Online Appendix
Bank Networks and Systemic Risk: Evidence from the National Banking Acts

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Abstract

In this Online Appendix we

A. describe methods to standardize the correspondent relationships data and balance-sheet data for state and national banks;

B. show evidence that the observed concentration of interbank deposits was not a mere reflection of an increase in the number of country banks;

C. show that the model has a unique best-case equilibrium solution and describe the algorithm that converges to this equilibrium solution;

D. analytically compare an N-bank chain network versus a two-tier pyramid, which underlines how the concentration of interconnections affects stability.

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Online Appendix A1: Correspondent Data Standardization

As state and national banks reported correspondent relationships data differently, we describe differences between the two and the standardization procedure to match them.

State banks' annual reports provided quarterly balance sheets and the amounts due to each state-chartered Pennsylvania bank by individual debtors annually. Balance-sheet information is available for March, June, September, and November, while correspondents' information is available for November of each year. We collect information on balance sheets and amounts due to each Pennsylvania state bank by individual debtor for November.

National banks did not report all of their correspondent banks because the primary purpose of examinations was to verify whether national banks met legal reserve requirements. Country banks selected the national banks in reserve cities with which they wished to keep a portion of their legal reserves, and sent the names of those banks to the comptroller. Once approved, they were known as *approved reserve agents*. Similarly, national banks in reserve cities selected national banks in central reserve cities. Hence, for both country banks and reserve city banks, only amounts due from approved reserve agents in reserve cities and the central reserve city were enumerated. Amounts due from other banks in reserve cities and the central reserve city were not reported. In addition, amounts due from other country banks did not need to be reported. For national banks in the central reserve city, no due-from information was reported since these banks had to hold all their reserves in cash.

Examiners' reports include three types of "due-from" payments from the banks with whom they had relationships: (1) amounts due from approved redeeming agents, (2) amounts due from other national banks, and (3) amounts due from other banks. For approved redeeming agents, each agent's name is recorded with the corresponding amount. For other national banks and other banks, only aggregate due-from amounts were reported. During this period, most national banks had one reserve agent to keep their legal reserves. These reserve agents tended to be the major holder of national banks' correspondent deposits. On average, national banks kept 50 percent of total interbank deposits in one bank.[1] However, a few Philadelphia banks kept their reserves in multiple banks in New York City, with about 20 percent of total interbank deposits in each bank. To make the data on state banks' correspondents comparable to that of national banks with their approved reserve agents, we list only correspondent banks that held more than 20 percent of total interbank deposits for each bank.

[1] Calomiris and Carlson (2017) study the interbank network from the panic of 1893; they find similar values of 56 percent.

Online Appendix A2: Balance-Sheet Standardization

Because state and national bank balance sheets report different items, we combine them to create a standardized list of six asset categories (cash; government securities; other securities; amounts due from other banks; loans; and other assets) and six liability categories (capital; notes; deposits; amounts due to other banks; surplus; and other liabilities). Table A1 and A2 report the balance-sheet categories for state banks and national banks, respectively.

Table A1. State Bank Balance-Sheet Structure

Assets	Standardized
Gold and silver in the vault of the bank	Cash
Current notes, checks, and bills of other banks	Cash
Uncurrent notes, checks, and bills of other banks	Cash
Other obligations of other banks	Due from
Bills and notes discounted, (not under protest)	Loans
Bills and notes discounted, (under protest)	Loans
Mortgages held and owned by the bank	Loans
Assessed value for the year 186- of the real estate bound by said mortgages	Loans
Judgments held and owned by the bank	Loans
Real estate held and owned by the bank	Loans
Due from solvent banks	Due from
Due from insolvent banks	Due from
Public and corporate stocks and loans	Other securities
Bonds held by the bank	Other securities
Treasury notes	Government securities
Claims against individuals or corporations, disputed or in controversy	Loans
All other debts and claims either due or to become due	Loans
Expenses	Other assets
Value of any other property of the bank, as the same stands charged on the books, or otherwise	Other assets
Liabilities	**Standardized**
Capital stock actually paid in	Capital
Notes in circulation	Notes
Deposits	Deposits
Certificates of deposit	Deposits
Due to the Commonwealth	Other liabilities
Due to corporations	Deposits
Due to banks	Due to
Due to individuals	Deposits
Claims against the bank, in controversy	Other liabilities
Surplus, contingent or sinking fund	Surplus
All other items of indebtedness not embraced in foregoing specifications	Other liabilities

Notes: This table lists the original and standardized balance-sheet items for state banks.
Source: Reports of the Several Banks and Savings Institutions of Pennsylvania (1863, 1868)

Table A2. National Bank Balance-Sheet Structure

Assets	Standardized
Loans and discounts	Loans
Overdrafts	Loans
U.S. bonds deposited to secure circulation	Government securities
U.S. bonds deposited to secure deposits	Government securities
U.S. bonds and securities on hand	Government securities
Other stocks, bonds, and mortgages	Other securities
Due from approved redeeming agents	Due from
Due from other national banks	Due from
Due from other banks and bankers	Due from
Real estate, furniture, etc.	Other assets
Current expenses	Other assets
Premiums	Other assets
Checks and other cash items	Cash
Bills of national banks	Cash
Bills of other banks	Cash
Specie	Cash
Fractional currency	Cash
Legal tender notes	Cash
Compound interest notes	Cash
Liabilities	**Standardized**
Capital stock	Capital
Surplus fund	Surplus
Undivided profits	Surplus
National bank notes outstanding	Notes
State bank notes outstanding	Notes
Individual deposits	Deposits
United States deposits	Deposits
Deposits of U.S. disbursing officers	Deposits
Due to national banks	Due to
Due to other banks and bankers	Due to
Amount due, not included under either of the above headings	Other liabilities

Notes: This table lists the original and standardized balance-sheet items for national banks. Due from approved redeeming agents, checks and other cash items, specie, fractional money, legal tender notes, and compound interest notes counted toward legal reserves (Bankers' Magazine, 1875).
Source: National Banks' Examination Reports (1867)

Online Appendix B: Bank Entry and the Concentration of Interbank Deposits

After the NBAs were passed, many new national banks entered the market, especially outside financial centers. The number of banks in Pennsylvania and New York City increased from 113 in 1862 to 198 in 1867. This was largely driven by a doubling of country banks from 64 to 132. The coincidence of the rule change and the increase in new bank entries raise the concern that the concentration of interbank deposits may have originated from the increased volume of the banking sector rather than regulation. In this Appendix, we show that regulation led to the concentration of interbank deposits. To do so, we examine the distribution of interbank deposits across converted national, new national, and state banks.

We begin by comparing the interbank deposits of converted national banks in 1867 to themselves as state banks in 1862. Since these banks did not have to comply with reserve requirements before the NBAs, this exercise allows us to document the direct effect of regulation. Seventy-five state banks converted into national banks after the NBAs. Table B1 compares the distribution of interbank deposits of these banks before and after the conversion.

We find that the distribution of interbank deposits varied significantly after the rule change. For country banks, the percentage of interbank deposits in Philadelphia and Pittsburgh went up from 68 percent to 77 percent, and the percentage of correspondent relationships went up from 60 percent to 76 percent. In particular, Pittsburgh became a major financial center after it was designated as a reserve city. The fraction of correspondent relationships between country banks and Pittsburgh climbed from 2 percent to 10 percent. For Philadelphia and Pittsburgh banks, the percentage of deposits and correspondent linkages with New York City banks increased from 72 percent to 96 percent and 46 percent to 94 percent, respectively. These findings suggest that the law caused the concentration of deposits.

Next, we compare the distribution of interbank deposits of new national banks to those of state banks in 1867. By doing so, we alleviate the concern that new bank entries alone could have caused the concentration of deposits. Without the NBAs, these new banks would have behaved similarly to the state banks, which were not under the reserve requirements in 1867. In Table B2, we compare the interbank deposits of 91 new national banks to 12 state banks in 1867.[2] The distribution of interbank deposits differed for these two groups. The deposits of an average state bank were more dispersed. For example, the Pittsburgh state banks allocated 42 percent of deposits outside of New York City, and the country state banks allocated 21 percent of deposits to non-reserve city banks. In comparison, these numbers for the new national banks

[2]The 91 new national banks included 87 new banks that entered under the national charters and four banks that entered initially as state banks between 1863 and 1866 and converted to national banks by 1867. The 12 state banks included nine original state banks and three new state banks.

