

The Treatise on Probability and Keynes critique of "the classics"

Massimo Cingolani - European Investment Bank - Participation on a personal basis

Executive Summary

Among the three major works of Keynes—*The Treatise on Probability*, *The Treatise on Money*, and *The General Theory*—the most important is arguably *The Treatise on Probability*, the work to which he devoted the most time. This early major book presents a logical critique of the then dominant frequentist approach to probability based on which Keynes develops his own logic of probability. Equipped with this logic, which follows the work of Boole, he can connect probabilistically theoretical assumptions and their ultimate consequences in terms of most likely predictions. This allows him in turn to interpret historical reality on an inferential basis. He then builds the two alternative and complementary models of the *Treatise on Money* and of the *General Theory*, which offer both different variants of the same "logical critique" of the classical approach. The latter is not deterministically false, but it is, in most cases, unlikely to be of policy relevance.

Keywords: Keynes, *Treatise on Probability*, *Treatise on Money*, *General Theory*, Boole

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Introduction

Judging simply by length, Keynes's three major books are the *Treatise on Probability* (Keynes, 1921), the *Treatise on Money* (Keynes, 1930), and *The General Theory* (Keynes, 1936). It is fair to say that the first is probably the least read of the three. Yet it was also the earliest to be written and the work on which Keynes spent the greatest number of years, making it a natural candidate for uncovering the roots of his later ideas.

This is, however, not an easy task, since the book addresses an intricate subject—the theory of probability—which is difficult to master given its complexity and the diversity of its interpretations. It is therefore not surprising that most commentators have focused primarily on a limited number of its chapters, at the risk of overlooking aspects that are central to the overall structure of this complex book.

A first point to note is that the *Treatise on Probability* essentially seeks to develop a logic of reasoning under uncertainty. This point was clearly recognised, by some of the scholars who read the book carefully and understood well its technical aspects, amongst which one can mention Russel (1922), de Finetti (1938) and Hishiyama (1969).

The main biographers of Keynes somewhat recognized the importance of the book in the genesis of Keynes ideas, but put the emphasis on the "ethical" aspects of probability rather than on the mathematical and statistical aspects. This brought many of them to accept that the "non-numerical" probability of Keynes was qualitative and therefore somehow linked to ethics.

In the broader narrative of The Life of John Maynard Keynes, Harrod's treatment of the *Treatise on Probability* is in several respects original. He takes the work seriously as a major intellectual contribution, recognises its grounding in logical analysis rather than psychology, and explicitly acknowledges the influence of Russell alongside Moore and Ramsey. In this sense, Harrod avoids some of the later tendencies to reduce the *Treatise* either to ethics or to a prelude to Ramsey's critique. At the same time, the originality of Harrod's reading remains confined to the early phase of Keynes's development. Within the overall architecture of the book, the *Treatise on Probability* is presented as a formative

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philosophical achievement, rather than as an analytical framework that continued to inform Keynes's subsequent work on money and macroeconomics.

O'Donnell (1989, 1991, 2021) has been a prominent reference point for Post-Keynesian interpretations of Keynes's *Treatise on Probability*, with consequences that have significantly shaped—and constrained—the subsequent literature. His work displays close attention to many of the *Treatise's* technical elements, including the distinction between primary and secondary propositions, the role of the conventional coefficient of risk and weight, what he sees as Keynes' structure of rational belief. The fact that in his 1989 book he aligns several of the main reviews of the *Treatise* grouping them between those that are favourable, those that are negative and those that are agnostic, shows that he is uncertain of which one is correct and in later chapters and other writings he seems to side with the critics such as Ramsey. O'Donnell's persistent framing of Keynes's probability theory primarily as a component of Keynes's philosophical outlook tends to misdirect his interpretation. In particular, it leads him to neglect the mathematical–logical tradition within which Keynes was working, most notably the connection with Boole's work on logical probability, discussed below. This omission is striking: in O'Donnell's revised monograph based on his doctoral dissertation, Boole appears only briefly once among Keynes's "philosophical influences," alongside mathematicians and actuaries, despite the fact that Boole is the second most frequently cited author in the *Treatise on Probability* after Laplace².

O'Donnell is also a leading proponent of the "continuity hypothesis," according to which Keynes did not abandon his fundamental views on probability, including in response to Ramsey's critique. While this position has considerable merit, O'Donnell's acceptance of Ramsey's objections as broadly well founded limits his subsequent analysis, particularly of the links between the *Treatise on Probability* and the *General Theory*. As Brady has argued, Ramsey's criticisms rest on serious misunderstandings of Keynes's probability theory, and their uncritical incorporation obscures the interval-valued and non-additive dimensions of Keynes's approach.

Skidelsky (2003) broadly follows O'Donnell in approaching Keynes's views on probability primarily through an ethical and philosophical lens. Particular emphasis is placed on Keynes's early paper "*Ethics in Relation to Conduct*" (1904), read to the Cambridge Apostles, and on the influence of Moore's *Principia Ethica*. This early ethical formation is central to Skidelsky's claim that philosophy provided the foundational framework of Keynes's thought, preceding both his economics and his analysis of means–ends relationships. As Skidelsky puts it, Keynes's philosophy was largely worked out between 1903 and 1906 within the Apostles' circle, as a response to questions that preoccupied that milieu (Skidelsky 2003, p. 85). While this biographical and ethical reconstruction is illuminating, it tends to reinforce a reading of Keynes's probability theory as primarily an extension of his moral philosophy.

Among biographers and scholars who have developed Keynes's ideas, several have acknowledged the importance of the *Treatise on Probability*, but without fully capturing its analytical role; discussion of the *Treatise* has generally been framed with reference to *The General Theory*. By contrast, analyses of the *Treatise on Money* have largely focused on its links to *The General Theory*, rather than on its relationship with the *Treatise on Probability*.

² Keynes index of the *Treatise* has 12 entries on Boole (second most indexed author after Laplace), while Venn, who was the main English logician after Boole, has only 10 entries.

Several Post-Keynesian authors, focusing on *The General Theory* as Keynes's central work, have stressed the importance of the *Treatise on Probability* for its interpretation. Among these, Minsky (1975, p. 65) notably observed: "In interpreting *The General Theory*, it should be kept in mind that Keynes was first the author of *A Treatise on Probability*." Similarly Davidson

Davidson, in John Maynard Keynes, characterises Keynes's position as resting on the polar opposite of the mainstream axioms—non-neutral money, the failure of gross substitution, and non-ergodicity. While analytically richer than the mainstream framework, this reconstruction treats these features as axiomatic opposites, whereas Keynes presented them as possibilities that undermine universal axiomatic closure rather than as alternative foundational assumptions. As a result, Davidson's account ultimately gravitates toward a Knightian conception of uncertainty.

It is however mainly Anna Carabelli (1983, 2021), who accorded the *Treatise* a central place in her interpretation of the subsequent development of Keynes's ideas. Her choice was highly original. Her intuition was that Keynes's early work on probability was not an isolated philosophical exercise, but rather the foundation of a coherent and continuous line of thought running through his entire intellectual production. Through a close reading of Keynes's "logic of probability," she reconstructed a substantial continuity between his early reflections on probability and ethics—dating back to the first version of the *Treatise on Probability* of 1907–08—and his later economic and policy writings, from the *Treatise on Money* and *The General Theory* to his proposals of the 1940s, including the Bancor plan and the commodity buffer stock scheme.

Carabelli convincingly showed that returning to Keynes's probabilistic vision makes it possible to clarify key aspects of *The General Theory* that are otherwise difficult to grasp. In particular, it sheds light on the way Keynes constructed a genuinely monetary macroeconomics, grounded in a logic that rejects the reduction of macroeconomic outcomes to the mere aggregation of individual behaviours.

Central to this interpretation is Keynes's critique of the classical assumption of the "logical independence" of economic variables. Whether applied to money, income, or employment, this assumption leads to the erroneous belief that the whole is merely the sum of its parts. What Keynes famously termed the "fallacy of composition" thus stems from a logical mistake: the uncritical application of *ceteris paribus* reasoning. In economic terms, this notably derives from the fact that variables treated as exogenous at the microeconomic level cease to be so at the macroeconomic level. Carabelli rightly traces this insight back to Keynes's theory of probability, whose most systematic expression is found in the *Treatise on Probability*.

A second, more intricate aspect—precisely because of its technical nature—concerns the type of probability theory proposed by Keynes, which is crucial for fully grasping the logical implications of his analysis. The key point here is that Keynes's probability theory is not "a-numerical," as it has often been argued, but is instead centred on his conventional coefficient of weight and risk, introduced in Chapter VI and developed in Chapter XXVI of

the *Treatise on Probability*. When viewed within the broader theory of knowledge outlined in the opening chapters of the book, Keynes framework draws the contours of a propositional logic closely related to that introduced by George Boole.

