Climbing the Energy Ladder: How Energy Resources Hinder, Facilitate, and Fuel Economic Growth*

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December 18, 2025

I show that the nature of the energy resources available to an economy qualitatively determines long-run growth outcomes. A harvested resource such as biomass drags on growth, a mined resource such as coal enables output per capita to hold constant, and both a tapped resource such as oil and a manufactured resource such as solar panels risk degrowth if energy return on energy invested (EROI) cannot stay above a threshold. The only energy resource that can fuel long-run growth is a manufactured resource such as solar panels. Either that resource must rely on substitutable energy inputs that have a sufficiently large EROI, or it must be produced by robots that are themselves produced from robots and energy. In ongoing work, I aim to numerically explore the implications for energy transitions in response to climate change.

JEL: O13, O41, Q43

Keywords: growth, energy, resources, solar, robots, EROI

*I thank Ryan Kellogg and seminar participants at Hawaii, UC Berkeley, and UC Santa Barbara for helpful comments.

1 Introduction

The ignition of sustained economic growth around the Industrial Revolution roughly coincided with the first sustained exploitation of fossil energy resources.[1] Over the ensuing centuries, both economic output and fossil resource use have continually grown, generating global wealth and global warmth without precedent in human history. Now the global economy is in the midst of a transition from fossil resources to renewable resources, in particular solar photovoltaics. Some fear that this transition will impede further economic growth by removing the fossil energy dividend that underpins it (e.g., King and van den Bergh, 2018; Jackson and Jackson, 2021).

I study the energetic basis of economic growth. In particular, I consider how fossil energy resources may have fostered growth and the implications of solar resources for growth. To this end, I extend a neoclassical Solow-Swan growth model to include energy as a factor of production. I show that the origin of that energy determines the types of growth outcomes that are possible. (In ongoing work, I aim to analytically explore the conditions under which one resource is used to develop another, explore the dynamics of that transition, and consider implications for renewable energy and climate change.)

Until the nineteenth century, all human societies were run on "harvested resources" in the form of biomass. These resources are extracted from an exogenous and renewable resource base via application of labor and capital. I show that the limited availability of land constituted a drag on growth in output per capita. I call such an outcome "energy-scarce degrowth". Output per capita could tread water or grow only if technological change was sufficiently rapid.

In the nineteenth century, several European economies began large-scale exploitation of "mined resources" in the form of coal. These resources require inputs of labor, capital, and energy, both to initially open a mine in a coal seam and to subsequently extract coal from that seam over time. I show that mined resources cannot drive growth in output per capita but do enable output per capita to tread water even in the absence of technological change. As a result, positive growth arises from even the slightest rate of technological change. I call such an outcome "energy-enabled growth". The essential element of growth around the Industrial Revolution may well have been some factor other than energy, but the increasing use of coal energy may have enabled that other factor to drive growth.

In the twentieth century, developed economies came to rely on "tapped resources" in the form of oil and gas. Initially tapping these resources requires an upfront input of labor, capital, and energy, but once tapped, energy production does not scale with further inputs. In particular, oil and gas wells need inputs at the point of drilling, but once a well has been drilled, the pressure in the well drives production, without needing to send workers or

[1] Some attribute economic growth to that shift in the energetic basis of the economy (e.g., Wrigley, 1988; Pomeranz, 2000; Allen, 2009; Wrigley, 2010), whereas others argue that energy was not critical to growth (e.g., Mokyr, 1990; Clark and Jacks, 2007; Clark, 2014; Mokyr, 2016).

machines into the well to coax the oil or gas out. I show that such resources can sustain output per capita, as mined resources do. But I also show that another outcome is possible when energy is complementary to capital and labor in deposit-tapping: if each deposit does not produce enough energy, then energy cannot be reinvested at a high enough rate to ensure the continued supply of energy. The limited availability of energy prevents the economy from reaching a balanced growth path. I call such an outcome "energy-scarce degrowth".[2]

In the twenty-first century, economies are increasingly reliant on solar energy. Solar panels have consistently outpaced expectations for installed capacity (Economist, 2024) and cost (Way et al., 2022), and solar power is on track to become the largest single source of electricity generation around 2030 (IEA, 2024). Solar panels are similar to oil and gas resources in that an initial investment produces energy for years on end. Solar panels' distinguishing feature is that they are "manufactured resources", with machines and energy combining to convert silicon and other minerals into a standardized product.[3] I show that a solar economy with panels produced from capital, labor, and energy contains the long-run outcomes described above for the economy with a tapped resource. In addition, I show that solar panels can drive long-run growth if energy inputs to their manufacture are substitutable with other inputs to their manufacture and the productivity of those energy inputs is sufficiently large. In that case, solar energy becomes the dominant input to solar panel production, with solar energy producing ever more panels. Energy becomes abundant and can cause output per capita to grow over time. I call such an outcome "energy-fueled growth".

Many companies are working to automate the production and installation of solar panels via use of robots and artificial intelligence (Liu, 2024; Plumer, 2024).[4] One day, these robots may themselves be produced in an automated fashion, from other robots and energy. In this case, I show that the economy can enter a regime of "energy-fueled growth" even when production functions are Cobb-Douglas. Again the critical condition is that the productivity of energy in making energy be sufficiently large.

I tie each of these outcomes to energy return on energy investment (EROI). This metric is much analyzed and discussed among scholars of energy systems.[5] EROI captures how much energy is produced per unit of energy invested in making energy. Energy analysts

[2]A mined resource avoids this outcome because production from existing mines scales with labor inputs to mining.

[3]Solar panels do require mined minerals, but there is a salient difference with respect to a "mined resource" such as coal: the minerals embodied in solar panels subsequently produce a flow of energy that does not scale with ongoing inputs, whereas coal produces energy only at the moment of combustion.

[4]Nemet (2019) describes the key role that automation played in the declining cost of solar panels. The scope for automating production of solar panels potentially exceeds that of other sectors of the economy. The manager of solar panel manufacturing plant in Florida boasted, "City and state officials who have seen our plant contend this is likely the most automated factory in Florida" (Mendenhall, 2022).

[5]See Murphy and Hall (2010) and Brandt and Dale (2011) for histories of the concept. A small prior literature formally relates EROI and growth in numerical simulations (Court et al., 2018; Fagnart et al., 2020) and Leontief production structures (Fagnart and Germain, 2016).

have been especially interested in the possibility that the economy is approaching a "net energy cliff": society might eventually have to invest so much energy into producing energy that it becomes hard (or even impossible) to generate the energy required for the rest of civilization.[6] EROI looms especially large in current discussions for two reasons. First, the EROI of fossil fuels has been declining over time.[7] Second, many analysts worry that a shift to renewable energy would reduce EROI and risk approaching the net energy cliff.[8]

I show that EROI is endogenous to an economy, in the sense that it depends on input choices. EROI can even decrease in the physical productivity of an energy resource, depending on how energy is substituted for other inputs to energy production. As a result of this endogeneity, EROI in an economy with a mined resource always exceeds unity plus the rate of population growth ($1+g_L$). In an economy with a tapped resource having complementary inputs in its tapping or a solar resource having complementary inputs in its manufacture, energy-scarce degrowth occurs when EROI cannot exceed $1 + g_L$ along a balanced growth path. This possibility substantiates fears of a net energy cliff.[9] In economies with energy-fueled growth, EROI exceeds $1 + g_L$ by a margin that increases with the rate of economic growth. When solar panels are manufactured from substitutable inputs or by robots, a high-EROI solar resource accumulates energy indefinitely, with each panel making enough energy to both accumulate more energy and devote energy to final good production. Such a solar resource promises a "net energy ramp", which is the optimistic counterpart of the dreaded net energy cliff.

Quantitative analysis in progress. Calibration and desired experiments described in text.

Energy resources are of potential importance to long-run economic growth because the second law of thermodynamics means that energy cannot be completely recycled. The ability to sustain energy consumption therefore ultimately depends on stocks of fossil resources and flows of renewable resources. Economists have long studied how the finitude of fossil resource

[6] Analysts disagree about the minimum EROI required to sustain society: Hall et al. (2009) suggest that the minimum EROI is 5; Hall and Klitgaard (2012) say the minimum is probably around 10; Brandt (2017) argues for a minimum between 1.1 and 15; and Fizaine and Court (2016) suggest 11.

[7] See, among others, Cleveland (1992), Gagnon et al. (2009), Guilford et al. (2011), Murphy (2014), Court and Fizaine (2017), and Brockway et al. (2019).

[8] There is much debate about the EROI of particular renewable technologies. Two analyses of EROI for a renewable energy system suggest that it would be around 5, engendering concern about reaching the net energy cliff (Trainer, 2018; Capellán-Pérez et al., 2019). Others argue that the EROI for renewables will fall as the energy system changes (Trainer, 2018; Fabre, 2019) and that a transition to a renewable energy system would require a period in which so much energy is devoted to renewable energy development that energy becomes less abundant for the rest of society and energy sector emissions may even increase (King and van den Bergh, 2018; Sers and Victor, 2018; Slameršak et al., 2022). On the other hand, some argue that the EROI of renewable energy is now relatively large and will increase further over time as technology improves (e.g., Steffen et al., 2018; Diesendorf and Wiedmann, 2020).

[9] But the cliff is not diagnostic. Numerical simulations show that EROI can exceed $1 + g_L$ even in the midst of energy-scarce degrowth and that EROI can decline towards $1+g_L$ even in the midst of energy-fueled growth.

stocks affects the economy's ability to sustain long-run consumption growth (e.g., Dasgupta and Heal, 1974; Solow, 1974; Stiglitz, 1974; Krautkraemer, 1998; Hassler et al., 2021) and can induce transitions between resources (e.g., Nordhaus, 1973; Chakravorty and Krulce, 1994; Tahvonen and Salo, 2001).[10] Motivated by evidence that fossil reserves are actually quite abundant (e.g., Rogner et al., 2012), I here focus on how each energy resource is accessed, albeit permitting extraction costs to depend on the availability of resource deposits (see Heal, 1976).[11] My analysis can thus describe nearer-term growth outcomes rather than the much longer-run outcomes for which resource exhaustion may become a concern.

The present paper analyzes the implications for growth of some aspects of energy production that have been studied in the applied microeconomics literature. First, some prior papers study the role of fixed costs in accessing deposits of energy resources (e.g., Hartwick et al., 1986; Holland, 2003; Venables, 2014). I integrate more general forms of fixed costs into a growth model. Second, recent work emphasizes that the flow of oil from a well is, for practical purposes, exogenous (Thompson, 2001; Mason and van't Veld, 2013; Anderson et al., 2018; Kellogg, 2024), based on a physical relationship known as Darcy's Law. The present paper's "tapped resource" is consistent with these papers' models.[12]

The "linearity critique" holds that sustained economic growth has fragile preconditions within macroeconomic models. To generate a trajectory with constant growth in output, economic models require a linear relationship somewhere under the hood: AK models place the linearity in physical capital accumulation and production, Lucas-type models place it in human capital accumulation, ideas-based endogenous growth models place it in productivity growth, and semi-endogenous growth models place it in population growth (Jones, 2005). These linear relationships are all knife-edge conditions, with small deviations ruling out sustained exponential growth (Growiec, 2010). The question is which type of linearity is well-founded, both in principle and empirically. Jones (2005) observes that there is no intuitive reason to prefer a model with linear growth in ideas to a model with either fishing-out or spillover effects, and Jones (1995) reports that linearity in idea accumulation is inconsistent with empirical evidence. Instead, Jones (2005) argues that population growth is sensibly linear since "people reproduce in proportion to their numbers", but others find such linearity contrary to both intuition and evidence (e.g., Solow, 2003; Growiec, 2010).

I show that energy can sustain growth in two ways. First, if energy is a substitutable input to solar panel manufacturing, then it can crowd out other inputs and thereby relieve

[10] Other literature endogenizes the role of innovation in determining the types of energy resources used (e.g., Acemoglu et al., 2012; Hart, 2019; Lemoine, 2024).

[11] I assume that new deposits of the tapped and mined resources are discovered at a constant, exogenous rate. Prior work has analyzed the equilibrium implications of endogenous exploration for new resource deposits (e.g., Pindyck, 1978; Gilbert, 1979; Arrow and Chang, 1982; Ekeland et al., 2022). Future work might synthesize the literatures on growth and endogenous exploration.

[12] Moreno-Cruz and Taylor (2017) model how a resource's energetic density per unit of land affects the spatial distribution of economic activity. I abstract from space and study how a resource's energy produced per unit energy input affects the evolution of energy and output over time.

that manufacturing of other constraints.[13] Energy production becomes linear in energy. This mechanism resembles the potential for capital to crowd out labor at high elasticities of substitution in aggregate production functions (Solow, 1956; Pitchford, 1960; de La Grandville, 1989).[14] In reality, capital requires energy, so the present analysis clarifies the preconditions for substitution from labor to drive growth. Second, if solar panels are produced by robots, then the stocks of robots and solar panels grow in tandem and the transition equation for solar panels becomes, in equilibrium, effectively linear. The mechanism has parallels in the recent literature on artificial intelligence and robots (see Trammell and Korinek, 2023). In particular, Mookherjee and Ray (2017) analyze when it is optimal for robots to crowd out labor in the production of robots and output. They show that this possibility depends on an additional robot producing sufficiently more than one extra robot even as robot production becomes entirely reliant on capital, which is an EROI-like condition (they refer to it as their "von Neumann singularity condition"). In that case, sustained growth is achievable, driven by capital accumulation. However, this literature generally ignores that operating robots requires energy. Incorporating a scarce energy input would tend to prevent sustained growth in economic output. I show that solar energy resources can restore the link between robots and growth if their production function permits self-sustaining feedback between energy and robots.[15]

The next section outlines the broader economic environment. Subsequent sections analyze harvested, mined, tapped, and manufactured resources. Section 7 analyzes resources manufactured by robots. Section 8 discusses actual energy transitions within the context of the model. Section 9 contains a quantitative investigation of implications for climate change policy. The final section concludes. The appendix contains additional analysis, numerical details, and proofs.

2 Setting

I begin by introducing the elements of the model that are common across types of resources.

At each instant t, the economy is populated by $L(t)$ households who discount future utility at rate ρ. The population grows at rate $g_L \geq 0$, with $g_L < \rho$. Households supply a fixed unit of labor to the economy. They save a fixed share $s \in (0, 1)$ of the final good $Y(t)$ and consume the rest. They have utility $u(\cdot)$ over time t per-capita consumption, with $u(\cdot)$ strictly increasing and concave.

[13]In particular, energy would crowd out labor along the solar panel manufacturing chain.

[14]And the potential for energy-scarce degrowth resembles the potential for degrowth at low elasticities of substitution in aggregate production functions.

[15]Nordhaus (2021) shows that artificial intelligence can increase the rate of economic growth if it makes capital substitutable for labor. This possibility is similar to my discussion of solar panels produced from substitutable energy inputs.

The aggregate capital stock is increased by savings:

$$\dot{K}(t) = sY(t) - \delta K(t),$$

with $\delta > 0$ the depreciation rate. A representative firm produces the final good $Y(t)$ by combining labor $L_Y(t)$, capital $K_Y(t)$, and energy $E_Y(t)$ in a Cobb-Douglas production function:

$$Y(t) = A(t)L_Y(t)^{\alpha_L} K_Y(t)^{\alpha_K} E_Y(t)^{\alpha_E},$$

where $\alpha_L, \alpha_K, \alpha_E > 0$ and $\alpha_L + \alpha_K + \alpha_E = 1$. $A(t) > 0$ is productivity, which grows at rate $g_A \geq 0$. After the first part of the paper, I fix $g_A = 0$ in order to focus on the role of energy in growth.[16]

Available energy $E(t)$ depends on the time t resource base $R(t)$. Subsequent sections will specify how energy is produced and how the resource base evolves. Depending on the section, the resource base will represent land, coal mines, oil deposits, or solar panels.

I analyze the economy's optimal path, ranked according to the following welfare criterion:

$$\int_0^\infty e^{-\rho t} L(t)\, u\Big((1-s)Y(t)/L(t)\Big)\, \mathrm{d}t. \tag{1}$$

I am interested in the long-run evolution of output per capita. In particular, I am interested in the existence of a balanced growth path along which output is strictly positive and output, capital, and energy each grow at constant rates while maximizing (1) subject to resource constraints. Throughout, lower-case symbols indicate variables in per-capita form.

3 Harvested Resources: Biomass

Begin by considering a pre-industrial world in which energy derives from harvesting the products of land and sun, such as crops or trees. Because land and solar radiation flows are fixed from year to year, the aggregated resource base R is fixed (so I here drop its time argument). Energy produced from the land depends on labor $L_E(t)$ and capital $K_E(t)$ devoted to harvesting the resource:

$$E(t) = Q_H L_E(t)^{\phi_{HL}} K_E(t)^{\phi_{HK}} R^{\phi_{HR}},$$

where $\phi_{HL}, \phi_{HK}, \phi_{HR} > 0$, $\phi_{HL} + \phi_{HK} + \phi_{HR} = 1$, and $Q_H > 0$. There is only one use for energy, so $E(t) = E_Y(t)$. The labor market clears when $L(t) = L_Y(t) + L_E(t)$, and the capital market clears when $K(t) = K_Y(t) + K_E(t)$.

[16]Ignoring productivity growth also means that I do not have to take a stand on how productivity growth depends on the availability of energy (see Suzuki, 1976).

Substituting for $E(t)$ and converting to per-capita form, the final good production function becomes

$$y(t) = A(t)\ell_Y(t)^{\alpha_L} k_Y(t)^{\alpha_K} \left[Q_H \ell_E(t)^{\phi_{HL}} k_E(t)^{\phi_{HK}}\right]^{\alpha_E} r(t)^{\alpha_E \phi_{HR}},$$

where $r(t) \triangleq R/L(t)$ decreases over time. From the capital transition equation, capital and output must grow at the same rate on a balanced growth path. However, growth in capital is not sufficient for continual growth in output. First, the labor supply is constrained by population growth. Second, the land resource is fixed. The greater the value share of the land resource (i.e., the larger is $\alpha_E \phi_{HR}$), the greater the drag on growth in output per capita.

The following proposition establishes the growth rate of output per capita:

Proposition 1. *Along a balanced growth path, the growth rate of output per capita is*

$$\frac{g_A - \alpha_E \phi_{HR} g_L}{\alpha_L + \alpha_E(\phi_{HL} + \phi_{HR})}.$$

Proof. See Appendix C. □

This economy exhibits two types of long-run behavior, depending on the pace of technical change.[17] First, output per capita eventually collapses to zero if $g_A < \alpha_E \phi_{HR} g_L$. If technical change is not sufficiently fast to overcome the diminishing returns in energy production, then output per capita collapses to zero. This is *energy-scarce degrowth* because the collapse is driven by lack of energy. Importantly, this case arises when technical change is nonexistent ($g_A = 0$). Second, output per capita grows forever if $g_A > \alpha_E \phi_{HR} g_L$. If technical change is sufficiently fast, then output per capita can grow forever, but this growth occurs in spite of, not because of, the availability of the energy resource.

4 Mined Resources: Coal

Now consider an economy fueled by coal mines. There are two stages to obtaining coal from a mine. First, the mine must be opened: the hole must be bored and the shaft set up.[18] Second, coal must be extracted from the mine. The resource base $R(t)$ represents mines already opened and available for production. Opening a mine at time t uses labor $L_R(t)$, capital $K_R(t)$, and energy $E_R(t)$ inputs, and mining coal uses labor $L_E(t)$ and capital $K_E(t)$ inputs.[19] The energy market clears when $E(t) = E_Y(t) + E_R(t)$, the labor market

[17] There is also a knife-edge case with constant output per capita.

[18] See Ashton and Sykes (1929, Chapter II) for a description of this process. Underground mines still constituted most of U.S. coal production as recently as 1970 (EIA, 2024).

[19] Fouquet (2008, 222) observes, "At the pit face, the principal method of extraction was the miner's pick and shovel." Capital in the form of steam engines kept mines clear of water: see Ashton and Sykes (1929, Chapter III), among others. And see footnote 21 below regarding the fuel for those steam engines.

clears when $L(t) = L_Y(t) + L_R(t) + L_E(t)$, and the capital market clears when $K(t) = K_Y(t) + K_R(t) + K_E(t)$.

