Online Appendices to "Cadet-branch Matching in a Kelso-Crawford Economy"

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The online appendices present example that illustrate how DA-equivalence relates to weakened substitutability conditions from the matching literature. Online Appendix 1 presents examples omitted from Section VI.A. Online Appendix 2 discusses the relationship of Section VI.B with the results of Hatfield and Kominers (2019).

1 DA-equivalence and unilateral substitutability: Examples

The following example shows that the law of aggregate demand for $\hat{C}^b$ is necessary in Theorem 4(a).[1] The law of aggregate demand is clearly necessary in Theorem 4(b).

Example 1 (Necessity of the law of aggregate demand in Theorem 4(a)). Let $I = \{i, j\}$, let $B = \{b\}$, and let $X = \{x, x', y\}$ with $\iota(x) = \iota(x') = d$ and $\iota(y) = e$. Let C^b be the choice function associated to the priority order

$$\{x', y\} \succ_b \{x\} \succ_b \{x'\} \succ_b \{y\} \succ_b \varnothing,$$

and let $\hat{C}^b$ be the choice function associated to the priority order

$$\{x\} \mathrel{\widehat{\succ}_b} \{x', y\} \mathrel{\widehat{\succ}_b} \{x'\} \mathrel{\widehat{\succ}_b} \{y\} \mathrel{\widehat{\succ}_b} \varnothing.$$

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[1] Aygün and Sönmez (2012, 2013) showed that substitutability and the law of aggregate demand together imply the irrelevance of rejected contracts condition. Example 1 shows that, even to deduce only unilateral substitutability, the hypothesis that $\hat{C}^b$ satisfy the law of aggregate demand cannot be weakened to require $\hat{C}^b$ to only satisfy the irrelevance of rejected contracts condition.

It is straightforward to verify that C^b and $\hat{C}^b$ are feasible and DA-equivalent, and that $\hat{C}^b$ is substitutable. However, C^b is not unilaterally substitutable because $y \in C^b(\{x, x', y\})$ but $y \notin C^b(\{x, y\})$. Note that $\hat{C}^b$ does not satisfy the law of aggregate demand because $|C^b(\{x, x', y\})| = |\{x\}| = 1$ while $|C^b(\{x', y\})| = |\{x', y\}| = 2$.

The following two examples show that the feasibility of $\hat{C}^b$ is necessary in both parts of Theorem 4. In the language of Section VI.B, the examples show that DA-strategy-proofness and the irrelevance of rejected contracts condition do not imply unilateral substitutability or the law of aggregate demand.

Example 2 shows furthermore that DA-strategy-proofness and the irrelevance of rejected contracts condition do not imply that the deferred acceptance mechanism is stable (see also Footnote 29). By the contrapositive of Theorem 4 in Hatfield and Kojima (2010), DA-strategy-proofness does not imply unilateral substitutability either.[2]

Example 2 (DA-strategy-proofness + irrelevance of rejected contracts does not imply that deferred acceptance is stable). Let $X = \{x, x', y, y'\}$ with $B = \{b\}$ and $I = \{i, j\}$. Define $\iota(x) = \iota(x') = i$ and $\iota(y) = j$. Define C^b to be the choice function induced by the priority order

$$\{x, y'\} \succ_b \{x', y'\} \succ_b \{y'\} \succ_b \{x'\} \succ_b \{y\} \succ_b \{x\} \succ_b \varnothing.$$

Note that if the preference of i is $x \succ_i x'$ and the preference of j is $y \succ_i y'$, then deferred acceptance with respect to C^b returns the allocation $\{x', y\}$, which is blocked by $\{x\}$. By the contrapositive of Theorem 4 in Hatfield and Kojima (2010), C^b is not unilaterally substitutable. More explicitly, we have that $x \in \{x, y'\} = C^b(\{x, y, y'\})$ but $x \notin \{y\} = C^b(\{x, y\})$, violating unilateral substitutability.

Let $\hat{C}^b$ be the choice function induced by the priority order

$$\{x', y'\} \mathrel{\widehat{\succ}_b} \{y, y'\} \mathrel{\widehat{\succ}_b} \{x, y'\} \mathrel{\widehat{\succ}_b} \{y'\} \mathrel{\widehat{\succ}_b} \{x'\} \mathrel{\widehat{\succ}_b} \{y\} \mathrel{\widehat{\succ}_b} \{x\} \mathrel{\widehat{\succ}_b} \varnothing.$$

Clearly $\hat{C}^b$ and C^b are DA-equivalent and $\hat{C}^b$ is substitutable and satisfies the law of aggregate demand. Hence, C^b is DA-strategy-proof. However, $\hat{C}^b$ is not feasible.