Table B1. Distribution of Interbank Deposits: Converted National Banks in 1862 vs. 1867

	Converted National Banks							
	All Banks		Philadelphia Banks		Pittsburgh Banks		Country Banks	
Year = 1862	Amount	Links	Amount	Links	Amount	Links	Amount	Links
New York City	41.5	36.1	75.6	38.1	68.6	53.8	30.3	32.6
Philadelphia	54.8	41.7	13.5	11.9	25.0	30.8	67.8	57.3
Pittsburgh	0.4	1.4	0.0	0.0	0.0	0.0	0.5	2.2
Other PA	2.4	11.8	5.9	23.8	4.7	7.7	1.3	6.7
Other U.S.	0.9	9.0	5.0	26.2	1.7	7.7	0.0	1.1
Year = 1867	Amount	Links	Amount	Links	Amount	Links	Amount	Links
New York City	58.6	47.1	100.0	100.0	91.2	88.9	20.8	22.4
Philadelphia	36.5	44.8	0.0	0.0	8.8	11.1	69.3	65.5
Pittsburgh	3.7	6.9	0.0	0.0	0.0	0.0	7.5	10.3
Other PA	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Other U.S.	1.2	1.1	0.0	0.0	0.0	0.0	2.4	1.7

Notes: This table compares the distribution of interbank deposits of the 75 state banks that converted to national banks in Pennsylvania for the years 1862 and 1867. All numbers are in percentages. The rows indicate the location of correspondent banks. The columns indicate the location of respondent banks. We classify respondent banks into three groups: Philadelphia, Pittsburgh, and country banks. The columns show the fraction of deposits held at different locations against total major due-from deposits in all the 75 converted Pennsylvania national banks, those in Philadelphia, in Pittsburgh, and converted country banks.

were only 8 percent and 2 percent, respectively. These findings further corroborate that the rule change was critical.

To conclude, reserve requirements led to the concentration of interbank deposits in financial centers. While significant bank entry occurred at the same time as the NBAs, our analysis shows that the same level and structure of concentration would not have appeared without the rule change by the NBAs.

Table B2. Distribution of Interbank Deposits: New National Banks vs. State Banks in 1867

	New National Banks							
	All Banks		Philadelphia Banks		Pittsburgh Banks		Country Banks	
Year = 1862	Amount	Links	Amount	Links	Amount	Links	Amount	Links
New York City	68.4	50.5	100.0	100.0	92.4	83.3	42.3	40.5
Philadelphia	26.4	38.1	0.0	0.0	7.6	16.7	47.7	45.2
Pittsburgh	4.3	8.6	0.0	0.0	0.0	0.0	8.3	10.7
Other PA	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Other U.S.	0.9	2.9	0.0	0.0	0.0	0.0	1.7	3.6
	State Banks							
Year = 1867	Amount	Links	Amount	Links	Amount	Links	Amount	Links
New York City	29.2	21.1	-	-	58.5	50.0	14.5	17.6
Philadelphia	42.9	42.1	-	-	0.0	0.0	64.4	47.1
Pittsburgh	13.8	5.3	-	-	41.5	50.0	0.0	0.0
Other PA	13.6	21.1	-	-	0.0	0.0	20.4	23.5
Other U.S.	0.5	10.5	-	-	0.0	0.0	0.8	11.8

Notes: This table compares the distribution of interbank deposits of 91 new national banks vs. 12 state banks in 1867. All numbers are in percentages. The rows indicate the location of correspondent banks. The columns indicate the location of respondent banks. We classify respondent banks into three groups: Philadelphia, Pittsburgh, and country banks. The columns show the fraction of deposits held at different locations against total major due-from deposits in all the Pennsylvania respondent banks, those in Philadelphia, in Pittsburgh, and country banks.

Online Appendix C: Best-Case Equilibrium

In this Appendix, we show that the model has a unique *best-case equilibrium solution*. This equilibrium outcome reflects the minimum set of possible withdrawals and defaults. We also show that an iterative algorithm converges to the best-case equilibrium solution.

As explained in the body of the paper, the two-period payment equilibrium is computed in two steps. We solve first for the $t = 1$ equilibrium taking into account the expected equilibrium outcome of $t = 2$, and then for the $t = 2$ equilibrium upon the realization of vector $\mathbf{R}^2$. The algorithm to compute the $t = 1$ equilibrium has an outer loop and an inner loop. The outer loop computes the withdrawals $\mathbf{W}^1$, and the inner loop computes the clearing system $\mathbf{X}^1$. As in Elliott, Golub and Jackson (2014) and Walden, Wallace and Stanton (2018), we focus on the best-case equilibrium, i.e., the outcome with the minimal set of possible withdrawals and defaults.

0. Initialization. Set iteration $m = 0$. Set $\overset{(0)}{\mathbf{W}^1}$ such that $\overset{(0)}{W^1_{ii}} = 1, \forall i \in \Omega_W$, $\overset{(0)}{W^1_{ii}} = 0, \forall i \notin \Omega_W$ and $\overset{(0)}{W^1_{ij}} = 0$, $\forall j \neq i$. Set $\overset{(0)}{\mathbf{X}^1} = \mathbf{W}^1\mathbf{D}$.

1. Finding equilibrium for $t = 1$ (outer loop for $\mathbf{W}^1$)
   (a) Set $m = m + 1$.
   (b) Given $\overset{(m-1)}{\mathbf{W}^1}$, solve for the unique payment matrix $\overset{(m)}{\mathbf{X}^1}$ using the Eisenberg-Noe fictitious default algorithm and $\overset{(m-1)}{\mathbf{X}^1}$ as the initial guess (inner loop for $\mathbf{X}^1$).
   (c) Update $\overset{(m)}{\mathbf{W}^1}$ according to the withdrawal conditions (8)–(12) and $\overset{(m)}{\mathbf{X}^1}$.
   (d) Terminate if $\overset{(m)}{\mathbf{W}^1} = \overset{(m-1)}{\mathbf{W}^1}$; otherwise, go back to Step 1.(a).
2. Finding equilibrium for $t = 2$ for the set of banks that survive from $t = 1$
   (a) Given $\overset{(m)}{\mathbf{W}^1}$ and $\overset{(m)}{\mathbf{X}^1}$, obtain $\mathbf{W}^2 = 1 - \overset{(m)}{\mathbf{W}^1}$ and $\mathbf{A}^2$ according to equation (5) in the paper.
   (b) Solve for the unique payment matrix $\mathbf{X}^2$ using the Eisenberg-Noe fictitious default algorithm.

To show that this algorithm converges to the best-case equilibrium solution, we begin by decomposing the payment variable X^t_{ki} in equation (6) to the product of two subcomponents, Π^t_{ki} and P^t_i. The first component, $\Pi^t_{ki} = W^t_{ki} D_{ki} / \left(\sum_j W^t_{ji} D_{ji}\right)$, is the nominal liability of bank i to depositor k as a fraction of bank i's total liabilities at time t; the second component, $P^t_i = \min\left\{\sum_j W^t_{ji} D_{ji}, A^t_i\right\}$, is the *total payment* by bank i to depositors at time t. The total payments thus satisfy the following mappings. For all $i = 1, \ldots, N$,

$$\text{(C1)} \qquad P^1_i = \Phi^1_i(\mathbf{P}^1) = \min\left\{\sum_j W^1_{ji} D_{ji}, C_i + \mathbb{1}^l_i \xi I_i + \sum_{j \neq i} \Pi^1_{ij} P^1_j\right\},$$

where the liquidation event $\mathbb{1}^l_i$ is given by

$$\text{(C2)} \qquad \mathbb{1}^l_i = \begin{cases} 1 & \text{if } C_i + \sum_{j\neq i} \Pi^1_{ij} P^1_j - \sum_j W^1_{ji} D_{ji} < 0 \text{ and } \sum_{j\neq i} W^1_{ij} D_{ij} = \sum_{j \neq i} D_{ij} \\ 0 & \text{otherwise} \end{cases}.$$

Denote the set of banks that survive at $t = 1$ by $\mathcal{S}$, i.e., $\mathcal{S} = \{i : \mathbb{1}^{d1}_i = 0\}$. For all $i \in \mathcal{S}$,

$$\text{(C3)} \qquad P^2_i = \Phi^2_i(\mathbf{P}^2) = \min\left\{\sum_{j\in\mathcal{S}} W^2_{ji} D_{ji}, A^1_i - \sum_{j\in\mathcal{S}} W^1_{ji} D_{ji} + (1 - \mathbb{1}^l_i) I_i R^2_i + \sum_{j\in\mathcal{S}, j\neq i} \Pi^2_{ij} P^2_j\right\}.$$

We start by analyzing the $t = 1$ equilibrium. From mapping Φ^1 in equation (C1), our model differs from the Eisenberg-Noe (2001) setting because of the endogenous withdrawals $\sum_j W^1_{ji} D_{ji}$. To understand the equilibrium properties, let us first consider the formulation in which the liquidity withdrawals $\mathbf{W}^1$ are exogenously given. This corresponds to the inner loop of the algorithm that solves the payment matrix $\mathbf{X}^1$. The following conclusions hold.

