Online Appendix to: *Is Government Spending at the Zero Lower Bound Desirable?* Full DSGE Model

Florin Bilbiie, University of Lausanne, PSE, and CEPR
Tommaso Monacelli, Bocconi University, IGIER and CEPR.
Roberto Perotti, Bocconi University, IGIER, NBER and CEPR.

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1 Optimal pricing: recursive structure

We assume a typical environment with a continuum of monopolistic producers, each of measure zero. We begin by illustrating the problem in the absence of capital accumulation and of price indexation. Our final description of the equilibrium, however, incorporates both features. The production function of each monopolistic producer is:

$$Y_t(i) = N_t(i)^{1-\alpha} \tag{1}$$

where $N_t(i)$ is total labor demand by individual producer i.

The optimal demand for the individual variety i reads:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p} Y_t \tag{2}$$

where Y_t is total demand for variety i.

In equilibrium, the following relationship between individual and *average* nominal marginal cost, MC_{t+k}, holds

$$\begin{aligned} MC_{t+k|t} &= \frac{W_{t+k}}{(1-\alpha)N_{t+k}^{-\alpha}}\left(\frac{N_{t+k}}{N_{t+k|t}}\right)^{-\alpha} \\ &= MC_{t+k}\left(\frac{N_{t+k}}{N_{t+k|t}}\right)^{-\alpha} \\ &= MC_{t+k}\left(\frac{P_{t+k}}{\overline{P}_t}\right)^{\frac{\varepsilon_p\alpha}{1-\alpha}} \end{aligned} \tag{3}$$

where $MC_{t+k|t}$ is the nominal marginal cost at $t+k$ of a firm that last reset its price at time t. Notice that the last equality follows from (2), and MC_{t+k} is the *average* nominal marginal cost. Similarly, notice that:

$$Y_{t+k|t} = \left(\frac{\overline{P}_t}{P_{t+k}}\right)^{-\varepsilon_p} Y_{t+k} \tag{4}$$

Optimal Pricing The first order condition with respect to $\overline{P}_t$ is (abstracting from index i):

$$\underbrace{\mathbb{E}_t \sum_{k=0}^{\infty} \theta_p^k \{Q_{t,t+k} \left(\prod_{s=1}^{k} \Pi_{t+s}\right)^{-1} Y_{t+k|t}\overline{P}_t\}}_{LHS} = \underbrace{\mathcal{M}_p(1-\mathcal{S})\mathbb{E}_t \sum_{k=0}^{\infty} \theta_p^k \{Q_{t,t+k} \left(\prod_{s=1}^{k} \Pi_{t+s}\right)^{-1} Y_{t+k|t}MC_{t+k|t}\}}_{RHS} \tag{5}$$

where $Q_{t,t+k} = \beta^k \frac{U_{c,t+k}}{U_{c,t}}$, $\mathcal{S} = 1 - \frac{1}{\mathcal{M}_p \mathcal{M}_w}$ is an employment subsidy financed via lump sum taxes, and $MC_{t+k|t}$ is the nominal marginal cost at $t+k$ of a firm that last reset its price at time t.

Dividing through by P_t we can write the LHS of the above equation as follows (using 3 and 4):

$$LHS \equiv \left(\frac{\overline{P}_t}{P_t}\right)^{1-\varepsilon_p} \mathbb{E}_t \sum_{k=0}^{\infty} \theta_p^k Q_{t,t+k} Y_{t+k} \left(\prod_{s=1}^{k} \Pi_{t+s}\right)^{\varepsilon_p - 1}$$

where $\Pi_{t+s} = P_{t+s}/P_{t+s-1}$.

Consider next the RHS of (5):