To the best of the author's knowledge, the scholar who has most fully uncovered the "Boole connection" is Michael Brady, who first addressed the issue in his PhD thesis (1983) and later developed it in a number of contributions—notably Brady and Arthmar (2012) and Brady (2021)—which stress the strong links between Keynes's *Treatise* (1921) and Boole's *The Laws of Thought* (1854), referred to as the "Boole connection" in his most recent writings.

The purpose of this paper is to examine the implications of Keynes's logic of probability—understood as a development of Boole's ideas—for the subsequent evolution of his thought. The argument broadly supports Brady's interpretation of Keynes, with minor departures, while also endorsing Carabelli's view regarding the centrality of the *Treatise on Probability* in reconstructing Keynes's intellectual itinerary. By emphasising the Boole connection—which was largely overlooked in her work, as well as in that of the great majority of Keynes's commentators—the paper reinforces her claim that the *Treatise on Probability* constitutes the foundational and unifying work of Keynes's oeuvre. If, through its logic of probability and its applications to rational economics and policy, Keynes's oeuvre may be read as a lifelong effort to expose the limits of Marshall's partial equilibrium method, the specific contribution of this paper is to make explicit the implications of this *aggiornamento* for economics and finance and, consequently, for contemporary assessments of the available menu of policy choices and actions.

The paper proceeds as follows. First, the ideas developed by Boole in *The Laws of Thought* are briefly recalled. The logic of probability developed in the *Treatise on Probability* is then introduced, with particular attention to Keynes's reading of Boole's problem of the probability of causes and to his development of this problem within a numerical, non-additive and non-linear propositional logic through the concepts of evidential weight and risk.

After illustrating this line of interpretation by means of Venn's diagrammatic apparatus and translating the propositional logic of Boole and Keynes into contemporary terms, the paper argues that, with his first major book, Keynes laid the foundations of a "general systems theory" more comprehensive than those later developed by von Bertalanffy, Parsons, Luhmann, Morin, and their followers. In Keynes's framework, discrete qualitative and quantitative elements are combined within a logically structured method, which is subsequently applied to criticise—and, in a sense, to extend—Marshall's partial equilibrium approach when it is mobilised to address the problems of money and unemployment in the *Treatise on Money* and *The General Theory*.

Boolean algebra and the algebra of George Boole

George Boole investigated mathematical logic and its application to probability in two books (Boole, 1847; 1854) and in several articles devoted to the subject (notably Boole, 1851, 1854, 1857). In these works, he gradually developed a method for applying probability

within a calculus of propositions. This approach was later taken up and substantially expanded by Keynes in *A Treatise on Probability*.

What is today called *Boolean algebra* refers instead to a later systematisation and generalisation of Boole's original—and rather unique—mode of thought. Boolean algebra is an axiomatized algebraic structure, widely used in digital computation and in the formal treatment of events in probability theory, and it differs in important respects from George Boole's own algebra (Hailperin, 1981). It is the latter that must be discussed in relation to Keynes's *A Treatise on Probability*. As noted by Brady (2025), Boolean algebra is based on binary logic and rests on a systematisation of Boole's ideas developed by William Stanley Jevons and Charles Sanders Peirce—a systematisation that Boole explicitly rejected in his correspondence with Jevons in 1863³.

Boole (1847) first developed his calculus for deterministic propositions and then extended it to probabilistic statements (Boole, 1854). For the purposes of the present paper, we set aside the specific way in which Boole applies propositional calculus to particular problems and instead mainly rely on the more intuitive linear-programming formulation, which Hailperin (1986) shows to be fully equivalent. In this context, it is noteworthy that the distinction between primary and secondary propositions in *A Treatise on Probability*—on the basis of which Keynes defines probability as a relation between two sets of propositions—derives directly from Boole's framework and is difficult to understand without reference to his two major books of 1847 and 1854 (see Cingolani 2022, 2023, 2024).

In his application of probability to the calculus of propositions, Boole expressed what he regarded as the main problem in the science of probability with the following words:

"Now one general problem, which the existing theory of Probabilities enables us to solve, is the following, viz. :—Given the probabilities of any simple events: required the probability of a given compound event, i. e. of an event compounded in a given manner out of the given simple events." (Boole, 1854, pp. 14–15).

Here it should be kept in mind that, contrary to Keynes, Boole uses "events" and "propositions" (about events) interchangeably. He further generalized this formulation, presenting it as the most comprehensive statement of the problem:

"[...] as the general problem of the science, viz.: Given the probabilities of any events, simple or compound, conditioned or unconditioned: required the probability of any other event equally arbitrary in expression and conception" (Boole, 1854, p. 15).

The method followed by Boole to solve his problems is extremely complex and ingenious. Hailperin (1986) explains why, despite some errors in the theorems from which it is derived, the method nevertheless yields correct results. Hailperin shows that once the method is recast in terms of the modern algebra of signed multisets (SM)—which he characterizes as

³ A crucial but often overlooked aspect of Boole's position—taken up explicitly by Keynes—is his rejection of Jevons's attempt to recast the algebra of logic as a purely truth-functional, two-valued system. Boole's own project was not to reduce logical reasoning to certainty, but to develop a calculus capable of handling inference under partial specification of the universe of discourse. Keynes's logic of probability, with its emphasis on bounds, admissibility, and evidential weight, should be read as a direct continuation of this Booleian programme rather than as an anticipation of later Boolean or subjective probabilistic frameworks. On Boole's explicit rejection of Jevons's truth-functional reinterpretation of the algebra of logic, see the correspondence between Boole and Jevons (1863–1864), critically edited and discussed by Ivor Grattan-Guinness (1991).

Boole's algebra, as opposed to the "Boolean algebra" of the so-called Boolean rings—it is possible to demonstrate the axioms that would have been required for Boole to apply the rules of symbolic algebra to the calculus of propositions in the way he actually did.

One of the most celebrated discussions of the "probability of causes" developed by Boole is the *Challenge Problem*, which he proposed without giving its solution in an article published in 1851. Thomas Cawley provided a solution in 1853, which was rejected by Boole, who proposed his own solution in 1854 that he also reproduced as Problem I in Chapter XX of *The Rules of Thought* entitled "Problems relating to the connection of cause and effects".

This solution was criticized by Richard Wilbraham in an article published in 1854, which appeared in print almost in parallel with the *Rules of Thought* and is thus probably based on the earlier exposition in Boole's dispute with Cawley, a dispute which incidentally went on until 1862 (Heilperin, 1986). In his reply to Wilbraham, Boole never argued that Wilbraham was wrong, but only that he had applied his method consistently. The criticism of Wilbraham was endorsed by Keynes in the *Treatise*, where he followed closely all these discussions on the probability of causes in chapters XV, XVI and XVII.

In the last of these chapters, he reviewed in detail the 10 problems of Boole's Chapter XX of the *Rules of Thought*, stating that for Problem I, Boole made an error, outlined by Wilbraham, and proposing a simplified solution for Problem VI, which is a generalisation of Problem 1 and for which, incidentally, nobody ever said that Boole was wrong. Indeed, in proposing his simplified solution for Problem VI, Keynes did not criticize the solution provided by Boole.

Now Problem VI is essentially the same as Problem I, except that instead of two possible causes it envisages n possible causes. Historically it is Problem VI and its solution which have remained known as Boole's "Challenge problem". This was later discussed in Hugh MacCall (1896), Edwin Wilson (1934) and Theodore Heilperin (1986).

In presenting the Challenge Problem, it is better to focus on the "simplified" version of *Problem I* with only two causes, because it was the main object of the debate between Henry Wilbraham and Boole and, through its later developments up to Wilson and Heilperin, it highlights an important issue on the "independence of assumptions", which is a key element of the argument developed in this paper.

Boole's *Problem I* examines the probability of causes when one observes an event E and seeks to infer the probability that it was produced by one of two possible causes, acknowledging that: i) the two causes A_1 and A_2 may co-occur; ii) the event E is observed; and, iii) it is assumed that E cannot occur unless at least one cause is present. In Boole's canonical formulation, the data consist of the marginal probabilities of the causes $P(A_1) = c_1$ and $P(A_2) = c_2$, together with the conditional probabilities of the effect given each cause $P(E|A_1) = p_1$ and $P(E|A_2) = p_2$ and the stipulation that E cannot occur when neither cause occurs (so $P[E \wedge \neg A_1 \wedge \neg A_2] = 0$).