Energy production from existing mines is

$$E(t) = Q_M L_E(t)^{\phi_{ML}} K_E(t)^{\phi_{MK}} R(t)^{\phi_{MR}}, \tag{2}$$

where $\phi_{ML}, \phi_{MK}, \phi_{MR} > 0$ and $\phi_{ML} + \phi_{MK} + \phi_{MR} = 1$. The multiplier $Q_M > 0$ accounts for both the productivity of mining and the energy contained in each unit of coal.[20] Through appropriate rescaling, this production function is consistent with requiring energy to extract coal from a mine and transport it to market.[21]

This energy production function is superficially similar to the one in Section 3, but there is an important difference: rather than being fixed by the availability of land and sun, the resource base is here dynamic and endogenous. The transition equation for the resource base is:

$$\dot{R}(t) = Z(t)^{\omega} F(L_R(t), K_R(t), E_R(t))^{1-\omega} - \lambda R(t), \tag{3}$$

with $\omega \in [0, 1)$. $\lambda > 0$ is the depreciation rate of mines. The quantity of new mines opened depends on inputs of labor, capital, and energy. $Z(t)$ represents the stock of available sites. When $\omega > 0$, mines are easier to open when more mine sites are available to choose from, in the spirit of models of resource extraction following Heal (1976), whereas when $\omega = 0$, mine sites are not depletable, as when the highest quality coal resources are abundant. The stock of available sites evolves as

$$\dot{Z}(t) = -Z(t)^{\omega} F(L_R(t), K_R(t), E_R(t))^{1-\omega} + \Omega Z(t). \tag{4}$$

Opening a mine reduces the stock of mine sites. New deposits are discovered at rate $\Omega \in [0, \rho)$.[22] This discovery rate can also be interpreted as technical change that increases the

[20]When describing the expansion of the coal industry, Wrigley (2010, 46) argues that labor productivity was static over 1700–1900, and Fouquet (2008, 57) writes, "...the main reason the industry expanded was simply because it used existing practices and multiplied the number of men and seams being exploited."

[21]First, imagine that time t energy production were instead $\tilde{Q}_M L_E(t)^{\tilde{\phi}_{ML}} K_E(t)^{\tilde{\phi}_{MK}} E_E(t)^{\tilde{\phi}_{ME}} R(t)^{\tilde{\phi}_{MR}}$, with $\tilde{\phi}_{ML} + \tilde{\phi}_{MK} + \tilde{\phi}_{ME} + \tilde{\phi}_{MR} = 1$. The representative energy producer's first-order condition for $E_E(t)$ would require that $E_E(t) = [\tilde{\phi}_{ME} \tilde{Q}_M L_E(t)^{\tilde{\phi}_{ML}} K_E(t)^{\tilde{\phi}_{MK}} R(t)^{\tilde{\phi}_{MR}}]^{1/(1-\tilde{\phi}_{ME})}$. Substituting, time t energy production becomes $\left[\tilde{Q}_M (\tilde{\phi}_{ME})^{\tilde{\phi}_{ME}} L_E(t)^{\tilde{\phi}_{ML}} K_E(t)^{\tilde{\phi}_{MK}} R(t)^{\tilde{\phi}_{MR}}\right]^{1/(\tilde{\phi}_{ML}+\tilde{\phi}_{MK}+\tilde{\phi}_{MR})}$. This is equivalent to our case with ϕ_{ML}, ϕ_{MK}, ϕ_{MR}, and Q_M appropriately defined. Second, imagine that production of energy does not require energy inputs but bringing energy to market imposes a fixed cost of c units of energy. If this production function has productivity $\tilde{Q}_M$, it is equivalent to our case if we define $Q_M \triangleq \tilde{Q}_M - c$. Moreno-Cruz and Taylor (2017) consider the implications of c varying with distance.

[22]For tractability, here deposits are easier to find when they are relatively abundant, but the opposite story is also plausible if economic incentives are stronger than geologic constraints on deposit discovery. Exhaustibility is not focus of my analysis, and I leave the study of endogenous discovery in a growth model to future work.

economically recoverable resources from known deposits. I assume that $\omega(\Omega - g_L) \geq 0$, so that either resources are not depletable or the discovery rate keeps up with population growth. This assumption is a good fit for coal resources, which remain physically abundant after hundreds of years of exploitation (Rogner et al., 2012, Section 7.4).

The representative firm's production function for opening mines has the constant elasticity of substitution (CES) form:

$$F(L_R(t), K_R(t), E_R(t)) \triangleq \begin{cases} \left((1-\kappa_E)(A_{LK} L_R(t)^{\kappa_L} K_R(t)^{\kappa_K})^{\frac{\sigma-1}{\sigma}} + \kappa_E (A_E E_R(t))^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} & \text{for } \sigma \neq 1 \\ (A_{LK} L_R(t)^{\kappa_L} K_R(t)^{\kappa_K})^{1-\kappa_E} (A_E E_R(t))^{\kappa_E} & \text{for } \sigma = 1 \end{cases}, \tag{5}$$

where $\kappa_L, \kappa_K > 0$, $\kappa_L + \kappa_K = 1$, and $\kappa_E \in (0, 1)$. $\sigma > 0$ is the elasticity of substitution between energy and labor inputs, and $A_{LK}, A_E > 0$ control the productivity of inputs in mine-opening.

Murphy and Hall (2010) define energy return on energy invested (EROI) as measuring energy gained over energy required to get it. But there are many possible metrics compatible with that definition—and the energy analysis literature has indeed used many types of metrics. My metric is in the spirit of the "net external energy ratio" (NEER), which is the ratio of the net energy produced in a system to the energy inputs from external sources (Brandt and Dale, 2011). I treat the system as the economy at a given instant t. The external input is the energy devoted to mine opening (i.e., diverted from uses that assist in contemporaneous final good production) in the previous instant. Formally,

$$EROI(t) \triangleq \frac{E_R(t) + \dot{E}(t)}{E_R(t)}. \tag{6}$$

When $EROI(t) \geq 1$, the energy committed to opening mines is technically sustainable for some finite interval of time, without reducing energy available for final good production.

The following proposition establishes a lower bound on EROI:

Proposition 2 (EROI Lower Bound)**.** *If* $Y(t) > 0$, *then*

$$EROI(t) > 1 + g_L + \frac{\dot{e}(t)}{e(t)}.$$

Proof. From definition (6) and the identity $\dot{E}(t)/L(t) = \dot{e}(t) + g_L e(t)$,

$$EROI(t) = 1 + \left(\frac{\dot{e}(t)}{e(t)} + g_L\right) \frac{e(t)}{e_R(t)}. \tag{7}$$

From $\alpha_E > 0$ and $e_Y(t) + e_R(t) = e(t)$, final good production $Y(t)$ is strictly positive only if $e(t) > e_R(t)$. □

Economy-wide EROI must be large enough to generate surplus energy that can be dedicated to final good production, not just reinvested in opening mines. If energy grows at the rate of population along a balanced growth path (i.e., if $\dot{e}(t) = 0$), then EROI must exceed $1 + g_L$ so that maintaining that rate of growth does not require substituting energy from final-good production. If energy is growing faster than population along a balanced growth path (i.e., if $\dot{e}(t) > 0$), then EROI must be even larger so as to generate the additional energy that can maintain investment in opening and extracting from ever more mines.

From here on, I eliminate exogenous productivity growth so as to highlight whether output growth is possible in the absence of technical change:

Assumption 1. $g_A = 0$.

The following propositions describe balanced growth paths. Begin with the Cobb-Douglas case:

Proposition 3 (Cobb-Douglas Mine-Opening)**.** *Fix $\sigma = 1$ and let Assumption 1 hold. An interior balanced growth path has all real variables growing at the same rate as population and the shadow prices of unmined deposits and operating mines growing at rate $\rho - g_L$. Such a path exists. Along that path,*

$$EROI(t) = 1 + \frac{1}{\phi_{MR}\kappa_E} \frac{\lambda + \rho}{\lambda + g_L} \frac{(1-\omega)(\rho - \Omega) + \omega(\rho - g_L)}{(1-\omega)(\rho - \Omega)} g_L.$$

The elasticity of output per capita on the balanced growth with respect to Q_M is strictly positive and constant in Q_M.

Proof. See Appendix E. □

Proposition 3 describes a case of *energy-enabled growth*. Whereas Proposition 1 showed that an economy with a harvested resource avoids a collapse in output per capita only if technical progress is fast enough, Proposition 3 shows that an economy with a mined resource can have output grow as fast as population even in the absence of any technical progress, as long as either resources are not depletable or deposit discoveries keep up with population growth. Because resource production has constant returns to scale, resources per capita are constant as long as inputs to mine opening increase at the same rate as population. And if the shares of labor and capital allocated to energy production are constant, then energy production will increase at the same rate as population. The foregoing implies constant per-capita inputs to final good production, so final good production grows at the rate of population growth. In this world, energy resources act like a second capital stock that must be produced from labor and capital inputs.

Although energy enables output growth to match population growth, it cannot drive growth. Technical progress is still key for growth. Energy per capita can grow no faster than capital per capita. Through savings, capital per capita grows as fast as output per capita.

However, labor constraints mean that output per capita cannot grow as fast as the growth rate of its inputs. As a result, output per capita must not be growing, and neither must capital per capita nor energy per capita.[23]

In equilibrium, inputs adjust to ensure that EROI remains above $1 + g_L$, as required by Proposition 2. The precise EROI observed depends on economic forces. One might expect that increasing the productivity of energy in energy production (i.e., increasing A_E or Q_M) would increase EROI, but increasing either term also leads the economy to substitute towards energy inputs to mine opening: increasing A_E makes energy more productive relative to other inputs to mine opening, and increasing Q_M makes energy more abundant relative to other inputs to mine opening. This substitution works to reduce EROI through the declining returns to scale of increasing only one input in a CES production function.

In a Cobb-Douglas case ($\sigma = 1$), the two effects exactly offset. Based on engineering considerations, we may have intuitively expected EROI to depend on A_E and Q_M, but EROI is in fact independent of these technical parameters. EROI increases in ρ because less patient agents require a greater return on their investments in mine opening, and it decreases in ϕ_{MR} and κ_E because agents will invest more of their energy supplies in mine opening when the marginal energetic return of mines is larger (ϕ_{MR} is large) or energy inputs carry a larger weight in mine opening (κ_E is large).

The next proposition describes economic growth and EROI under more general production functions for mine opening, in the special case that capital and labor inputs have the same relative weights in mine opening, energy production, and final good production:

Assumption 2. $\kappa_K/\kappa_L = \phi_{MK}/\phi_{ML} = \alpha_K/\alpha_L$.

Proposition 4 (CES Mine Opening)**.** *Let Assumptions 1 and 2 hold. Any interior balanced growth path has all real variables growing at the same rate as population and the shadow prices of unmined deposits and operating mines growing at $\rho - g_L$. Such a path exists. Along that path,*

i $EROI(t)$ is constant and is strictly greater than $1 + \frac{1}{\phi_{MR}} \frac{\lambda+\rho}{\lambda+g_L} \frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)} g_L$.

ii $EROI(t)$ increases in A_E and Q_M if $\sigma < 1$.

iii $EROI(t)$ decreases in A_E and Q_M if $\sigma > 1$.

Proof. See Appendix F. □

We again have energy-enabled growth. As before, EROI must be large enough to compensate for population growth (and indeed is along a balanced growth path), but now EROI does depend on the technical parameters A_E and Q_M. When $\sigma < 1$, substitution towards energy

[23] Formally, we have $g_e \leq g_k$, $g_k = g_y$, and $g_y = \alpha_K g_k + \alpha_E g_e$, which implies $g_k \leq (1 - \alpha_L) g_k$. The only solution has $g_k = 0$, which implies $g_e, g_y = 0$.

inputs in mine opening is weak, leading the direct effects of A_E and Q_M on productivity to dominate the EROI calculation. EROI thus increases in A_E and Q_M, as engineering intuition would suggest. But when $\sigma > 1$, substitution towards energy inputs in mine opening is strong, so that the effects of diminishing returns to energy investment dominate the direct effects of A_E and Q_M. Contrary to engineering intuition, EROI decreases in A_E and Q_M.

Among much else, the Industrial Revolution is known for a shift towards coal use. Many have debated whether this shift towards coal use spurred the takeoff in economic growth (see footnote 1 above). We have obtained a more nuanced result. In the present setting, discovering the ability to use a coal resource saves the economy from the possibility that population growth erodes standards of living over time and does so irrespective of the productivity of that coal resource. Sustained growth in standards of living still requires technical change, but now any degree of technical change suffices, however small.

5 Tapped Resources: Oil

Now consider an economy with oil resources. These require inputs when they are first tapped but do not require ongoing inputs to produce once they have been tapped: an oil well must be drilled and connected, but pressure within the well then forces oil to flow without requiring labor to haul it out of the well.[24]

The resource base $R(t)$ now represents deposits already tapped and producing, evolving as in (3). Tapping a resource deposit at time t uses labor inputs $L_R(t)$, capital inputs $K_R(t)$, and energy inputs $E_R(t)$, following (5). The energy market clears when $E(t) = E_Y(t) + E_R(t)$, the labor market clears when $L(t) = L_Y(t) + L_R(t)$, and the capital market clears when $K(t) = K_Y(t) + K_R(t)$. I again apply Assumption 1 (fixing $g_A = 0$) in order to highlight the potential for growth in the absence of technical change.

Because oil resources are depletable, I fix $\omega > 0$. In keeping with Section 4, I also fix $\Omega > g_L$.[25] Each tapped deposit produces $Q_D > 0$ units of energy:

$$E(t) = Q_D R(t). \tag{8}$$

Through appropriate rescaling, this production function is consistent with requiring energy to transport oil to market (see footnote 21 above). The economy's EROI is as in (6), and

[24] Anderson et al. (2018, 987) write, "...oil extraction is more akin to a "keg-tapping" problem than a cake-eating problem: extractors choose when to drill their wells (or tap their kegs), but the flow from these wells is (like the libation from a keg) constrained because of pressure and decays toward zero as more oil is extracted." Anderson et al. (2018) assume zero marginal costs of extracting from a tapped well, up to the physically determined constraint.

[25] Oil resources should be fixed over some sufficiently long timescale, but even now the oil resource is rather large (Rogner et al., 2012, Section 7.2) and reserves continue to grow (Sorrell et al., 2012). The present model of continued discoveries is an adequate approximation over timescales of decades or more.

Proposition 2 still holds.

This production function does not require that tapped deposits maintain constant productivity over time. Darcy's Law implies that the flow of oil from a tapped deposit decays over time, which is captured here by $\lambda > 0$ in (3).[26] Additional capital or labor inputs may be used to offset the decline in productivity by tapping new deposits. Importantly, though, energy production from already-tapped deposits does not scale with any such inputs.

The following proposition describes outcomes with this energy resource:

Proposition 5. *Let Assumption 1 hold and define*

$$\chi \triangleq (\lambda + g_L)\left(\frac{\lambda+\rho}{\kappa_E(\lambda+g_L)}\right)^{\frac{\sigma}{\sigma-1}}\left(\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}\right)^{\frac{\sigma}{\sigma-1}}(\Omega-g_L)^{\frac{\omega}{1-\omega}}.$$

Any balanced growth path has all real variables growing at the rate of population and the shadow prices of untapped deposits and operating wells growing at rate $\rho - g_L$.

i If $\sigma = 1$, such a balanced growth path exists and the elasticity of output per capita on the balanced growth with respect to Q_D is strictly positive and constant in Q_D.

ii If $\sigma < 1$, such a balanced growth path exists if and only if $A_E Q_D > \chi$.

iii If $\sigma > 1$, such a balanced growth path exists if and only if $A_E Q_D < \chi$.

iv Along such a balanced growth path,

$$EROI(t) = 1 + \left(\frac{Q_D A_E}{\chi}\right)^{1-\sigma} g_L.$$

Proof. See Appendix G. □

The proposition describes several possibilities, which depend on σ and $A_E Q_D$. Using its results, Figure 1 divides $(\sigma, A_E Q_D)$ space into three regions, with χ from the proposition forming the borders between them.

The middle region (in white) is a case of *energy-enabled growth*, in which the energy resource enables output to keep up with population but cannot drive output to grow faster than population. When $\sigma = 1$, this region encompasses all permissible values of $A_E Q_D$, as it did for the mined resource.[27] However, whereas the middle region would have encompassed the entire plot for the mined resource, we here see that the middle region requires that $A_E Q_D$ not be too small when $\sigma < 1$ and not be too large when $\sigma > 1$. From part iv of the proposition, this middle region ensures that EROI is greater than $1 + g_L$. As we approach

[26] The exponential decline is consistent with prior work (Mason and van't Veld, 2013; Anderson et al., 2018; Kellogg, 2024).

[27] And taking the limit as σ goes to 1 in part iv of Proposition 5 yields the EROI from Proposition 3.

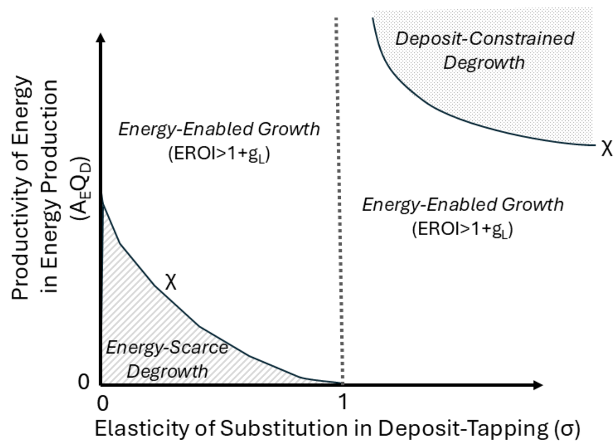


Figure 1: Schematic of long-run outcomes for tapped resources.

either boundary from the middle region, EROI approaches $1 + g_L$ and, from the proof of Proposition 5, the share of energy devoted to deposit-tapping approaches 1.

It is intuitively plausible that labor, capital, and energy are complementary inputs to tapping oil wells, so that $\sigma < 1$.[28] In that case, we obtain a different outcome when $A_EQ_D < \chi$ (lower-left shaded region in Figure 1): the economy cannot maintain constant output per capita because energy resources are not sufficiently productive in making energy. If a balanced growth path cannot have EROI greater than $1 + g_L$, then the inability to generate enough energy for deposit-tapping eventually binds the entire economy. With $\sigma < 1$, energy is an essential input, and with A_EQ_D small, energy is scarce. The proof of Proposition 5 shows that holding output per capita constant would require more energy to be allocated to deposit-tapping than is produced in the entire economy. Because such an allocation is infeasible, the economy instead experiences *energy-scarce degrowth*, as seen with the harvested resource. This is the type of outcome that analysts have in mind when they express concerns about a "net energy cliff" in EROI (e.g., Murphy and Hall, 2010). We here see that a net energy cliff requires that the resource be a tapped deposit and not a mined resource (because labor and capital can be used to produce more energy from already-opened mines) and that energy be an essential input in deposit-tapping (because otherwise the economy could substitute capital and labor for energy).[29]

[28]Indeed, empirical work suggests that the elasticity of substitution between energy and value-added is strictly less than 1 in the energy supply sector: Koesler and Schymura (2015) estimate the elasticity of substitution between energy and value-added in resource mining as a (noisy) 0.42.

[29]Some premodern societies managed quite sophisticated drilling operations without modern energy inputs (see Kuhn, 2004). This suggests that energy may not be truly essential to deposit-tapping and that society would return to energy-enabled growth before collapse was complete. However, this premodern production had relatively low productivity, so that energy-enabled growth would sustain a relatively low level of consumption.

If $\sigma > 1$, we again obtain a different outcome, but now only if energy resources are *too* productive. EROI declines towards $1 + g_L$ as $A_E Q_D$ increases to χ. As described following Proposition 4, making energy resources more productive induces substitution towards energy inputs in deposit-tapping that works to reduce EROI. As $A_E Q_D$ becomes large (upper-right shaded region in Figure 1), demand for energy as an input to deposit-tapping eventually outstrips the economy's ability to generate that energy from scarce deposits.[30] Appendix A shows that, in this region, a balanced growth path exists in which energy crowds out all other inputs to deposit-tapping and output per capita shrinks at a constant rate.[31] This is a case of *deposit-constrained degrowth*, in which the high productivity of energy resources makes energy abundant but that very abundance means that the pace of deposit-tapping proceeds faster than deposits are found.