$$\begin{aligned}
RHS &\equiv \mathcal{M}_p(1-\mathcal{S})\frac{1}{P_t}\mathbb{E}_t \sum_{k=0}^{\infty} \theta_p^k \left\{ Q_{t,t+k} \left(\prod_{s=1}^{k} \Pi_{t+s}\right)^{-1} \left(\frac{\bar{P}_t}{P_{t+k}}\right)^{-\varepsilon_p} Y_{t+k} MC_{t+k} \left(\frac{P_{t+k}}{\bar{P}_t}\right)^{\frac{\alpha\varepsilon_p}{1-\alpha}} \right\} \\
&= \mathcal{M}_p(1-\mathcal{S})\frac{1}{P_t}\mathbb{E}_t \sum_{k=0}^{\infty} \theta_p^k \left\{ Q_{t,t+k} \left(\prod_{s=1}^{k} \Pi_{t+s}\right)^{-1} \left(\frac{\bar{P}_t}{P_{t+k}}\right)^{-\varepsilon_p} Y_{t+k} mc_{t+k} P_t \left(\prod_{s=1}^{k} \Pi_{t+s}\right) \left(\frac{P_{t+k}}{\bar{P}_t}\right)^{\frac{\alpha\varepsilon_p}{1-\alpha}} \right\} \\
&= \mathcal{M}_p(1-\mathcal{S}) \left(\frac{\bar{P}_t}{P_t}\right)^{-\frac{\varepsilon_p}{1-\alpha}} \mathbb{E}_t \sum_{k=0}^{\infty} \theta_p^k \left\{ Q_{t,t+k} Y_{t+k} mc_{t+k} \left(\prod_{s=1}^{k} \Pi_{t+s}\right)^{\frac{\varepsilon_p}{1-\alpha}} \right\}
\end{aligned}$$

where $mc_{t+k} \equiv MC_{t+kt}\ /P_{t+k}$ is the average real marginal cost

Equating LHS and RHS and rearranging we finally obtain:

$$\left(\frac{\overline{P}_t}{P_t}\right)^{\frac{1-\alpha+\varepsilon_p\alpha}{1-\alpha}} \underbrace{\mathbb{E}_t \sum_{k=0}^{\infty} \theta_p^k Q_{t,t+k} Y_{t+k} \left(\prod_{s=1}^{k} \Pi_{t+s}\right)^{\varepsilon_p - 1}}_{\mathcal{K}_t^p} = \mathcal{M}_p(1-\mathcal{S})\, \underbrace{\mathbb{E}_t \sum_{k=0}^{\infty} \theta_p^k Q_{t,t+k} Y_{t+k} mc_{t+k} \left(\prod_{s=1}^{k} \Pi_{t+s}\right)^{\frac{\varepsilon_p}{1-\alpha}}}_{\mathcal{Z}_t^p}$$

Recursive representation Define

$$\mathcal{K}_t^p \equiv \mathbb{E}_t \sum_{k=0}^{\infty} \theta_p^k Q_{t,t+k} Y_{t+k} \left(\prod_{s=1}^{k} \Pi_{t+s}\right)^{\varepsilon_p - 1}$$

$$\mathcal{Z}_t^p \equiv \mathbb{E}_t \sum_{k=0}^{\infty} \theta_p^k Q_{t,t+k} Y_{t+k} mc_{t+k} \left(\prod_{s=1}^{k} \Pi_{t+s}\right)^{\frac{\varepsilon_p}{1-\alpha}}$$

Express recursively as:

$$\mathcal{K}_t^p = Y_t + \theta_p \underbrace{\left(\beta \frac{U_{c,t+1}}{U_{c,t}}\right)}_{Q_{t,t+1}} \Pi_{t+1}^{\varepsilon_p - 1} \mathcal{K}_{t+1}^p$$

Similarly:

$$\mathcal{Z}_t^p = Y_t mc_t + \theta_p \left(\beta \frac{U_{c,t+1}}{U_{c,t}} \right) \Pi_{t+1}^{\frac{\varepsilon_p}{1-\alpha}} \mathcal{Z}_{t+1}^p .$$

We also have:

$$1 = \theta_p \Pi_t^{\varepsilon_p - 1} + (1-\theta_p) \left(\frac{\overline{P}_t}{P_t} \right)^{1-\varepsilon_p} . \tag{6}$$

Summarizing, the pricing block comprises the following set of equilibrium conditions:

$$\mathcal{K}_t^p = Y_t + \theta_p \left(\beta \frac{U_{c,t+1}}{U_{c,t}} \right) \Pi_{t+1}^{\varepsilon_p - 1} \mathcal{K}_{t+1}^p$$

$$\mathcal{Z}_t^p = Y_t mc_t + \theta_p \left(\beta \frac{U_{c,t+1}}{U_{c,t}} \right) \Pi_{t+1}^{\frac{\varepsilon_p}{1-\alpha}} \mathcal{Z}_{t+1}^p$$

$$\overline{p}_t^{\frac{1-\alpha+\varepsilon_p \alpha}{1-\alpha}} \mathcal{K}_t^p = \mathcal{M}_p (1-\mathcal{S}) \mathcal{Z}_t^p$$

$$1 = \theta_p \Pi_t^{\varepsilon_p - 1} + (1-\theta_p) \overline{p}_t^{1-\varepsilon_p} .$$

where $\overline{p}_t \equiv \overline{P}_t / P_t$.