The problem of the "probability of causes" asks for the inverse probabilities such as $P(A_1|E)$, $P(A_2|E)$ as well as, implicitly, the probability of co-causation $P(A_1 \wedge A_2 | E)$. To answer these

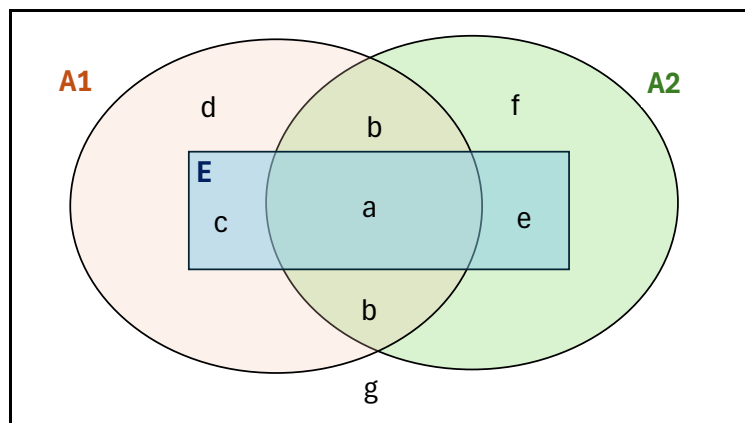
questions, one must know how probability mass is distributed in the two slices of the universe picked out by E and by $\neg E$, and—crucially—how much of the observed E -mass lies in the overlap $A_1 \wedge A_2$.

The given “data” constrain that distribution, yet they do not pin it down uniquely: they define a feasible region of joint probability tables compatible with the same marginals and conditionals, so that the inverse probabilities are in general set-valued (interval-valued), not single numbers.

The linear programming translation of Boole’s solution consists in **maximizing and minimizing the probability of E** , given the data (c_1, c_2, p_1, p_2) and the conditions that probability rules logically impose on atomic probabilities, under which the “data of the problem” become linear constraints. The **maximand/minimand $P(E) = a + c + e$** is also linear in the atoms. The **linear-programming objective** is simply: **minimize and maximize $P(E) = a + c + e$** .

Figure 1 below provides an illustration of Boole’s *Problem I* in line with Hailperin’s re-reading of its solution in terms of linear programming. The two possible causes A_1 and A_2 intersect and the event E is represented by a square entirely included in the $(A_1 \vee A_2)$ region, intersecting all its subregions. There are seven mutually exclusive “atomic” probabilities for the joint partition of (A_1, A_2, E) : $a = P(A_1 A_2 E)$, $b = P(A_1 A_2 \neg E)$, $c = P(A_1 \neg A_2 E)$, $d = P(A_1 \neg A_2 \neg E)$, $e = P(\neg A_1 A_2 E)$, $f = P(\neg A_1 A_2 \neg E)$, $g = P(\neg A_1 \neg A_2 \neg E)$, with an eighth condition given by $P(\neg A_1 \neg A_2 E) = 0$ by assumption.

Figure 1: Probability of causes in Boole’s problem I of Chapter XX⁴



$a = P(A_1 \wedge A_2 \wedge E)$ $b = P(A_1 \wedge A_2 \wedge \neg E)$ $c = P(A_1 \wedge \neg A_2 \wedge E)$
 $d = P(A_1 \wedge \neg A_2 \wedge \neg E)$ $e = P(\neg A_1 \wedge A_2 \wedge E)$ $f = P(\neg A_1 \wedge A_2 \wedge \neg E)$
 $g = P(\neg A_1 \wedge \neg A_2 \wedge \neg E)$ (and $P(\neg A_1 \wedge \neg A_2 \wedge E) = 0$)

Objective: minimize (or maximize) $P(E) = a + c + e$. **Variables:** $a, b, c, d, e, f, g \geq 0$
 $P(A_1) = c_1 = 0.6$ $P(A_2) = c_2 = 0.5$ $P(E|A_1) = p_1 = 0.4$ $P(E|A_2) = p_2 = 0.15$
 $P(U) = a + b + c + d + e + f + g = 1$ $P(A_1) = a + b + c + d = 0.6$ $P(A_2) = a + b + e + f = 0.5$
 $P(E \cap A_1) = a + c = 0.6 \cdot 0.4 = 0.24$ $P(E \cap A_2) = a + e = 0.5 \cdot 0.3 = 0.15$

General solution: $0.24 \leq P(E) \leq 0.39$

Probability of causes:

$$\begin{aligned}
 P(A_1/E) &= \frac{P(A_1 \cap E)}{P(E)} = \frac{a+c}{a+c+e} \\
 &= \frac{0.24}{[0.39, 0.24]} \in \left[\frac{0.24}{0.39}, \frac{0.24}{0.24} \right] \\
 &\approx [0.615, 1]
 \end{aligned}$$

$$\begin{aligned}
 P(A_2/E) &= \frac{P(A_2 \cap E)}{P(E)} = \frac{a+e}{a+c+e} \\
 &= \frac{0.15}{[0.39, 0.24]} \in \left[\frac{0.15}{0.39}, \frac{0.15}{0.24} \right] \\
 &\approx [0.3846, 0.625]
 \end{aligned}$$

Boole additional constraints:

$$\begin{cases} \frac{a}{g} = \frac{c}{e} \\ \frac{b}{g} = \frac{d}{f} \end{cases} \longrightarrow \begin{cases} P(E) = 0.291144 \\ P(A_1/E) = 0.824334 \\ P(A_2/E) = 0.515209 \end{cases}$$

⁴ The chart is an illustration, and the areas of the sets are not proportional to the numerical assumptions used in the example.

As shown in the figure, the variables are the atomic probabilities: **Variables⁵: a, b, c, d, e, f, g ≥ 0**. In the selected example, the data are given by:

$$c_1 = 0.6, \quad c_2 = 0.5, \quad p_1 = 0.4, \quad p_2 = 0.3.$$

The **constraints** that follow logically from the setup are:

$$a + b + c + d + e + f + g = 1$$

$$a + b + c + d = c_1 = 0.6$$

$$a + b + e + f = c_2 = 0.5$$

$$a + c = c_1 \cdot p_1 = 0.6 \cdot 0.4 = 0.24$$

$$a + e = c_2 \cdot p_2 = 0.5 \cdot 0.3 = 0.15$$

The general solution of the problem found by Boole (1854 pp. 309-310 of Chapter XIX) is:

$$\max(c_1 p_1, c_2 p_2) \leq P(E) \leq \min[c_1 p_1 + c_2 p_2, 1 - c_1(1 - p_1), 1 - c_2(1 - p_2)]$$

and in Chapter XX p. 324 he refers to this result and confirms that $P(E)$ “must exceed each of the quantities $c_1 p_1$, $c_2 p_2$, and fall short of each of the quantities $1 - c_1(1 - p_1)$, $1 - c_2(1 - p_2)$ and $c_1 p_1 + c_2 p_2$.” He then goes on by further applying the steps of his method and finally finds a specific expression which provides a **single value** for $P(E)$.

As shown by Wilbraham (1854), this is precisely where Boole, who was not using linear programming but solving complicated systems by hand⁶ implicitly introduced two additional assumptions in order to lift the indeterminacy. Wilbraham, who provided a fully algebraic solution of the Challenge Problem, showed that the second part of Boole’s method, applied in Chapter XX, was equivalent to adding two assumptions that Wilbraham said had been “hidden” by Boole.

Those “hidden” assumptions can be written as equalities between ratios linking the **E** to **¬E**—forcing particular proportionalities across corresponding cells. The first of these two conditions of “proportionality” (the one that Wilbraham found arbitrary but also “anomalous”) states that the ratio of the intersection between the three events (**A₁, A₂, E**) – region **a** in **Figure 1** - and the region outside the causal events $\neg(\mathbf{A}_1 \vee \mathbf{A}_2)$ – region **g** in **Figure 1** - is the same as that between the two parts of **E** not included in the intersection: (**A₁, ¬A₂, E**) and (**¬A₁, A₂, E**) – regions **c** and **e** in **Figure 1**.

The second condition (the one that Wilbraham found purely arbitrary) states that the part of the intersection between the causes that is not covered by the event (when the two causes act together but do not produce the effect) – region **b** in **Figure 1**-, relates to the “outside union of causes” $\neg(\mathbf{A}_1 \vee \mathbf{A}_2)$ – always **g** in **Figure 1** - like the parts in which cause acts alone [(**A₁, ¬A₂**) and (**¬A₁, A₂**)] relate to each other – regions **d** and **f** in **Figure 1**.