Proposition C1 For a given liquidity withdrawal matrix $\mathbf{W}^1$, a payment equilibrium of $t = 1$ is a pair of payment vector $\mathbf{P}^1$ and liquidation vector $\mathbb{1}^l$ that simultaneously solve (C1) and (C2). The payment equilibrium exists and is unique. Furthermore, the payment matrix $\mathbf{X}^1 = \mathbf{\Pi}^1 \mathbf{P}^1$ can be obtained via an iterative algorithm in at most N iterations.

Proof. The proof builds on Eisenberg and Noe (2001) and Acemoglu, Ozdaglar and Tahbaz-Salehi (2015). Based on how the liquidation event is defined in equation (2), for a given liquidity withdrawal matrix $\mathbf{W}^1$, we can separate the banks into two sets. Denote by $\mathcal{A}$ the set of banks that withdraw all interbank deposits due from correspondents, i.e., $\mathcal{A} = \{i : \sum_{j \neq i} W^1_{ij} D_{ij} = \sum_{j \neq i} D_{ij}\}$. Then for all $i \in \mathcal{A}$, liquidation occurs if and only if $C_i + \sum_{j \neq i} \Pi^1_{ij} P^1_j < \sum_j W^1_{ji} D_{ji}$. The mapping satisfies[3]

$$\begin{aligned}
P^1_i = \Phi^1_i(\mathbf{P}^1) &= \min\left\{\sum_j W^1_{ji} D_{ji}, C_i + \sum_{j \neq i} \Pi^1_{ij} P^1_j + \mathbb{1}^l_i \xi I_i\right\} \\
&= \sum_j W^1_{ji} D_{ji} + \min\left\{0, C_i + \sum_{j \neq i} \Pi^1_{ij} P^1_j - \sum_j W^1_{ji} D_{ji} + \mathbb{1}_{\{C_i + \sum_{j\neq i} \Pi^1_{ij} P^1_j < \sum_j W^1_{ji} D_{ji}\}} \xi I_i\right\} \\
&= \sum_j W^1_{ji} D_{ji} + \min\left\{0, C_i + \sum_{j \neq i} \Pi^1_{ij} P^1_j - \sum_j W^1_{ji} D_{ji} + \xi I_i\right\} \\
&= \min\left\{\sum_j W^1_{ji} D_{ji}, C_i + \xi I_i + \sum_{j \neq i} \Pi^1_{ij} P^1_j\right\}, \forall i \in \mathcal{A}, \qquad \text{(C4)}
\end{aligned}$$

where $\mathbb{1}_E$ denotes the indicator function of event E.

For all $i \notin \mathcal{A}$, liquidation does not happen according to equation (2), so $\mathbb{1}^l_i = 0$. Consequently, the mapping reduces to

$$\begin{aligned}
P^1_i = \Phi^1_i(\mathbf{P}^1) &= \min\left\{\sum_j W^1_{ji} D_{ji}, C_i + \sum_{j \neq i} \Pi^1_{ij} P^1_j + \mathbb{1}^l_i \xi I_i\right\} \\
&= \min\left\{\sum_j W^1_{ji} D_{ji}, C_i + \sum_{j \neq i} \Pi^1_{ij} P^1_j\right\}, \forall i \notin \mathcal{A}. \qquad \text{(C5)}
\end{aligned}$$

[3] The last but two equality follows from a general result: suppose that $l > 0$ and $a \in \mathbb{R}$; then $\min\{0, a + \mathbb{1}_{\{a<0\}} l\} = \min\{0, a + l\}$.

Combining equations (C4) and (C5) gives a new mapping

$$P_i^1 = \Psi_i^1(\mathbf{P}^1) = \min\left\{\sum_j W_{ji}^1 D_{ji}, C_i + \mathbb{1}_{i\in\mathcal{A}}\xi I_i + \sum_{j\neq i}\Pi_{ij}^1 P_j^1\right\}, \tag{C6}$$

in which $\mathbb{1}_{i\in\mathcal{A}}$ is exogenous for a given $\mathbf{W}^1$. We then have the following result. Suppose that $(\mathbf{P}^1, \mathbb{1}^l)$ is a payment equilibrium of $t = 1$. Then, from (C4) and (C5), $\mathbf{P}^1$ satisfies (C6). Conversely, if $\mathbf{P}^1 \in \mathbb{R}^N$ satisfies (C6), then there exists $\mathbb{1}^l \in \{0,1\}^N$ such that $(\mathbf{P}^1, \mathbb{1}^l)$ is a payment equilibrium of $t = 1$. To see this, let $\mathbb{1}_i^l = \mathbb{1}_{\left\{\left(C_i+\sum_{j\neq i}\Pi_{ij}^1 P_j^1 < \sum_j W_{ji}^1 D_{ji}\right)\wedge(i\in\mathcal{A})\right\}}$. Then by construction, (C2) is satisfied. And again from (C4) and (C5), $\Psi^1(\mathbf{P}^1) = \Phi^1(\mathbf{P}^1)$, which verifies that $(\mathbf{P}^1, \mathbb{1}^l)$ is indeed a payment equilibrium.

The above result contains an important insight. The mapping $\mathbf{P}^1 = \Phi^1(\mathbf{P}^1)$ given in equation (C1) is a function of the liquidation event $\mathbb{1}^l$, which is itself endogenous to $\mathbf{P}^1$. Consequently, the appropriate mapping Φ^1 whose fixed points determine the equilibrium should take the endogeneity of the liquidation event into account. In the analysis above, we borrow the approach of Acemoglu, Ozdaglar and Tahbaz-Salehi (2015) and show that the mapping $\Phi^1(\mathbf{P}^1)$ can be reduced to an equivalent mapping $\Psi^1(\mathbf{P}^1)$ which is independent of the vector $\mathbb{1}^l$. As a result, it is sufficient to show the existence and uniqueness of $\mathbf{P}^1 \in \mathbb{R}^N$ that satisfies $\mathbf{P}^1 = \Psi^1(\mathbf{P}^1)$.

We proceed to show existence by analyzing the properties of mapping Ψ^1. Denote $\mathbf{1}$ as the N-dimensional vector with all components equal to 1. The vector of total withdrawal requests is $\mathbf{1}^T(\mathbf{W}^1 \circ \mathbf{D}) < \mathbf{1}^T\mathbf{D}$, in which $\circ$ represents the Hadamard product of two matrices. It follows that $\mathbf{P}^1 \in [0, \mathbf{1}^T\mathbf{D}] \subset \mathbb{R}^N$. The set $[0, \mathbf{1}^T\mathbf{D}]$ is bounded and, with the pointwise ordering induced by the lattice operations, forms a complete lattice. Hence, the equilibrium payment vector $\mathbf{P}^1$ is a fixed point of the mapping $\Psi^1 : [0, \mathbf{1}^T\mathbf{D}] \to [0, \mathbf{1}^T\mathbf{D}]$ defined by (C6). As shown in Theorem 1 of Eisenberg and Noe (2001), the mapping Ψ^1 is continuous, positive, increasing, concave, and nonexpansive. Tarski's fixed-point theorem (1955) implies that the set of fixed points is nonempty and forms a complete lattice.

We next show uniqueness. Suppose for contradiction that there are two distinct payment equilibria, denoted by $\mathbf{P}^1$ and $\hat{\mathbf{P}}^1$ such that $\mathbf{P}^1 \neq \hat{\mathbf{P}}^1$. From the above analysis, both $\mathbf{P}^1$ and $\hat{\mathbf{P}}^1$ must satisfy (C6). Next we introduce an auxiliary lemma.

Lemma B1 of Acemoglu, Ozdaglar and Tahbaz-Salehi (2015) *Suppose that* $\beta > 0$. *Then*

$$\left|[\min\{\alpha,\beta\}]^+ - [\min\{\hat{\alpha},\beta\}]^+\right| \leq |\alpha - \hat{\alpha}|\,.$$

Following this lemma, for any bank i,

$$
\begin{aligned}
&\left|P_i^1 - \hat{P}_i^1\right| \\
&= \left|\min\left\{\sum_j W_{ji}^1 D_{ji}, C_i + \mathbb{1}_{i\in\mathcal{A}}\xi I_i + \sum_{j\neq i}\Pi_{ij}^1 P_j^1\right\} - \min\left\{\sum_j W_{ji}^1 D_{ji}, C_i + \mathbb{1}_{i\in\mathcal{A}}\xi I_i + \sum_{j\neq i}\Pi_{ij}^1 \hat{P}_j^1\right\}\right| \\
&\leq \left|\sum_{j\neq i}\Pi_{ij}^1 P_j^1 - \sum_{j\neq i}\Pi_{ij}^1 \hat{P}_j^1\right|.
\end{aligned}
$$

Here $[\cdot]^+$ is dropped because the term inside is non-negative by construction in our setting.

