Slow Moving Debt Crises

Guido Lorenzoni Iván Werning

Online Appendix

Proof of Proposition 1

The argument is by backward induction. The functions X_T, Q_{T-1} and m_{T-1} are uniquely defined. The first step at which multiple equilibria can arise is in the selection of b_T when constructing the bond issuance function $B_T(b_{T-1}, s^T)$. However, when $\delta = 1$, the bond issuance function B_T does not affect the construction of the repayment function X_{T-1} and of the pricing function Q_{T-2}, as repayment only depends on the maximum of the function $Q_{T-1}\left(b_T, s^{T-1}\right) b_T$ and the term $(1-\delta)\, Q_{T-1}\left(B_T(b_{T-1}, s^{T-1}), s^{T-1}\right)$ in (3) disappears when $\delta = 1$. The same argument applies in all previous periods.

Proof of Proposition 4

In the case considered, the Laffer curve takes the form $\left[1 - F\left((1+r)\, b - m\right)\right] b$ (omitting time subscripts and dependence on s^t to simplify notation). The slope of the Laffer curve is

$$1 - F\left((1+r)\, b - m\right) - (1+r)\, f\left((1+r)\, b - m\right) b$$

which has the same sign of

$$1 - (1+r)\, \frac{f\left((1+r)\, b - m\right)}{1 - F\left((1+r)\, b - m\right)} b.$$

So if $f/\left(1-F\right)$ is monotone non-decreasing, the derivative can only change sign once.

Proof of Lemma 1

Since $Q_{T-1}(b_T)$ is non-increasing in b_T, we need to show that

$$(1-\delta') \, b'_{T-1} > (1-\delta) \, b_{T-1}. \tag{22}$$

Using (10) we have

$$(1-\delta') \, b'_{T-1} = \frac{1-\delta'}{r+\delta'+(1-\delta') \, Q_{T-1}(b_T^*)} \left(r+\delta+(1-\delta) \, Q_{T-1}(b_T^*)\right) b_{T-1}$$

and inequality (22) follows from the fact that the right-hand side is decreasing in δ'.

More on boundary conditions in Section 5.1

Let $v = qb$ denote the value of debt. Multiplying both sides of (12) by b, substituting

$$\kappa b = z + q\left(\dot{b} + \delta b\right),$$

and rearranging, yields

$$(r+\lambda) \, qb = z + \lambda \Psi(b) \, b + \dot{q} b + q\dot{b} = z + \lambda \Psi(b) \, b + \dot{v}.$$

Suppose b is large enough that $z = \bar{z}$ and $\Psi(b) \, b = \phi E[Z]$. Then we can characterize the dynamics of (q, b) for b large enough by studying the following ODEs in (q, v)

$$\begin{aligned}(r+\delta+\lambda) \, q &= \kappa + \lambda\phi E[Z] + \dot{q}, \\ (r+\lambda) \, v &= \bar{z} + \frac{q}{v}\lambda\phi E[Z] + \dot{v},\end{aligned}$$

with terminal conditions

$$q\left(T\right) = 0,$$
$$v\left(T\right) = \phi\frac{\bar{z} + \lambda E[Z]}{r + \lambda}.$$

Proof of Lemma 2

Using steady-state conditions, the Jacobian can be written as

$$J = \begin{bmatrix} \frac{\kappa - h'(b)}{q} - \delta & -\frac{\delta b}{q} \\ -\lambda\Psi'\left(b\right) & r + \delta + \lambda \end{bmatrix}.$$

A necessary and sufficient condition for a saddle is a negative determinant of J, i.e., $J_{11}J_{22} < J_{12}J_{21}$. Since $J_{12} < 0$ and $J_{22} > 0$, this is equivalent to $-J_{11}/J_{12} < -J_{21}/J_{22}$, which means that the $\dot{b} = 0$ locus is downward sloping and steeper than the $\dot{q} = 0$ locus. Condition (16) then follows.