⁵ Although the joint universe generated by the three binary propositions A_1 , A_2 , and E contains eight logical atoms (2^3), one of them— $P(\neg A_1 \wedge \neg A_2 \wedge E)$ —is set to zero by assumption. As a result, the linear program involves seven non-negative variables corresponding to the remaining atomic probabilities.

⁶ Talking about Boole’s method for the solution of the problems in Chapter XX, Keynes notes: “the difficulty of his method is so great that I do not know of any one but himself who has ever attempted to use it”. (Keynes 1921, p. 206).

$$\begin{cases} \frac{P(A_1 \wedge A_2 \wedge E)}{P[\neg(A_1 \vee A_2)]} = \frac{P(A_1 \wedge \neg A_2 \wedge E)}{P(\neg A_1 \wedge A_2 \wedge E)} \Rightarrow \left(\frac{a}{g} = \frac{c}{e}\right) \\ \frac{P(A_1 \wedge A_2 \wedge \neg E)}{P[\neg(A_1 \vee A_2)]} = \frac{P(A_1 \wedge \neg A_2)}{P(\neg A_1 \wedge A_2)} \Rightarrow \left(\frac{b}{g} = \frac{d}{f}\right) \end{cases}$$

These were internally consistent with his general formalism, based on “disjunctive or”. Retaining them amounted to taking a “slice” of the possible values that can be taken by the probability of causes, and this is why Wilbraham criticized them for being “arbitrary” and also, for one of them “anomalous”.

The “slices” retained by Boole are constraints that determine how the total mass of probability in E is internally split between the three regions $A_1 \wedge \neg A_2$, $\neg A_1 \wedge A_2$, and $A_1 \wedge A_2$. Different slices (different “closures”), yield different posterior causal probabilities, with Cawley presenting another variant. This is where Wilbraham’s objection bites. In the *Philosophical Magazine* exchange triggered by Cawley’s note and Boole’s reply, Wilbraham argued that Boole’s procedure smuggles in extra structural assumptions that are not contained in the data, thereby collapsing an indeterminate inverse problem to a single numerical answer.

Taking the example of **Figure 1**, the general solution provided by Boole in Chapter XIX and recalculated by Wilbraham, once replicated with linear programming yields the two bounds for $P(E)$:

$$0.24 \leq P(E) \leq 0.39,$$

so even the inverse quantities such as $P(A_1 | E) = \frac{P(A_1 E)}{P(E)} = \frac{0.24}{P(E)}$ cannot be calculated other than as **intervals**. With the same numerical data, imposing **Boole’s closure** selects a unique feasible joint table and yields a single value $P(E) \approx 0.2911$, hence the probabilities of the causes are single valued: $P(A_1|E) \approx 0.24 / 0.2911 \approx 0.824$. By contrast, imposing the alternative “Cawley closure” (also discussed by Wilbraham as a distinct way of closing the system) produces a different unique joint table with $P(E) \approx 0.2604$; hence $P(A_1|E) \approx 0.922$.

One can note that in *Problem VI*, Boole proceeds in the same manner. He finds a particular solution and then shows that it falls between the bounds of the relevant intervals; thus he finds an interval of probabilities for the posteriors and his calculation is methodologically correct, as demonstrated by Wilson (1934) and Hailperin (1986).

In 1934 Wilson examined the whole issue of the Challenge Problem afresh, and, while confirming the correctness of Boole calculations, his purely algebraic analysis allows to go further in the comprehension of the problem by making explicit the degrees of freedom of the Challenge problem. Initially there are **eight logical atoms** (regions **a** to **g** in **Figure 1**, together with $P(\neg A_1 \wedge \neg A_2 \wedge E)$). However, the first condition on the closure of the universe ($a+b+c+d+e+f+g = 1$), reduces the degrees of freedom to **seven**. Then the further condition imposed by Boole that E lies in the union of A and B ($P(\neg A_1 \cap \neg A_2 \cap E) = 0$) further reduces the degrees of freedom to **six**. The data considered by Boole as “given” add another **four further linear constraints**:

$$P(A_1) = a + b + c + d = c_1 = 0.6$$

$$\begin{aligned}
P(A_2) &= a + b + e + f = c_2 = 0.5 \\
P(E/A_1) &= a + c = c_1 \cdot p_1 = 0.6 \cdot 0.4 = 0.24 \\
P(E/A_2) &= a + e = c_2 \cdot p_2 = 0.5 \cdot 0.3 = 0.15
\end{aligned}$$

These **six** equations allow one to express six parameters of the system as a function of the other two. It is then natural to choose to express the remaining parameters as a function of g and a , because these represent the two main logical kinds of indeterminacy:

- i) $g = P[\neg(A_1 \vee A_2)]$
 - g controls how much of the universe lies outside of the causal system
 - g does not enter $P(E)$
 - g determines the scope of the causal model
- ii) $a = P(A_1 \wedge A_2 \wedge E)$
 - a determines the joint causality of A_1 and A_2 on E .
 - a determines $P(E) : P(E) = c_1 p_1 + c_2 p_2 - a$.⁷
 - a captures what is missing in the evidence on causation

On this basis one can argue that the two degrees of freedom in the Challenge Problem correspond respectively to an “**exogenous**” value to be assigned to the parameter g , which, once set at a particular value through the addition of proportionality conditions (such as Boole’s or Cawley’s closures), defines “slices” of uncertainty for the probability of causes and an “**endogenous**” weight of the argument that must be added to account for the fact that a includes both a probability and its weight.

Figure 1 can thus also be read as an illustration of the problem of hidden assumptions, which hinge both on **(non-)independence of causes** and on the **choice of the universe of reference**, two key points for Keynes’ subsequent work:

1. The issue of hidden assumptions arising from the **non-independence of causes** can be addressed in two main ways.
 - a. One may introduce additional assumptions on their co-occurrence. The latter could either be:
 - i. assuming that the causes are mutually exclusive ($a=0$); or,
 - ii. assuming a particular **proportionality structure** governing their co-occurrence ($a = \bar{a}$), as in the specific solution proposed by Bowley and Cawley for the *Challenge Problem*.

In the first case, the risk is to neglect interactions between causes, which may in practice reverse causal chains. In the second, the arbitrariness of the retained proportionality implies that the point estimates rest on a specific and untestable structural assumption; alternative proportionality choices—equally compatible with the data—would generate different posterior probabilities and potentially reverse causality interpretations.

⁷ This identity, made explicit by Wilson, shows that Boole’s upper bound for $P(E)$ is obtained at the minimal admissible value of $a = P(A_1 \cap A_2 \cap E)$, namely $a = 0$, corresponding to the absence of concurring causes with respect to the effect. Claims in the literature that Boole committed an error typically stem from a confusion between the upper bound on $P(E)$ and the value of $P(E)$ itself, as well as from a failure to recognize that Boole had already introduced in his *Mathematical Analysis of Logic* of 1847—arguably for the first time in the literature—the logical operator **or** as an **exclusive disjunction** (“one or the other but not both”), implemented through algebraic addition (denoted by “+”) under a 0/1 semantics and corresponding, in modern notation, to exclusive disjunction (XOR, sometimes written $\underline{\vee}$ or \oplus). Boole consistently adhered to this convention throughout his later work.

- b. Alternatively, one may recognize that the limits of the available information contained in E —that is, ignorance about a —naturally point toward an **interval interpretation of probability**, which is numerical, non-linear, and non-additive.

In this case, interval probabilities, handled through Keynes's notions of **weight** and **risk**, explain why investment is likely to fall below its actuarially optimal level (thus providing also a probability-based explanation of why no-arbitrage conditions fail), and why output and employment tend to remain below full-capacity levels.

2. The issue of hidden assumptions embedded in the **size and shape of the universe of reference** is often neglected, yet it is crucial for policy-relevant economic modelling. Re-interpreting **Figure 1** in terms of a “**moving universe**” provides an intuitive representation of several dimensions of Keynesian uncertainty that go beyond **Frank Knight's** classical distinction between risk and uncertainty.

In axiomatically defined theories, the universe is often implicitly identified with what, in **Figure 1**, corresponds to the union of the known causes $A_1 \vee A_2$, with the effect E postulated—as in Boole's framework—to be a subset of this union. Yet beyond known causes, the limits of the universe are largely unknown. As a result, even estimates of marginal probabilities such as $P(A_1)$ and $P(A_2)$ depend critically on how the universe is delimited: if g changes, all other parameters in **Figure 1** are affected. Moreover, the larger the universe relative to the set of known causes, the more likely it becomes that unforeseen causes may intervene and interact with known ones in unpredictable ways⁸.