Let matrix $\mathbf{\Lambda} = \mathbf{\Pi} - \mathbf{\Pi} \circ diag(1, \ldots, 1)$ such that the diagonal elements of $\mathbf{\Lambda}$ are zero, and the off-diagonal elements of $\mathbf{\Lambda}$ equal those of matrix $\mathbf{\Pi}$. The above inequality is equivalent to

$$\left|P_i^1 - \hat{P}_i^1\right| \leq \left|(\mathbf{\Lambda P})_i - (\mathbf{\Lambda \hat{P}})_i\right|.$$

Because $\Pi_{ii} > 0, \forall i$ (since a bank must have a positive retail deposit, i.e., $D_{ii} > 0$), the column sums of $\mathbf{\Lambda}$ are all less than one, i.e. $\|\mathbf{\Lambda}\|_1 < 1$. Summing both sides of the above inequality over all banks i and making use of $\|\mathbf{\Lambda}\|_1 < 1$, we have

$$\|\mathbf{P}^1 - \hat{\mathbf{P}}^1\|_1 \leq \|\mathbf{\Lambda}(\mathbf{P}^1 - \hat{\mathbf{P}}^1)\|_1 \leq \|\mathbf{\Lambda}\|_1 \cdot \|\mathbf{P}^1 - \hat{\mathbf{P}}^1\|_1 < \|\mathbf{P}^1 - \hat{\mathbf{P}}^1\|_1.$$

Hence, we obtain a contradiction and thus complete the proof for uniqueness.

This unique equilibrium payment vector $\mathbf{P}^1$ can be obtained via the *fictitious default algorithm* in at most N iterations.[4] The algorithm starts with the assumption that no banks default. If all obligations being satisfied is indeed a feasible outcome, the algorithm terminates. If some banks default when all other banks pay fully, we update the payment vector given the defaults in the previous step and check for new defaults. The algorithm terminates when no new defaults occur. ■

Having established the equilibrium properties under exogenous withdrawals, next, we incorporate the withdrawal conditions (8)–(12) in the main paper and analyze how they affect the equilibrium characterization. To begin with, note that the exogenous withdrawal shocks by retail depositors Ω_W do not affect the above results. We thus focus on the endogenous withdrawal decisions.

Condition (8) states that bank i withdraws all its deposits due from correspondent banks when bank i itself faces withdrawals that could not be met by its liquid assets. Depositors' withdrawals from bank i are characterized by conditions (9)–(10) and (11)–(12). From (9)–(10),

[4] This result is shown in Lemma 3 of Eisenberg and Noe (2001).

respondent banks of bank i withdraw if bank i defaults early or is expected to default. From (11)–(12), the retail depositor withdraws from bank i when other depositors of her bank do so or when her bank's correspondent defaults. Other than exogenous reasons that bank i has a low expected return R_i^1, this happens when bank i's correspondents face large withdrawals that cannot be met. In other words, significant withdrawals at its correspondents lead to withdrawals at bank i.

The contagious withdrawals give rise to an important feature: depositors face strategic complementarities in their withdrawal decisions. Following Bulow, Geanakoplos and Klemperer (1985), the marginal payoff of any depositor's withdrawal increases with other depositors' withdrawals. Specifically, a respondent bank's marginal payoff to withdraw increases as other depositors withdraw under (8)–(10), and flat otherwise; a retail depositor's marginal payoff to withdraw increases as other depositors withdraw under (11)–(12), and flat otherwise. Supermodular games provide the appropriate framework to model strategic interactions in the presence of complementarities (Topkis, 1979; Milgrom and Roberts, 1990; Vives, 1990). The following lemma establishes the supermodularity property.

Lemma C1 *The game of depositors' strategic withdrawals at $t = 1$ is supermodular.*

Proof. The proof is based on Milgrom and Roberts (1990). This non-cooperative game has $2N$ players: N bank depositors and N retail depositors, denoted respectively by b_i and r_i, $i \in \{1, 2, \ldots, N\}$. A bank depositor b_i has an $(N-1)$-dimensional strategy set: $W_{i,\cdot}^b \in \{0,1\}^{N-1}$ where $W_{i,j}^b = W_{ij}^1 \in \{0,1\}$, $\forall j \neq i$. A retail depositor r_i has a one-dimensional strategy set: $W_i^r = W_{ii}^1 \in \{0,1\}$. The strategy set of each player is finite, compact, and forms a complete lattice in the Euclidean space with the usual vector ordering.

Denote the payoff function of the bank depositor as $f_i^b(W_i^b; W_{-i}^b \times \mathbf{W}^r)$ and the retail depositor as $f_i^r(W_i^r; W_{-i}^r \times \mathbf{W}^b)$. Since the players' strategy sets are finite, the payoff functions are continuous with respect to the strategy sets.

Next we show that the payoff functions satisfy increasing differences and supermodularity. For a bank depositor b_i, the payoff function satisfies $f_i^b(W_{i,j}^b = 1, \forall j \neq i; W_{-i}^b \times \mathbf{W}^r) - f_i^b(W_{i,j}^b = 0, \exists j \neq i; W_{-i}^b \times \mathbf{W}^r)$ is positive if condition (8) holds, and is negative otherwise; $f_i^b(W_{i,j}^b = 1; W_{i,k\neq j}^b; W_{-i}^b \times \mathbf{W}^r) - f_i^b(W_{i,j}^b = 0; W_{i,k\neq j}^b; W_{-i}^b \times \mathbf{W}^r)$ is positive if condition (9) or (10) holds for bank j, and is negative otherwise. Given the nature of conditions (8)–(10), an element of $W_{-i}^b \times \mathbf{W}^r$ under which any of these conditions hold cannot be smaller than an element under which conditions (8)–(10) do not hold. Hence, $\forall \hat{W}_i^b \geq W_i^b$, $\forall \hat{W}_{-i}^b \times \hat{\mathbf{W}}^r \geq W_{-i}^b \times \mathbf{W}^r$, we have

$$f_i^b(\hat{W}_i^b; \hat{W}_{-i}^b \times \hat{\mathbf{W}}^r) - f_i^b(W_i^b; \hat{W}_{-i}^b \times \hat{\mathbf{W}}^r) \geq f_i^b(\hat{W}_i^b; W_{-i}^b \times \mathbf{W}^r) - f_i^b(W_i^b; W_{-i}^b \times \mathbf{W}^r).$$

That is, f_i^b has increasing differences in W_i^b and $W_{-i}^b \times \mathbf{W}^r$. In a similar fashion, f_i^b satisfies increasing differences with respect to any pair of $W^b_{i,j}$ and $W^b_{i,k}$ for a given $\{W^b_{i,-\{j,k\}} \times W^b_{-i} \times \mathbf{W}^r\}$. Equivalently, f^b_i is supermodular in W^b_i for any given $W^b_{-i} \times \mathbf{W}^r$.

For a retail depositor r_i, the payoff function satisfies: $f^r_i(W^r_i = 1; W^r_{-i} \times \mathbf{W}^b) - f^r_i(W^r_i = 0; W^r_{-i} \times \mathbf{W}^b)$ is positive if $\Theta_i = \{0, 1\}$ and is negative otherwise. Given the nature of conditions (11)–(12), an element in $\{W^r_{-i} \times \mathbf{W}^b : \Theta_i(W^r_{-i} \times \mathbf{W}^b) = \{0, 1\}\}$ cannot be smaller than any element in $\{W^r_{-i} \times \mathbf{W}^b : \Theta_i(W^r_{-i} \times \mathbf{W}^b) = \{0\}\}$. Hence, $\forall \hat{W}^r_{-i} \times \hat{\mathbf{W}}^b \geq W^r_{-i} \times \mathbf{W}^b \in \{0, 1\}^{N^2-1}$, we have

$$f^r_i(1; \hat{W}^r_{-i} \times \hat{\mathbf{W}}^b) - f^r_i(0; \hat{W}^r_{-i} \times \hat{\mathbf{W}}^b) \geq f^r_i(1; W^r_{-i} \times \mathbf{W}^b) - f^r_i(0; W^r_{-i} \times \mathbf{W}^b).$$

This establishes that f^r_i has increasing differences in W^r_i and $W^r_{-i} \times \mathbf{W}^b$. Furthermore, since W^r_i is one-dimensional, $\forall W^r_i, \hat{W}^r_i \in \{0, 1\}$ and $\forall W^r_{-i} \times \mathbf{W}^b \in \{0, 1\}^{N^2-1}$ we have

$$f^r_i(W^r_i, W^r_{-i} \times \mathbf{W}^b) + f^r_i(\hat{W}^r_i, W^r_{-i} \times \mathbf{W}^b) \leq f^r_i(\inf\{W^r_i, \hat{W}^r_i\}, W^r_{-i} \times \mathbf{W}^b) + f^r_i(\sup\{W^r_i, \hat{W}^r_i\}, W^r_{-i} \times \mathbf{W}^b).$$

This establishes that f^r_i is supermodular in W^r_i.