Formally, the cutoff χ on $A_E Q_D$ arises for $\sigma \neq 1$ due to an inability to maintain incentives for deposit-tapping without violating the economy's energy constraint. A balanced growth path must have deposits being tapped. For that case to be optimal, the marginal value of a tapped resource needs to exceed the marginal value of a deposit in the ground. In equilibrium, the premium for tapped vs untapped deposits depends on the marginal productivity of energy inputs to deposit-tapping. For $\sigma < 1$, the equilibrium marginal productivity of energy inputs to deposit-tapping is large when $A_E Q_D$ is small, and for $\sigma > 1$, the equilibrium marginal productivity of energy inputs to deposit-tapping is large when $A_E Q_D$ is large. On the other hand, the equilibrium marginal productivity of energy inputs to tapping tends to be small when a greater share of energy is used in deposit-tapping, via the logic of diminishing returns. To maintain a given premium for tapped vs untapped deposits, the share of energy used in deposit-tapping must move opposite to $A_E Q_D$ when $\sigma < 1$ and must move with $A_E Q_D$ when $\sigma > 1$. But the share of energy used in deposit-tapping is bounded above by 1. Therefore $A_E Q_D$ can only become so small if a balanced growth path is to exist for $\sigma < 1$ and $A_E Q_D$ can only become so large if a balanced growth path is to exist for $\sigma > 1$.[32]

Figure 2 simulates outcomes for $\sigma = 0.5$ and $\sigma = 2$ over 300 years, using a model discretized with an annual timestep and calibrated to prior literature (details in Appendix B).[33]

[30] In the case of the mined resource, labor and capital inputs constrain energy production from already-opened mines and limit demand for energy as an input to mine-opening.

[31] In particular, such a path exists when ρ is close to Ω, Ω is not too close to g_L, and $A_E Q_D > X$, where $X \leq \chi$, χ/X is strictly decreasing in σ, and $\lim_{\sigma\to\infty} X = \lim_{\sigma\to\infty} \chi$. At any given finite σ, there is a region of $A_E Q_D$ just below χ in which multiple balanced growth paths coexist when ρ is close to Ω and Ω is not too close to g_L. One path has energy-enabled growth, and the other path has deposit-constrained degrowth.

[32] From equation (A-31) and $e_{ss} = Q_D r_{ss}$ (with subscript ss indicating evaluation at a steady state),

$$\frac{e_{Rss}}{e_{ss}} = \left(\frac{\phi_{MR}\kappa_E}{\lambda + \rho} \frac{(1-\omega)(\rho - \Omega)}{(1-\omega)(\rho-\Omega) + \omega(\rho - g_L)} \right)^{\sigma} (\lambda + g_L)(\Omega - g_L)^{\frac{\omega}{1-\omega}(1-\sigma)} [A_E Q_D]^{\sigma - 1}.$$

The bounds on $A_E Q_D$ derive from recognizing that $e_{Rss}/e_{ss} \leq 1$. There were no such bounds in the case of the mined resource because, from equation (2), the endogenous mining inputs imply $e_{ss} \neq Q_M r_{ss}$.

[33] I solve the economy over 400 years but cut the plots off at 300 years in order to minimize effects of the

The top left panel plots output per capita, the top right panel plots tapped deposits per capita (which are proportional to energy per capita), the lower left panel plots untapped deposits per capita, and the lower right panel plots EROI. The solid lines show cases of energy-enabled growth, calibrated to the current EROI of tapped resources. The black dashed line reduces $A_E Q_D$ to half of χ with $\sigma = 0.5$, putting the economy in a regime of energy-scarce degrowth. The gray dashed line increases $A_E Q_D$ to twice χ with $\sigma = 2$, putting the economy in a regime of deposit-constrained degrowth.

In line with the theory, output per capita approaches a constant level for both cases of energy-enabled growth. When $\sigma = 0.5$, the economy does so by maintaining enough resources per capita to generate the energy needed to tap new deposits and by maintaining enough deposits to tap in the future. EROI becomes large because the need to preserve deposits keeps the marginal returns to energy inputs in deposit-tapping high. In contrast, when $\sigma = 2$, the economy substitutes away from the tapped resource as deposits become scarce. Both tapped and untapped deposits approach zero. EROI is negative while the deposits are being run down but eventually stabilizes well above $1 + g_L = 1.01$ (and well below its value when $\sigma = 0.5$).

In the other two cases, output per capita declines at a constant rate over much of the interval. In the case of energy-scarce degrowth, that decline rate is 0.03% per year. The productivity of deposits is too low to sustain a significant rate of deposit-tapping: the stock of tapped deposits initially declines rapidly over an interval with negative EROI before declining more slowly over an interval with EROI above $1 + g_L$, albeit with EROI small and declining. Ironically, this case of energy-scarce degrowth has the most deposits per capita, because deposit-tapping is constrained by the availability of energy. And this case arises even with EROI above the net energy cliff of $1 + g_L$.

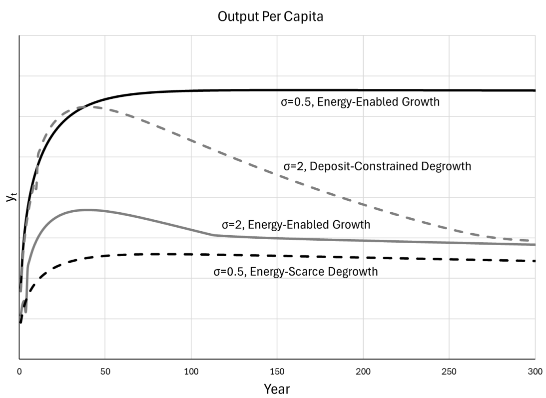
In the case of deposit-constrained degrowth, output per capita is initially high but soon declines by around 0.3% per year. Untapped deposits quickly dwindle to essentially zero, at which point EROI becomes negative and the stock of tapped deposits begins dwindling. The dwindling energy supplies eventually cause output per capita to start declining.

6 Manufactured Resources: Solar Photovoltaics

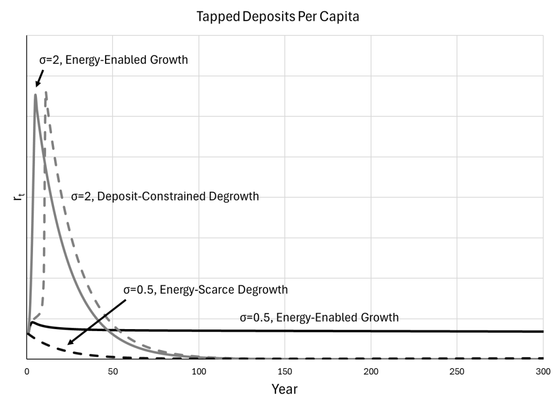
A solar resource is similar to oil resources in that, once installed, solar photovoltaic cells produce energy for a period of time without ongoing inputs of labor or capital. However, a solar resource is different in that photovoltaic cells are manufactured, not found. The transition equation for the resource base becomes:

$$\dot{R}(t) = F(L_R(t), K_R(t), E_R(t)) - \lambda R(t). \tag{9}$$

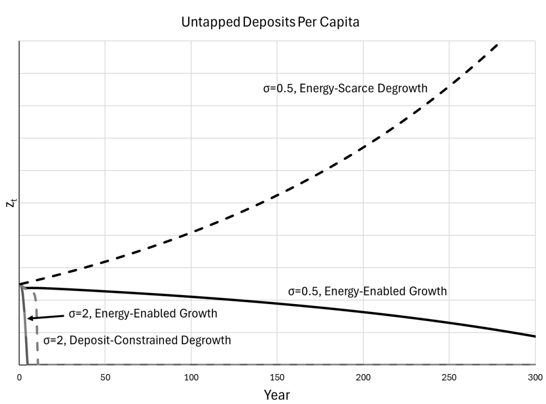
terminal horizon.



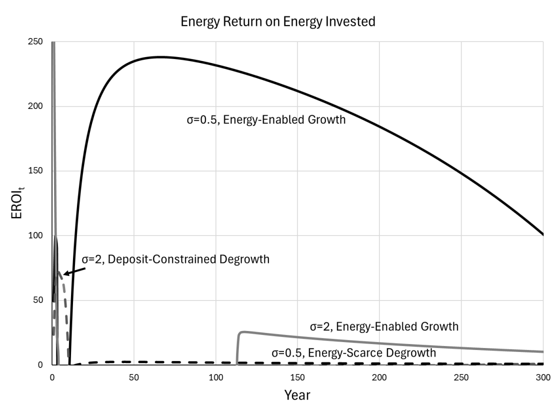
(a) y_t



(b) r_t



(c) z_t



(d) $EROI_t$

Figure 2: Simulated outcomes for a tapped resource.

Energy production is analogous to (8):

$$E(t) = Q_S R(t), \tag{10}$$

with $Q_S > 0$. Equation (9) is the same as equation (3) with $\omega = 0$ (i.e., in the special case of abundant resources). Solar panels do require surface area, whether on Earth or in space, so the solar resource might in principle become scarce, but the stock of sites with high-quality solar fluxes is likely so large that resource scarcity may be a second-order concern for a long time (Jacobson and Delucchi, 2011; Rogner et al., 2012).

The following corollary describes the potential for constant output per capita in this economy:

Corollary 6. *Let Assumption 1 hold and define*

$$\chi \triangleq (\lambda + g_L) \left(\frac{\lambda + \rho}{\kappa_E (\lambda + g_L)} \right)^{\frac{\sigma}{\sigma - 1}}.$$

Any interior balanced growth path has all real variables growing at the rate of population and the shadow price of solar panels growing at rate $\rho - g_L$.

i If $\sigma = 1$, such a balanced growth path exists. The elasticity of output per capita on the balanced growth with respect to Q_S is strictly positive and constant in Q_S.

ii If $\sigma < 1$, such a balanced growth path exists if and only if $A_E Q_S > \chi$.

iii If $\sigma > 1$, such a balanced growth path exists if and only if $A_E Q_S < \chi$.

iv Along such a balanced growth path,

$$EROI(t) = 1 + \left(\frac{A_E Q_S}{\chi} \right)^{1-\sigma} g_L.$$

Proof. Follows proof of Proposition 5, with $\omega = 0$. □

Figure 3 depicts the set of possible outcomes described by the corollary. These results are largely familiar from the case with a tapped resource. We again have a balanced growth path with constant output per capita in the Cobb-Douglas case. If inputs to solar panel manufacturing are instead complementary and solar panels are not sufficiently productive, then we again risk energy-scarce degrowth. Many analysts have expressed concern about the lower EROI of solar photovoltaics compared to fossil energy resources (e.g., Trainer, 2018; Capellán-Pérez et al., 2019). We here see that while a decline in the EROI achievable along a balanced growth path may affect the level of consumption per capita, it does not affect the long-run growth rate of consumption as long as EROI remains above $1 + g_L$. However,

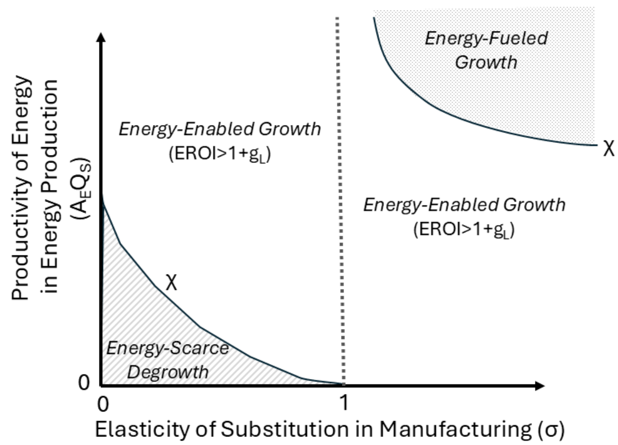


Figure 3: Schematic of long-run outcomes for manufactured resources.

if the productivity of solar panels is low enough and inputs to solar panel manufacturing are complementary, then the economy does not generate enough additional energy to have energy and output both keep pace with population growth in the absence of technical change.

A new possibility arises if energy inputs are substitutable in solar panel manufacturing and $A_E Q_S$ is sufficiently large, as in the upper-right shaded region of Figure 3:

Assumption 3 (Log Utility)**.**

$$u(Y_t/L_t) = \ln(Y_t/L_t).$$

Proposition 7 (Energy-Fueled Growth with Manufactured Resources)**.** *Let Assumptions 1 and 3 hold and fix $\sigma > 1$. Define g_e as the growth rate of energy per capita, g_y as the growth rate of output per capita, g_k as the growth rate of capital per capita, and χ as in Corollary 6. If $A_E Q_S > \chi$, then there exists a balanced growth path with $L_R(t), K_R(t) = 0$ and with $R(t)/L(t)$, $E_R(t)/L(t)$, and $E_Y(t)/L(t)$ all growing at rate $g_e > 0$. Along such a path:*

i $g_y = g_k = \frac{\alpha_E}{\alpha_E + \alpha_L} g_e$.

ii $g_e = A_E Q_S \kappa_E^{\frac{\sigma}{\sigma-1}} - (\rho + \lambda) > 0$.

iii $EROEI(t) = 1 + \frac{A_E Q_S \kappa_E^{\frac{\sigma}{\sigma-1}}}{A_E Q_S \kappa_E^{\frac{\sigma}{\sigma-1}} - (\rho - g_L)} (g_e + g_L)$

Proof. See Appendix H. □

We have *energy-fueled growth*: the economy features strictly positive long-run growth in output per capita, despite the absence of technical progress. When energy crowds out other resources in solar panel manufacturing, the production function for solar panels becomes, from (5),

$$F(L_R(t), K_R(t), E_R(t)) = A_E \kappa_E^{\frac{\sigma}{\sigma-1}} E_R(t).$$

Differentiating (10) with respect to time and using the previous equation and (9), the instantaneous change in energy production at time t is:

$$\dot{E}(t) = A_E Q_S \kappa_E^{\frac{\sigma}{\sigma-1}} E_R(t) - Q_S \lambda R(t).$$

If $E_R(t)$ is a constant share s_E of $E(t)$, this becomes:

$$\dot{E}(t) = \left[A_E Q_S \kappa_E^{\frac{\sigma}{\sigma-1}} s_E - \lambda\right] E(t).$$

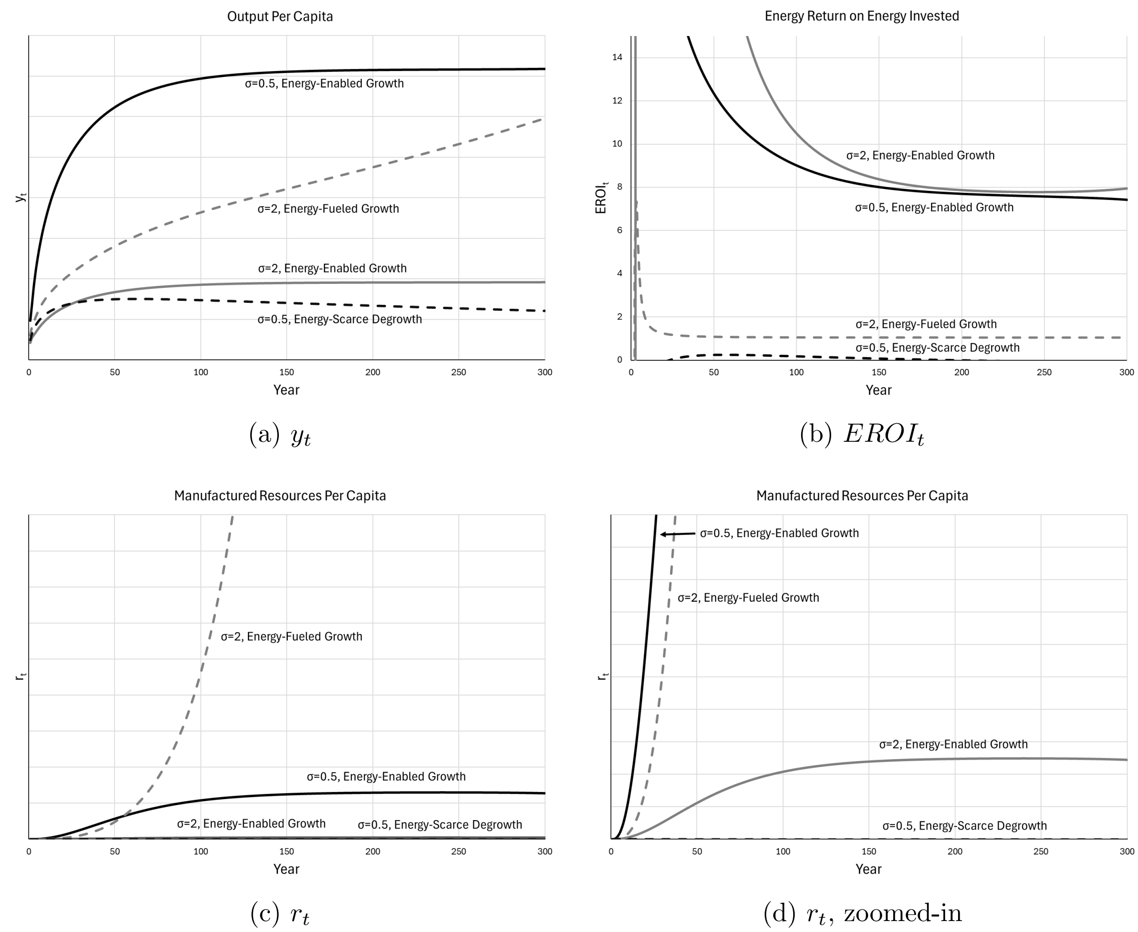
We have a linear accumulation equation. The economy can grow by reinvesting energy in making energy as long as the return on that investment is sufficiently large. The growth rate of output per capita is proportional to the growth rate of energy per capita (part i). That growth rate of energy per capita increases in the productivity Q_S of solar panels and the productivity A_E of energy in manufacturing solar panels (part ii). Economy-wide EROI increases in both productivity parameters and, consistent with Proposition 2, is greater than $1 + g_L + g_e$ (part iii).

Figure 4 simulates outcomes for $\sigma = 0.5$ and $\sigma = 2$ (details again in Appendix B). The top left panel plots output per capita over time, the top right panel plots EROI over time, and the lower panels plot manufactured resources per capita over time, with the lower right panel a zoomed-in version of the lower-left panel. The solid lines again show cases of energy-enabled growth, calibrated to the current EROI of solar photovoltaic resources. The black dashed line again reduces $A_E Q_D$ to half of χ with $\sigma = 0.5$, putting the economy in a regime of energy-scarce degrowth. And the gray dashed line again increases $A_E Q_D$ to twice χ with $\sigma = 2$, now putting the economy in a regime of energy-fueled growth.

Output per capita again approaches a constant level for both cases of energy-enabled growth, in line with the theory. As before, the economy maintains more resources per capita when energy is an essential input to resource production. EROI steadily declines but remains well above the cutoff $1 + g_L = 1.01$.

In the case of energy-scarce degrowth, output and resources per capita steadily decline and resources per capita become very scarce. The small $A_E Q_D$ means that EROI is actually negative at first. EROI increases to around 0.25 before declining, so that it is well below 1 even at its peak. In this case, the small EROI reflects the low productivity of energy resources.

In the case of energy-fueled growth, output per capita steadily increases, with growth of 0.2–0.3% per year. Resources per capita explode, as the high productivity of the energy resource and the substitutability of energy for other inputs to resource production create a



(a) y_t (b) $EROI_t$

(c) r_t (d) r_t, zoomed-in

Figure 4: Simulated outcomes for a manufactured resource.

self-sustaining loop of creating energy to make energy. EROI is quite high in early periods and declines to around 1.05 as energy becomes abundant. In this case, EROI declining towards a level only slightly above $1 + g_L$ reflects ongoing growth in energy and output per capita rather than a scarcity of energy. The distance of EROI from the net energy cliff represented by $1+g_L$ is not on its own a good guide to whether energy is fueling the economy or will continue fueling the economy.

7 Self-Replicating Resources: Roboticized Solar

I now assess the potential for interactions between solar panels and robotics to generate a qualitatively different set of outcomes. Let energy production follow (10). The resource base of solar photovoltaics is now manufactured from energy $E_R(t)$ and robots $B_R(t)$:

$$\dot{R}(t) = A_E B_R(t)^{\kappa_B} E_R(t)^{\kappa_E} - \lambda R(t),$$

with $\kappa_E, \kappa_B > 0$ and $\kappa_E + \kappa_B = 1$. Robots themselves are produced by combining energy $E_B(t)$ with robots $B_B(t)$. The total stock $B(t)$ of robots evolves as:

$$\dot{B}(t) = A_B B_B(t)^{\beta_B} E_B(t)^{\beta_E} - \Psi B(t),$$

with $A_B > 0$, $\beta_E, \beta_B > 0$, and $\beta_E + \beta_B = 1$. Robots depreciate at rate $\Psi > 0$. The robot market clears when $B_R(t) + B_B(t) = B(t)$, and the energy market clears when $E_Y(t) + E_R(t) + E_B(t) = E(t)$.