The dispute over Boole's *Challenge Problem*, originated in the nineteenth century, was followed by Keynes with great attention and detail and continued for much of the twentieth century. It illustrates the point made by Brady that the concept of probabilistic causality is essentially a matter of probability intervals, rather than of probability point estimates and this helps to understand some of the key ideas that Keynes derived from Boole and further developed concerning the nature of the logic of probability and uncertainty.

The next and last section provides the missing step needed to argue that in *A Treatise on Probability* Keynes constructs a logic of probability that he subsequently uses to criticize the received economic doctrine for its largely deterministic character. Keynes relaxes different groups assumptions in a selective manner in *A Treatise on Money* and in *The General Theory*, showing that this changes the structure of policy causality—what today would be discussed under the heading of theory of change.

Three main aspects will be developed: (i) the critique of **Alfred Marshall's** partial equilibrium framework, which includes the joint critique of the static assumption and ii) the neutrality of money in the *Treatise on Money*; and (iii) the implications of the **coefficient of weight** and **risk** for handling uncertainty within a static analytical framework in the *General Theory*.

⁸ By way of illustration, if life were to be discovered on another planet, it is clear that our knowledge on humanity, intended as collection of known inferences linked by probability relations—that is as a network of causally ordered propositions—will be redefined.

The Treatise on Probability as a key tool for the logical critique of the mainstream in the *Treatise on Money* and the *General Theory*: with focus on Augusto Graziani.

This section presents Keynes' critique of the mainstream from a logical viewpoint, drawing on the discussion of the "probability of causes" developed in the previous section with reference to Boole's *Challenge Problem*. For reasons of space and focus, the discussion is limited to a comparison of the different sets of assumptions underlying *A Treatise on Money* and *The General Theory*, relying on the interpretations of a small number of selected Keynes scholars. The analysis focuses on the relation between the relaxation of these assumptions and the "open degrees of freedom" identified in the *Challenge Problem*. The purpose is to argue that Keynes' position—that a decentralised economy, left to itself, tends to underinvest and generate unemployment, a theme common to *A Treatise on Money* and *The General Theory* (Roncaglia, 2005b)—has a firm logical basis in the *Treatise on Probability*.

Keynes was trained first as a mathematician and turned to economics only later, encouraged by Marshall, about whose *Principles* he famously remarked that "the formation of a good economist only requires the *Principles*, accompanied by the careful reading of the economic pages of a good newspaper" (quoted in Roncaglia, 2005b, p. 352).

Keynes' critique of the mainstream was therefore, first and foremost, a critique of the Marshallian tradition that dominated Cambridge at the time and that, besides Marshall and Keynes himself, included Pigou, Robertson, and Hawtrey, with whom Keynes engaged in frequent and intense debates. In particular, this critique sought to address the limits of Marshall's partial equilibrium method—limits of which Marshall himself was well aware (Roncaglia, 2005a; De Vroey, 1999; Sraffa, 2011).

There is a clear methodological continuity in Keynes' critique of the classical approach between *A Treatise on Money* and *The General Theory*, consisting in the progressive relaxation of different sets of assumptions. In the *Treatise on Money*, this takes the form of questioning the static assumption through a sequential, dynamic approach in which uncertainty remains largely in the background. In *The General Theory*, by contrast, Keynes relaxes the assumption of certainty within an analytical framework that may be described as "para-static", in the sense that it remains formally static in structure, while allowing variables initially treated as independent to become mutually interdependent.

Before entering into a more detailed discussion, it is useful to illustrate these logical issues by means of a minimal analytical example. The purpose of the example is not to provide a comprehensive model of the economy, but to show how relaxing a partial equilibrium assumption and allowing for intertemporal consistency immediately introduces memory and alters the direction of causality. The example is therefore deliberately simple and should be read as a logical transposition of the probabilistic arguments developed above.

Specifically, it aims to illustrate the logical implications of relaxing a partial equilibrium assumption and introducing time into an otherwise static framework, with reference to the “scope” and “independence of assumptions” degrees of freedom discussed in the previous section. Taken together, these two open degrees of freedom suggest an interval interpretation of probability, which in turn calls into question policy prescriptions based on a single equilibrium presumed to be unique and stable. Beyond the static–dynamic divide, partial equilibrium analysis also takes for granted further assumptions—most notably the reduction of uncertainty to measurable risk—that are relaxed in Keynes’ logical framework. In the *Treatise on Probability*, uncertainty is not characterised by the absence of structure, but by partial comparability and non-additivity, corresponding to a lattice rather than a scalar measure of probability. When such assumptions are relaxed, the direction of causality may be modified independently of the static or dynamic nature of the analysis. In the resulting framework, shaped by interval probability and open degrees of freedom, a decentralised economy, left to itself, is therefore likely to follow a divergent trajectory. This conclusion finds its logical foundations in the probabilistic framework developed in the *Treatise on Probability*.

An illustrated example

Consider an economy producing a single good. The variables are:

L_t	: labour employed (flow)	Π_t	: profits (flow)
Y_t	: output (flow)	I_t	: E_t : investment (event/flow)
W_t	: wage bill (flow)	K_t	: capital stock (state variable, memory)

The parameters are:

$\nu > 0$: labour requirement per unit of output	$\delta \in [0,1]$: depreciation rate
$w > 0$: money wage	$\alpha > 0$: capacity output – capital ratio

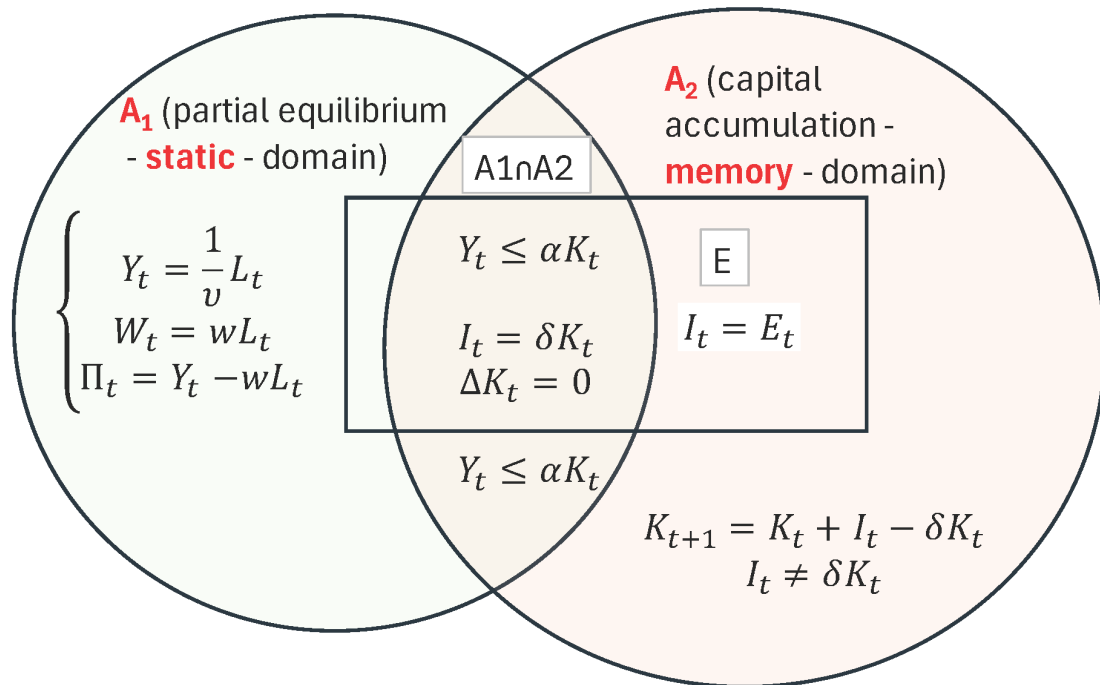
The equations are:

1) <i>Labour – based production</i>	Y_t	=	$\frac{1}{\nu} L_t$
2) <i>Wage bill</i>	W_t	=	wL_t
3) <i>Profits (residual)</i>	Π_t	=	$Y_t - wL_t$
4) <i>Investment (event E)</i>	I_t	=	E_t
5) <i>Capital accumulation (memory)</i>	K_{t+1}	=	$K_t + I_t - \delta K_t$
6) <i>Capacity constraint</i>	$Y_t \leq \alpha K_t$	\Rightarrow	$L_t \leq \nu \alpha K_t$