2 Optimal wage setting: recursive form

In this section we introduce nominal wage rigidity along the lines of Erceg et al. (2000). The economy is populated by a continuum of households, each supplying a differentiated labor type j, and by a continuum of firms. .

Deriving total demand for each labor type Each firm i employs *all* differentiated labor types. Hence total labor demand by firm i can be written:

$$N_t(i) = \left(\int_0^1 N_t(i,j)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dj \right)^{\frac{\varepsilon_w}{\varepsilon_w - 1}}$$

where $N_t(i,j)$ is demand by firm i of labor type j.

Optimal demand for labor type j by firm i reads:

$$N_t(i,j) = \left(\frac{W_t(j)}{W_t} \right)^{-\varepsilon_w} N_t(i) \tag{7}$$

Integrating across firms, we can derive the equilibrium total demand for each labor type j (using (7) above):

$$
\begin{aligned}
\underbrace{N_t(j)}_{\substack{\text{total demand}\\ \text{for labor type j}}} &= \underbrace{\int_0^1 N_t(i,j)di}_{\substack{\text{integrating}\\ \text{across firms}}} \qquad (8)\\
&= \int_0^1 \left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_w} N_t(i)di\\
&= \left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_w} \int_0^1 N_t(i)di\\
&= \left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_w} N_t
\end{aligned}
$$

Optimal wage setting problem Next, consider the optimal wage setting problem for household j:

$$
\max \mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta_w)^k U(\widetilde{C}_{t+k|t}(j), \mathcal{N}_{t+k|t}(j))
$$

where $\mathcal{N}_{t+k|t}(j)$ is time $t+k$ labor supply by household type j who last reset her wage in time t.

At the chosen wage $\overline{W}_t(j)$, household type j is assumed to supply enough labor to satisfy demand. The constraint reads, using (8):

$$
\begin{aligned}
\underbrace{\mathcal{N}_{t+k|t}(j)}_{\substack{\text{total supply}\\ \text{of labor type j}}} &= \underbrace{N_{t+k|t}(j)}_{\substack{\text{total demand for}\\ \text{for labor type j}}}\\
&= \left(\frac{\overline{W}_t(j)}{W_{t+k}}\right)^{-\varepsilon_w} N_{t+k|t}
\end{aligned}
$$

Notice that N_{t+k} bears the index $t+k$ (and *not* $t+k|t$) because it corresponds to aggregate (or average) labor demand.

The additional household's constraint is the budget constraint:

$$
P_{t+k}C_{t+k|t}(j) + \mathbb{E}_t\left\{Q_{t+k,t+k+1}B_{t+k+1|t}\right\} \leq B_{t+k|t} + \overline{W}_t(j)\mathcal{N}_{t+k|t}(j) - T_{t+k}
$$

Each household j reoptimizing the wage at a given time t will choose the same optimal wage. It is therefore convenient to *abstract from index j.*

Household problem The (relevant portion of the) Lagrangian of the household's problem is

$$
\mathcal{L}^w = \mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \left\{U\left(\widetilde{C}_{t+k|t},\ \mathcal{N}_{t+k|t}\right) - \lambda_{t+k|t}\left[P_{t+k}C_{t+k|t} - \overline{W}_t\ \mathcal{N}_{t+k|t}\right]\right\}. \qquad (9)
$$

The FOC of the problem with respect to $\overline{W}_t$ is:

$$
\sum_{k=0}^{\infty} (\beta\theta_w)^k \mathbb{E}_t \left\{U_{\mathcal{N},t+k|t}\frac{\partial \mathcal{N}_{t+k|t}}{\partial \overline{W}_t} + \lambda_{t+k|t}\left(\mathcal{N}_{t+k|t} + \overline{W}_t \frac{\partial \mathcal{N}_{t+k|t}}{\partial \overline{W}_t}\right)\right\} = 0
$$

Notice:

$$\begin{aligned}\frac{\partial \mathcal{N}_{t+k|t}}{\partial \overline{W}_t} &= -\varepsilon_w \left(\frac{\overline{W}_t}{W_{t+k}}\right)^{-\varepsilon_w-1} \frac{N_{t+k}}{W_{t+k}} \\ &= -\varepsilon_w \mathcal{N}_{t+k|t} \frac{1}{\overline{W}_t}\end{aligned}$$