The following proposition describes outcomes in this economy:

Proposition 8 (Self-Replicating Resources)**.** *Let Assumptions 1 and 3 hold. Define g_e as the growth rate of energy per capita, g_y as the growth rate of output per capita, and g_k as the growth rate of capital per capita. There exists a balanced growth path with $R(t)/L(t)$, $E_R(t)/L(t)$, $E_Y(t)/L(t)$, and $B(t)/L(t)$ all growing at rate $g_e > 0$ if and only if*

$$A_E Q_S > \chi_0, \text{ where } \chi_0 \triangleq \frac{\rho + \lambda}{\kappa_E} \left(\frac{\kappa_E \beta_B}{\kappa_B \beta_E} \left(\frac{\rho + \Psi}{\beta_B A_B} \right)^{\frac{1}{\beta_E}} \right)^{\kappa_B}.$$

Along such a path:

i $g_y = g_k = \frac{\alpha_E}{\alpha_E + \alpha_L} g_e$.

ii g_e *increases in* Q_S, A_E, *and* A_B *and decreases in* ρ, λ, *and* Ψ.

iii There exists $\chi_1 > \chi_0$ *such that the prices of solar panels and robots decrease over time if and only if* $A_E Q_S > \chi_1$.

iv $EROI(t)$ *increases in* A_E, Q_S, *and* A_B.

v $EROI(t) > 1 + \frac{1}{\kappa_E} \frac{\rho + \lambda}{g_L + \lambda} g_L$.

Proof. See Appendix I. □

We have *energy-fueled growth* when solar photovoltaic panels are sufficiently productive: the economy features strictly positive long-run growth in output per capita, despite the absence of technical progress. Moreover, in contrast to Section 6, this outcome arises even when the elasticities of substitution in solar panel production and robot production are each unity (i.e., even with Cobb-Douglas production functions).

Proposition 8 shows that the growth rate of output per capita is proportional to the growth rate of energy per capita (part i). The growth rates of energy per capita and output per capita increase in the productivity of the energy resource and in the productivity of robot production and decrease in each depreciation rate and in the discount rate (part ii). The prices of solar panels and robots increase over time if solar panels are not too productive but decrease over time if solar panels are sufficiently productive (part iii). EROI increases in the productivities of the energy resource and robot production (part iv), and strictly positive growth in energy per capita and output per capita requires that EROI exceed $1 + g_L$ by a sufficiently large margin (part v). Solar photovoltaics must generate enough surplus energy to outpace population growth.[34]

To see why long-run growth is possible in the absence of technical progress, consider the dynamic equations governing the growth rates of the solar resource and of robots. The growth rate of the solar photovoltaic resource base is:

$$\frac{\dot{R}(t)}{R(t)} = A_E Q_S \left[\frac{B_R(t)}{E_R(t)}\right]^{\kappa_B} \frac{E_R(t)}{E(t)} - \lambda. \tag{11}$$

If robots could not keep up with the growth rate of energy, then the term in brackets would go to zero over time and the growth rate of energy must become negative. If the shares s_R^B and s_R^E of robots and energy allocated to solar panel production are constant, then equation (11) becomes:

$$\frac{\dot{R}(t)}{R(t)} = A_E Q_S \left[\frac{s_R^B}{s_R^E} \frac{E(t)}{B(t)}\right]^{\kappa_B} s_R^E - \lambda. \tag{12}$$

$\dot{R}(t)$ is linear in $R(t)$ when energy and robots grow at the same rate.

The growth rate of robots is:

$$\frac{\dot{B}(t)}{B(t)} = A_B \left[\frac{E_B(t)}{B_B(t)}\right]^{\beta_E} \frac{B_B(t)}{B(t)} - \Psi. \tag{13}$$

If energy could not keep up with the growth rate of robots, then the term in brackets would go to zero over time and the growth rate of robots must become negative. Combining this result with the result from (11) that robots must grow at least as fast as energy when growth is positive, solar photovoltaics and robots must grow at the same rate on a balanced growth path with positive growth. If the shares s_B^B and s_B^E of robots and energy allocated to robot production are constant, then equation (13) becomes:

$$\frac{\dot{B}(t)}{B(t)} = A_B \left[\frac{s_B^E}{s_B^B} \frac{B(t)}{E(t)}\right]^{\beta_E} s_B^B - \Psi.$$

[34] Defining the robot return on robot investment (RROI) by analogy to EROI in (6), we have, from (A-65), that $g_e > 0$ only if $RROI(t) > 1 + \frac{1}{\beta_B} \frac{\rho+\Psi}{g_L+\Psi} g_L$.

$\dot{B}(t)$ is linear in $B(t)$ when energy and robots grow at the same rate. So, combining with (12), when energy and robots grow at the same rate and maintain constant shares, we have an equilibrium with linear accumulation equations for solar panels and robots and therefore featuring the potential for strictly positive growth.

Combining equations (11) and (13) (by way of equation (A-58)), we find:

$$\frac{\dot{R}(t)}{R(t)} = A_E Q_S \left(\frac{A_B \frac{B_B(t)}{B(t)}}{\frac{\dot{B}(t)}{B(t)} + \Psi} \right)^{\kappa_B/\beta_E} \left(\frac{\beta_E \kappa_B}{\beta_B \kappa_E} \right)^{\kappa_B} \frac{E_R(t)}{E(t)} - \lambda.$$

The growth rate of the solar photovoltaic resource base declines in the growth rate of robots, because an increasing share of energy must be devoted toward robot production in order to sustain its high growth rate. The common growth rate g_e solves:

$$g_e = A_E Q_S \left(\frac{A_B \frac{B_B(t)}{B(t)}}{g_e + \Psi} \right)^{\kappa_B/\beta_E} \left(\frac{\beta_E \kappa_B}{\beta_B \kappa_E} \right)^{\kappa_B} \frac{E_R(t)}{E(t)} - \lambda.$$

If the solar resource is sufficiently productive, then the right-hand side is greater than zero when evaluated at the optimized factor allocation, in which case the unique g_e that solves the equation at that allocation is strictly positive. In line with this analysis, Proposition 8 shows that the growth rate of energy per capita is strictly positive if and only if the solar resource is sufficiently productive, where the threshold is lower when robot production is more productive and robots depreciate more slowly.

In this economy, the solar resource becomes self-replicating. Its energy creates the robots that combine with solar energy to produce more solar photovoltaics. Those additional solar panels generate additional energy that, once combined with robots, helps produce more robots, more photovoltaics, and more output. If the solar resource is sufficiently productive, then its energy can generate economic growth. This story requires highly automated processes for fabricating solar photovoltaics, but once these exist, it does not require any ongoing technical change in any part of the economy. It also does not assume that robots directly affect final good production. Energy production here fuels its own growth and thereby fuels the growth of output per capita.

8 Implications for Real-World Energy Transitions

At the turn of the nineteenth century, the global economy was driven by biomass, a harvested resource. Within the model of Section 3, technical change was necessary just to hold output per capita constant. Over the nineteenth century, the global economy came to be dominated by coal, a mined resource (Smil, 2010, Chapter 2). Within the model of Section 4, this transition generated a qualitative shift in behavior: energy-enabled growth became possible, with any degree of technical change now sufficient to induce growth in output per capita.

Over the course of the twentieth century, the global economy came to be driven by oil and gas (Smil, 2010, Chapter 2). Rather than an economy that produced energy from coal with substantial ongoing labor and capital inputs, the economy became one that required inputs upfront, at the point of tapping an oil or gas well. These inputs were plausibly essential to the tapping process. Within the model of Section 5, the high EROI of oil and gas kept the economy within a regime of energy-enabled growth. However, the EROI of oil and gas deposits has declined over time—and may decline further—as oil and gas deposits have become increasingly scarce (see footnote 7 above). And a further reduction in EROI could occur as power plants are paired with technology to capture carbon emissions, whether onsite or through direct air capture. Within the model of Section 5, the oil and gas economy risks entering a regime of energy-scarce degrowth if EROI declines a lot.

Solar energy may someday dominate the economy. There is currently no shortage of locations with high-quality solar resources. Once solar panels are produced, they generate energy with little to no ongoing inputs, much as oil and gas resources do. But producing solar panels does require energy, and some literature suggests that solar panels have a smaller EROI than do contemporary oil and gas resources (see footnote 8 above). Two qualitative shifts are possible for growth behavior in a solar economy.

First, if energy inputs are complementary to labor and capital inputs in the production of solar panels, then the reduction in EROI could shift the economy from a regime in which any technical change generates growth to a regime in which technical change would be necessary just to overcome the drag from energy-scarce degrowth (Section 6). This pessimistic scenario manifests the "net energy cliff" that some analysts fear will be induced by a transition to a solar economy (e.g., Trainer, 2018; Capellán-Pérez et al., 2019).

Second, if energy can substitute for other inputs in the production of solar panels, or if solar panels come to be produced with automatic robotic inputs, then solar panels may shift the economy from relying on resources with scarce deposits to relying on resources produced from ever-more abundant factors of production. In that case, a solar economy may experience energy-fueled growth, even in the absence of further technical change (Section 7). A fraction of the electricity generated from existing solar panels would be reinvested in producing more solar panels in factories, with potentially another fraction reinvested in producing more automated panel-making robots. Energy would become increasingly abundant, in line with some speculation (Economist, 2024). This optimistic scenario manifests a "net energy ramp", which is the energetic counterpart of the growth takeoff scenarios pondered by scholars of artificial intelligence (e.g., Sandberg, 2013; Nordhaus, 2021).

9 Quantitative Analysis

I next explore the implications of these resource types and of EROI for climate change. I combine the mined, tapped, and manufactured resources in a single setting, corresponding to coal, gas, and solar. Total energy is the sum of these three types. Each non-solar resource generates emissions, which accumulate in the atmosphere as carbon dioxide and cause warming. That warming in turn reduces economic output.

Appendix B describes the calibration. I here highlight four aspects of the calibration. First, I back out Q_M, Q_D, and Q_S from prior literature's estimates of the EROI of each resource and from analytic results relating EROI and productivity. Second, I set the initial stocks of opened mines, tapped deposits, and manufactured panels (i.e., each R_0) so that production of energy from each resource in the first ten-year timestep matches global production over 2010–2019. Third, I apply Assumption 1 for consistency with the theoretical results. Finally, I assume that the coal resource is sufficiently abundant that I can treat it as non-depletable (having $\omega = 0$).

The solid lines in Figure **??** plot the cutoffs χ implied by the calibrations of the tapped (left) and manufactured (right) resources as a function of the elasticity of substitution σ. From Proposition 5 and Corollary 6, this cutoff determines the growth regimes in economies with only a tapped or a manufactured resource. The dashed lines in Figure **??** depict the calibrated productivities of the tapped and manufactured energy resources as functions of σ. We see that, across σ, the calibration implies that the economy would be in a regime of energy-fueled growth with either resource.

Figure **??** depicts simulated outcomes in this economy. It assumes that the elasticity of substitution σ is the same for each energy resource and plots a case with complements ($\sigma = 0.50$) and substitutes ($\sigma = 2$). *The four panels will show (a) energy production from each resource, (b) the stocks of r for each and of z for the tapped resource, (c) EROI for each resource, and (d) temperature and the growth rate of consumption per capita.*

I next conduct experiments in which I change the EROI of either the tapped or the manufactured resource. Reducing EROI corresponds to bringing the productivity of a resource towards χ. In the case of complements, I conduct two experiments: one in which I bring the productivity of tapped resources below the threshold χ, and another in which I bring the productivity of manufactured resources below the threshold χ. In a single-resource setting, these correspond to cases of energy-scarce degrowth. *What find*

I also conduct two experiments in the case of substitutes: one in which I bring the productivity of tapped resources above the threshold χ, and another in which I bring the productivity of manufactured resources above the threshold χ. In a single-resource setting, these correspond to cases of deposit-constrained degrowth and energy-fueled growth, respectively. *What find*

Plot temperature and cons growth against EROI of solar, for case of complements and case of substitutes

10 Conclusions

All economic activity requires energetic inputs. We have seen that the characteristics of energy resources used by an economy shape its growth possibilities. In particular, the characteristics of energy resources determine how fast technical change needs to be in order to achieve sustained growth. The important characteristics of energy resources are the nature of resource production and the productivity of energy in making additional energy, known in the literature as energy return on energy invested (EROI). We have seen that the EROI of a coal resource exceeds unity plus the growth rate of population regardless of the productivity of coal mining. And the EROI of oil or solar resources produced from complementary inputs must exceed unity plus the growth rate of population for the economy to avoid energy-scarce degrowth.

We have also seen that the step-change from tapped oil resources to manufactured solar resources may generate energy-fueled growth, even in the absence of further technical change, if either energy is substitutable for other inputs to solar panel manufacturing or there is an additional step-change in using robots to produce and install solar panels. The latter mechanism for sustained growth is consistent with robots requiring energy to operate. In reality, the production of solar panels is indeed increasingly automated, with further automation on the horizon. If these efforts succeed, the transition to low-carbon energy resources may end up stimulating an improvement in the economy's long-run growth prospects.

This analysis does not consider how energy may affect the pace of technical change, whether by directly fueling research (see Schurr, 1984) or by absorbing scientists who may otherwise work on improving general productivity (see Arkolakis and Walsh, 2024). If a solar resource makes energy cheaper and more abundant, then interactions with technical change could constitute a second growth dividend.

It may seem like a waste to have burned so much carbon when lower-carbon resources might actively drive growth, but accessing knowledge- and capital-intensive lower-carbon resources may have required burning fossil resources in order to raise output high enough. Future work should quantitatively assess historical transitions between resource types in order to understand how much earlier the world could have reached a solar economy. The emissions in excess of that point may have special significance when attributing historical responsibility for climate change.

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Appendix

The first section contains additional analysis. The second section provides details for the numerical examples. Subsequent sections contain proofs. Throughout, I use ℓ, k, e, r, and y to denote per-capita variables, I use a *ss* subscript to denote a steady state, and I use g_x to indicate the instantaneous growth rate of variable x.

A Analysis of deposit-constrained degrowth, for tapped resources with $\sigma > 1$

Consider the setting of Section 5, with $\sigma > 1$. When inputs are gross substitutes, the optimal path may not be interior. Instead, it may be a corner solution in which energy inputs to deposit-tapping completely crowd out other inputs to deposit-tapping (i.e., $L_R(t), K_R(t) = 0$). I analyze these balanced growth paths under log utility:

Proposition A-1 (Deposit-Constrained Degrowth with Tapped Resources)**.** *Let Assumptions 1 and 3 hold and fix $\sigma > 1$. Define g_e as the growth rate of energy per capita, g_y as the growth rate of output per capita, g_k as the growth rate of capital per capita, and χ as in Proposition 5. If $\rho - \Omega$ is not too large and $\Omega - g_L > \frac{\omega}{1-\omega}(\lambda + g_L)$, then there exists $X \leq \chi$ such that χ/X is strictly decreasing in σ, $\lim_{\sigma\to\infty} X = \lim_{\sigma\to\infty} \chi$, and $A_E Q_D > X$ implies the existence of a balanced growth path with $L_R(t), K_R(t) = 0$ and with $R(t)/L(t)$, $E_R(t)/L(t)$, $E_Y(t)/L(t)$, and $Z(t)/L(t)$ all growing at rate $g_e \neq 0$. Along this path:*

i $g_y = g_k = \frac{\alpha_E}{\alpha_E+\alpha_L} g_e$.

ii $g_e \in \big(-(\rho - \Omega), 0\big)$.

iii g_e *and* $Z(t)/E_R(t)$ *each decrease in* A_E *and* Q_D.

iv $EROI(t) = 1 + A_E Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} \frac{g_e+g_L}{[g_e+g_L+\lambda](\Omega-g_L-g_e)^{\frac{\omega}{1-\omega}}}$.

Proof. See Appendix J. □

When energy resources are sufficiently productive (i.e., $A_E Q_D$ is large), the growth rates of output and energy are negative (parts i and ii). Reinvesting growing energy stocks in deposit-tapping at a constant rate eats up the available deposits, as $Z(t)/E_R(t)$ falls in the productivity of energy resources (part iii). The increasing scarcity of deposits restricts the ability to produce further energy, so the growth rate of energy must also fall until the rate of energy reinvestment comes into equilibrium with the rate of deposit finds (part iii).

This is a scenario of *deposit-constrained degrowth*. The economy accumulates a lot of spare energy. Deposits are scarce relative to energy, so their availability constrains how much of the spare energy can be devoted to tapping new deposits. As a result, a large share of the energy produced is devoted to final good production ($e_R(t)/e(t)$ is small) and thus to consumption. Nonetheless, the available deposits decline over time as they are exploited via energy inputs faster than they are found. Energy production falls with deposits, and consumption falls over time as available energy falls, albeit from a potentially high level that was enabled by the high productivity of energy deposits.

B Numerical Implementation

I first describe the calibration for the numerical examples. I then describe the calibration for the quantitative application.

B.1 Numerical Examples in Sections 5 and 6

I implement the model by discretizing the transition equations over a one-year timestep. I determine the economy's path over 400 years by solving the social planner's problem. The control variables comprise the factor allocations and the state variables, with the transition and market-clearing equations serving as constraints. I use the Knitro solver's interior-point algorithm in Matlab. I provide an analytic gradient and an analytic Hessian.

I fix the annual growth rate of population to $g_L = 0.01$, which is roughly in keeping with the recent growth rate of world population and only a bit higher than the pre-1900 growth rate.[35] I fix the saving rate to $s = 0.25$, which is in keeping with the global rate.[36] I set the value share of energy in final good production to $\alpha_E = 0.05$, which is around the value in the U.S. over the past decades (Hassler et al., 2021; Orak and Çakır Melek, 2021). I then set $\alpha_K/(1 - \alpha_E) = 0.2632$ to match the capital share relative to labor in Hassler et al. (2021), which implies $\alpha_K = 0.25$ and $\alpha_L = 0.7$. Following Hassler et al. (2021), the annual utility discount rate is $\rho = 0.0152$ and the annual depreciation rate is $\delta = 0.05$. I normalize $A(t) = 1$ and assume log utility.

For the tapped resource, I calibrate the annual discovery rate Ω to the 11% growth in global probable and proved oil reserves between 2000 and 2007 (Sorrell et al., 2012, 716). This implies $\Omega = 0.015$. I fix $\lambda = 0.041$ to the production-weighted aggregate annual decline rate of all oil fields (Sorrell et al., 2012, 718). For the manufactured resource, I set $\lambda = 0.005$ based on the median annual degradation rate for silicon panels in Jordan and Kurtz (2013) and Jordan et al. (2016).

For both the tapped and the manufactured resource, I fix $\kappa_E = 0.5$ and $\kappa_K = 0.9$ and I normalize $A_E = 1$ and $A_{LK} = 1$. For the tapped resource, I fix $\omega = 0.5$.

[35] https://en.wikipedia.org/wiki/Human_population_projections#/media/File:World_population_growth,_1700-2100,_2022_revision.png

[36] https://data.worldbank.org/indicator/NE.GDI.FTOT.ZS

I calibrate Q_D and Q_S to the EROI of oil and solar resources. First, I fix the EROI of each resource to an average from Supplementary Table 1 of Slameršak et al. (2022): I set the EROI of the tapped resource to 10.65 following their analysis of gas-to-electricity, and I set the EROI of the manufactured resource to 7.75 following their analysis of solar photovoltaics. Second, I use equation (A-20) to obtain the e_{Rss}/e_{ss} that must hold in steady state. Finally, I back out Q_D or Q_S from that and (A-34). When exploring values of Q_D or Q_S that do not yield energy-enabled growth, I set the relevant parameter to twice or half of the boundary value χ, depending on which boundary I want to violate.

I set the initial resource stock to $R_0 = 100$ and the initial deposit stock to $Z_0 = 1000$. I follow DICE-2016R (Nordhaus, 2017) in setting the initial capital stock to $K_0 = 223$, in trillion year 2010 dollars.