This simple system illustrates the logical implications of relaxing a partial equilibrium assumption and introducing time into an otherwise static framework. Labour is the sole source of production and value, while capital appears only as a historically accumulated stock that constrains activity and carries memory across periods. In the static case, defined by the absence of accumulation, investment is reduced to replacement investment, and the economy can reproduce the same configuration indefinitely, yielding a reduced-form relation between output and capital. Once this condition is relaxed, investment decisions affect the future productive capacity of the economy, altering the direction of causality independently of the static or dynamic nature of the analysis. In the presence of uncertainty

characterised by interval probability and open degrees of freedom, decentralised investment decisions are therefore unlikely to ensure intertemporal consistency, making divergent trajectories a potentially likely logical possibility rather than an anomaly

Figure 2: Illustration of the model (1)-(6)



Graphically, Figure 2 illustrates the model for the case in which the universe coincides with $A_1 \cup A_2$. The left-hand domain (A_1) represents a partial-equilibrium, static closure in which within-period relations between output, employment, wages, and profits are fully specified, but no intertemporal consistency is imposed. The right-hand domain (A_2) introduces capital accumulation as a state variable and thus memory, allowing past investment decisions to affect future productive capacity. The intersection of the two domains identifies the unique configuration in which within-period flows and intertemporal stocks are mutually consistent. In this region, stationary reproduction obtains when investment equals depreciation ($I_t = \delta K_t$), so that the capital stock remains constant and the same configuration can be reproduced period after period. Outside the intersection, investment decisions may still generate accumulation and memory, but without imposing a within-period closure, so that the notion of stationary reproduction is no longer defined. The figure thus makes explicit that statics corresponds to the special case in which memory is neutralised, while dynamics arises whenever accumulation departs from this condition, even though the explanatory universe remains closed. Divergent trajectories therefore reflect the logical structure of the model rather than the presence of omitted causes or exogenous shocks.

Comparing the sets of assumptions of *A Treatise on Money* and *The General Theory*

Graziani (1981) proposes an explicit and systematic comparison between *A Treatise on Money* and *The General Theory* based on the sets of assumptions underlying each work. His

central claim is that, contrary to the standard interpretation, it is the *Treatise* that contains the genuinely innovative elements of Keynes's thought, while the *General Theory* appears more attenuated and partially intertwined with traditional modes of reasoning. The comparison is conducted in terms of the distance of each work from the neoclassical, Marshallian framework and its proximity to heterodox economic reasoning. In particular, Graziani identifies as "traditional" those approaches grounded in methodological individualism, the neutrality of money, static or para-static closure, and a saving-first view of accumulation. From this perspective, the originality of the *Treatise on Money* lies in its ability to escape several of these hidden assumptions simultaneously, thereby anticipating core propositions—such as the determination of saving by investment—that are often attributed exclusively to the *General Theory*.

It is worth noting that explicit references to probability are extremely rare in *A Treatise on Money*, with only a handful of occurrences of the term and a single direct reference to the *Treatise on Probability*; significantly, this reference points to Chapter 3 and arises in the context of the impossibility of defining a unique price level, a difficulty that strikes at the core of the quantity theory of money.

Beyond the issue of price indices—where the reference to Chapter 3 of the *Treatise on Probability* clearly concerns the degree of freedom associated with the weighting of arguments—the innovative assumptions of the *Treatise on Money* primarily relate to what, in the terminology of the Challenge Problem, corresponds to the residual domain g : that is, to dimensions of economic reasoning that are typically excluded from the explanatory scope of traditional equilibrium analysis.

The analytical structure of the *Treatise on Money*:

In the comparison, Graziani devotes most of his analysis to the *Treatise on Money*, arguing that its originality lies in the analytical structure generated by its assumptions. At the core of this structure is the separation of economic agents into workers-consumers and entrepreneurs-capitalists, and the parallel distinction between consumption goods and investment goods. These distinctions correspond to different market circuits. Consumption goods are transferred from entrepreneurs to workers, while investment goods circulate within the entrepreneurial sector itself. As a result, prices, income distribution, and profits cannot be understood as the outcome of a single, unified market mechanism.

Prices, indices, and the rejection of a unique price level:

On this basis, Graziani shows that the *Treatise* determines separately the prices of consumption goods and the prices of investment goods, each governed by different mechanisms. The price of consumption goods serves to reconcile independently taken decisions by entrepreneurs (who choose quantities produced) and workers-consumers (who decide how much of their monetary income to spend). By contrast, the price of investment goods depends on expectations of future profitability, the interest rate, and the availability of alternative liquid assets. Since investment goods circulate exclusively among entrepreneurs, their prices do not perform the standard equilibrium function of reconciling autonomous demand and supply. From this follows a crucial implication: the *Treatise on Money* contains no equation determining a general price level. Any such level can only be

constructed ex post as an arithmetic average of heterogeneous prices and is therefore downgraded to a purely numerical device, devoid of analytical significance. Graziani emphasises that this rejection of the general price level does not deny the empirical regularities highlighted by the quantity theory, but it abandons the latter as an explanation of the price formation mechanism itself. What matters analytically are relative prices and their non-uniform, non-simultaneous adjustment to monetary changes.

Investment, profits, and the independence of saving

A further decisive departure from traditional theory concerns the relation between investment, saving, and profits. In the *Treatise*, investment and saving are decided by different agents and are therefore logically independent. When, by coincidence, investment equals saving, profits vanish and entrepreneurs receive only a “normal” remuneration. When the two diverge—which Graziani stresses is the more likely case—profits emerge as the residual difference between investment and saving. In this framework, profits are not a marginal reward for a factor of production, but the direct consequence of entrepreneurs’ control over investment decisions. As Graziani notes, this implies a reversal of the standard causal ordering: profits depend on investment, not the other way around, and saving adjusts ex post to income changes induced by investment. There exists no market mechanism capable of ensuring the spontaneous equality of saving and investment, nor of stabilising a marginalist distribution of income once it is momentarily realised.

Money, time, and sequential causality

Finally, Graziani highlights the fundamentally monetary and sequential nature of the economic process described in the *Treatise on Money*. Production unfolds in successive phases that cannot be collapsed into a single moment of exchange: entrepreneurs must first obtain finance from the banking system, then hire labour, then produce goods, which are only subsequently offered on the market. Workers, in turn, must first sell labour to obtain money before they can exercise demand. Money is therefore not a neutral medium or a mere store of value, but the necessary initial condition that sets the entire process in motion. This sequential structure introduces genuine historical time and assigns a privileged causal role to entrepreneurial and financial decisions. It also explains why fluctuations in investment—rather than in saving—are identified as the primary source of instability and cyclical dynamics, and why, in certain circumstances, monetary policy alone may be insufficient to restore full employment without direct public investment.

Graziani on the General Theory

When turning to *The General Theory*, Graziani’s guiding question is the extent to which the analytical framework of the *Treatise on Money* is preserved or, instead, modified. His central conclusion is that the genuinely innovative elements of the *General Theory* largely consist in results already established in the *Treatise*, while the points at which the *General Theory* departs from the *Treatise* correspond to aspects where Keynes deliberately returned to a more traditional and academically conventional mode of exposition. From this perspective, the *General Theory* does not represent a logical break with the *Treatise*, but rather a partial rearticulation of its results within a framework more easily assimilable to prevailing economic theory.

In particular, Graziani stresses that *The General Theory*, like the *Treatise*, preserves the structural distinction between consumption and investment and, correspondingly, between

workers-consumers and entrepreneurs-capitalists. Consumption and investment are analysed as fundamentally different components of aggregate demand, subject to distinct motives and constraints. Investment remains an autonomous component of demand, in the precise sense that entrepreneurs are not constrained by previously earned income but rely on access to credit and financial markets. This autonomy does not imply irrationality, but reflects the specific nature of investment decisions, which are based on expectations of future profitability and involve elements of uncertainty that cannot be reduced to quantifiable risk. In this respect, Graziani emphasises that the *General Theory* merely confirms a position already fully developed in the *Treatise*, rather than introducing a new analytical principle.

The main difference between the two works, according to Graziani, concerns the treatment of money and prices. While the *Treatise* assigns money a primary and explicit role as the indispensable condition for initiating production—financing both the supply of consumption goods and investment goods—the *General Theory* places greater emphasis on money as a store of value and on liquidity preference. As a result, the financing role of money on the supply side, which is central and explicit in the *Treatise*, becomes less visible in the *General Theory*. A similar attenuation occurs in price theory: whereas the *Treatise* maintains a clear separation between the pricing mechanisms of consumption goods and capital goods, the *General Theory* adopts a unified, marginalist theory of prices based on marginal costs. This return to a single price mechanism eliminates a key analytical distinction and shifts adjustment from prices to quantities through the multiplier, thereby reintroducing elements of traditional equilibrium reasoning.