Proof of Proposition 5

Consider the functions on the right-hand sides of (13) and (14), which are both continuous for $b > 0$. If there is a saddle-path stable steady state at b', the second function is steeper, from Lemma 2, and so is below the first function at $b' + \epsilon$ for some $\epsilon > 0$. Taking limits for $b \to \infty$ the the second function yields $q \to \kappa/\delta$ and the first yields

$$q \to \frac{\kappa + \lambda\Psi\left(\overline{S}\right)}{r + \delta + \lambda} < \frac{\kappa}{\delta},$$

where the inequality can be proved using $\Psi\left(\overline{S}\right) < 1$ and $\kappa = r + \delta$. Therefore, the second function is above the first for some b'' large enough. The intermediate value theorem implies that a second steady state exists in $(b' + \epsilon, b'')$.

Proof of Proposition 6

Consider the path that solves our ODE system going backwards in time, starting on the saddle path converging to the low-debt steady state, at some value of $b = b' + \epsilon$. Given a small enough $\epsilon > 0$ the saddle path must lie above the $\dot{q} = 0$ locus. Moreover, between b' and b'' the $\dot{q} = 0$ locus lies strictly above the $\dot{b} = 0$ locus. Therefore, the path can never cross the $\dot{q} = 0$ locus because along the path $\dot{b} < 0$ and $\dot{q} > 0$. Therefore, it is possible to solve the ODE backwards until b approaches b'' from below. This implies that for all $b(0) < b''$ we can select a path with $\dot{b} < 0$ and $b \to b'$. Consider next the path that solves the ODE going backwards starting at $\left(\hat{b}, \hat{q}\right)$. By construction the point $\left(\hat{b}, \hat{q}\right)$ must lie in the region of the phase diagram below both the $\dot{b} = 0$ locus and the $\dot{q} = 0$ locus (to see this notice that at the definition of $\hat{b}$ implies that $\dot{b} > 0$ at $\left(\hat{b}, \hat{q}\right)$ and the constancy of qb implies $\dot{q} < 0$). If $\hat{b} < b''$ the path with $qb = \hat{v}$ is an equilibrium for all initial conditions in $[\hat{b}, \infty)$, so the interesting case is $\hat{b} > b''$. In this case, we can solve backward the ODE. As long as $b > b''$ the $\dot{b} = 0$ locus lies strictly above the $\dot{q} = 0$ locus. Therefore, the path can never cross the $\dot{q} = 0$ locus, because along the path $\dot{b} > 0$ and $\dot{q} < 0$. Therefore, it is possible to solve the ODE backwards until b approaches b'' from above. This implies that for all $b(0) > b''$ we can select a path with $\dot{b} > 0$ and $b \to \infty$.

Turning to multiplicity, consider the first path constructed above. As we approach b'' two possibilities arise. Either q remains bounded away from its steady state value q'' or q converges to q''. In the first case, $\dot{b}$ is bounded above by a negative value, so we must cross b'' and can extend the solution in some interval $[b'', b'' + \epsilon)$. In this case, we have multiple equilibria because for some $b > b''$ we can select both an equilibrium path with $\dot{b} < 0$ and an equilibrium path with $\dot{b} > 0$. In the second case, the path converges to the steady state (b'', q'') along a monotone path with $\dot{b} < 0$. However, if the local dynamics near (b'', q'') are characterized by a spiral, we reach a contradiction (since the path must cross the arms of the spiral and then convergence can no longer be monotone).

Proof of Proposition 9

To prove the proposition, we construct an equilibrium which implements the desired outcome. The equilibrium pricing function satisfies $\mathcal{Q}\left(d^i, q^{i-1}\right) = q^*$ for any history $\left(d^i, q^{i-1}\right)$ with $q^{i-1} = \{q^*, ..., q^*\}$. The strategy of the government is to issue $b^* - (1-\delta)\, b_- - \sum_{j=0}^{i} d_j$ and consume after any history with $q^{i-1} = \{q^*, ..., q^*\}$. The government strategy is optimal following any history with $q^{i-1} = \{q^*, ..., q^*\}$ because the maximum utility the government can reach following any future deviation is

$$\max_b u\left(\bar{y} + q^*\left(b - (1-\delta)\, b_-\right) - \kappa b_-\right) + \beta W\left(b\right)$$

and issuing b^* reaches the maximum by construction. The pricing function satisfy rational expectations because the government will reach a total stock of debt b^* independently of the past history. It is not difficult to complete the description of the equilibrium constructing continuation strategies after histories with $q^{i-1} \neq \{q^*, ..., q^*\}$. However, given the atomistic nature of investors, these off-equilibrium paths are irrelevant for the borrower's maximization problem. The resulting equilibrium play is that the government issues b^* in the first auction and no further auction takes place.