Example 3 (DA-strategy-proofness + irrelevance of rejected contracts does not imply the law of aggregate demand). The choice function C^b in this example is taken from

[2]Example 1 in Kominers and Sönmez (2016) provides another example of the necessity of feasibility in Theorem 4(a), when, as in Example 4, $\hat{C}^b$ is the *substitutable completion* of C^b defined in the proof of Theorem F.1 in Hatfield and Kominers (2019).

Example 2 in Kominers and Sönmez (2016). Let $X = \{x, x', y\}$ with $B = \{b\}$ and $I = \{i, j\}$. Define $\iota(x) = \iota(x') = i$ and $\iota(y) = j$. Define C^b to be the choice function induced by the priority order

$$\{x\} \succ_b \{x', y\} \succ_b \{y\} \succ_b \{x'\} \succ_b \varnothing.$$

As $|C^b(\{x, x', y\})| = |\{x\}| < |\{x', y\}| = |C^b(\{x', y\})|$, the choice function C^b does not satisfy the law of aggregate demand.

Let $\hat{C}^b$ be the choice function induced by the priority order[3]

$$\{x, x'\} \mathrel{\widehat{\succ}_b} \{x', y\} \mathrel{\widehat{\succ}_b} \{x\} \mathrel{\widehat{\succ}_b} \{y\} \mathrel{\widehat{\succ}_b} \{x'\} \mathrel{\widehat{\succ}_b} \varnothing.$$

Clearly $\hat{C}^b$ and C^b are DA-equivalent and $\hat{C}^b$ is substitutable and satisfies the law of aggregate demand. However, $\hat{C}^b$ is not feasible.

The following example shows that one possible converse to Theorem 4 is not true. More precisely, the example shows that feasibility, unilateral substitutability, the law of aggregate demand, and the irrelevance of rejected contracts condition do not together imply DA-equivalence to a feasible, substitutable choice function. This provides a counterexample to a converse to Theorem 4.

Example 4 (Unilateral substitutability + law of aggregate demand does not imply DA-equivalence to a feasible, substitutable choice function). Let $B = \{b\}$, let $I = \{i, j, k\}$, and let $X = \{x, x', y, z\}$ with $\iota(x) = \iota(x') = i$, $\iota(y) = j$, and $\iota(z) = k$. Let C^b be the choice function induced by the priority order

$$\{y, z\} \succ_b \{x', y\} \succ_b \{y\} \succ_b \{x, z\} \succ_b \{x\} \succ_b \{z\} \succ_b \{x'\} \succ_b \varnothing.$$

It is straightforward to verify that C^b is unilaterally substitutable.

However, C^b is not DA-equivalent to a feasible, substitutable choice function that satisfies the irrelevance of rejected contracts condition. Suppose for the sake of deriving a contradiction that C^b is DA-equivalent to $\hat{C}^b$, where $\hat{C}^b$ is feasible, substitutable, and satisfies the irrelevance of rejected contracts condition. To obtain a contradiction, we divide into cases based on the value of $\hat{C}^b(\{x, x'\})$.

Case 1: $\hat{C}^b(\{x, x'\}) = \{x\}$. Note that $\hat{C}^b(\{x, y\}) = \{y\}$ because $\hat{C}^b$ is DA-equivalent to C^b. As $\hat{C}^b$ is substitutable, it follows that $\hat{C}^b(\{x, x', y\}) \subseteq \{y\}$. By

[3] The choice function $\hat{C}^b$ is the *substitutable completion* of C^b defined in the proof of Theorem F.1 in Hatfield and Kominers (2019).

the irrelevance of rejected contracts condition, we have that $\hat{C}^b(\{x', y\}) \subseteq \{y\}$, contradicting the assumption that $\hat{C}^b$ is DA-equivalent to C^b.

Case 2: $\hat{C}^b(\{x, x'\}) = \{x'\}$. Note that $\hat{C}^b(\{x', z\}) = \{z\}$ because $\hat{C}^b$ is DA-equivalent to C^b. As $\hat{C}^b$ is substitutable, it follows that $\hat{C}^b(\{x, x', z\}) \subseteq \{z\}$. By the irrelevance of rejected contracts condition, we have that $\hat{C}^b(\{x, z\}) \subseteq \{z\}$, contradicting the assumption that $\hat{C}^b$ is DA-equivalent to C^b.

Case 3: $\hat{C}^b(\{x, x'\}) = \varnothing$. By the irrelevance of rejected contracts condition, we have that $\hat{C}^b(\{x\}) = \varnothing$, contradicting the assumption that $\hat{C}^b$ is DA-equivalent to C^b.

As $\hat{C}^b$ was assumed to be feasible, the cases exhaust all possible values of $\hat{C}^b(\{x, x'\})$, and we have therefore produced the desired contradiction. Thus, we can conclude that C^b is not DA-equivalent to a feasible, substitutable choice function that satisfies the irrelevance of rejected contracts condition.