Taken together, all conditions in Milgrom and Roberts (1990) for a supermodular game satisfy. ■

Supermodular games have nice properties. The following result characterizes the equilibrium.

Proposition C2 The set of pure strategy Nash equilibria of withdrawals $\mathbf{W}^1$ is non-empty and forms a complete lattice. Let the best-case equilibrium be the one with the minimum withdrawals. The best-case equilibrium can be obtained via an iterative algorithm with finite steps.

Proof. The proof applies results established in Tarski (1955), Topkis (1979), Milgrom and Roberts (1990), and Vives (1990). The following theorem is central to our results.

Theorem 5 of Milgrom and Roberts (1990) *Let Γ be a supermodular game. For each player n, there exist largest and smallest serially undominated strategies, $\overline{x}_n$ and $\underline{x}_n$. Moreover, the strategy profiles $(\underline{x}_n; n \in N)$ and $(\overline{x}_n; n \in N)$ are pure Nash equilibrium profiles.*

This theorem says that all serially undominated strategies form a complete lattice, whose extreme points are the largest and smallest Nash equilibria. Moreover, the theorem establishes that the extreme points can be obtained using the iterated elimination process which produces a series of monotone strategies.

From Lemma C1, the game of depositors' withdrawals at $t = 1$ is supermodular. Applying Milgrom and Roberts (1990), the set of pure strategy Nash equilibria exists and forms a complete lattice. The best-case equilibrium has the minimum withdrawals and thus is the smallest Nash equilibrium in the complete lattice.

Equilibria of games with supermodular payoffs, yielding monotone increasing best responses, have nice stability properties. In particular, the smallest Nash equilibrium can be found by an iterative elimination of strictly dominated strategies starting from the smallest action profile. This algorithm is based on the one proposed in Topkis (1979). The algorithm corresponds to the iterative decision-making process by which each of the players concurrently and individually chooses the next payoff-optimizing strategy under the assumption that the other players will hold their decisions unchanged. A new joint decision is put together by combining these individually determined decisions, and the next iteration then begins. For finite games, the iteration converges in finite steps (Topkis, 1979, pg. 784). This algorithm is formalized as the "best response dynamics" in Milgrom and Roberts (1990) and the "Cournot tâtonnement" in Vives (1990). Theorem 5.1 in Vives (1990) establishes monotone convergence to an equilibrium point of the game whenever the starting point is 'below' or 'above' all the best reply correspondences of the players. ■

Once the $t = 1$ equilibrium is determined and the returns $\mathbf{R}^2$ are realized, the $t = 2$ payment equilibrium for the set of banks that survive from $t = 1$ can be characterized following Eisenberg and Noe (2001).

Proposition C3 Once the $t = 1$ equilibrium $(\mathbf{W}^1, \mathbf{X}^1)$ is determined, the final date payment equilibrium characterized by $\mathbf{X}^2$ exists and is unique. Furthermore, $\mathbf{X}^2$ can be obtained via an iterative algorithm in at most N iterations.

Proof. The $t = 2$ clearing system matches that in Eisenberg and Noe (2001). This is because (1) once the returns $\mathbf{R}^2$ are realized, the term $A_i^1 - \sum_{j\in\mathcal{S}} W_{ji}^1 D_{ji} + (1 - \mathbb{1}_i^l) I_i R_i^2 \geq 0$ is exogenously given for each bank; (2) default does not create extra costs that would affect the clearing outcome. The proof works similarly to Proposition C1. Since $\mathbf{X}^2 = \mathbf{\Pi}^2 \mathbf{P}^2$, it is equivalent to analyzing the properties of the payment vector $\mathbf{P}^2$. It follows that $\mathbf{P}^2 \in [0, \mathbf{1}^T\mathbf{D}] \subset \mathbb{R}^N$. The set $[0, \mathbf{1}^T\mathbf{D}]$ is bounded and forms a complete lattice. The equilibrium payment vector is a fixed point of the mapping $\Phi^2 : [0, \mathbf{1}^T\mathbf{D}] \to [0, \mathbf{1}^T\mathbf{D}]$ defined by equation (C3). As shown in Theorem 1 of Eisenberg and Noe (2001), the mapping Φ^2 is continuous, positive, increasing, concave, and nonexpansive. Tarski's fixed-point theorem (1955) implies that the set of fixed points is nonempty and forms a complete lattice. Furthermore, since $\mathbf{\Pi}^2$ shares the same properties as $\mathbf{\Pi}^1$, using the same technique as in Proposition C1, we obtain the uniqueness of $\mathbf{P}^2$ at equilibrium. This unique equilibrium payment vector $\mathbf{P}^2$ can be obtained via the Eisenberg-Noe *fictitious default algorithm* in at most N iterations, in the same way as $\mathbf{P}^1$ is computed. ■

Online Appendix D: Analytical Results for Stylized Networks

The NBAs led to changes in both interbank networks and bank balance sheets, e.g., New York City (NYC) banks held more cash after the acts' introduction. To evaluate the effect of network changes in isolation, we provide analytical results for a pair of stylized networks. We compare an N-bank chain network versus a two-tier pyramid, which underlines how the concentration of interconnections affects stability. We also extend the results to a stylized network of seven banks, which resembles the emergence of the pyramiding structure after the NBAs. To simplify the analysis, we normalize banks' balance sheets. Such normalizations guarantee that any variation in the stability of the system is due to changes in the distribution of interbank liabilities, while abstracting away from other features of the network. To ease readability, we move the proof to the end of this appendix.

Balance-Sheet Normalization

We let bank 1 solely receive deposits, which resembles an NYC bank. We normalize banks' balance sheets such that (1) the size of cash equals the size of equity capital for all banks; (2) the retail deposits are the same across all banks; and (3) the size of cash and the size of investment, respectively, are the same across all the non-NYC banks. Let us also fix the investment returns for all the non-NYC banks to 1. Formally, we make the following assumption.

Assumption D1 We normalize banks' balance sheets such that $C_i = K_i$, $D_{ii} = d$, for all i. Let bank $i = 1$ represents the NYC bank and banks $i \geq 2$ represent the non-NYC banks; $C_i = c$, $I_i = a$, $R_i^2 = 1$, for all $i \geq 2$. Moreover, $0 < a < d$.

An N-Bank Chain vs. a Two-Tier Pyramid

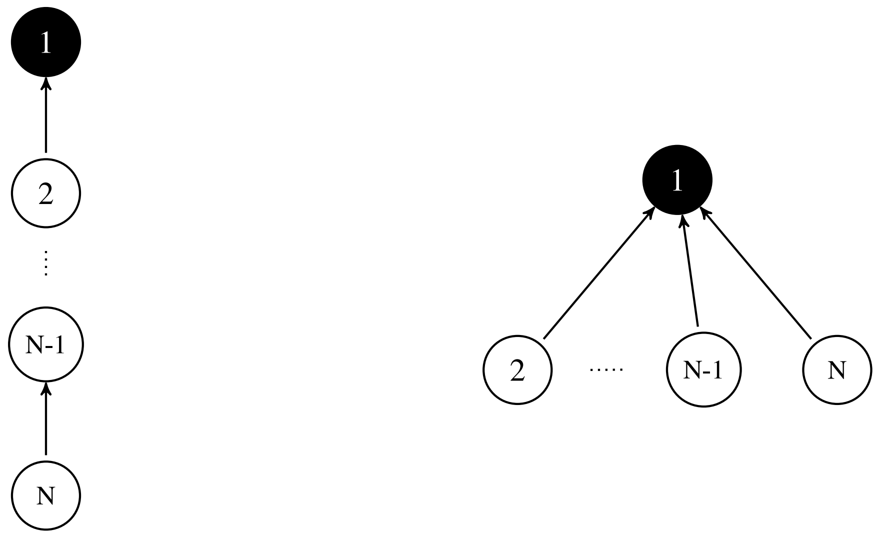
Denote a bank's total liability as $D_i = \sum_j D_{ji}$. The interbank deposits $\mathbf{D}$ are determined by the network structure and the balance-sheet equality. We proceed to introduce the two stylized networks.