Hence we can write:

$$-\sum_{k=0}^{\infty} (\beta\theta_w)^k \mathbb{E}_t \left\{ U_{\mathcal{N},t+k|t}\ \varepsilon_w \mathcal{N}_{t+k|t} \frac{1}{\overline{W}_t} + \lambda_{t+k|t} \mathcal{N}_{t+k|t} \left(\varepsilon_w - 1\right)\right\} = 0$$

Under complete markets and separable utility we have $U_{c,t+k}(C_{t+k|t}, \mathcal{N}_{t+k|t}) = U_{c,t+k}(C_{t+k})$. In addition, equilibrium implies $U_{c,t+k} = \lambda_{t+k} P_{t+k}$ (since λ_{t+k} is the shadow value of one unit of *nominal* income at $t+k$).

Hence we have:

$$-\sum_{k=0}^{\infty} (\beta\theta_w)^k \mathbb{E}_t \left\{ U_{\mathcal{N},t+k|t} \mathcal{N}_{t+k|t} \mathcal{M}_w + U_{c,t+k} \mathcal{N}_{t+k|t} \frac{\overline{W}_t}{P_{t+k}} \right\} = 0$$

where $\mathcal{M}_w \equiv \varepsilon_w/(\varepsilon_w - 1)$.

The above expression can be rewritten:

$$\sum_{k=0}^{\infty} (\beta\theta_w)^k \mathbb{E}_t \left\{ U_{c,t+k} \mathcal{N}_{t+k|t} \left[\frac{\overline{W}_t}{P_{t+k}} + \frac{U_{\mathcal{N},t+k|t}}{U_{c,t+k}} \mathcal{M}_w \right] \right\} = 0 \qquad (10)$$

Recursive representation Condition (10) reads:

$$\underbrace{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \mathcal{N}_{t+k|t} U_{c,t+k} \frac{\overline{W}_t}{P_{t+k}}}_{LHS} = \underbrace{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta_w)^k \mathcal{N}_{t+k|t} \mathcal{M}_w (-U_{\mathcal{N},t+k|t})}_{RHS}$$

Using the optimal labor demand condition

$$\mathcal{N}_{t+k|t} = \left(\frac{\overline{W}_t}{W_{t+k}}\right)^{-\varepsilon_w} N_{t+k}, \qquad (11)$$

we can write the LHS as follows:

$$\begin{aligned} LHS &\equiv \left(\frac{\overline{W}_t}{P_t}\right)^{1-\varepsilon_w} \left\{ \begin{array}{c} \left(\frac{W_t}{P_t}\right)^{\varepsilon_w} N_t U_{c,t} + \beta\theta_w \left(\frac{W_{t+1}}{P_{t+1}}\right)^{\varepsilon_w} \Pi_{t+1}^{\varepsilon_w - 1} N_{t+1} U_{c,t+1} + \\ + (\beta\theta_w)^2 \left(\frac{W_{t+2}}{P_{t+2}}\right)^{\varepsilon_w} \left(\Pi_{t+1}\Pi_{t+2}\right)^{\varepsilon_w-1} N_{t+2} U_{c,t+2} + \ldots \end{array} \right\} \\ &= \overline{w}_t^{1-\varepsilon_w} \mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta_w)^k w_{t+k}^{\varepsilon_w} \left(\prod_{s=1}^{k} \Pi_{t+s}\right)^{\varepsilon_w - 1} N_{t+k} U_{c,t+k}, \end{aligned}$$

where $\overline{w}_t \equiv \overline{W}_t / P_t$.

Next consider RHS:

$$RHS \equiv -\left(\frac{\overline{W}_t}{P_t}\right)^{-\varepsilon_w} \left\{ \begin{array}{c} \left(\frac{W_t}{P_t}\right)^{\varepsilon_w} N_t \mathcal{M}_w U_{\mathcal{N},t|t} \\ +\beta\theta_w \left(\frac{W_{t+1}}{P_{t+1}}\right)^{\varepsilon_w} N_{t+1} \Pi_{t+1}^{\varepsilon_w} \mathcal{M}_w U_{\mathcal{N},t+1|t} + \ldots \end{array} \right\}$$