Finally, Graziani argues that the theory of crisis and unemployment presented in the *General Theory* is largely anticipated in the *Treatise on Money*. The rejection of Say's Law, the determination of income by effective demand, and the central role of unstable investment expectations are all already present in the *Treatise*. What changes in the *General Theory* is not the analytical diagnosis of crisis, but the discussion of policy instruments—most notably the emphasis on the limits of monetary policy and the consequent justification for direct public investment. Even here, however, Graziani notes that the underlying conception of the economic system, based on the asymmetry between entrepreneurs and workers and on the autonomy of investment, is inherited directly from the *Treatise*, where it is articulated in an even more immediate and coherent form.

In Graziani's interpretation, the *General Theory* innovates primarily at the level of policy conclusions, while its analytical core—and in particular the reversal of the saving–investment causality and the rejection of a self-equilibrating market mechanism—is already fully established in the *Treatise on Money*, which therefore represents the more radical break with traditional economic theory.

The Treatise on Probability as a key tool for the logical critique of the mainstream in the Treatise on Money and the General Theory: with a focus on Michael Brady.

The *General Theory* is Keynes's most celebrated work, though it has often been criticised for a lack of clarity and, at times, a lack of rigour. This section discusses two related issues: first, the link between The *General Theory* and the *Treatise on Probability*, which allows such criticisms to be reconsidered on more precise grounds; and second, the picture that emerges of the role of *The General Theory* within the broader architecture of Keynes's three major books, thereby allowing its original and innovative elements to be more clearly identified.

The main reference for the links between *The General Theory* (GT) and the *Treatise on Probability* (ToP) is provided by the extensive body of work of Michael Brady on the subject, beginning with his 1983 doctoral thesis and continuing to the present. In essence, Brady argues that once it is recognised that the *Treatise on Probability* is grounded in Boole's *Laws of Thought*, together with the associated apparatus of interval probability derived from the *Challenge Problem* and related examples discussed in the second section above, the next decisive step consists in grasping the meaning of the conventional coefficient of weight and risk, which proves central to interpreting several key aspects of *Treatise on Money* and *The General Theory*.

The concept of the weight of the evidence is introduced by Keynes in Chapter VI of the ToP, where he writes:

"The conclusion, that the 'weight' and the 'probability' of an argument are independent properties, may possibly introduce a difficulty into the discussion of the application of probability to practice." (p. 83)

"One argument has more weight than another if it is based upon a greater amount of relevant evidence; but it is not always, or even generally, possible to say of two sets of propositions that one set embodies more evidence than the other. It has a greater probability than another if the balance in its favour, of what evidence there is, is greater than the balance in favour of the argument with which we compare it; but it is not always, or even generally, possible to say that the balance in the one case is greater than the balance in the other. The weight, to speak metaphorically, measures the sum of the favourable and unfavourable evidence, the probability measures the difference." (p. 84)

The link to Chapter XXVI on "*The application of probability to conduct*" is explicitly made through footnote 1 following "practice" at the end of the first quotation (p. 83), where Keynes refers to §7 of this chapter (pp. 345-46). Here, additional comments are offered on the difficulties arising from the independence of weight and probability, before introducing the key §8, which opens by noting that:

"other things being equal, that course of action is preferable, which involves least risk and about the results of which we have the most complete knowledge." p. 347

Keynes then introduces the coefficient of risk and weight defined as (pp. 346-48).

$$c = \frac{2pw}{(1+q)(1+w)}$$

Where:

- c = Conventional coefficient of risk and weight
- A = Amount at stake ("amount of good which may result")
- p = Probability of gaining the amount A
- q = 1-p = Probability of not gaining the amount A
- E = A = Mathematical expectation "the net immediate sacrifice which should be made in the hope of obtaining A"
- w = Weight "which is equal to unity when p = 1 and w = 1, to zero when p = 0 or w = 0, and has an intermediate value in other cases."

For Keynes, under conditions of uncertainty and risk, the decision-maker maximises the "weighted gain" given by the product cA. As noted by Brady (1994, p. 99) the coefficient of risk and weight can also be expressed as the product of two weights, p/(1+q) and 2w/(1+w), which represent respectively "risk" and "evidence":

$$c = \frac{p}{1+q} * \frac{2w}{1+w}$$

This formalises the fact that decision makers under uncertainty weight both "risk" and "evidence". If there is no uncertainty, the weight of evidence parameter is w=1, and the coefficient c depends only on the risk weight p/(1+q). Alternatively, if there is no risk (q=0), the decision depends only on "uncertainty", and the coefficient reduces to c = 2w/(1+w). Taking into account the Keynes quotation p. 347 reproduced above, together with the explanations given in footnote 2 on p. 348, one can illustrate the way in which the coefficient c operates by comparing its value in two decisions, A and A', that have the same probability (p=p'), but different weights of the argument and risks:

	A has lower or equal risk: $q_A \leq q_{A'}$	A has higher risk: $q_A > q_{A'}$
A has higher weight: $w_A > w_{A'}$	$c_A > c_{A'}$	Not comparable
Equal weight: $w_A = w_{A'}$	$c_A > c_{A'}$ if $q_A < q_{A'}$ Indifferent if $q_A = q_{A'}$	$c_A < c_{A'}$

In words, for the same probability (p=p'), if the risk is lower or equal for A, A is preferred to A' if it has a strictly higher weight, or, for equal weight, if it has a lower risk (the decision maker is indifferent if the risk is the same). If the risk is higher for A, and A has a higher weight, the situation cannot be decided. By contrast if the risk is higher for A while A and A' have the same weight, A' is chosen.

Thus, c is a function of two variables: risk and weight. Interestingly, risk is defined by Keynes as $R = qE$. Since he defines $E = pA$ as "the net immediate sacrifice which should be made in the hope of obtaining A", risk is therefore: "the probability that this sacrifice will be made in vain". Today, after several decades of dominance of the expected utility paradigm, it may

appear counter-intuitive that Keynes effectively applies p twice in the evaluation of risk [$R=p(A-E)=p(1-p)A=pqA=qE$].

One may first note that in this respect in footnote 1 p. 348 Keynes refers to the German actuarial literature to explain that he defines risk is as the first terms of a geometric progression whose first term is E and whose common ratio is q . Since $q \leq 1$ by definition, this progression converges to $E/(1-q)=E/p=A$. Thus, what Keynes defines as risk is, by his own explanation, the “first round of reinsurance” equal to qE ⁹.

Another way to understand this definition is to compare Keynes’ notion of risk to de Finetti’s notion of probability based on bets. For de Finetti, (and, in a sense, also for Keynes) “probability does not exist”, he means that there can be no such thing as an “objective probability” grounded in frequency. Probability is therefore purely subjective for de Finetti and is defined as the amount p that an individual would be prepared to pay for a lottery linked to the realisation of a random event which, if the event occurs, yields a gain of one, and if it does not occur, entails a loss of p . Now, if within Keynes’ framework one re-interprets the expected value pA as the “gain” in de Finetti’s lottery, the “risk” defined by Keynes corresponds to the loss in de Finetti’s lottery, that is to the subjective probability p .¹⁰

The numerous works of Brady show that the main route followed by Keynes to apply his logic of probability under uncertainty to economics is to base expectations on both risk and weight, rather than on probability alone, through the use of the coefficient c in the decision rule cA ¹¹. This is clearly expressed in Brady (2025, p. 367):

“It is easy to see that Keynes’s Evidential Weight of the Argument, V , in Keynes (1921, 1973) is a technical way of being able to deal exactly with those problems in decision making that have insufficient data while circumventing the difficult interval approach to probability, so that $V(a/h)=w$, where $0 < w < 1$, allows one to estimate a decision weight answer that is numerical, while also incorporating BOTH risk and uncertainty. Uncertainty for Keynes (1936) is carefully discussed on pp. 148-153. Uncertainty is defined as an inverse function of $V = w$, so that problems impacted by the confidence one has in the data, discussed by Boole on pp. 381, 398, and 403 of Boole (1854), usually result from decision making situations whenever the decision maker is faced with ‘insufficient data.’” (p. 367).

Brady’s thesis is anchored in several references to Keynes’s writings. For instance, Brady (1987) notes that:

“The concept of weight is the basis for Keynes’ discussions, in Chapters 5, 12, 13 and 22 of the General Theory (GT), of ‘uncertainty’, ‘confidence’, ‘liquidity preference’, ‘conventions’, ‘animal spirits’, and ‘short and long run expectations’” (p. 38),

⁹ It goes without saying that the similarity between the expression in this footnote and the Keynes-Kahn multiplier is striking.