In the N-bank chain network, bank N places a deposit $D_{N,N-1}$ at bank $N-1$, who then places a deposit $D_{N-1,N-2}$ at bank $N-2$, etc. Denote $\mathbf{D}^{\text{N-chain}}$ the N-bank chain network; formally, $\mathbf{D}^{\text{N-chain}} = \{D_{ij} : D_{ij} > 0, \forall i, j = 1, \ldots, N : i = j \text{ or } i = j + 1; D_{ij} = 0, \text{otherwise}\}$. Using Assumption D1 and the balance-sheet equality, we have

$$\text{(D1)} \quad K_N + d = C_N + a + D_{N,N-1} \Rightarrow D_{N,N-1} = d - a,$$

$$\text{(D2)} \quad K_i + d + D_{i+1,i} = C_i + a + D_{i,i-1} \Rightarrow D_{i,i-1} = (N - i + 1)(d - a), i = 2, \ldots, N - 1,$$

$$\text{(D3)} \quad K_1 + d + D_{2,1} = C_1 + I_1 \Rightarrow I_1 = D_1 = (N - 1)(d - a) + d.$$



(a) N-bank chain (b) Two-tier pyramid

Figure D1. An N-Bank Chain vs. a Two-Tier Pyramid. This figure shows two stylized networks of N banks. Subfigure D1a illustrates an N-bank chain network in which bank i places a deposit $D_{i,i-1}$ at bank $i-1$. Subfigure D1b illustrates a two-tier pyramid in which banks $i = 2, \dots, N$ all place a deposit $D_{i,1}$ at bank 1.

In the two-tier pyramid, banks $i = 2, \dots, N$ all place a deposit $D_{i,1}$ at bank 1 where interbank deposits are highly concentrated. Denote $\mathbf{D}^{\text{2-tier}}$ the two-tier pyramid network; formally, $\mathbf{D}^{\text{2-tier}} = \{D_{ij} : D_{ij} > 0, \forall i, j = 1, \dots, N : i = j \text{ or } j = 1; D_{ij} = 0, \text{otherwise}\}$. We have

$$K_i + d = C_i + a + D_{i,1} \quad \Rightarrow \quad D_{i,1} = d - a, i = 2, \dots, N, \tag{D4}$$

$$K_1 + d + \sum_{i=2}^{N} D_{i,1} = C_1 + I_1 \quad \Rightarrow \quad I_1 = D_1 = (N-1)(d-a) + d. \tag{D5}$$

Illustrative of a "top-to-bottom crisis," the NYC bank (bank 1) defaults after incurring losses in investment. Equivalently, the bank's asset is less than its liability, i.e., $C_1 + R_1^2 I_1 < D_1 = I_1$ (recall that $D_1 = I_1$ from equations (D3) and (D5)). Denote $\Delta R_1 = 1 - (C_1 + R_1^2 I_1)/I_1$ as the rate of bank 1's asset shortfall; bank 1 defaulting thus implies that $\Delta R_1 > 0$. Variations in ΔR_1 represent the size of the negative asset shock to bank 1. We evaluate financial stability by comparing the number of bank defaults across the two stylized networks.

Proposition D1 Suppose that Assumption D1 holds. Denote $\Delta R_1 = 1 - (C_1 + R_1^2 I_1)/I_1$ as the rate of bank 1's asset shortfall in a "top-to-bottom crisis." There exists a threshold value of shock size, $\Delta R_1^{\text{N-chain}}(N)$, such that $\sum_i \mathbb{1}_i^d(\mathbf{D}^{\text{N-chain}}) > \sum_i \mathbb{1}_i^d(\mathbf{D}^{\text{2-tier}})$ for $\Delta R_1 \in (c/((N-1)(d-a)), c/(d-a)]$; $\sum_i \mathbb{1}_i^d(\mathbf{D}^{\text{N-chain}}) < \sum_i \mathbb{1}_i^d(\mathbf{D}^{\text{2-tier}})$ for $\Delta R_1 \in \left(c/(d-a), \Delta R_1^{\text{N-chain}}(N)\right]$, where $\Delta R_1^{\text{N-chain}}(N) > c/(d-a)$ is given by

$$\text{(D6)} \qquad \Delta R_1^{\text{N-chain}}(N) = \frac{c}{(N-1)(d-a)}\left[1 + \sum_{k=2}^{N-1}\prod_{j=2}^{k}\frac{(N-j)(d-a)+d}{(N-j)(d-a)}\right].$$

This proposition shows that the stability of different networks against a top-to-bottom crisis depends on the size of the shocks. If the shock sizes are mild, $\Delta R_1 \in (c/((N-1)(d-a)), c/(d-a)]$, the N-bank chain network has multiple defaults, whereas the two-tier pyramid is limited to a single default at bank 1; hence, the more concentrated two-tier pyramid is more robust. However, if the shock sizes are large, $\Delta R_1 \in \left(c/(d-a), \Delta R_1^{\text{N-chain}}(N)\right]$, the N-bank chain network incurs no more than $N-1$ defaults, whereas the two-tier pyramid suffers from a simultaneous default of N banks; hence, the more concentrated two-tier pyramid is more fragile.

Illustrative of a "bottom-to-top crisis," a set of country banks faces exogenous withdrawals by the retail depositor, i.e., $W_{ii}^1 = 1, i \in \Omega_W$. Let the size of Ω_W be ω, so that $\Omega_W = \{N-\omega+1, N-\omega+2, \ldots, N\}$ and $\omega = 1, \ldots, N-2$. The size of the set Ω_W represents the size of the "bottom-to-top" withdrawal shocks. Since liquidation is the direct consequence of withdrawals, we evaluate financial stability by comparing the number of liquidations across the two stylized networks. In line with the regulations brought about by the NBAs, we introduce the following assumption.

Assumption D2 *Let $c < d$ so a country bank experiences cash shortage when facing retail withdrawals; assume $C_1 \geq (1-\xi)I_1$ so the NYC bank stays solvent after liquidation.*

Proposition D2 Let the size of the set Ω_W be $\omega = 1, \ldots, N-2$, so that $\Omega_W = \{N-\omega+1, N-\omega+2, \ldots, N\}$. Suppose that Assumptions D1 and D2 hold. Then $\sum_i \mathbb{1}_i^l(\mathbf{D}^{\text{N-chain}}) \geq \sum_i \mathbb{1}_i^l(\mathbf{D}^{\text{2-tier}})$. In particular, $\sum_i \mathbb{1}_i^l(\mathbf{D}^{\text{N-chain}}) > \sum_i \mathbb{1}_i^l(\mathbf{D}^{\text{2-tier}})$ when $c < a$, or, when $c \geq a$ and $C_1 \in [\max\{(1-\xi)I_1, \omega(d-a)+d\}, I_1)$.

This proposition shows that, for a bottom-to-top crisis, the two-tier pyramid is always more robust. As long as not all country banks face the initial exogenous withdrawal shocks ($\omega < N-1$) and that bank 1 has enough cash assets to stay solvent, the two-tier pyramid generates no more liquidations than the N-bank chain, a result insensitive to the size of the withdrawal shock ω.

The mechanism is as follows. In an N-bank chain network, the withdrawal shock at the bottom is contagious upwards along the chain. Banks that are subject to withdrawals suffer from a cash shortage, and thus need to redeem its interbank deposits, causing further panic withdrawals at all other banks. In contrast, in a two-tier pyramid, as long as bank 1 has enough cash and does not default, the panic withdrawals are contained within the exogenously shocked

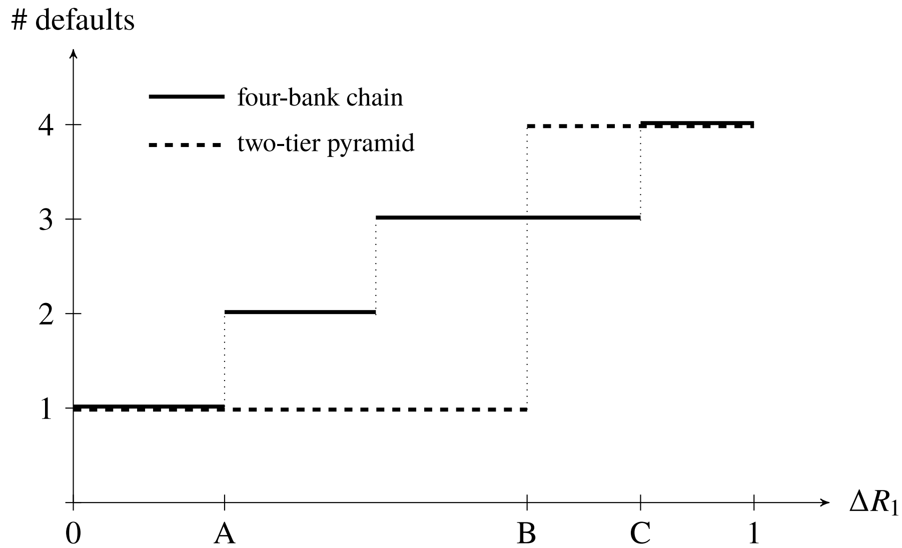


Figure D2. Number of Defaults and Size of Negative Shock ΔR_1. This figure compares the number of defaults in the four-bank chain and a two-tier pyramid as we vary the size of the negative asset shock ΔR_1. Compared to the four-bank chain, the two-tier pyramid is more robust when the shock size is mild (in interval AB) and is more fragile when the shock size is severe (in interval BC).

country banks, rather than spreading to the other country banks. Hence, the two-tier pyramid is more robust to withdrawal shocks originating from country banks.