B.2 Quantitative Application in Section 9

Much of the calibration follows the foregoing. I here describe the differences and novel parts.[37]

Let E_t^M, E_t^D, and E_t^S indicate production of energy from mines, tapped deposits, and solar panels, respectively. Total energy available to the economy at time t is now:

$$E_t = E_t^M + E_t^D + E_t^S.$$

First, I here need to calibrate the coal resource. I set ϕ_{ML}, the labor share of coal, to wages in the U.S. coal sector divided by its value of production. Wages in the year 2000 were \$4.107 billion,[38] and using production and prices by rank, the value of U.S. coal production in the year 2000 was \$17.95 billion.[39] These imply a labor share of 0.23. The capital and resource shares (ϕ_{MK} and ϕ_{MR}), as well as the capital and labor shares in mine opening κ_L and κ_K), follow from these and other calibrations in common with Assumption 2. The annual depreciation rate λ for coal mines comes from values in Watson et al. (2023) for the decline in labor productivity within mines. The implied annual decline rate for underground mines is 0.026 and for surface mines is 0.0307. I average these, obtaining $\lambda = 0.0283$ for the mined resource. The EROI of the coal resource comes from averaging values in Supplementary Table 1 of Slameršak et al. (2022), yielding an EROI of 8.95. The productivity Q_M of the coal resource then follows from equation (A-33) below. Finally, I assume that the coal resource is sufficiently abundant that I can treat it as non-depletable (having $\omega = 0$).

Second, I here calibrate the initial stocks of capital and of each resource type. The initial stock of capital K_0 is \$238.6 trilliong year 2014 dollars, from the calibration in Lemoine (2024). The initial stock of the gas resource Z_0 is 300 EJ, which is consistent with Table 7.11 in Rogner et al. (2012). I calibrate the initial stocks of the developed resources R_0 so that

[37] Some of the above will be reconciled with this.

[38] https://fred.stlouisfed.org/series/B4109C0A144NBEA

[39] https://www.eia.gov/coal/annual/

equilibrium production of each type of energy over the first timestep matches the calibration in Lemoine (2024) to global data: 1617 EJ per decade for the mined (coal) resource, 1278 EJ per decade for the tapped (gas) resource, and 224 EJ per decade for the manufactured (solar) resource.

Third, I develop a climate module and connect that back to the economy. Let ζ^M, ζ^d, and ζ^S be the emission intensity of each resource and ζ_0 be exogenous emissions (as from oil for transportation). Then total time t emissions are:

$$\Phi_t = \zeta^M E_t^M + \zeta^D E_t^D + \zeta^S E_t^S + \zeta_0.$$

Following Lemoine (2024), $\zeta^M = 0.0250$ Gt C per EJ, $\zeta^D = 0.0139$ Gt C per EJ, $\zeta^S = 0$ Gt C per EJ, and $\zeta_0 = 3.77$ Gt C per year. Cumulative carbon emissions as of time t are

$$\Lambda_t = \Lambda_0 + \sum_{j=1}^{t-1} \Phi_j.$$

From [IPCC AR6 WG1 Table SPM.2 (pg 29)], cumulative emissions over 1850-2019 were 2390 Gt CO_2-equivalent, implying $\Lambda_0 = 652$ Gt C. Following a cumulative emissions model of warming (Dietz et al., 2021), warming as of time t is $T_t = \Gamma \Lambda_t$. Following [IPCC AR6 WGI Table 5.7 (pg 748)], $\Gamma = 0.0017$ degrees Celsius per Gt C. And following the latest version of the DICE model (Barrage and Nordhaus, 2024), warming reduces output from Y_t to $(1 - d[T_t]^2)Y_t$, where $d = 0.003467$.

The solution of the model is as described above, except here with a 10-year timestep.

C Proof of Proposition 1

Equilibrium solves the following maximization problem:

$$\begin{aligned}
\max_{L_Y(\cdot), K_Y(\cdot), L_E(\cdot), K_E(\cdot)} & \int_0^\infty e^{-\rho t} L(t) u\left(\frac{(1-s)A(t)L_Y(t)^{\alpha_L} K_Y(t)^{\alpha_K} E(t)^{\alpha_E}}{L(t)}\right) \mathrm{d}t \\
\text{s.t. } \dot{A}(t) =& g_A A(t) \\
\dot{L}(t) =& g_L L(t) \\
\dot{K}(t) =& sAL_Y(t)^{\alpha_L} K_Y(t)^{\alpha_K} E(t)^{\alpha_E} - \delta K(t) \\
E(t) =& Q_H L_E(t)^{\phi_{HL}} K_E(t)^{\phi_{HK}} R^{\phi_{HR}} \\
L(t) =& L_Y(t) + L_E(t) \\
K(t) =& K_Y(t) + K_E(t).
\end{aligned}$$

Converting to per-capita and substituting from the market-clearing conditions, this is equivalent to:

$$\begin{aligned}\max_{\ell_Y(\cdot),k_Y(\cdot)} \int_0^\infty & e^{-(\rho-g_L)t} u\Bigg((1-s)A(t)\ell_Y(t)^{\alpha_L} k_Y(t)^{\alpha_K}\left[Q_H[1-\ell_Y(t)]^{\phi_{HL}}[k(t)-k_Y(t)]^{\phi_{HK}} r(t)^{\phi_{HR}}\right]^{\alpha_E}\Bigg)\,\mathrm{d}t \\ \text{s.t. } \dot{A}(t) =& g_A A(t) \\ \dot{k}(t) =& sA\ell_Y(t)^{\alpha_L} k_Y(t)^{\alpha_K}\left[Q_H[1-\ell_Y(t)]^{\phi_{HL}}[k(t)-k_Y(t)]^{\phi_{HK}} r(t)^{\phi_{HR}}\right]^{\alpha_E} - (\delta+g_L)k(t) \\ \dot{r}(t) =& -g_L r(t).\end{aligned}$$

The current-value Hamiltonian is:

$$\begin{aligned}& u\Bigg((1-s)A(t)\ell_Y(t)^{\alpha_L} k_Y(t)^{\alpha_K}\left[Q_H[1-\ell_Y(t)]^{\phi_{HL}}[k(t)-k_Y(t)]^{\phi_{HK}} r(t)^{\phi_{HR}}\right]^{\alpha_E}\Bigg) \\ & + \nu(t)\left[sA\ell_Y(t)^{\alpha_L} k_Y(t)^{\alpha_K}\left[Q_H[1-\ell_Y(t)]^{\phi_{HL}}[k(t)-k_Y(t)]^{\phi_{HK}} r^{\phi_{HR}}\right]^{\alpha_E} - (\delta+g_L)k(t)\right] \\ & - \mu(t) g_L r(t) + \psi(t) g_A A(t).\end{aligned}$$

The conditions to maximize the Hamiltonian are:

$$\begin{aligned}0 =& \Big[(1-s)u'((1-s)y(t)) + s\nu(t)\Big]\left[\frac{\alpha_L}{\ell_Y(t)} - \frac{\alpha_E\phi_{HL}}{1-\ell_Y(t)}\right] y(t), \\ 0 =& \Big[(1-s)u'((1-s)y(t)) + s\nu(t)\Big]\left[\frac{\alpha_K}{k_Y(t)} - \frac{\alpha_E\phi_{HK}}{k(t)-k_Y(t)}\right] y(t).\end{aligned}$$

Together, these imply:

$$\ell_Y(t) = \frac{\alpha_L}{\alpha_L+\alpha_E\phi_{HL}}, \qquad k_Y(t) = \frac{\alpha_K}{\alpha_K+\alpha_E\phi_{HK}} k(t).$$

Substituting, we find

$$\begin{aligned}Y(t) =& A(t)\left(\frac{\alpha_L}{\alpha_L+\alpha_E\phi_{HL}}\right)^{\alpha_L}\left(\frac{\alpha_K}{\alpha_K+\alpha_E\phi_{HK}}\right)^{\alpha_K} \\ & \left[Q_H\left(\frac{\alpha_E\phi_{HL}}{\alpha_L+\alpha_E\phi_{HL}}\right)^{\phi_{HL}}\left(\frac{\alpha_E\phi_{KL}}{\alpha_K+\alpha_E\phi_{HK}}\right)^{\phi_{HK}} R^{\phi_{HR}}\right]^{\alpha_E} L(t)^{\alpha_L+\alpha_E\phi_{HL}} K(t)^{\alpha_K+\alpha_E\phi_{HK}}.\end{aligned}$$

Define

$$A_H(t) \triangleq A(t)^{\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}}.$$

Defining $y_H(t) \triangleq Y(t)/[A_H(t)L(t)]$ and analogously for the other variables, we find:

$$\begin{aligned}
y_H(t) =& \left(\frac{\alpha_L}{\alpha_L+\alpha_E\phi_{HL}}\right)^{\alpha_L}\left(\frac{\alpha_K}{\alpha_K+\alpha_E\phi_{HK}}\right)^{\alpha_K} \\
& \left[Q_H\left(\frac{\alpha_E\phi_{HL}}{\alpha_L+\alpha_E\phi_{HL}}\right)^{\phi_{HL}}\left(\frac{\alpha_E\phi_{KL}}{\alpha_K+\alpha_E\phi_{HK}}\right)^{\phi_{HK}} r_H(t)^{\phi_{HR}}\right]^{\alpha_E} k_H(t)^{\alpha_K+\alpha_E\phi_{HK}}, \\
\dot{k}_H(t) =& s y_H(t) - \left(\delta + g_L + \frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right)k_H(t), \\
\dot{r}_H(t) =& -\left(g_L + \frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right)r_H(t).
\end{aligned}$$

The growth rate of $k_H(t)$ is:

$$\frac{\dot{k}_H(t)}{k_H(t)} = s\chi_H Q_H^{\alpha_E} r_H(t)^{\alpha_E\phi_{HR}} k_H(t)^{\alpha_K+\alpha_E\phi_{HK}-1} - \left(\delta + g_L + \frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right),$$

where

$$\chi_H \triangleq \left(\frac{\alpha_L}{\alpha_L+\alpha_E\phi_{HL}}\right)^{\alpha_L}\left(\frac{\alpha_K}{\alpha_K+\alpha_E\phi_{HK}}\right)^{\alpha_K}\left[\left(\frac{\alpha_E\phi_{HL}}{\alpha_L+\alpha_E\phi_{HL}}\right)^{\phi_{HL}}\left(\frac{\alpha_E\phi_{KL}}{\alpha_K+\alpha_E\phi_{HK}}\right)^{\phi_{HK}}\right]^{\alpha_E}.$$

That growth rate is constant if and only if

$$r_H(t)^{\alpha_E\phi_{HR}} k_H(t)^{\alpha_K+\alpha_E\phi_{HK}-1} = \frac{1}{s\chi_H Q_H^{\alpha_E}}\left[g_k + \delta + g_L + \frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right] \tag{A-1}$$

is constant. Observe that:

$$r_H(t) = e^{-\left(g_L+\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right)t} r_H(0).$$

Substitute into (A-1):

$$k_H(t) = \left[s\chi_H Q_H^{\alpha_E}\frac{e^{-\alpha_E\phi_{HR}\left(g_L+\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right)t}\left(\frac{R}{L(0)A_H(0)}\right)^{\alpha_E\phi_{HR}}}{g_k+\delta+g_L+\frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A}\right]^{\frac{1}{1-\alpha_K-\alpha_E\phi_{HK}}}.$$

And thus:

$$\frac{\dot{k}_H(t)}{k_H(t)} = -\frac{\alpha_E \phi_{HR}}{1-\alpha_K-\alpha_E\phi_{HK}}\left(g_L + \frac{1}{1-\alpha_K-\alpha_E(\phi_{HK}+\phi_{HR})}g_A\right) < 0.$$

The growth rate of output per capita is:

$$\begin{aligned}\frac{\dot{y}(t)}{y(t)} =& \alpha_E\phi_{HR}\frac{\dot{r}_H(t)}{r_H(t)} + [\alpha_K + \alpha_E\phi_{HK}]\frac{\dot{k}_H(t)}{k_H(t)} + \frac{\dot{A}_H(t)}{A_H(t)}\\ =& \frac{g_A - \alpha_E\phi_{HR}g_L}{\alpha_L + \alpha_E(\phi_{HL}+\phi_{HR})}.\end{aligned}$$

The proposition follows.

D Preliminaries for Propositions 3 and 4

Equilibrium solves the following maximization problem:

$$\begin{aligned}\max_{L_Y(\cdot),K_Y(\cdot),E_Y(\cdot),L_E(\cdot),K_E(\cdot),L_R(\cdot),K_R(\cdot),E_R(\cdot)} & \int_0^\infty e^{-\rho t}L(t)u\left(\frac{(1-s)AL_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E_Y(t)^{\alpha_E}}{L(t)}\right)\mathrm{d}t\\ \text{s.t. } \dot{L}(t) =& g_L L(t)\\ \dot{K}(t) =& sAL_Y(t)^{\alpha_L}K_Y(t)^{\alpha_K}E_Y(t)^{\alpha_E} - \delta K(t)\\ \dot{Z}(t) =& \Omega Z(t) - Z(t)^\omega F(L_R(t),K_R(t),E_R(t))^{1-\omega}\\ \dot{R}(t) =& Z(t)^\omega F(L_R(t),K_R(t),E_R(t))^{1-\omega} - \lambda R(t)\\ E(t) =& Q_M L_E(t)^{\phi_{ML}}K_E(t)^{\phi_{MK}}R(t)^{\phi_{MR}}\\ L(t) =& L_Y(t) + L_E(t) + L_R(t)\\ K(t) =& K_Y(t) + K_E(t) + K_R(t)\\ E(t) =& E_Y(t) + E_R(t).\end{aligned}$$

Converting to per-capita and substituting from the market-clearing conditions, this is equivalent to:

$$
\begin{aligned}
\max_{\ell_E(\cdot),k_E(\cdot),\ell_R(\cdot),k_R(\cdot),e_R(\cdot)} \int_0^\infty e^{-(\rho-g_L)t} u\Bigg(&(1-s)A\Big[1-\ell_E(t)-\ell_R(t)\Big]^{\alpha_L}\Big[k(t)-k_E(t)-k_R(t)\Big]^{\alpha_K} \\
&\Big[Q_M \ell_E(t)^{\phi_{ML}} k_E(t)^{\phi_{MK}} r(t)^{\phi_{MR}} - e_R(t)\Big]^{\alpha_E}\Bigg)\,\mathrm{d}t \\
\text{s.t. } \dot{k}(t) =& sA\Big[1-\ell_E(t)-\ell_R(t)\Big]^{\alpha_L}\Big[k(t)-k_E(t)-k_R(t)\Big]^{\alpha_K} \\
&\Big[Q_M \ell_E(t)^{\phi_{ML}} k_E(t)^{\phi_{MK}} r(t)^{\phi_{MR}} - e_R(t)\Big]^{\alpha_E} - (\delta+g_L)k(t) \\
\dot{z}(t) =& (\Omega - g_L)z(t) - z(t)^{\omega} F(\ell_R(t),k_R(t),e_R(t))^{1-\omega} \\
\dot{r}(t) =& z(t)^{\omega} F(\ell_R(t),k_R(t),e_R(t))^{1-\omega} - (\lambda+g_L)r(t).
\end{aligned}
$$

The following lemma establishes that any balanced growth path with interior solutions must have variables growing at the rate of population:

Lemma A-2. *A balanced growth path with $L_R(t), K_R(t), E_R(t) > 0$ must have all variables growing at the rate of population.*

Proof. Use g_x to indicate the instantaneous growth rate of variable x. From the transition equation for $z(t)$, $\dot{z}(t)/z(t)$ is constant if and only if $F(\ell_R(t),k_R(t),e_R(t))/z(t)$ is constant. From the transition equation for $r(t)$, $\dot{r}(t)/r(t)$ can be constant with $F(\ell_R(t),k_R(t),e_R(t))/z(t)$ constant if and only if $F(\ell_R(t),k_R(t),e_R(t))/r(t)$ is constant. Therefore, if g_z and g_r are each constant, then $g_z = g_r = g_{F/L}$.

If $g_{F/L} \neq 0$ and e_R and k_R grow at constant rates, then ℓ_R must grow at a constant rate. The constant growth rate of ℓ_R cannot be strictly positive because ℓ_R is bounded above by 1. The constant growth rate of ℓ_R cannot be strictly negative because the (full employment) labor constraint implies that ℓ_Y and/or ℓ_E must grow at a strictly positive rate yet these variables are each bounded above by 1. Therefore a constant growth rate of ℓ_R must be zero. If $g_{F/L}$, g_{k_R}, and g_{e_R} are also to be constant, then $g_{F/L}, g_{k_R}, g_{e_R} = 0$. That implies $g_z, g_r = 0$. It follows straightforwardly that the remaining variables (in per capita form) also grow at rate zero. □

So we seek a steady state in per-capita variables. The current-value Hamiltonian is:

$$
\begin{aligned}
&u\Bigg((1-s)A\Big[1-\ell_E(t)-\ell_R(t)\Big]^{\alpha_L}\Big[k(t)-k_E(t)-k_R(t)\Big]^{\alpha_K}\Big[Q_M\ell_E(t)^{\phi_{ML}}k_E(t)^{\phi_{MK}}r(t)^{\phi_{MR}}-e_R(t)\Big]^{\alpha_E}\Bigg)\\
&+\nu(t)\Bigg(sA\Big[1-\ell_E(t)-\ell_R(t)\Big]^{\alpha_L}\Big[k(t)-k_E(t)-k_R(t)\Big]^{\alpha_K}\Big[Q_M\ell_E(t)^{\phi_{ML}}k_E(t)^{\phi_{MK}}r(t)^{\phi_{MR}}-e_R(t)\Big]^{\alpha_E}\\
&\qquad-(\delta+g_L)k(t)\Bigg)\\
&+\gamma(t)\Big[(\Omega-g_L)z(t)-z(t)^{\omega}F(\ell_R(t),k_R(t),e_R(t))^{1-\omega}\Big]\\
&+\mu(t)\Big[z(t)^{\omega}F(\ell_R(t),k_R(t),e_R(t))^{1-\omega}-(\lambda+g_L)r(t)\Big].
\end{aligned}
$$

The costate equations are:

$$\dot{\nu}(t)=-\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\frac{\alpha_K y(t)}{k_Y(t)}+(\rho+\delta)\nu(t), \tag{A-2}$$

$$\dot{\gamma}(t)=\gamma(t)(\rho-\Omega)-\Big[\mu(t)-\gamma(t)\Big]\omega\left(\frac{F(\ell_R(t),k_R(t),e_R(t))}{z(t)}\right)^{1-\omega}, \tag{A-3}$$

$$\dot{\mu}(t)=\mu(t)(\rho+\lambda)-\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\frac{\alpha_E\phi_{MR}y(t)}{e_Y(t)}\frac{e(t)}{r(t)}. \tag{A-4}$$

The conditions to maximize the Hamiltonian are:

$$0=\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\left[\frac{\alpha_E\phi_{ML}}{e_Y(t)}\frac{e(t)}{\ell_E(t)}-\frac{\alpha_L}{\ell_Y(t)}\right]y(t), \tag{A-5}$$

$$0=\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\left[\frac{\alpha_E\phi_{MK}}{e_Y(t)}\frac{e(t)}{k_E(t)}-\frac{\alpha_K}{k_Y(t)}\right]y(t), \tag{A-6}$$

$$
\begin{aligned}
0=&-\frac{\alpha_L y(t)}{\ell_Y(t)}\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\\
&+\Big[\mu(t)-\gamma(t)\Big](1-\omega)\left(\frac{z(t)}{F(\ell_R(t),k_R(t),e_R(t))}\right)^{\omega}\frac{\partial F(\ell_R(t),k_R(t),e_R(t))}{\partial \ell_R(t)},
\end{aligned}
\tag{A-7}$$

$$
\begin{aligned}
0=&-\frac{\alpha_K y(t)}{k_Y(t)}\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\\
&+\Big[\mu(t)-\gamma(t)\Big](1-\omega)\left(\frac{z(t)}{F(\ell_R(t),k_R(t),e_R(t))}\right)^{\omega}\frac{\partial F(\ell_R(t),k_R(t),e_R(t))}{\partial k_R(t)},
\end{aligned}
\tag{A-8}$$

$$
\begin{aligned}
0=&-\frac{\alpha_E y(t)}{e_Y(t)}\Big[(1-s)u'((1-s)y(t))+s\nu(t)\Big]\\
&+\Big[\mu(t)-\gamma(t)\Big](1-\omega)\left(\frac{z(t)}{F(\ell_R(t),k_R(t),e_R(t))}\right)^{\omega}\frac{\partial F(\ell_R(t),k_R(t),e_R(t))}{\partial e_R(t)}.
\end{aligned}
\tag{A-9}$$