This can be written

$$RHS \equiv \overline{w}_t^{-\varepsilon_w} \mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta_w)^k w_{t+k}^{\varepsilon_w} \left(\prod_{s=1}^{k} \Pi_{t+s} \right)^{\varepsilon_w} N_{t+k} \mathcal{M}_w (-U_{N,t+k|t})$$

Under the assumption that $U_{\mathcal{N}}(\bullet)$ is homogenous of degree φ in $\mathcal{N}$ we have (using (11)):

$$\begin{aligned} -U_{\mathcal{N},t+k|t} &= \left(\frac{\overline{W}_t}{W_{t+k}} \right)^{-\varepsilon_w \varphi} (-U_{\mathcal{N},t+k}(N_{t+k})) \\ &= \left(\frac{\overline{W}_t / P_t}{W_{t+k}/P_{t+k}} \right)^{-\varepsilon_w \varphi} \left(\prod_{s=1}^{k} \Pi_{t+s} \right)^{\varepsilon_w \varphi} (-U_{\mathcal{N},t+k}(N_{t+k})) \end{aligned}$$

Substituting:

$$RHS \equiv \overline{w}_t^{-\varepsilon_w(1+\varphi)} \mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta_w)^k w_{t+k}^{\varepsilon_w(1+\varphi)} N_{t+k} \mathcal{M}_w \left(\prod_{s=1}^{k} \Pi_{t+s} \right)^{\varepsilon_w(1+\varphi)} (-U_{\mathcal{N},t+k}(N_{t+k}))$$

Combining LHS and RHS we obtain:

$$\begin{aligned} & \overline{w}_t^{1+\varepsilon_w \varphi} \underbrace{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta_w)^k w_{t+k}^{\varepsilon_w} \left(\prod_{s=1}^{k} \Pi_{t+s} \right)^{\varepsilon_w - 1} N_{t+k} U_{c,t+k}}_{\mathcal{K}_t^w} \\ = \; & \mathcal{M}_w \underbrace{\mathbb{E}_t \sum_{k=0}^{\infty} (\beta\theta_w)^k w_{t+k}^{\varepsilon_w(1+\varphi)} N_{t+k} \left(\prod_{s=1}^{k} \Pi_{t+s} \right)^{\varepsilon_w(1+\varphi)} (-U_{\mathcal{N},t+k}(N_{t+k}))}_{\mathcal{Z}_t^w} \end{aligned}$$

We can rewrite recursively:

$$\mathcal{K}_t^w = w_t^{\varepsilon_w} N_t U_{c,t} + \beta\theta_w \Pi_{t+1}^{\varepsilon_w - 1} \mathcal{K}_{t+1}^w$$

$$\mathcal{Z}_t^w = w_t^{\varepsilon_w(1+\varphi)} N_t (-U_{\mathcal{N},t}(N_t)) + \beta\theta_w \Pi_{t+1}^{\varepsilon_w(1+\varphi)} \mathcal{Z}_{t+1}^w$$

Hence the first order condition can be written in compact form:

$$\overline{w}_t^{1+\varepsilon_w \varphi} \mathcal{K}_t^w = \mathcal{M}_w \mathcal{Z}_t^w$$

Summary of wage setting equilibrium conditions

$$w_t^{1-\varepsilon_w} = \theta_w \left(\frac{w_{t-1}}{\Pi_t} \right)^{1-\varepsilon_w} + (1-\theta_w) \overline{w}_t^{1-\varepsilon_w}$$

$$\mathcal{K}_t^w = w_t^{\varepsilon_w} N_t U_{c,t} + \beta\theta_w \Pi_{t+1}^{\varepsilon_w - 1} \mathcal{K}_{t+1}^w$$

$$\mathcal{Z}_t^w = w_t^{\varepsilon_w(1+\varphi)} N_t (-U_{\mathcal{N},t}(N_t)) + \beta\theta_w \Pi_{t+1}^{\varepsilon_w(1+\varphi)} \mathcal{Z}_{t+1}^w$$

$$\overline{w}_t^{1+\varepsilon_w \varphi} \mathcal{K}_t^w = \mathcal{M}_w \mathcal{Z}_t^w$$

2.1 Price dispersion, wage dispersion, and equilibrium

Market clearing for each individual variety implies:

$$\underbrace{N_t(i)^{1-\alpha}}_{\substack{\text{supply of} \\ \text{variety i}}} = \underbrace{\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} Y_t}_{\substack{\text{demand of} \\ \text{variety i}}} \tag{12}$$

where $N_t(i)$ denotes the total amount of labor employed by firm i. Rearranging:

$$N_t(i) = \left[\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} Y_t\right]^{\frac{1}{1-\alpha}}$$