Example for Section 7

Consider the economy in Section 7. The optimality condition for the maximization problem in Proposition 9 can be written as follows

$$qu'\left(\bar{y} + q\left(b - (1-\delta)\, b_-\right) - \kappa b_-\right) = \frac{\beta}{1-\beta} r \int_{\frac{rb}{1-\eta}}^{\infty} U'\left(\max\{Y - rb, \eta Y\}\right) dH\left(Y\right).$$

To construct an example with multiple equilibria, we consider a simple case in which the utility function $u\left(c\right) = Ac - \frac{1}{2}Bc^2$ and $U\left(c\right) = \log c$. We use the following parameters

$$\beta = 0.95, \quad \phi = 0.7, \quad \eta = 0.8,$$

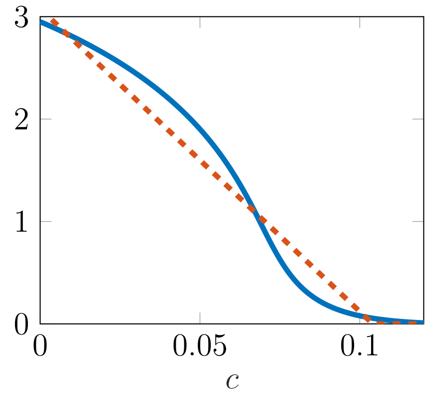

Figure 15: An example for the microfounded model of Section 7

$$A = 3, \quad B = -27, \quad \log Y \sim N(0.1 + \log\left(r/\left(1-\eta\right)\right), 0.2),$$

setting $r = \frac{1}{\beta} - 1$.

Define the functions

$$J\left(b\right) = \frac{\beta}{1-\beta} r \int_{\frac{rb}{1-\eta}}^{\infty} U'\left(\max\{Y - rb, \eta Y\}\right) dH\left(Y\right),$$

and

$$C\left(b\right) = \bar{y} + Q\left(b\right)\left(b - \left(1-\delta\right) b_{-}\right) - \kappa b_{-}.$$

Equilibria can be found solving the equation $u'\left(C\left(b\right)\right) = J\left(b\right)/Q\left(b\right)$. The solid blue line in Figure 15 represents the pairs $\left(C\left(b\right), J\left(b\right)/Q\left(b\right)\right)$ for $b \in [1, 1.5]$. The red dashed line represents the marginal utility of consumption in the first subperiod $u'\left(c\right)$ choosing the parameters of $u'\left(c\right)$ so that it crosses the blue line more than once. It can be shown that the middle point at which the two lines cross does not satisfy second order conditions for a maximum. It can also be shown that the other two points identify global optima, so they represent two equilibria.

The interpretation of the two equilibria is as follows. There is a low debt equilibrium in which the country defaults with low probability, the future marginal cost of debt $J\left(b\right)$ is high and so is the price $Q\left(b\right)$. There is a high debt equilibrium in which the country defaults with high probability and the future marginal cost of debt $J(b)$ and the price of debt $Q(b)$ are both low. The ratio $J(b)/Q(b)$ is higher in the first equilibrium. This reflects the presence of recovery which limits the reduction in $Q(b)$ in the low b equilibrium. Therefore, the marginal incentive to reduce debt is higher in the low debt equilibrium, which is reflected in a lower value of c.

Here, we have chosen an example in which c is fairly sensitive to the different equilibria to emphasize the novel forces that arise in a fully optimizing setup. However, it is also easy to construct examples that are closer to the two-period model of Section 4.2, by making the function $u'(c)$ be very steep near some $\bar{c}$ that delivers a given primary surplus $\bar{y} - \bar{c}$.