¹⁰ Following this line of reasoning, one may ask whether it is possible in principle to apply de Finetti’s betting approach to probability to a situation in which, in order to gain a “sure amount” of pA if an event occurs, one would be prepared to pay “ c ” (the coefficient of risk and weight), rather than “ p ”, the subjective (or, in Keynes’s framework, in principle objective) probability. This would however probably not work as it would amount to transforming a partial order into a complete order – thereby reintroducing additive (if not linear) probabilities. The issue may therefore be left open for future analysis.

¹¹ Following Keynes’s verbal explanation in chapters V and VI of the ToP, Brady (1983, pp. 38-39) initially defines three arguments of a rational decision in the case of uncertainty: V (the evidential weight), ψ (the relevance of the information) and P (the probability). He then collapses the first two arguments into a single function called the weight of the argument, which has both V and ψ as arguments and which he denotes by \bar{V} . This composite function then enters the decision function together with the probability P . In later writings Brady uses only the symbol V for the weight of the argument, inclusive of the relevance factor ψ , and this convention is followed in the present text as well.

whereas Brady (1993) refers to chapter 17 (p. 371) of the GT for the liquidity premium and Brady (2025) quotes the ToM in chapter 6, 7 and 8.

One may also mention references to some key footnotes in the GT that were overlooked by many scholars, in particular:

- Ftn. 3 p. 24, where Keynes defines the aggregate supply price of the output associated to a given level of employment as the expectation of proceeds that will just make it worthwhile for entrepreneurs to provide that employment:
“An entrepreneur, who has to reach a practical decision as to his scale of production, does not, of course, entertain a single undoubting expectation of what the sale-proceeds of a given output will be, but several hypothetical expectations held with varying degrees of probability and definiteness. By his expectation of proceeds I mean, therefore, that expectation of proceeds which, if it were held with certainty, would lead to the same behaviour as does the bundle of vague and more various possibilities which actually makes up his state of expectation when he reaches his decision”;
- Ftn. 1 p. 46, where he notes that:
“the entrepreneur (including both the producer and the investor in this description) has to form the best expectations() he can as to what the consumers will be prepared to pay when he is ready to supply them (directly or indirectly) after the elapse of what may be a lengthy period; and he has no choice but to be guided by these expectations, if he is to produce at all by processes which occupy time.”*
and where in referring to expectations Keynes directs the reader to the footnote above p. 26.
- Ftn. 1 p. 148 of Chapter XII where Keynes addresses producers’ long-term expectations noting that:
“It would be foolish, in forming our expectations, to attach great weight to matters which are very uncertain”.

To these GT references, one may add the sole direct reference to the ToP in the ToM, which appears in footnote 2 page 88 of volume I. There in discussing the possibility of comparing “the change in purchasing power for a community as a whole[...] averaged over populations which are not homogeneous in respect of their real incomes”, Keynes refers to chapter III § 7-16 (pp. 30-40) of the ToP. Chapter III is the one that concludes with the well-known chart p. 42 which, as noted by Brady, was correctly understood only by Kyburg (2003, 2011). Kyburg showed that if Keynes’s logical notion of probability is interpreted as a lattice¹², probabilities must take the form of intervals, as implied by their Boolean origin. Under this interpretation, the otherwise rather mysterious chart p. 42 of the ToP becomes a straightforward illustration of this fact (see also Brady & Arthmar, 2012). The fact that this reference concerns a key element of Keynes’s argument in the ToM on non-neutral money once again underscores the importance of the ToP.

¹² A lattice is a partially ordered set for which, for any two elements A and B, one can always define:

- their greatest lower bound ($\text{glb} = A \wedge B$), and
- their least upper bound ($\text{lub} = A \vee B$).

Many of Brady's points on the relationships between the ToP and the ToM and the GT were first made in Brady (1983) and notably the claims that:

- Keynes prospective returns on investment (the marginal efficiency of investment, MOI) defined in chapter 11 of the GT which must be interpreted in the light of the coefficient of risk and weight. This coefficient must be taken as an argument of all variables entering the MOI and therefore yields an investment level below that which would result under linear and additive expectations. In criticising the views of A. Leijonhufvud and D. Patinkin—according to whom Keynes advances a sketchy, point-based theory of expectations—Brady notes (p. 58) that: “The entire Treatise on Probability represents Keynes's theory of expectation formation.”
- In his analysis of speculation, Keynes shows that the weight of the argument is higher for speculators than for investors, in part because of the need to finance production until it first generates revenues (Keynes' finance motive for money demand). Because of the predominance of speculation in investors' choices, it is unlikely that a market economy will fulfill dynamic stability conditions such as full employment, a discussion that refers to chapters XII to XIV of the GT.
- Consumption is subordinated to investment, which constitutes the *causa movens* of the business cycle. Here Brady refers to the criticism that Keynes raises against Pigou pp. 275-276 of the GT (see also Brady, 1994).
- This criticism of Keynes in the GT is later generalized on pp. 297-298 in terms of an “independence of assumptions” argument which recalls the discussion on Boole's degrees of freedom in the *Challenge Problem* in the previous section:
“It is a great fault of symbolic pseudo-mathematical methods of formalising a system of economic analysis, such as we shall set down in section VI of this chapter, that they expressly assume strict independence between the factors involved and lose all their cogency and authority if this hypothesis is disallowed.”
This argument is further extended on p. 305 of the TP to the relationship between money and prices. Brady then applies this reasoning to show that the discussion of consumption and the multiplier must be linked to that of short- and long-term expectations, for which the integration of the weight of the argument defines a range of possible “consumption–multiplier models”.
- The availability of credit is an important determinant of effective demand, as it enters the weight of the argument of both producers and consumers. Keynes used his TP theories of risk, probability, evidential weight, relevance and time to establish a firm foundation for his monetary theories of congestion, lenders' risk and finance.
- This logic also applies to Keynes' theory of Finance. In the short-run demand prevails, in the long-run it is supply (and credit). Brady (p. 121) concludes that: “Thus, there is no conflict between the TM and GT, merely a change in emphasis (short run as opposed to the long run) and exposition (m.e.c. instead of credit).

Brady provides further clarification of Patinkin's misinterpretations of the GT (Brady , [2001] 2017a; Brady, [2001] 2017b); see also Arthmar & Brady (2009).

Therefore, through his analysis of Keynes ToP, Brady shows that the coefficient of risk and weight systematically determines an outcome that lies below the “maximum efficiency” (see

Brady, 1993, fig. 1, P. 364) which forms the basis of Keynes's consistent arguments in the TM and the GT and that:

"In summary, no general equilibrium exists for Keynes, for general equilibrium (competitive) analysis involves a world of certainty or a barter (non-monetary) economy." (Brady, 1983, p. 155)

Conclusion

This paper has examined the implications of Keynes's logic of probability—read through the prism of Boole's Laws of Thought—for the subsequent evolution and internal coherence of Keynes's economic thought. The analysis broadly supports Brady's interpretation of Keynes, albeit with some minor departures, while also endorsing Carabelli's emphasis on the centrality of the Treatise on Probability in reconstructing Keynes's intellectual itinerary. By placing the Boole connection at the centre of the analysis—a connection largely overlooked both in Carabelli's work and in the vast majority of the Keynesian secondary literature—the paper reinforces the view that the Treatise on Probability constitutes not merely an early philosophical exercise, but the foundational and unifying work of Keynes's oeuvre.

Seen from this perspective, Keynes's later contributions in the Treatise on Money and The General Theory appear not as partial breaks or pragmatic departures, but as systematic applications of a pre-existing logical framework designed to address decision-making under irreducible uncertainty. The logic of probability developed in the Treatise on Probability—with its rejection of additive probabilities, its emphasis on evidential weight, and its recognition of partial ordering—provides the conceptual infrastructure for Keynes's treatment of expectations, finance, liquidity, and effective demand. Read in this way, Keynes's oeuvre may be interpreted as a sustained effort to expose the analytical and policy limits of Marshall's partial-equilibrium method when confronted with monetary economies operating in historical time.

The specific contribution of this paper is to make explicit the implications of this logical aggiornamento for economics and finance. In particular, it shows how the integration of risk, uncertainty, and evidential weight systematically leads to outcomes that fall short of notional benchmarks of "maximum efficiency," thereby grounding Keynes's monetary and macroeconomic arguments in a coherent theory of rational choice under uncertainty. This has direct consequences for contemporary economic analysis and policy design. It suggests that many policy frameworks implicitly relying on additive expectations, representative-agent rationality, or complete probabilistic ordering remain structurally ill-equipped to deal with uncertainty-driven instability. Reconstructing Keynes's logic of probability thus not only clarifies longstanding interpretative disputes, but also provides a more robust foundation for assessing the feasible menu of economic policies and institutional arrangements in modern monetary economies.

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