q^{i-1} \neq \{q^*, ..., q^*\} \). However, given the atomistic nature of investors, these off-equilibrium paths are irrelevant for the borrower’s maximization problem. The resulting equilibrium play is that the government issues \( b^* \) in the first auction and no further auction takes place.

Example for Section 7

Consider the economy in Section 7. The optimality condition for the maximization problem in Proposition 9 can be written as follows

\[
qu'(ar{y} + q(b - (1 - \delta) b_\bot) - \kappa b_\bot) = \frac{\beta}{1 - \beta} r \int_{\frac{rb}{1 - \eta}}^\infty U'(\max\{Y - rb, \eta Y\}) \ dH(Y).
\]

To construct an example with multiple equilibria, we consider a simple case in which the utility function \( u(c) = Ac - \frac{1}{2} Bc^2 \) and \( U(c) = \log c \). We use the following parameters

\[
\beta = 0.95, \quad \phi = 0.7, \quad \eta = 0.8,
\]
Figure 15: An example for the microfounded model of Section 7

\[ A = 3, \quad B = -27, \quad \log Y \sim N(0.1 + \log (r / (1 - \eta)), 0.2), \]

setting \( r = \frac{1}{\beta} - 1. \)

Define the functions

\[ J(b) = \frac{\beta}{1 - \beta} r \int_{\frac{rb}{1 - \eta}}^{\infty} U'(\max\{Y - rb, \eta Y\}) \, dH(Y), \]

and

\[ C(b) = \bar{y} + Q(b)(b - (1 - \delta)b_-) - \kappa b_- . \]

Equilibria can be found solving the equation \( u'(C(b)) = J(b) / Q(b). \) The solid blue line in Figure 15 represents the pairs \((C(b), J(b) / Q(b))\) for \( b \in [1, 1.5]. \)

The red dashed line represents the marginal utility of consumption in the first subperiod \( u'(c) \) choosing the parameters of \( u'(c) \) so that it crosses the blue line more than once. It can be shown that the middle point at which the two lines cross does not satisfy second order conditions for a maximum. It can also be shown that the other two points identify global optima, so they represent two equilibria.

The interpretation of the two equilibria is as follows. There is a low debt equilibrium in which the country defaults with low probability, the future marginal cost of debt \( J(b) \) is high and so is the price \( Q(b). \) There is a high debt equilibrium in which the country defaults with high probability and the
future marginal cost of debt $J(b)$ and the price of debt $Q(b)$ are both low. The ratio $J(b)/Q(b)$ is higher in the first equilibrium. This reflects the presence of recovery which limits the reduction in $Q(b)$ in the low $b$ equilibrium. Therefore, the marginal incentive to reduce debt is higher in the low debt equilibrium, which is reflected in a lower value of $c$.

Here, we have chosen an example in which $c$ is fairly sensitive to the different equilibria to emphasize the novel forces that arise in a fully optimizing setup. However, it is also easy to construct examples that are closer to the two-period model of Section 4.2, by making the function $u'(c)$ be very steep near some $\bar{c}$ that delivers a given primary surplus $\bar{y} - \bar{c}$. 