Example 4 and the main result of Kadam (2017) imply that *substitutable completability* (in the sense of Hatfield and Kominers, 2019) does not imply DA-equivalence to a feasible, substitutable choice function either.[4]

2 DA-substitutability and substitutable completability: Examples

Hatfield and Kominers (2019) introduced a notion of *completing* a (usually feasible) choice function to an unfeasible choice function to restore substitutability. Recall that a choice function $\hat{C}^b$ *completes* C^b if $\hat{C}^b(Y)$ is unfeasible whenever $\hat{C}^b(Y) \neq C^b(Y)$. A choice function C^b is *substitutably completable* if C^b has a completion that is substitutable. The existence of a substitutable completion of C^b satisfying the law of aggregate demand for all $b \in B$ implies that $\mathcal{DA}_C$ is stable and strategy-proof (Hatfield and Kominers, 2019).

Clearly, a choice function $\hat{C}^b$ is DA-equivalent to C^b if $\hat{C}^b$ completes C^b. Thus, substitutatable completability implies DA-substitutability. Similarly, DA-strategy-proofness is implied by the existence of a completion that is substitutable and satisfies the law of aggregate demand. The following example shows that DA-strategy-proofness does not imply substitutable completability, so that DA-strategy-proofness

[4]The main result of Kadam (2017) asserts that unilateral substitutability implies substitutable completability. See also Proposition 2 in Zhang (2016).

(and hence DA-substitutability) is a strictly weaker condition than requiring the existence of a completion that is substitutable and satisfies the law of aggregate demand.

Example 5 (DA-strategy-proofness does not imply substitutable completability). This example is Example 2 in Hatfield et al. (2019). Let $B = \{b\}$, let $I = \{i, j, k\}$, and let $X = \{x, x', y, z, z'\}$ with $\iota(x) = \iota(x') = i$, $\iota(y) = j$, and $\iota(z) = \iota(z') = k$. Let C^b be the choice function induced by the priority order

$$\begin{aligned} \{x', z\} \succ_b \{z', x\} \succ_b \{z', y\} \succ_b \{x', y\} \succ_b \{x, y\} \succ_b \{z, y\} \succ_b \{x', z'\} \\ \succ_b \{x, z\} \succ_b \{y\} \succ_b \{z'\} \succ_b \{x'\} \succ_b \{x\} \succ_b \{z\} \succ_b \varnothing. \end{aligned}$$

Let $\hat{C}^b$ be the choice function induced by the priority order[5]

$$\begin{aligned} \{x, z'\} \mathrel{\widehat{\succ}_b} \{x, x'\} \mathrel{\widehat{\succ}_b} \{x, y\} \mathrel{\widehat{\succ}_b} \{x, z'\} \mathrel{\widehat{\succ}_b} \{x\} \mathrel{\widehat{\succ}_b} \{z, z'\} \mathrel{\widehat{\succ}_b} \{x', z\} \mathrel{\widehat{\succ}_b} \{y, z\} \mathrel{\widehat{\succ}_b} \{z\} \\ \mathrel{\widehat{\succ}_b} \{y, z'\} \mathrel{\widehat{\succ}_b} \{x', y\} \mathrel{\widehat{\succ}_b} \{y\} \mathrel{\widehat{\succ}_b} \{x', z'\} \mathrel{\widehat{\succ}_b} \{x'\} \mathrel{\widehat{\succ}_b} \varnothing. \end{aligned}$$

It is straightforward to verify that $\hat{C}^b$ is DA-equivalent to C^b, substitutable, and satisfies the law of aggregate demand. Thus, C^b is DA-strategy-proof.

However, as Hatfield et al. (2019) observed, the choice function C^b is not substitutably completable. I review their argument for the sake of completeness. Suppose for the sake of deriving a contradiction that $\tilde{C}^b$ is a substitutable completion of C^b. Clearly $\tilde{C}^b$ is DA-equivalent to C^b. Hence, we have that

$$\begin{aligned} x' \notin C^b(\{x', y, z\}) &\implies x' \notin \tilde{C}^b(\{x', y', z'\}) \\ z \notin C^b(\{x, y, z\}) &\implies z \notin \tilde{C}^b(\{x, y, z\}) \\ y \notin C^b(\{x', y, z\}) &\implies y \notin \tilde{C}^b(\{x', y, z\}). \end{aligned}$$

As $\tilde{C}^b$ is substitutable, it follows that $\tilde{C}^b(X) \subseteq \{x, z'\}$, contradicting the assumption that $\tilde{C}^b$ completes C^b.

[5]I could equivalently define $\tilde{C}^b$ by the following iterative process. Given a set of contracts $Y \subseteq X$, apply the following two steps.

- Step 1: If one of x, z, y, x' is in Y, accept the first one in the list that is available. Regardless, proceed to the next step.
- Step 2: If one of z', x', y, z is in Y and was not selected in the first step, accept the first one in the list that is available. Regardless, terminate the process.

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