Examples

We illustrate the above results in an example of $N = 4$ banks. We start by analyzing the top-to-bottom crises. In a four-bank chain, the conditions for the simultaneous default of two, three, and four banks are $\Delta R_1 > \frac{c}{D_{2,1}} = \frac{c}{3(d-a)}$, $\Delta R_1 > \frac{c}{D_{2,1}}\left(1 + \frac{D_2}{D_{3,2}}\right)$, and $\Delta R_1 > \frac{c}{D_{2,1}}\left(1 + \frac{D_2}{D_{3,2}} + \frac{D_2}{D_{3,2}}\frac{D_3}{D_{4,3}}\right)$, respectively. In a two-tier pyramid, banks $i = 2, 3, 4$ are direct respondents of bank 1—the condition for the simultaneous default of all banks is $\Delta R_1 > \frac{c}{D_{2,1}} = \frac{c}{d-a}$. Figure D2 illustrates the number of defaults in the two stylized networks when we vary the size of ΔR_1. Comparing across networks, under a mild negative shock when $\Delta R_1 \in \left(\frac{c}{3(d-a)}, \frac{c}{d-a}\right]$ (corresponding to interval AB in Figure D2), the two-tier pyramid is more robust because default is limited to only the shocked bank 1, whereas the four-bank chain has multiple defaults caused by contagion. However, under a severe negative shock $\Delta R_1 \in \left(\frac{c}{d-a}, \frac{c}{D_{2,1}}\left(1 + \frac{D_2}{D_{3,2}} + \frac{D_2}{D_{3,2}}\frac{D_3}{D_{4,3}}\right)\right]$ (corresponding to interval BC in Figure D2), the two-tier pyramid in which all four banks default is more fragile. In comparison, the four-bank chain has fewer defaults.

The comparison is different when it comes to a bottom-to-top crisis. The two-tier pyramid is always more robust and generates no more liquidations than the N-bank chain. In the four-bank

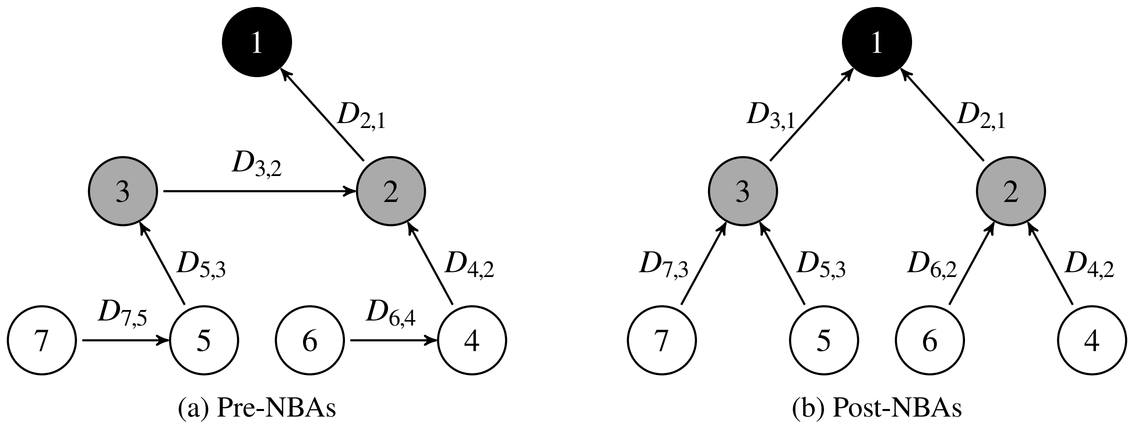


Figure D3. A Stylized Network of Seven Banks. This figure shows two stylized networks that resemble the structural changes brought by the NBAs. Subfigure D3a illustrates the pre-NBAs network. Subfigure D3b illustrates the post-NBAs network.

chain network, the exogenous withdrawal shock at the bottom of the chain is contagious along the chain, affecting all banks. Facing an exogenous withdrawal shock, bank 4 suffers from cash shortage and withdraws from bank 3 (following condition (8) in the paper), who then withdraws from bank 2, etc. Thus, bank 1 receives a total withdrawal request of $D_1 = I_1 = 3(d - a) + d$. Since $C_1 \geq (1 - \xi)I_1$, bank 1 is solvent. When the country banks do not have significant cash to meet depositors' run ($c < a$), all country banks would suffer from liquidation. This result holds no matter whether one or two country banks are hit with the withdrawal shock simultaneously. The two-tier pyramid is different because the depositors' run is contained to only the exogenously shocked country banks, as long as bank 1 stays solvent. The two-tier structure effectively avoids the propagation of withdrawal shocks along the chain and is thus more robust.

The above insights carry through to a stylized network of seven banks. As shown in Figure D3, this example resembles the structural changes brought about by the NBAs. The pre-NBAs network is summarized in Figure D3a: both the NYC bank (bank 1) and the Philadelphia bank (bank 2) are major correspondent banks. The Pittsburgh bank (bank 3) and other country banks (4 and 5) serve as local correspondents, taking deposits from country banks (6 and 7). The NBAs led to a three-tier reserve pyramid which had fewer numbers of correspondents. As in Figure D3b, all country banks ($i = 4, 5, 6, 7$) place deposits at reserve city banks in Philadelphia and Pittsburgh ($i = 2, 3$), which then place deposits at the NYC bank ($i = 1$) at the top.

The stylized network of seven banks is a stacked version of the two-tier model. The results in Propositions D1–D2 follow through qualitatively (the proofs are similar to those of Propositions D1–D2 and are thus omitted). For a top-to-bottom crisis, the size of negative asset shock matters. Under mild asset shocks, the post-NBAs three-tier pyramid is more robust than the pre-NBAs network because the asset shock is less likely to spread to respondent banks 2 and 3. For a bottom-to-top crisis, the post-NBAs three-tier pyramid is always more robust because the chains are shorter so the country banks that are not directly shocked can avoid liquidations.

Importance of Top-to-Bottom vs. Bottom-to-Top Crises

We have compared the stability of stylized networks during crises that originate from the top and from the bottom of the pyramid. The analysis also provides insights into what types of crises are more relevant to a concentrated network. Let us focus on the two-tier pyramid. For a top-to-bottom crisis, as long as the return shock to the NYC bank (bank 1 in the stylized model) is large enough, insolvency spreads to the entire network, so all banks default simultaneously. In contrast, for a bottom-to-top crisis, as long as the NYC bank has enough cash and stays solvent against withdrawals from country banks, liquidation will not occur at banks that do not directly face withdrawal shocks.

Furthermore, the two-tier pyramid is more robust than the chain network regardless of the size of withdrawal shocks to country banks Ω_W. This result suggests that a severe bottom-to-top crisis becomes less probable in a pyramid structure. These theoretical predictions are in line with evidence from the National Banking era that banking crises mainly originated from financial centers (see, e.g., Wicker, 2006).

Proof of Proposition D1

For the top-to-bottom crises, we compare the number of defaults across the two networks when varying ΔR_1, the size of the negative shock to bank 1. Denote $Q_i = X_{i,i}/D_{i,i}$ the fraction of payment over the nominal liability by bank i. Denote $\Delta R_1^{\text{N-chain}}(n)$ and $\Delta R_1^{\text{2-tier}}(n)$ as the minimum of the shock sizes that can cause the simultaneous defaults of n banks in the N-bank chain network, and the two-tier network, respectively.

We start from the *N-bank chain network*. From the definition of ΔR_1, bank 1 defaults if and only if $\Delta R_1 > 0$, and so $Q_1 = 1 - \Delta R_1 < 1$. Similarly, banks $i = 1, 2$ default if and only if $c + a + Q_1 D_{2,1} < d + D_{3,2}$; plugging in the relation, $D_{2,1} = D_{3,2} + d - a$, gives

$$\Delta R_1 > \Delta R_1^{\text{N-chain}}(2) = \frac{c}{D_{2,1}} = \frac{c}{(N-1)(d-a)}. \tag{D7}$$

Using the same method, we can show that banks $i = 1, 2, 3$ default if and only if[5]

$$\Delta R_1 > \Delta R_1(3) = \frac{c}{D_{2,1}} \left[1 + \frac{D_2}{D_{3,2}} \right]. \tag{D8}$$

[5]From the balance sheet equation of bank $i = 3$, banks $i = 1, 2, 3$ default if and only if $1 - Q_2 > c/D_{3,2}$. Using the relations that $Q_2 = (c + a + Q_1 D_{2,1})/D_2$ and $D_2 = d + D_{3,2} = D_{2,1} + a$, we obtain condition (D8).