Integrating across all producers:

$$\int_0^1 N_t(i)di = \int_0^1 \left[\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p} Y_t\right]^{\frac{1}{1-\alpha}} di \tag{13}$$

$$= Y_t^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\varepsilon_p}{1-\alpha}} di = Y_t^{\frac{1}{1-\alpha}} \Delta_{p,t} \tag{14}$$

where $\Delta_{p,t} \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\varepsilon_p}{1-\alpha}} di$ measures the dispersion of relative prices across producers. In a more compact form:

$$N_t = Y_t^{\frac{1}{1-\alpha}} \Delta_{p,t} \tag{15}$$

where $N_t = \int_0^1 N_t(i)di$.

Equilibrium in the market for the final good requires:

$$Y_t = C_t + G_t \tag{16}$$

Hence conditions (15) and (16) describe aggregate market clearing.[1]

Expressing $\Delta_{p,t}$ in recursive form:

$$\begin{aligned}
\Delta_{p,t} &= \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\varepsilon_p}{1-\alpha}} di \\
&= \int_{1-\theta_p} \left(\frac{\overline{P}_t}{P_t}\right)^{-\frac{\varepsilon_p}{1-\alpha}} di + \left(\frac{P_{t-1}}{P_t}\right)^{-\frac{\varepsilon_p}{1-\alpha}} \int_{\theta_p} \left(\frac{P_{t-1}(i)}{P_{t-1}}\right)^{-\frac{\varepsilon_p}{1-\alpha}} di \\
&= (1-\theta_p)\left(\frac{\overline{P}_t}{P_t}\right)^{-\frac{\varepsilon_p}{1-\alpha}} + \theta_p \Pi_t^{\frac{\varepsilon_p}{1-\alpha}} \Delta_{p,t-1}
\end{aligned}$$

[1] Equivalently, let $y_t^s(i) \equiv A_t N_t^d(i)^{1-\alpha}$ denote the supply of variety i. In equilibrium:

$$y_t^s(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} Y_t$$

Integrating across i:

$$Y_t^s \equiv \int_0^1 y_t^s(i)di = \Delta_{p,t} Y_t$$

From this notation it is clear the interpretation of $Y_t = C_t$ as an index of aggregate demand.

Let $\mathcal{N}_t(j)$ denote labor supply by each differentiated household. Since each household is assumed to satisfy labor demand at the given posted wage, equilibrium in the labor market requires:

$$\mathcal{N}_t(j) = N_t(j)$$

Aggregating across each household j one obtains, using (8):

$$\begin{aligned} \mathcal{N}_t &\equiv \int_0^1 \mathcal{N}_t(j)dj = \int_0^1 N_t(j) \\ &= \int_0^1 \left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_w} dj \ N_t \end{aligned}$$

where $\mathcal{N}_t$ is an index of *aggregate* labor supply. By defining $\Delta_{w,t} \equiv \int_0^1 \left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_w}$ as wage dispersion, the above equation becomes.

$$\mathcal{N}_t = \Delta_{w,t} N_t \tag{17}$$

Notice that by substituting (17) into (15) one obtains:

$$N_t = \frac{\mathcal{N}_t}{\Delta_{w,t}} = Y_t^{\frac{1}{1-\alpha}} \Delta_{p,t} \tag{18}$$

which shows that the relationship between aggregate employment $\mathcal{N}_t$ and aggregate output Y_t depends on both price and wage dispersion.

3 Capital accumulation

Suppose each monopolistic firm i produces a homogenous good according to the production function:

$$Y_t(i) = \left[N_t(i)^{1-\alpha} K_t^{\alpha}(i)\right]^{\xi} \tag{19}$$

where is a labor productivity shifter (common across firms). Notice that parameter $\xi \geq 1$ measures the degree of returns to scale in production.

The cost minimizing choice of labor and capital input implies:

$$\frac{W_t}{P_t(i)} = \frac{MC_t}{P_t(i)} (1-\alpha) \left(\frac{K_t(i)}{N_t(i)}\right)^{\alpha\xi} \tag{20}$$

$$\frac{Z_t}{P_t(i)} = \frac{MC_t}{P_t(i)} \alpha \left(\frac{N_t(i)}{K_t(i)}\right)^{(1-\alpha)\xi} \tag{21}$$

where Z_t is the nominal rental cost of capital.