Equations (A-5) and (A-6) imply:

$$\ell_Y(t) = \frac{\alpha_L}{\alpha_E \phi_{ML}} \frac{e_Y(t)}{e(t)} \ell_E(t), \tag{A-10}$$

$$k_Y(t) = \frac{\alpha_K}{\alpha_E \phi_{MK}} \frac{e_Y(t)}{e(t)} k_E(t). \tag{A-11}$$

Take ratios of equations (A-7), (A-8), and (A-9), substitute for $\ell_Y(t)$ and $k_Y(t)$, and cancel the $e_Y(t)$:

$$1 = \frac{\kappa_L}{\phi_{ML}} \frac{1-\kappa_E}{\kappa_E} \left(\frac{A_{LK}}{A_E}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{k_R(t)}{\ell_R(t)}\right)^{\kappa_K \frac{\sigma-1}{\sigma}} \left(\frac{e_R(t)}{\ell_R(t)}\right)^{\frac{1}{\sigma}} \frac{\ell_E(t)}{e(t)}, \tag{A-12}$$

$$1 = \frac{\kappa_K}{\phi_{MK}} \frac{1-\kappa_E}{\kappa_E} \left(\frac{A_{LK}}{A_E}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{\ell_R(t)}{k_R(t)}\right)^{\kappa_L \frac{\sigma-1}{\sigma}} \left(\frac{e_R(t)}{k_R(t)}\right)^{\frac{1}{\sigma}} \frac{k_E(t)}{e(t)}. \tag{A-13}$$

These two equations imply:

$$\begin{aligned} \ell_R(t) = & \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \\ & e_R(t) \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)+1}, \end{aligned} \tag{A-14}$$

$$\begin{aligned} k_R(t) = & \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)} \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \\ & e_R(t) \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)+1}. \end{aligned} \tag{A-15}$$

Substitute from (A-10) and (A-14) into the labor resource constraint and solve for $\ell_E(t)$:

$$\begin{aligned} \ell_E(t) = \Bigg[& 1 + \frac{\alpha_L}{\alpha_E \phi_{ML}} \frac{e_Y(t)}{e(t)} \\ & + \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \\ & \frac{e_R(t)}{e(t)} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)} \Bigg]^{-1}. \end{aligned} \tag{A-16}$$

Solve for $e_R(t)/e(t)$:

$$\frac{e_R(t)}{e(t)} = \frac{[1 - \ell_E(t) - \frac{\alpha_L}{\alpha_E \phi_{ML}} \ell_E(t)]/\ell_E(t)}{-\frac{\alpha_L}{\alpha_E \phi_{ML}} + \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}}. \tag{A-17}$$

Substitute from (A-11) and (A-15) into the capital constraint:

$$\frac{k_E(t)}{k(t)} = \Bigg[1 + \frac{\alpha_K}{\alpha_E \phi_{MK}} \frac{e_Y(t)}{e(t)} + \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)} \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \frac{e_R(t)}{e(t)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)}\Bigg]^{-1}. \tag{A-18}$$

Substitute for $e_R(t)/e(t)$ from (A-17):

$$\frac{k_E(t)}{k(t)} = \Bigg[1 - \frac{\alpha_K}{\alpha_E \phi_{MK}} \frac{1}{\ell_E(t)} \frac{1 - \ell_E(t) - \ell_E(t)\left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}}{-\frac{\alpha_L}{\alpha_E\phi_{ML}} + \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}}$$
$$+ \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)} \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \frac{[1 - \ell_E(t) - \frac{\alpha_L}{\alpha_E\phi_{ML}}\ell_E(t)]/\ell_E(t)}{-\frac{\alpha_L}{\alpha_E\phi_{ML}} + \left(\frac{\kappa_K}{\phi_{MK}}\right)^{\kappa_K(\sigma-1)} \left(\frac{\kappa_L}{\phi_{ML}}\right)^{\kappa_L(\sigma-1)+1} \left(\frac{1-\kappa_E}{\kappa_E}\right)^{\sigma} \left(\frac{A_{LK}}{A_E}\right)^{\sigma-1} \left(\frac{k_E(t)}{e(t)}\right)^{\kappa_K(\sigma-1)} \left(\frac{\ell_E(t)}{e(t)}\right)^{\kappa_L(\sigma-1)}}\Bigg]^{-1}. \tag{A-19}$$

Finally, at a steady state, $\dot{e}(t) = 0$ in equation (7). Therefore,

$$EROI_{ss} = 1 + \frac{e_{ss}}{e_{Rss}} g_L. \tag{A-20}$$

E Proof of Proposition 3

Section D contains preliminaries. By Lemma A-2, any interior balanced growth path must have per-capita variables constant. If $\sigma = 1$, we have, from (A-16) and (A-18):

$$\ell_E(t) = \left[1 + \frac{\alpha_L}{\alpha_E\phi_{ML}}\left(1 - \frac{e_R(t)}{e(t)}\right) + \frac{\kappa_L}{\phi_{ML}} \frac{1-\kappa_E}{\kappa_E} \frac{e_R(t)}{e(t)}\right]^{-1},$$
$$\frac{k_E(t)}{k(t)} = \left[1 + \frac{\alpha_K}{\alpha_E\phi_{MK}}\left(1 - \frac{e_R(t)}{e(t)}\right) + \frac{\kappa_K}{\phi_{MK}} \frac{1-\kappa_E}{\kappa_E} \frac{e_R(t)}{e(t)}\right]^{-1}. \tag{A-21}$$

These are both interior as long as $e_R(t)/e(t) < 1$. With these:

$$\begin{aligned} \ell_Y(t) =& \frac{\alpha_L}{\alpha_E \phi_{ML}} \frac{e_Y(t)}{e(t)} \ell_E(t), \\ \frac{k_Y(t)}{k(t)} =& \frac{\alpha_K}{\alpha_E \phi_{MK}} \frac{e_Y(t)}{e(t)} \frac{k_E(t)}{k(t)}. \end{aligned} \tag{A-22}$$

And:

$$\begin{aligned} \ell_R(t) =& \frac{\kappa_L}{\phi_{ML}} \frac{1-\kappa_E}{\kappa_E} \frac{e_R(t)}{e(t)} \ell_E(t), \\ \frac{k_R(t)}{k(t)} =& \frac{\kappa_K}{\phi_{MK}} \frac{1-\kappa_E}{\kappa_E} \frac{e_R(t)}{e(t)} \frac{k_E(t)}{k(t)}. \end{aligned} \tag{A-23}$$

At a steady state, (A-4) becomes:

$$y_{ss}\Big[(1-s)u'((1-s)y_{ss}) + s\nu_{ss}\Big] = \mu_{ss}(\rho+\lambda)\frac{r_{ss}}{\alpha_E \phi_{MR}}\left(1 - \frac{e_{Rss}}{e_{ss}}\right).$$

Substitute into the steady-state version of (A-9):

$$0 = -\frac{1}{\phi_{MR}}\mu_{ss}(\rho+\lambda)\frac{r_{ss}}{e_{ss}} + \Big[\mu_{ss} - \gamma_{ss}\Big](1-\omega)\left(\frac{z_{ss}}{F(\ell_{Rss}, k_{Rss}, e_{Rss})}\right)^{\omega}\frac{F_{ss}}{e_{Rss}}\kappa_E,$$

where

$$F_{ss} \triangleq F(\ell_{Rss}, k_{Rss}, e_{Rss}).$$

Solve for γ_{ss}:

$$\gamma_{ss} =\mu_{ss}\left(1 - \frac{\frac{1}{\phi_{MR}}(\rho+\lambda)\frac{r_{ss}}{e_{ss}}}{(1-\omega)\left(\frac{z_{ss}}{F(\ell_{Rss}, k_{Rss}, e_{Rss})}\right)^{\omega}\frac{F_{ss}}{e_{Rss}}\kappa_E}\right).$$

Substitute into (A-3):

$$\begin{aligned} 0 =&(\rho-\Omega)\left((1-\omega)\left(\frac{z_{ss}}{F(\ell_{Rss}, k_{Rss}, e_{Rss})}\right)^{\omega}\frac{F_{ss}}{e_{Rss}}\kappa_E - \frac{1}{\phi_{MR}}(\rho+\lambda)\frac{r_{ss}}{e_{ss}}\right) \\ &-\omega\left(\frac{F(\ell_{Rss}, k_{Rss}, e_{Rss})}{z_{ss}}\right)^{1-\omega}\frac{1}{\phi_{MR}}(\rho+\lambda)\frac{r_{ss}}{e_{ss}}. \end{aligned}$$

From the transition equation for $z(t)$,

$$\frac{z_{ss}}{F_{ss}} =(\Omega - g_L)^{-\frac{1}{1-\omega}}. \tag{A-24}$$

Using (A-24) in the expression prior to it yields:

$$0 =(\rho-\Omega)(1-\omega)\left(\Omega-g_L\right)^{-\frac{\omega}{1-\omega}}\kappa_E - \left[(\rho-\Omega)+\omega(\Omega-g_L)\right]\frac{1}{\phi_{MR}}(\rho+\lambda)\frac{r_{ss}}{F_{ss}}\frac{e_{Rss}}{e_{ss}}.$$

The transition equation for $r(t)$ and (A-24) yield:

$$(\lambda+g_L)\frac{r_{ss}}{F_{ss}} =(\Omega-g_L)^{-\frac{\omega}{1-\omega}}. \tag{A-25}$$

Substitute into the prior expression:

$$\frac{e_{Rss}}{e_{ss}} =\phi_{MR}(1-\omega)\kappa_E\frac{\rho-\Omega}{(\rho-\Omega)+\omega(\Omega-g_L)}\frac{g_L+\lambda}{\rho+\lambda}. \tag{A-26}$$

The right-hand side is strictly positive because $\rho > \Omega, g_L$.

It remains to verify that the steady state is feasible. Equation (A-25) shows that a steady state with other variables strictly positive has $r_{ss} > 0$ if and only if either $\omega = 0$ or $\Omega > g_L$. From (A-24), $z_{ss} > 0$ with other variables strictly positive requires $\Omega > g_L$. Because $e_{Yss} = e_{ss} - e_{Rss}$, a steady state has strictly positive final good production only if $e_{Rss}/e_{ss} < 1$. Using $\rho > g_L$ and $\kappa_E, \phi_{MR} < 1$, it is clear from (A-26) that $e_{Rss}/e_{ss} < 1$ when either $\omega = 0$ or $\Omega > g_L$. Finally, from the capital transition equation, $y_{ss}/k_{ss} = (\delta + g_L)/s$, so $k_{ss} > 0$ if $y_{ss} > 0$, and from the final good production function and (A-22), $y_{ss} > 0$ when $\ell_{Yss}, e_{Yss} > 0$ if $k_{ss} > 0$. So it is internally consistent for all variables to be strictly positive if and only if either $\omega = 0$ or $\Omega > g_L$.

The claims about prices are implied by the existence of steady states in per-capita, current value terms for $\gamma(t)$ and $\mu(t)$. The claim about $EROI_{ss}$ follows from (A-20) and (A-26).

Finally, consider the elasticity of y_{ss} with respect to Q_M. From (A-25), (A-23), (A-21), (A-26), and the transition equation for $k(t)$:

$$\frac{\mathrm{d}r_{ss}}{\mathrm{d}Q_M} =(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\left[\kappa_K(1-\kappa_E)\frac{F_{ss}}{k_{Rss}}\frac{\mathrm{d}k_{Rss}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M}+\kappa_E\frac{F_{ss}}{e_{Rss}}\frac{\mathrm{d}e_{Rss}}{\mathrm{d}e_{ss}}\frac{\mathrm{d}e_{ss}}{\mathrm{d}Q_M}\right]. \tag{A-27}$$

From (2), (A-21), (A-26), and the transition equation for $k(t)$:

$$\frac{\mathrm{d}e_{ss}}{\mathrm{d}Q_M} =\frac{e_{ss}}{Q_M}+\phi_{MK}\frac{e_{ss}}{k_{Ess}}\frac{\mathrm{d}k_{Ess}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M}+\phi_{MR}\frac{e_{ss}}{r_{ss}}\frac{\mathrm{d}r_{ss}}{\mathrm{d}Q_M}.$$

Substitute from (A-27) and solve for $\mathrm{d}e_{ss}/\mathrm{d}Q_M$:

$$\frac{\mathrm{d}e_{ss}}{\mathrm{d}Q_M} =\frac{\frac{e_{ss}}{Q_M}+\phi_{MK}\frac{e_{ss}}{k_{Ess}}\frac{\mathrm{d}k_{Ess}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M}+\phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_K(1-\kappa_E)\frac{F_{ss}}{k_{Rss}}\frac{\mathrm{d}k_{Rss}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M}}{1-\phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E\frac{F_{ss}}{e_{Rss}}\frac{\mathrm{d}e_{Rss}}{\mathrm{d}e_{ss}}}. \tag{A-28}$$

From the final good production function, (A-22), (A-21), (A-26), and the transition equation for $k(t)$:

$$\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} = \alpha_K \frac{y_{ss}}{k_{Yss}} \frac{\mathrm{d}k_{Yss}}{\mathrm{d}k_{ss}} \frac{s}{\delta + g_L} \frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} + \alpha_E \frac{y_{ss}}{e_{Yss}} \frac{\mathrm{d}e_{Yss}}{\mathrm{d}e_{ss}} \frac{\mathrm{d}e_{ss}}{\mathrm{d}Q_M}.$$

Substitute from (A-28) and rearrange to solve for $\mathrm{d}y_{ss}/\mathrm{d}Q_M$:

$$\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} = \frac{\alpha_E \frac{y_{ss}}{e_{Yss}} \frac{\mathrm{d}e_{Yss}}{\mathrm{d}e_{ss}} \frac{\frac{e_{ss}}{Q_M}}{1-\phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E \frac{F_{ss}}{e_{Rss}}\frac{\mathrm{d}e_{Rss}}{\mathrm{d}e_{ss}}}}{1 - \alpha_K \frac{y_{ss}}{k_{Yss}} \frac{\mathrm{d}k_{Yss}}{\mathrm{d}k_{ss}} \frac{s}{\delta+g_L} - \alpha_E \frac{y_{ss}}{e_{Yss}} \frac{\mathrm{d}e_{Yss}}{\mathrm{d}e_{ss}} \frac{\phi_{MK}\frac{e_{ss}}{k_{Ess}}\frac{\mathrm{d}k_{Ess}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L} + \phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_K(1-\kappa_E)\frac{F_{ss}}{k_{Rss}}\frac{\mathrm{d}k_{Rss}}{\mathrm{d}k_{ss}}\frac{s}{\delta+g_L}}{1-\phi_{MR}\frac{e_{ss}}{r_{ss}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E\frac{F_{ss}}{e_{Rss}}\frac{\mathrm{d}e_{Rss}}{\mathrm{d}e_{ss}}}}.$$

Substitute for y_{ss}/k_{ss} from the transition equation for $k(t)$ and simplify some other terms, using constancy of shares from (A-21), (A-22), (A-23), and (A-26):

$$\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M} = \frac{\alpha_E y_{ss} \frac{1}{Q_M} \frac{1}{1-\phi_{MR}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E\frac{F_{ss}}{r_{ss}}}}{1 - \alpha_K - \alpha_E \frac{\phi_{MK}+\phi_{MR}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}(1-\kappa_E)\kappa_K\frac{F_{ss}}{r_{ss}}}{1-\phi_{MR}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}(\lambda+g_L)^{-1}\kappa_E\frac{F_{ss}}{r_{ss}}}}.$$

Substitute for F_{ss}/r_{ss} from (A-25):

$$\frac{\mathrm{d}y_{ss}}{\mathrm{d}Q_M}\frac{Q_M}{y_{ss}} = \frac{\alpha_E \frac{1}{1-\phi_{MR}\kappa_E}}{1 - \alpha_K - \alpha_E \frac{\phi_{MK}+\phi_{MR}(1-\kappa_E)\kappa_K}{\phi_{MK}+\phi_{ML}+\phi_{MR}(1-\kappa_E)}}.$$

The left-hand side is the elasticity of y_{ss} with respect to Q_M. The right-hand side is constant in Q_M and strictly positive. We have established the claim in the proposition about the elasticity.

F Proof of Proposition 4

Section D contains preliminaries. By Lemma A-2, any interior balanced growth path must have per-capita variables constant.

Use (A-4) in (A-9) and evaluate at a steady state:

$$0 = -\frac{\alpha_E}{e_{Yss}}\mu_{ss}(\rho+\lambda)\frac{e_{Yss}}{\alpha_E \phi_{MR}}\frac{r_{ss}}{e_{ss}} + \left[\mu_{ss} - \gamma_{ss}\right](1-\omega)\left(\frac{z_{ss}}{F_{ss}}\right)^{\omega}\frac{\partial F_{ss}}{\partial e_{Rss}},$$

where

$$F_{ss} \triangleq F(\ell_{Rss}, k_{Rss}, e_{Rss}).$$

Solve for μ_{ss}:

$$\mu_{ss} =\frac{(1-\omega)\left(\frac{z_{ss}}{F_{ss}}\right)^{\omega}\frac{\partial F_{ss}}{\partial e_{Rss}}}{(1-\omega)\left(\frac{z_{ss}}{F_{ss}}\right)^{\omega}\frac{\partial F_{ss}}{\partial e_{Rss}}-(\rho+\lambda)\frac{1}{\phi_{MR}}\frac{r_{ss}}{e_{ss}}}\gamma_{ss}.$$

Substitute into the steady-state version of (A-3):

$$0=(\rho-\Omega)\left[(1-\omega)\left(\frac{z_{ss}}{F_{ss}}\right)^{\omega}\frac{\partial F_{ss}}{\partial e_{Rss}}-(\rho+\lambda)\frac{1}{\phi_{MR}}\frac{r_{ss}}{e_{ss}}\right]-\omega\left(\frac{F_{ss}}{z_{ss}}\right)^{1-\omega}(\rho+\lambda)\frac{1}{\phi_{MR}}\frac{r_{ss}}{e_{ss}}.$$

From the transition equation for $z(t)$:

$$\frac{z_{ss}}{F_{ss}} = (\Omega - g_L)^{-\frac{1}{1-\omega}}. \tag{A-29}$$

Use (A-29) in the prior expression and substitute for the partial derivative of F_{ss}:

$$0 =(\rho-\Omega)(1-\omega)(\Omega-g_L)^{-\frac{\omega}{1-\omega}}\kappa_E A_E^{\frac{\sigma-1}{\sigma}}-\Big[(\rho-\Omega)+\omega(\Omega-g_L)\Big](\rho+\lambda)\frac{1}{\phi_{MR}}\frac{r_{ss}}{F_{ss}}\frac{e_{Rss}}{e_{ss}}\left(\frac{F_{ss}}{e_{Rss}}\right)^{\frac{\sigma-1}{\sigma}}.$$

And use (A-29) in the transition equation for $r(t)$, evaluated at a steady state:

$$F_{ss} =(\Omega-g_L)^{\frac{\omega}{1-\omega}}(\lambda+g_L)r_{ss}. \tag{A-30}$$

Substitute that into the prior expression and rearrange:

$$r_{ss}=\left(\frac{\phi_{MR}\kappa_E}{\lambda+\rho}\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}(\lambda+g_L)^{\frac{1}{\sigma}}\frac{e_{ss}}{e_{Rss}}\right)^{\frac{\sigma}{\sigma-1}}(\Omega-g_L)^{-\frac{\omega}{1-\omega}}A_E e_{Rss}. \tag{A-31}$$

When $\omega > 0$, equation (A-31) shows that a steady state requires $\Omega > g_L$. Using $\sigma \neq 1$, substitute for r_{ss} in (A-31) from (A-30), substitute for F_{ss} from (5), and rearrange:

$$e_{Rss}=\left(\frac{1-\kappa_E}{\kappa_E}\right)^{\frac{\sigma}{\sigma-1}}\frac{A_{LK}}{A_E}\left(\frac{1}{\frac{\lambda+g_L}{\lambda+\rho}\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}\phi_{MR}\frac{e_{ss}}{e_{Rss}}-1}\right)^{\frac{\sigma}{\sigma-1}}\ell_{Rss}^{\kappa_L}k_{Rss}^{\kappa_K}. \tag{A-32}$$