More generally, among the defaulting banks along the chain network, the series of payment fraction $\{Q_i\}$ obeys a recursive relation

(D9)
$$Q_i = \frac{a + c + Q_{i-1} D_{i,i-1}}{D_i}.$$

Plugging in the balance-sheet relation $D_i = D_{i,i-1} + a$, for all $i \geq 2$, we arrive at the following recursive form,

(D10)
$$(1 - Q_i) = \frac{D_{i,i-1}}{D_i}(1 - Q_{i-1}) - \frac{c}{D_i}.$$

From the balance sheet equation of bank $i+1$, bank $i+1$ defaults if and only if $1 - Q_i > c/D_{i+1,i}$. Using (D10), we obtain the following threshold condition for bank $i + 1$ to default,

(D11)
$$\Delta R_1^{\text{N-chain}}(i + 1) = \frac{c}{D_{2,1}} \left[1 + \sum_{k=2}^{i} \prod_{j=2}^{k} \frac{D_j}{D_{j+1,j}} \right].$$

Hence, the threshold condition for the simultaneous default of all N banks satisfies

(D12)
$$\Delta R_1^{\text{N-chain}}(N) = \frac{c}{D_{2,1}} \left[1 + \sum_{k=2}^{N-1} \prod_{j=2}^{k} \frac{D_j}{D_{j+1,j}} \right] > \frac{c}{D_{2,1}} \left[1 + \sum_{k=2}^{N-1} \prod_{j=2}^{k} 1 \right] = \frac{c(N-1)}{D_{2,1}} = \frac{c}{d-a}.$$

Next we turn to the *two-tier pyramid*. As in the previous case, bank 1 defaults if and only if $\Delta R_1 > 0$. In the second tier of the network, banks $i = 2, \ldots, N$ simultaneously default if and only if

(D13)
$$\Delta R_1 > \Delta R_1^{\text{2-tier}}(N) = \frac{c}{D_{i,1}} = \frac{c}{d-a}.$$

From (D7) and (D13), we conclude that $\Delta R_1^{\text{2-tier}}(N) > \Delta R_1^{\text{N-chain}}(2)$. This result suggests that the threshold value for the negative shock size to generate contagion in the N-chain network, $\mathbf{D}^{\text{N-chain}}$, is lower than that in the two-tier pyramid $\mathbf{D}^{\text{2-tier}}$.

From (D12) and (D13), we conclude that $\Delta R_1^{\text{N-chain}}(N) > \Delta R_1^{\text{2-tier}}(N)$. This result suggests that the threshold value for the negative shock size to generate N simultaneous defaults in the N-chain network, $\mathbf{D}^{\text{N-chain}}$, is higher than that in the two-tier pyramid $\mathbf{D}^{\text{2-tier}}$. Summarizing the above results, we have

$$\begin{cases} \sum_i \mathbb{1}_i^d(\mathbf{D}^{\text{N-chain}}) > \sum_i \mathbb{1}_i^d(\mathbf{D}^{\text{2-tier}}), & \text{for } \Delta R_1 \in \left(\frac{c}{(N-1)(d-a)}, \frac{c}{d-a} \right] \\ \sum_i \mathbb{1}_i^d(\mathbf{D}^{\text{N-chain}}) < \sum_i \mathbb{1}_i^d(\mathbf{D}^{\text{2-tier}}), & \text{for } \Delta R_1 \in \left(\frac{c}{d-a}, \Delta R_1^{\text{N-chain}}(N) \right] \end{cases}$$

which completes the proof of the proposition.

Proof of Proposition D2

For the bottom-to-top crises, we compare the number of liquidations across the two networks when varying the size of the exogenous withdrawal set Ω_W. Recall that C_1 is the level of cash at bank 1 and c the level of cash at all other banks.

We start with the *N-bank chain network*. Facing the retail depositor's withdrawal, bank N does not have enough cash to meet the withdrawal request (recall from Assumption D2 that $c < d$); hence, bank N withdraws from bank $N-1$ under condition (8) in the paper. As a result, bank $N-1$ faces a total withdrawal request of $D_{N,N-1} + d = 2d - a$ and has to withdraw from bank $N-2$, etc. Even if bank $i = N - \omega$ does not face an exogenous withdrawal shock, the retail depositor still withdraws because she follows bank depositors according to condition (12) in the paper. The same holds for all other banks $i < N - \omega$. Hence, bank 1 receives a total withdrawal of $D_1 = (N-1)(d-a) + d$ (recall that $D_1 = I_1$).

Two cases are relevant depending on the level of C_1.

If $C_1 \geq I_1 = D_1 = (N-1)(d-a) + d$, bank 1 has enough cash to cover the withdrawal requests so it does not liquidate. We further have that, under the condition $c \geq a$, all payments are paid in full so $\mathbf{X} = \mathbf{D}$ and no banks liquidate, i.e., $\sum_i \mathbb{1}_i^l = 0$.[6] If $c < a$, then even if a bank redeems all its interbank due-from deposits in full, it still does not have enough cash to honor the total withdrawal request; hence, liquidation occurs at all banks other than bank 1, i.e., $\sum_i \mathbb{1}_i^l = N - 1$.

If $C_1 \in [(1-\xi)I_1, I_1)$, bank 1 does not have enough cash to cover the withdrawal requests unless it liquidates, so $\mathbb{1}_1^l = 1$ and $\mathbb{1}_1^{d1} = 0$. Under the condition $c \geq a$, all payments are paid in full so $\mathbf{X} = \mathbf{D}$ and none of the non-NYC banks liquidate, i.e., $\sum_i \mathbb{1}_i^l = 1$. If $c < a$, then even if a bank redeems all its interbank due-from deposits in full, it still does not have enough cash to honor the total withdrawal request; hence, liquidation occurs at all banks, i.e., $\sum_i \mathbb{1}_i^l = N$.

On balance, the number of liquidations, $\sum_i \mathbb{1}_i^l$, depends on C_1, the cash level at bank 1, and c, the cash level at all other banks; see a summary in Table D1. Notably, the number of liquidations does not depend on the shock size ω.

We now move on to the case of the two-tier pyramid. Banks $i \in \Omega_W = \{N-\omega+1, N-\omega+2, \ldots, N\}$ face withdrawal shocks and have to redeem deposits $D_{i,1}$ from bank 1. Together with the retail depositor's withdrawal based on condition (12) in the main paper, bank 1 receives a total withdrawal of $\omega(d-a) + d$ in total.

[6]To see this, notice that $X_{2,1} = D_{2,1}$. Since $c \geq a$ and $D_{i,i-1} = D_{i+1,i} + d - a$, we have $c + X_{2,1} \geq d + D_{3,2}$, i.e., bank 2 does not liquidate, and so on. More generally, as long as bank 1 does not default and pays deposits in full, we have that all banks $i \geq 2$ avoid liquidation if and only if $c + D_{i,i-1} \geq d + D_{i+1,i}$. This condition is further reduced to $c \geq a$ when plugging in the relation that $D_{i,i-1} = D_{i+1,i} + d - a$.

Table D1. Comparing the Number of Liquidations

$\sum_i \mathbb{1}_i^l$	N-Bank Chain		Two-Tier Pyramid	
	$C_1 \geq I_1$	$C_1 \in [(1-\xi)I_1, I_1)$	$C_1 \geq \omega(d-a)+d$	$C_1 \in [(1-\xi)I_1, \omega(d-a)+d)$
$c \geq a$	0	1	0	1
$c < a$	$N-1$	N	ω	$\omega+1$

If $C_1 \geq \omega(d-a)+d$, bank 1 has enough cash to cover the withdrawal requests so it does not liquidate. Accordingly, the other respondents do not withdraw. Under the condition that $c \geq a$, no banks liquidate (as in the case above), i.e., $\sum_i \mathbb{1}_i^l = 0$. If $c < a$, then even if a bank redeems all its interbank due-from deposits in full, liquidation still occurs at all the banks suffering from the exogenous withdrawal shock, i.e., $\sum_i \mathbb{1}_i^l = \omega$.

If $C_1 \in [(1-\xi)I_1, \omega(d-a)+d)$ (if $(1-\xi)I_1 < \omega(d-a)+d$), bank 1 does not have enough cash to cover the withdrawal requests unless it liquidates, so $\mathbb{1}_1^l = 1$ and $\mathbb{1}_1^{d1} = 0$. Under the condition that $c \geq a$, all payments are paid in full and none of the non-NYC banks liquidate, i.e., $\sum_i \mathbb{1}_i^l = 1$. If $c < a$, then even if a bank redeems all its interbank due-from deposits in full, liquidation still occurs at all banks suffering from withdrawals, i.e., $\sum_i \mathbb{1}_i^l = \omega + 1$.

Suppose instead that $(1-\xi)I_1 \geq \omega(d-a)+d$, then the case discussed in the previous paragraph does not exist, so bank 1 never liquidates.

Overall, the number of liquidations $\sum_i \mathbb{1}_i^l$ therefore depends on banks' cash holdings C_1 and c; see a summary in Table D1.

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