Notice that the above conditions imply:

$$MC_t(i) = \frac{W_t^{(1-\alpha)\xi} Z_t^{\alpha\xi}}{(\alpha\xi)^{\alpha\xi}(\xi(1-\alpha))^{\xi(1-\alpha)}} Y_t(i)^{\frac{1-\xi}{\xi}}. \tag{22}$$

Constant returns to scale. We assume $\xi = 1$ Hence we have $MC_t(i) = MC_t$ for all i, i.e., the nominal marginal cost is identical across firms. Notice also that we can write:

$$MC_t(i) = \frac{W_t}{\xi(1-\alpha)\left(\frac{K_t(i)}{N_t(i)}\right)^{\xi\alpha}} = \frac{W_t}{\xi(1-\alpha)} \frac{N_t(i)}{Y_t(i)} \tag{23}$$

and

$$MC_t(i) = \frac{Z_t}{\xi\alpha} \frac{K_t(i)}{N_t(i)} \tag{24}$$

In the case $\xi = 1$, since $MC_t(i) = MC_t$ for all i, we also have $K_t(i)/N_t(i) = K_t/N_t$ for all i. In other words, under constant returns to scale, the capital labor ratio is equalized across firms.

Market clearing Henceforth we assume $\xi = 1$. Market clearing for each individual variety implies:

$$N_t(i)^{1-\alpha} K_t^{\alpha}(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} Y_t \tag{25}$$

Equilibrium in the market for the final good requires:

$$Y_t = C_t + I_t + G_t \tag{26}$$

Integrating (25) across i, and combining with (26):

$$\left(\frac{K_t}{N_t}\right)^{\alpha} \int_0^1 N_t(i) di = \Delta_{p,t} Y_t$$

or alternatively:

$$K_t^{\alpha} N_t^{1-\alpha} = \Delta_{p,t} Y_t$$

4 Equilibrium

Let $\Delta_{p,t} \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p} di$ and $\Delta_{w,t} \equiv \int_0^1 \left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_w}$ denote price and wage dispersion respectively. Let $\overline{p}_t \equiv \overline{P}_t/P_t$, $\overline{w}_t \equiv \overline{W}_t/P_t$, $z_t \equiv Z_t/P_t$ and mc_t be the real marginal cost of production (equal for all firms). Let $\mathcal{K}_t^p$, $\mathcal{Z}_t^p$, $\mathcal{K}_t^w$, $\mathcal{Z}_t^w$ be recursive objects in the optimal pricing and wage setting problems. For any given exogenous processes $\{\varrho_t,\ G_t\}$, an equilibrium is a set of endogenous variables $\{\lambda_t,\ C_t,\ Y_t,\ N_t,\ \mathcal{N}_t,\ mc_t,\ i_t,\ \Pi_t,\ \overline{p}_t,\ I_t,\ K_t,\ \bar{K}_t,\ u_t,\ q_t,\ z_t,\ w_t,\ \bar{w}_t,\ \Delta_{w,t},\ \Delta_{p,t},\ \mathcal{K}_t^p,\ \mathcal{Z}_t^p, \mathcal{K}_t^w,$ $\mathcal{Z}_t^w\}$ solving the following set of conditions:

1. Marginal utility of consumption:

$$\lambda_t = (C_t - hC_{t-1})^{-\sigma}$$

2. Euler equation:

$$\lambda_t = \beta_t (1+i_t)(1+\rho_t) \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\pi_{t+1}}\right]$$

3. Production:

$$\Delta_{p,t} Y_t = K_t^{\alpha} N_t^{1-\alpha}$$

4. Optimal labour demand:

$$w_t = mc_t(1-\alpha)\frac{Y_t}{N_t}$$

5. Optimal demand for capital:

$$z_t = mc_t\alpha\frac{Y_t}{K_t}$$

6. Price of capital:

$$q_t = \beta_t\mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}(z_{t+1}u_{t+1} - a\left(u_{t+1}\right) + (1-\delta)q_{t+1})\right]$$

7. Optimal investment:

$$q_t\left[1 - \Omega(\cdot) - \Omega'(\frac{I_t}{I_{t-1}})\frac{I_t}{I_{t-1}}\right] = 1 - \beta_t\mathbb{E}_t\left[q_{t+1}\frac{\lambda_{t+1}}{\lambda_t}\left(\frac{I_{t+1}}{I_t}\right)^2\Omega'\left(\frac{I_{t+1}}{I_t}\right)\right]$$