Rearrange (A-12) and (A-13) and evaluate at a steady state:

$$\ell_{Ess} =\frac{\phi_{ML}}{\kappa_L}\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{-\frac{\sigma-1}{\sigma}}\left(\frac{k_{Rss}}{\ell_{Rss}}\right)^{-\kappa_K\frac{\sigma-1}{\sigma}}\left(\frac{e_{Rss}}{\ell_{Rss}}\right)^{-\frac{1}{\sigma}}e_{ss},$$

$$k_{Ess} =\frac{\phi_{MK}}{\kappa_K}\frac{\kappa_E}{1-\kappa_E}\left(\frac{A_{LK}}{A_E}\right)^{-\frac{\sigma-1}{\sigma}}\left(\frac{\ell_{Rss}}{k_{Rss}}\right)^{-\kappa_L\frac{\sigma-1}{\sigma}}\left(\frac{e_{Rss}}{k_{Rss}}\right)^{-\frac{1}{\sigma}}e_{ss}.$$

Substitute into the per-capita version of (2) evaluated at the steady state, substitute for r_{ss} from (A-31), substitute for e_{Rss} from (A-32), and rearrange to obtain:

$$\begin{aligned}\frac{e_{ss}}{e_{Rss}} =&(\Omega - g_L)^{\frac{\omega}{1-\omega}(\sigma-1)} \left[(1-\kappa_E) A_{LK}^{\frac{\sigma-1}{\sigma}}\right]^{\sigma \frac{\phi_{ML}+\phi_{MK}}{\phi_{MR}}} \\ &\left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_M \left(\frac{\phi_{ML}}{\kappa_L}\right)^{\phi_{ML}} \left(\frac{\phi_{MK}}{\kappa_K}\right)^{\phi_{MK}} \ell_{Rss}^{\kappa_K \phi_{ML} - \kappa_L \phi_{MK}} k_{Rss}^{\kappa_L \phi_{MK} - \kappa_K \phi_{ML}}\right]^{\frac{-(\sigma-1)}{\phi_{MR}}} \\ &\left(\frac{1}{\lambda+\rho} \phi_{MR} \frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)} (\lambda+g_L)^{\frac{1}{\sigma}}\right)^{-\sigma} \\ &\left[\frac{1}{\frac{\lambda+g_L}{\lambda+\rho} \frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)} \phi_{MR} \frac{e_{ss}}{e_{Rss}} - 1}\right]^{\frac{\phi_{ML}+\phi_{MK}}{\phi_{MR}}} .\end{aligned}$$

Apply Assumption 2 to eliminate ℓ_{Rss} and k_{Rss}:

$$\begin{aligned}\frac{e_{ss}}{e_{Rss}} =&(\Omega - g_L)^{\frac{\omega}{1-\omega}(\sigma-1)} \left[(1-\kappa_E) A_{LK}^{\frac{\sigma-1}{\sigma}}\right]^{\sigma \frac{\phi_{ML}+\phi_{MK}}{\phi_{MR}}} \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_M \left(\frac{\phi_{ML}}{\kappa_L}\right)^{\phi_{ML}} \left(\frac{\phi_{MK}}{\kappa_K}\right)^{\phi_{MK}}\right]^{\frac{-(\sigma-1)}{\phi_{MR}}} \\ &\left(\frac{1}{\lambda+\rho} \phi_{MR} \frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)} (\lambda+g_L)^{\frac{1}{\sigma}}\right)^{-\sigma} \\ &\left[\frac{1}{\frac{\lambda+g_L}{\lambda+\rho} \frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)} \phi_{MR} \frac{e_{ss}}{e_{Rss}} - 1}\right]^{\frac{\phi_{ML}+\phi_{MK}}{\phi_{MR}}} . \qquad \text{(A-33)}\end{aligned}$$

A strictly positive solution requires either $\omega = 0$ or $\Omega > g_L$. If $e_{ss}/e_{Rss} \in \left(0, \frac{\lambda+\rho}{\lambda+g_L} \frac{1}{\phi_{MR}} \frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}\right)$, the left-hand side is strictly positive but the right-hand side is strictly negative, so the equation cannot hold. For $e_{ss}/e_{Rss} > \frac{\lambda+\rho}{\lambda+g_L} \frac{1}{\phi_{MR}} \frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}$ and either $\omega = 0$ or $\rho > \Omega$, the right-hand side of (A-33) monotonically decreases in e_{ss}/e_{Rss}, going to 0 as $e_{ss}/e_{Rss} \to \infty$ and going to infinity as $e_{ss}/e_{Rss} \to \frac{\lambda+\rho}{\lambda+g_L} \frac{1}{\phi_{MR}} \frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}$ from above. The left-hand side monotonically increases in e_{ss}/e_{Rss} and is strictly positive. So when either $\omega = 0$ or $\Omega > g_L$, there is a unique $e_{ss}/e_{Rss} > 0$ that solves the equation. That solution has:

$$\frac{e_{ss}}{e_{Rss}} > \frac{\lambda+\rho}{\lambda+g_L} \frac{1}{\phi_{MR}} \frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}.$$

From (A-31), the steady state has $r_{ss} > 0$ if and only if $\Omega > g_L$. Because $e_{Yss} = e_{ss} - e_{Rss}$, the steady state has strictly positive final good production if and only if $e_{ss}/e_{Rss} > 1$, which holds because $\rho > \Omega, g_L$. From (A-20), $EROI(t)$ is constant along the balanced growth path and

$$EROI_{ss} > 1 + \frac{\lambda+\rho}{\lambda+g_L}\frac{1}{\phi_{MR}}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}g_L.$$

We have proved part i of the proposition.

The right-hand side of equation (A-33) increases in A_E and Q_M if $\sigma < 1$ and decreases in A_E and Q_M if $\sigma > 1$. So the e_{ss}/e_{Rss} that solves (A-33)—and, by (A-20), $EROI_{ss}$—increases in A_E and Q_M if $\sigma < 1$ and decreases in A_E and Q_M if $\sigma > 1$. We have proved parts ii and iii of the proposition.

The claims about prices are implied by the existence of steady states in per-capita, current value terms for $\gamma(t)$ and $\mu(t)$.

G Proof of Proposition 5

The setting of Proposition 5 matches those of Propositions 3 and 4 as $\phi_{MR} \to 1$ (which implies $\phi_{ML}, \phi_{MK} \to 0$) and with Q_D in place of Q_M. Lemma A-2 still applies here. Therefore any interior balanced growth path must have per-capita variables constant and prices growing at rate $\rho - g_L$.

The case with $\sigma = 1$ (i.e., part i of the proposition) follows from taking the limit as $\phi_{MR} \to 1$ in the proof of Proposition 3.

Now consider $\sigma \neq 1$. Using $e_{ss} = Q_D r_{ss}$ in equation (A-31) and taking $\phi_{MR} \to 1$ (and $\phi_{ML}, \phi_{MK} \to 0$) yields:

$$\frac{e_{ss}}{e_{Rss}} = \left(\frac{(\lambda+\rho)\left[(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)\right]}{(\rho-\Omega)(1-\omega)\kappa_E}\right)^{\sigma}\frac{[A_E Q_D]^{1-\sigma}}{(\lambda+g_L)(\Omega-g_L)^{\frac{\omega}{1-\omega}(1-\sigma)}}. \tag{A-34}$$

From (A-30), an interior solution requires either $\omega = 0$ or $\Omega > g_L$. In either case, e_{ss}/e_{Rss} is interior.

The steady state has strictly positive final good production if and only if $e_{ss}/e_{Rss} > 1$. From (A-34), $e_{ss}/e_{Rss} > 1$ if and only if

$$\kappa_E^{-\sigma}[A_E Q_D]^{1-\sigma} > (\lambda+g_L)^{1-\sigma}\left(\frac{\lambda+g_L}{\lambda+\rho}\right)^{\sigma}(\Omega-g_L)^{\frac{\omega}{1-\omega}(1-\sigma)}\left(\frac{(1-\omega)(\rho-\Omega)}{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}\right)^{\sigma}. \tag{A-35}$$

For $\sigma < 1$, inequality (A-35) holds if and only if

$$A_E Q_D > (\lambda+g_L)\left(\frac{\lambda+\rho}{\kappa_E(\lambda+g_L)}\right)^{\frac{\sigma}{\sigma-1}}(\Omega-g_L)^{\frac{\omega}{1-\omega}}\left(\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}\right)^{\frac{\sigma}{\sigma-1}},$$

where the right-hand side is χ from the proposition. We have proved part ii of the proposition. For $\sigma > 1$, inequality (A-35) holds if and only if

$$A_E Q_D < (\lambda + g_L) \left(\frac{\lambda + \rho}{\kappa_E(\lambda + g_L)} \right)^{\frac{\sigma}{\sigma-1}} (\Omega - g_L)^{\frac{\omega}{1-\omega}} \left(\frac{(1-\omega)(\rho - \Omega) + \omega(\rho - g_L)}{(1-\omega)(\rho - \Omega)} \right)^{\frac{\sigma}{\sigma-1}},$$

where the right-hand side is χ from the proposition. We have proved part iii of the proposition.

$EROI(t)$ along a balanced growth path follows from equations (A-20) and (A-34). We have proved part iv of the proposition.

H Proof of Proposition 7

Consider a case with $\ell_R(t), k_R(t) = 0$ and $e_R(t) > 0$. By labor and capital market-clearing, $\ell_Y(t) = 1$ and $k_Y(t) = k(t)$. Equilibrium then solves the following maximization problem:

$$\begin{aligned}
\max_{E_Y(\cdot), E_R(\cdot)} \int_0^\infty & e^{-\rho t} L(t) u\left(\frac{(1-s) A L(t)^{\alpha_L} K(t)^{\alpha_K} E_Y(t)^{\alpha_E}}{L(t)} \right) \mathrm{d}t \\
\text{s.t. } \dot{L}(t) =& g_L L(t) \\
\dot{K}(t) =& s A L(t)^{\alpha_L} K(t)^{\alpha_K} E_Y(t)^{\alpha_E} - \delta K(t) \\
\dot{R}(t) =& \kappa_E^{\frac{\sigma}{\sigma-1}} A_E E_R(t) - \lambda R(t) \\
E(t) =& Q_S R(t) \\
E(t) =& E_Y(t) + E_R(t).
\end{aligned}$$

Converting to per-capita and substituting from the market-clearing condition, this is equivalent to:

$$\begin{aligned}
\max_{e_Y(\cdot)} \int_0^\infty & e^{-(\rho - g_L)t} u\left((1-s) A [k(t)]^{\alpha_K} [e_Y(t)]^{\alpha_E} \right) \mathrm{d}t \\
\text{s.t. } \dot{k}(t) =& s A [k(t)]^{\alpha_K} [e_Y(t)]^{\alpha_E} - (\delta + g_L) k(t) \\
\dot{r}(t) =& \kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_S r(t) - e_Y(t)] - (\lambda + g_L) r(t).
\end{aligned}$$

The current-value Hamiltonian is:

$$\begin{aligned}
& u\left((1-s) A [k(t)]^{\alpha_K} [e_Y(t)]^{\alpha_E} \right) + \nu(t) \left(s A [k(t)]^{\alpha_K} [e_Y(t)]^{\alpha_E} - (\delta + g_L) k(t) \right) \\
& + \mu(t) \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_S r(t) - e_Y(t)] - (\lambda + g_L) r(t) \right].
\end{aligned}$$

The costate equations are:

$$\dot{\nu}(t) =-\left[(1-s)u'((1-s)y(t))+s\nu(t)\right]\frac{\alpha_K y(t)}{k(t)}+(\rho+\delta)\nu(t), \tag{A-36}$$

$$\dot{\mu}(t) =\mu(t)\left[\rho+\lambda-\kappa_E^{\frac{\sigma}{\sigma-1}}A_E Q_S\right]. \tag{A-37}$$

The condition to maximize the Hamiltonian is:

$$0 =\left[(1-s)u'((1-s)y(t))+s\nu(t)\right]\frac{\alpha_E y(t)}{e_Y(t)}-\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E. \tag{A-38}$$

Substituting into (A-36), we obtain:

$$\dot{\nu}(t) =-\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\frac{\alpha_K e_Y(t)}{\alpha_E k(t)}+(\rho+\delta)\nu(t). \tag{A-39}$$

Time-differentiating (A-38), we obtain:

$$\begin{aligned}0 =&\left[(1-s)u'((1-s)y(t))+s\nu(t)\right]\alpha_E\left[\frac{\dot{y}(t)}{e_Y(t)}-\frac{y(t)}{e_Y(t)}\frac{\dot{e}_Y(t)}{e_Y(t)}\right]\\&+(1-s)^2u''((1-s)y(t))\frac{\alpha_E y(t)}{e_Y(t)}\dot{y}(t)+s\dot{\nu}(t)\frac{\alpha_E y(t)}{e_Y(t)}-\dot{\mu}(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E.\end{aligned}$$

Substitute from (A-37), (A-38), and (A-39):

$$\begin{aligned}0 =&\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[\frac{\dot{y}(t)}{y(t)}-\frac{\dot{e}_Y(t)}{e_Y(t)}\right]+\alpha_E(1-s)^2u''((1-s)y(t))\frac{y(t)}{e_Y(t)}\dot{y}(t)\\&+sy(t)\left[-\alpha_K\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\frac{\mu(t)}{k(t)}+\alpha_E(\rho+\delta)\frac{\nu(t)}{e_Y(t)}\right]-\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[(\rho+\lambda)-\kappa_E^{\frac{\sigma}{\sigma-1}}A_E Q_S\right].\end{aligned}$$

Time-differentiate the final good production function and substitute for $\dot{y}(t)$ in the previous equation:

$$\begin{aligned}0 =&\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[\frac{\dot{y}(t)}{y(t)}-\frac{\dot{e}_Y(t)}{e_Y(t)}\right]+\alpha_E(1-s)^2u''((1-s)y(t))\frac{y(t)}{e_Y(t)}y(t)\left[\alpha_K\frac{\dot{k}(t)}{k(t)}+\alpha_E\frac{\dot{e}_Y(t)}{e_Y(t)}\right]\\&+sy(t)\left[-\alpha_K\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\frac{\mu(t)}{k(t)}+\alpha_E(\rho+\delta)\frac{\nu(t)}{e_Y(t)}\right]-\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[(\rho+\lambda)-\kappa_E^{\frac{\sigma}{\sigma-1}}A_E Q_S\right].\end{aligned}$$

Use Assumption 3 and rearrange:

$$\begin{aligned}0 =&\mu(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[\frac{\dot{y}(t)}{y(t)}-\frac{\dot{e}_Y(t)}{e_Y(t)}-(\rho+\lambda)+\kappa_E^{\frac{\sigma}{\sigma-1}}A_E Q_S-s\alpha_K\frac{y(t)}{k(t)}\right]\\&-\alpha_E\frac{1}{e_Y(t)}\left[\alpha_K\frac{\dot{k}(t)}{k(t)}+\alpha_E\frac{\dot{e}_Y(t)}{e_Y(t)}\right]+\nu(t)s\alpha_E(\rho+\delta)\frac{y(t)}{e_Y(t)}.\end{aligned} \tag{A-40}$$

Now consider a balanced growth path. $\dot{k}(t)/k(t)$ is constant over time if and only if

$$\frac{y(t)}{k(t)} = \frac{g_k + \delta + g_L}{s} \tag{A-41}$$

Therefore $g_k = g_y$, which, in the final good production function, implies that

$$\begin{aligned} g_y =& \alpha_K g_k + \alpha_E g_{e_Y} \\ \Leftrightarrow g_y =& \frac{\alpha_E}{\alpha_E + \alpha_L} g_{e_Y}. \end{aligned}$$

The resource transition equation implies:

$$\frac{\dot{r}(t)}{r(t)} = \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[Q_S - \frac{e_Y(t)}{r(t)} \right] - (\lambda + g_L).$$

The growth rate of r is constant if and only if $e_Y(t)$ grows at g_r. The growth rate of r is the same as the growth rate of e. Therefore $e_R(t)$ grows at rate g_e and

$$g_y = \frac{\alpha_E}{\alpha_E + \alpha_L} g_e. \tag{A-42}$$

Using this and $g_k = g_y$ in (A-41),

$$\frac{y(t)}{k(t)} = \frac{\frac{\alpha_E}{\alpha_E + \alpha_L} g_e + \delta + g_L}{s} \tag{A-43}$$

Use $g_k = g_y$ in (A-40) and substitute from (A-42) and (A-43):

$$\begin{aligned} 0 =& \mu(t) \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[-\frac{\alpha_L}{\alpha_E + \alpha_L} g_e - (\rho + \lambda) + \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - \alpha_K \left(\frac{\alpha_E}{\alpha_E + \alpha_L} g_e + \delta + g_L \right) \right] \\ & - \alpha_E \frac{1}{e_Y(t)} \frac{\alpha_E}{\alpha_E + \alpha_L} g_e + \nu(t) s \alpha_E (\rho + \delta) \frac{y(t)}{e_Y(t)}. \end{aligned}$$

Multiply through by $e_Y(t)$ and multiply the final term by $k(t)/k(t)$, again using (A-43):

$$\begin{aligned} 0 =& \mu(t) e_Y(t) \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left[-\frac{\alpha_L}{\alpha_E + \alpha_L} g_e - (\rho + \lambda) + \kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - \alpha_K \left(\frac{\alpha_E}{\alpha_E + \alpha_L} g_e + \delta + g_L \right) \right] \\ & - \alpha_E \frac{\alpha_E}{\alpha_E + \alpha_L} g_e + \nu(t) k(t) \alpha_E (\rho + \delta) \left(\frac{\alpha_E}{\alpha_E + \alpha_L} g_e + \delta + g_L \right). \end{aligned} \tag{A-44}$$

$\nu(t)k(t)$ is constant if and only if $\dot{\nu}(t)/\nu(t) = -g_k$. In that case, (A-39) implies

$$\nu(t) k(t) = \mu(t) e_Y(t) \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \frac{\alpha_K}{\alpha_E} \frac{1}{g_k + \rho + \delta}. \tag{A-45}$$

Substitute into (A-44) and use (A-42):

$$0 =\mu(t)e_Y(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[-\frac{\alpha_L}{\alpha_E+\alpha_L}g_e-(\rho+\lambda)+\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S-\alpha_K\frac{\frac{\alpha_E}{\alpha_E+\alpha_L}g_e+g_L+\delta}{\frac{\alpha_E}{\alpha_E+\alpha_L}g_e+\rho+\delta}\frac{\alpha_E}{\alpha_E+\alpha_L}g_e\right] - \alpha_E\frac{\alpha_E}{\alpha_E+\alpha_L}g_e. \tag{A-46}$$

If g_e is constant, so too is $\mu(t)e_Y(t)$. From (A-37), $\mu(t)$ grows at rate $\rho+\lambda-\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S$, so $\mu(t)e_Y(t)$ constant implies:

$$g_e =\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S-(\rho+\lambda). \tag{A-47}$$

Substituting into (A-46), we find:

$$0 =\frac{\alpha_E}{\alpha_E+\alpha_L}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S-(\rho+\lambda)\right) \left\{\mu(t)e_Y(t)\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[1-\alpha_K\frac{\frac{\alpha_E}{\alpha_E+\alpha_L}\left[\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S-(\rho+\lambda)\right]+g_L+\delta}{\frac{\alpha_E}{\alpha_E+\alpha_L}\left[\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S-(\rho+\lambda)\right]+\rho+\delta}\right]-\alpha_E\right\}.$$

This holds if either $g_e=0$ or

$$\mu(t)e_Y(t) =\frac{\alpha_E}{\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left[1-\alpha_K\frac{\frac{\alpha_E}{\alpha_E+\alpha_L}\left[\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S-(\rho+\lambda)\right]+g_L+\delta}{\frac{\alpha_E}{\alpha_E+\alpha_L}\left[\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S-(\rho+\lambda)\right]+\rho+\delta}\right]}.$$

The right-hand side is strictly positive for $g_e\neq 0$ if and only if:

$$1 >\alpha_K\frac{\frac{\alpha_E}{\alpha_E+\alpha_L}\left[\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S-(\rho+\lambda)\right]+g_L+\delta}{\frac{\alpha_E}{\alpha_E+\alpha_L}\left[\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S-(\rho+\lambda)\right]+\rho+\delta}. \tag{A-48}$$

Observe that $y(t)/k(t)>0$ requires, from (A-43), that the numerator on the right-hand side of (A-48) be strictly positive and that $\nu(t)k(t)>0$ requires, from (A-45), that the denominator on the right-hand side of (A-48) be strictly positive. Using $\rho>g_L$, these two conditions are jointly satisfied if and only if the numerator in (A-48) is strictly positive, so if and only if

$$\kappa_E^{\frac{\sigma}{\sigma-1}}A_EQ_S>\rho+\lambda-\frac{\alpha_E+\alpha_L}{\alpha_E}(g_L+\delta). \tag{A-49}$$

If this last inequality holds, then inequality (A-48) also holds, because $\alpha_K<1$. If $\mu(t)e_Y(t)$ is weakly negative, then (A-48) would not hold and so (A-49) would not hold, which means that the path is not feasible. So $\mu(t)e_Y(t) > 0$ on a feasible path, as would be expected. Finally, observe that $r(t)$ grows at a constant rate if and only if

$$\frac{e_R(t)}{e(t)} = \frac{g_e + \lambda + g_L}{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S}. \tag{A-50}$$

Substituting from (A-47), we have $e_R(t)/e(t) \in (0,1)$ if and only if

$$\frac{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S - (\rho - g_L)}{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S} \in (0,1).$$

Because $\rho > g_L$, this condition holds if and only if

$$\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S > \rho - g_L. \tag{A-51}$$

We have found a feasible path along which $\ell_R(t), k_R(t) = 0$ with $e_R(t) > 0$ and all variables growing at a constant rate. If $A_E Q_S > \chi$, then $\kappa_E^{\frac{\sigma}{\sigma-1}} A_E Q_S > \rho + \lambda$ (observing that $(\lambda + g_L)[(\lambda + \rho)/(\lambda + g_L)]^{\frac{\sigma}{\sigma-1}}$ decreases in σ for $\sigma > 1$) and thus inequalities (A-49) and (A-51) hold. Part i of the proposition follows from $g_y = g_k$ and (A-42). Part ii follows from (A-47). Part iii follows from (7), (A-47), and (A-50).