8. Capital accumulation:

$$\bar{K}_t = (1-\delta)\bar{K}_{t-1} + I_t\left[1 - \Omega\left(\frac{I_t}{I_{t-1}}\right)\right]$$

9. Utilization transformation

$$K_t = u_t\bar{K}_{t-1}$$

10. Optimal utilization

$$z_t = a'\left(u_t\right)$$

11. Equilibrium in the final good market:

$$Y_t = C_t + I_t + a\left(u_t\right)\bar{K}_{t-1} + G_t$$

12. Monetary policy rule:

$$1 + i_t = \max\left\{1, (1+i_{t-1})^{\phi_i}\cdot\left[\left(\frac{\pi}{\beta(1+\rho_t)}\right)\cdot\left(\frac{\pi_t\pi_{t-1}\pi_{t-2}\pi_{t-3}}{\pi}\right)^{\phi_p}\cdot\left(\frac{Y_t}{Y_{t-1}}\right)^{\phi_y}\right]^{1-\phi_i}\right\} \tag{27}$$

13. Recursive representation for $\mathcal{K}_t^p$:

$$\mathcal{K}_t^p = Y_t + \theta_p\beta_t\mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\pi_{t+1}^{\epsilon_p-1}\pi_t^{\chi_p(1-\epsilon_p)}\mathcal{K}_{t+1}^p\right]$$

14. Recursive representation for $\mathcal{Z}_t^p$:

$$\mathcal{Z}_t^p = Y_t mc_t + \theta_p\beta_t\mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\pi_{t+1}^{\epsilon_p}\pi_t^{-\chi_p\epsilon_p}\mathcal{Z}_{t+1}^p\right]$$

15. Optimal pricing:

$$\bar{p}\mathcal{K}_t^p = \frac{\epsilon_p}{\epsilon_p - 1}\mathcal{Z}_t^p$$

16. Inflation:

$$1 = \theta_p \left(\frac{\pi_t}{\pi_{t-1}^{\chi_p}} \right)^{\epsilon_p - 1} + (1 - \theta_p) \bar{p}_t^{1-\epsilon_p}$$

17. Price dispersion:

$$\Delta_{p,t} = (1 - \theta_p) \bar{p}_t^{-\epsilon_p} + \theta_p \left(\frac{\pi_t}{\pi_{t-1}^{\chi_p}} \right)^{\epsilon_p} \Delta_{p,t-1}$$

18. Recursive representation for $\mathcal{K}_t^w$:

$$\mathcal{K}_t^w = w_t^{\epsilon_w} N_t \lambda_t + \theta_w \beta_t \mathbb{E}_t \left[\pi_{t+1}^{\epsilon_w - 1} \left(\frac{w_t}{w_{t-1}} \pi_t \right)^{-\chi_w \epsilon_w} \mathcal{K}_{t+1}^w \right]$$

19. Recursive representation for $\mathcal{Z}_t^w$:

$$\mathcal{Z}_t^w = w_t^{\epsilon_w(1+\varphi)} \chi_N N_t^{1+\varphi} + \theta_w \beta_t \mathbb{E}_t \left[\pi_{t+1}^{\epsilon_w(1+\varphi)} \left(\frac{w_t}{w_{t-1}} \pi_t \right)^{-\chi_w \epsilon_w (1+\varphi)} \mathcal{Z}_{t+1}^w \right]$$

20. Optimal wage:

$$\bar{w}^{1+\epsilon_w \varphi} \mathcal{K}_t^w = \frac{\epsilon_w}{\epsilon_w - 1} \mathcal{Z}_t^w$$

21. Wage level:

$$w_t^{1-\epsilon_w} = \theta_w \left(w_{t-1} \frac{\pi_t^{\chi_w}}{\pi_t} \right)^{1-\epsilon_w} + (1 - \theta_w) \bar{w}_t^{1-\epsilon_w}$$

22. Wage dispersion:

$$\Delta_{w,t} = (1 - \theta_w) \left(\frac{\bar{w}_t}{w_t} \right)^{-\epsilon_w} + \theta_w \left(\frac{w_t}{w_{t-1}} \frac{\pi_t}{\pi_{t-1}^{\chi_p}} \right)^{\epsilon_w} \Delta_{w,t-1}$$

23. Optimal labour supply:

$$\mathcal{N}_t = \Delta_{w,t} N_t$$