I Proof of Proposition 8

Equilibrium solves the following maximization problem:

$$\max_{L_Y(\cdot),K_Y(\cdot),E_Y(\cdot),E_R(\cdot),E_B(\cdot),B_R(\cdot),B_B(\cdot)} \int_0^\infty e^{-\rho t} L(t) u\left(\frac{(1-s) A L_Y(t)^{\alpha_L} K_Y(t)^{\alpha_K} E_Y(t)^{\alpha_E}}{L(t)} \right) \mathrm{d}t$$
$$\begin{aligned}
\text{s.t. } \dot{L}(t) =& g_L L(t) \\
\dot{K}(t) =& s A L(t)^{\alpha_L} K_Y(t)^{\alpha_K} E_Y(t)^{\alpha_E} - \delta K(t) \\
\dot{R}(t) =& A_E B_R(t)^{\kappa_B} E_R(t)^{\kappa_E} - \lambda R(t) \\
\dot{B}(t) =& A_B B_B(t)^{\beta_B} E_B(t)^{\beta_E} - \Psi B(t) \\
E(t) =& Q_S R(t) \\
L(t) =& L_Y(t) \\
K(t) =& K_Y(t) \\
E(t) =& E_Y(t) + E_R(t) + E_B(t) \\
B(t) =& B_B(t) + B_R(t).
\end{aligned}$$

Converting to per-capita and substituting from the market-clearing conditions, this is equivalent to:

$$\max_{e_Y(\cdot),e_R(\cdot),b_R(\cdot)} \int_0^\infty e^{-(\rho-g_L)t} u\bigg((1-s)A[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E}\bigg)\,\mathrm{d}t$$
$$\text{s.t. } \dot{k}(t) = sA[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E} - (\delta+g_L)k(t)$$
$$\dot{r}(t) = A_E b_R(t)^{\kappa_B} e_R(t)^{\kappa_E} - (\lambda+g_L)r(t)$$
$$\dot{b}(t) = A_B[b(t)-b_R(t)]^{\beta_B}[Q_S r(t) - e_Y(t) - e_R(t)]^{\beta_E} - (\Psi+g_L)b(t).$$

$\dot{k}(t)/k(t)$ is constant over time if and only if $y(t)/K(t)$ is constant over time. Therefore $g_k = g_y$, which, in the final good production function, implies that

$$g_y = \alpha_K g_k + \alpha_E g_{e_Y}$$
$$\Leftrightarrow g_y = \frac{\alpha_E}{\alpha_E+\alpha_L} g_{e_Y}.$$

We have established part i of the proposition.

The current-value Hamiltonian is:

$$u\bigg((1-s)A[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E}\bigg) + \nu(t)\bigg(sA[k(t)]^{\alpha_K}[e_Y(t)]^{\alpha_E} - (\delta+g_L)k(t)\bigg)$$
$$+ \mu(t)\bigg[A_E b_R(t)^{\kappa_B} e_R(t)^{\kappa_E} - (\lambda+g_L)r(t)\bigg]$$
$$+ \Upsilon(t)\left[A_B[b(t)-b_R(t)]^{\beta_B}[Q_S r(t) - e_Y(t) - e_R(t)]^{\beta_E} - (\Psi+g_L)b(t)\right].$$

The costate equations are:

$$\dot{\nu}(t) = -\frac{\alpha_K y(t)}{k(t)}\left[(1-s)u'((1-s)y(t)) + s\nu(t)\right] + (\rho+\delta)\nu(t), \tag{A-52}$$

$$\dot{\mu}(t) = -Q_S\beta_E \frac{\Upsilon(t)}{e_B(t)} A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E} + \mu(t)[\rho+\lambda], \tag{A-53}$$

$$\dot{\Upsilon}(t) = -\beta_B \frac{\Upsilon(t)}{b_B(t)} A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E} + \Upsilon(t)[\rho+\Psi]. \tag{A-54}$$

The conditions to maximize the Hamiltonian are:

$$0 = \bigg[(1-s)u'((1-s)y(t)) + s\nu(t)\bigg]\frac{\alpha_E y(t)}{e_Y(t)} - \beta_E \frac{\Upsilon(t)}{e_B(t)} A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E}, \tag{A-55}$$

$$0 = \kappa_E \frac{\mu(t)}{e_R(t)} A_E b_R(t)^{\kappa_B} e_R(t)^{\kappa_E} - \beta_E \frac{\Upsilon(t)}{e_B(t)} A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E}, \tag{A-56}$$

$$0 = \kappa_B \frac{\mu(t)}{b_R(t)} A_E b_R(t)^{\kappa_B} e_R(t)^{\kappa_E} - \beta_B \frac{\Upsilon(t)}{b_B(t)} A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E}. \tag{A-57}$$

Equations (A-56) and (A-57) imply:

$$\frac{\kappa_E b_R(t)}{\kappa_B e_R(t)} = \frac{\beta_E b_B(t)}{\beta_B e_B(t)}. \tag{A-58}$$

Rearrange equation (A-55):

$$\frac{\Upsilon(t)}{e_B(t)} A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E} = \Big[(1-s)u'((1-s)y(t)) + s\nu(t)\Big] \frac{\alpha_E y(t)}{\beta_E e_Y(t)}. \tag{A-59}$$

Time-differentiating yields:

$$\begin{aligned}
&\left[\frac{\dot{\Upsilon}(t)}{\Upsilon(t)} - \frac{\dot{e}_B(t)}{e_B(t)} + \beta_B \frac{\dot{b}_B(t)}{b_B(t)} + \beta_E \frac{\dot{e}_B(t)}{e_B(t)}\right] \frac{\Upsilon(t)}{e_B(t)} A_B b_B(t)^{\beta_B} e_B(t)^{\beta_E} \\
=&\Big[(1-s)^2 \dot{y}(t) u''((1-s)y(t)) + s\dot{\nu}(t)\Big] \frac{\alpha_E y(t)}{\beta_E e_Y(t)} \\
&+ \Big[(1-s)u'((1-s)y(t)) + s\nu(t)\Big] \frac{\alpha_E y(t)}{\beta_E e_Y(t)} \left[\frac{\dot{y}(t)}{y(t)} - \frac{\dot{e}_Y(t)}{e_Y(t)}\right].
\end{aligned}$$

Substitute from (A-52), (A-54), and (A-59):

$$\begin{aligned}
&- \beta_B A_B \left(\frac{e_B(t)}{b_B(t)}\right)^{\beta_E} + \rho + \Psi - \frac{\dot{e}_B(t)}{e_B(t)} + \beta_B \frac{\dot{b}_B(t)}{b_B(t)} + \beta_E \frac{\dot{e}_B(t)}{e_B(t)} \\
=&\frac{(1-s)^2 \dot{y}(t) u''((1-s)y(t)) + s(\rho+\delta)\nu(t)}{(1-s)u'((1-s)y(t)) + s\nu(t)} - s\frac{\alpha_K y(t)}{k(t)} + \frac{\dot{y}(t)}{y(t)} - \frac{\dot{e}_Y(t)}{e_Y(t)}.
\end{aligned}$$

Substitute from the transition equation for $b(t)$, for $\nu(t)$ from (A-55), and for $\Upsilon(t)$ from (A-56), apply Assumption 3, and rearrange:

$$\begin{aligned}
&\frac{\kappa_E}{\alpha_E} \mu(t) e_Y(t) A_E \left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B} \\
&\left[-\left(\frac{\dot{b}(t)}{b(t)} + \Psi + g_L\right)\beta_B \frac{b(t)}{b_B(t)} + \Psi - \delta - \frac{\dot{e}_B(t)}{e_B(t)} + \beta_B \frac{\dot{b}_B(t)}{b_B(t)} + \beta_E \frac{\dot{e}_B(t)}{e_B(t)} + s\frac{\alpha_K y(t)}{k(t)} - \frac{\dot{y}(t)}{y(t)} + \frac{\dot{e}_Y(t)}{e_Y(t)}\right] \\
=& - \left(\frac{\dot{y}(t)}{y(t)} + \rho + \delta\right).
\end{aligned}$$

Substitute constant growth rates (recognizing that $e(t)$ and $b(t)$ must grow at the same rate g_e), substitute for $y(t)/k(t)$ from the transition equation for $k(t)$, use $g_y = g_k = \frac{\alpha_E}{1-\alpha_K} g_e$, and substitute from (A-58):

$$\begin{aligned}&\frac{\kappa_E}{\alpha_E}\mu(t)e_Y(t)A_E\left(\frac{\kappa_B\beta_E b_B(t)}{\kappa_E\beta_B e_B(t)}\right)^{\kappa_B}\\&\qquad\left[-\left(g_e+\Psi+g_L\right)\beta_B\frac{b(t)}{b_B(t)}+\Psi-\delta+\alpha_K\left(\delta+g_L\right)+(1-\alpha_E)g_E\right]\\&=-\left[\frac{\alpha_E}{1-\alpha_K}g_e+\rho+\delta\right].\end{aligned}\tag{A-60}$$

For this to hold, $\mu(t)e_Y(t)$ must be constant. Therefore $g_\mu=-g_e$. Use this in (A-53):

$$g_e+\rho+\lambda=Q_S\beta_E\frac{\Upsilon(t)}{\mu(t)}A_B\left(\frac{b_B(t)}{e_B(t)}\right)^{\beta_B}.\tag{A-61}$$

For this to hold, $\Upsilon(t)$ must grow at the same rate as $\mu(t)$. Use that in (A-54) and rearrange:

$$\frac{e_B(t)}{b_B(t)}=\left(\frac{g_e+\rho+\Psi}{\beta_B A_B}\right)^{\frac{1}{\beta_E}}.\tag{A-62}$$

Equation (A-56) implies:

$$\frac{\Upsilon(t)}{e_B(t)}A_B b_B(t)^{\beta_B}e_B(t)^{\beta_E}=\frac{\kappa_E}{\beta_E}\frac{\mu(t)}{e_R(t)}A_E b_R(t)^{\kappa_B}e_R(t)^{\kappa_E}.$$

Substitute from (A-61) and (A-58), and then substitute from (A-62):

$$g_e+\rho+\lambda=\kappa_E A_E Q_S\left(\frac{\kappa_B\beta_E}{\kappa_E\beta_B}\left(\frac{g_e+\rho+\Psi}{\beta_B A_B}\right)^{-\frac{1}{\beta_E}}\right)^{\kappa_B}.\tag{A-63}$$

The left-hand side monotonically increases in g_e, and the right-hand side monotonically decreases in g_e. As $g_e\to\infty$, the left-hand side goes to ∞ and the right-hand side goes to zero. As $g_e\to-(\rho+\Psi)$ from above, the left-hand side goes to a finite value and the right-hand side goes to $-\infty$. So there is exactly one intersection at some $g_e\in\big(-[\rho+\Psi],\infty\big)$. That intersection has $g_e>0$ if and only if the right-hand side is greater than the left-hand side at $g_e=0$, so if and only if

$$\kappa_E A_E Q_S>(\rho+\lambda)\left(\frac{\kappa_E\beta_B}{\kappa_B\beta_E}\left(\frac{\rho+\Psi}{\beta_B A_B}\right)^{\frac{1}{\beta_E}}\right)^{\kappa_B}.$$

This is equivalent to the condition on χ_0 given in the proposition. Because the right-hand side of (A-63) increases in Q_S while the left-hand side of (A-63) is independent of Q_S, g_e

increases in Q_S. Similar analysis holds that g_e increases in A_E and A_B and decreases in Ψ, ρ, and λ. We have established part ii of the proposition.

We established that $\mu(t)$ and $\Upsilon(t)$ grow at rate $-g_e$. Therefore the current-value prices of $R(t)$ and $B(t)$ each grow at rate $\rho-(g_e+g_L)$. These prices increase over time if and only if $g_e<\rho-g_L$. Part iii of the proposition follows from that observation and, from (A-63), that g_e increases in $A_E Q_S$.

Now consider the feasibility of the solution. From the transition equation for $r(t)$,

$$g_e = Q_S A_E \left(\frac{b_R(t)}{e_R(t)}\right)^{\kappa_B} \frac{e_R(t)}{e(t)} - (\lambda + g_L).$$

Substitute from (A-58):

$$g_e = Q_S A_E \left(\frac{\kappa_B \beta_E b_B(t)}{\kappa_E \beta_B e_B(t)}\right)^{\kappa_B} \frac{e_R(t)}{e(t)} - (\lambda + g_L).$$

Rearrange:

$$\frac{e(t)}{e_R(t)} = Q_S A_E \left(\frac{\kappa_B \beta_E b_B(t)}{\kappa_E \beta_B e_B(t)}\right)^{\kappa_B} \frac{1}{g_e + \lambda + g_L}.$$

And substitute from (A-62):

$$\frac{e(t)}{e_R(t)} = Q_S A_E \left(\frac{\kappa_B \beta_E}{\kappa_E \beta_B} \left(\frac{g_e + \rho + \Psi}{\beta_B A_B}\right)^{\frac{-1}{\beta_E}}\right)^{\kappa_B} \frac{1}{g_e + \lambda + g_L}. \tag{A-64}$$

Feasibility requires $g_e > -(\lambda + g_L)$ and

$$g_e + \lambda + g_L < Q_S A_E \left(\frac{\kappa_B \beta_E}{\kappa_E \beta_B} \left(\frac{g_e + \rho + \Psi}{\beta_B A_B}\right)^{\frac{-1}{\beta_E}}\right)^{\kappa_B}.$$

Substituting from (A-63), this is equivalent to:

$$g_e > -\frac{1}{1-\kappa_E}(\rho - \kappa_E g_L) - \lambda.$$

This condition is satisfied when $g_e > 0$.

From the transition equation for $b(t)$,

$$g_e = A_B \left(\frac{e_B(t)}{b_B(t)}\right)^{\beta_E} \frac{b_B(t)}{b(t)} - (\Psi + g_L).$$

Rearrange and substitute from (A-62):

$$\frac{b(t)}{b_B(t)} = \frac{g_e + \rho + \Psi}{\beta_B} \frac{1}{g_e + \Psi + g_L}. \tag{A-65}$$

Feasibility on the robot side requires

$$1 < \frac{1}{\beta_B} \frac{g_e + \Psi + \rho}{g_e + \Psi + g_L},$$

while holds for all $g_e > 0$.

Substitute (A-64) into (7) and substitute from (A-63):

$$EROI(t) = 1 + \frac{1}{\kappa_E} \frac{g_e + g_L}{g_e + \lambda + g_L}(g_e + \rho + \lambda). \tag{A-66}$$

$EROI(t)$ depends on Q_S, A_E, and A_B only through g_e in (A-66). $EROI(t)$ increases in g_e in (A-66) because the fraction and the final term both increase in g_e. Part iv of the proposition follows from this observation and part ii. Part v also follows from using $g_e = 0$ in (A-66) and recalling that $EROI(t)$ increases in g_e in (A-66).

J Proof of Proposition A-1

Consider a case with $\ell_R(t), k_R(t) = 0$ and $e_R(t) > 0$. By labor and capital market-clearing, $\ell_Y(t) = 1$ and $k_Y(t) = k(t)$. Equilibrium solves the following maximization problem:

$$\begin{aligned}
&\max_{E_Y(\cdot)} \int_0^\infty e^{-\rho t} L(t) u\left((1-s)\frac{AL(t)^{\alpha_L} K(t)^{\alpha_K} E_Y(t)^{\alpha_E}}{L(t)}\right) \mathrm{d}t \\
\text{s.t. } \dot{L}(t) =& g_L L(t) \\
\dot{K}(t) =& sAL(t)^{\alpha_L} K(t)^{\alpha_K} E_Y(t)^{\alpha_E} - \delta K(t) \\
\dot{Z}(t) =& \Omega Z(t) - Z(t)^\omega \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E E_R(t)\right)^{1-\omega} \\
\dot{R}(t) =& Z(t)^\omega \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E E_R(t)\right)^{1-\omega} - \lambda R(t) \\
E(t) =& Q_D R(t) \\
E(t) =& E_Y(t) + E_R(t).
\end{aligned}$$

Converting to per-capita, substituting, and applying Assumption 3, this is equivalent to:

$$\begin{aligned}
&\max_{e_Y(\cdot)} \int_0^\infty e^{-(\rho - g_L)t} \ln\left((1-s)Ak(t)^{\alpha_K} e_Y(t)^{\alpha_E}\right) \mathrm{d}t \\
\text{s.t. } \dot{k}(t) =& sAk(t)^{\alpha_K} e_Y(t)^{\alpha_E} - (\delta + g_L)k(t) \\
\dot{z}(t) =& (\Omega - g_L)z(t) - z(t)^\omega \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{1-\omega} \\
\dot{r}(t) =& z(t)^\omega \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{1-\omega} - (\lambda + g_L)r(t).
\end{aligned}$$

The current-value Hamiltonian is:

$$\ln\left((1-s)Ak(t)^{\alpha_K}e_Y(t)^{\alpha_E}\right) + \nu(t)\left[sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E} - (\delta+g_L)k(t)\right]$$
$$+\gamma(t)\left[(\Omega-g_L)z(t) - z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}\right]$$
$$+\mu(t)\left[z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega} - (\lambda+g_L)r(t)\right].$$

The costate equations are:

$$\dot{\nu}(t) = (\rho+\delta)\nu(t) - \alpha_K k(t)^{-1} - \alpha_K \nu(t) sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}k(t)^{-1} \tag{A-67}$$

$$\dot{\gamma}(t) = \gamma(t)(\rho-\Omega) - [\mu(t)-\gamma(t)]\omega z(t)^{\omega-1}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega} \tag{A-68}$$

$$\dot{\mu}(t) = \mu(t)(\lambda+\rho) - [\mu(t)-\gamma(t)](1-\omega)z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{-\omega}\kappa_E^{\frac{\sigma}{\sigma-1}}A_E Q_D. \tag{A-69}$$

The condition to maximize the Hamiltonian is:

$$0 = \alpha_E e_Y(t)^{-1} - [\mu(t)-\gamma(t)](1-\omega)z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{-\omega}\kappa_E^{\frac{\sigma}{\sigma-1}}A_E$$
$$+\alpha_E\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}e_Y(t)^{-1}. \tag{A-70}$$

Time-differentiate (A-70):

$$0 = -\alpha_E e_Y(t)^{-1}\frac{\dot{e}_Y(t)}{e_Y(t)} - [\dot{\mu}(t)-\dot{\gamma}(t)](1-\omega)z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{-\omega}\kappa_E^{\frac{\sigma}{\sigma-1}}A_E$$
$$-[\mu(t)-\gamma(t)]\omega(1-\omega)z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{-\omega}\kappa_E^{\frac{\sigma}{\sigma-1}}A_E$$
$$\left\{\frac{\dot{z}(t)}{z(t)} - \frac{Q_D\dot{r}(t)-\dot{e}_Y(t)}{Q_D r(t)-e_Y(t)}\right\}$$
$$+\alpha_E\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}e_Y(t)^{-1}\left\{\frac{\dot{\nu}(t)}{\nu(t)} + \alpha_K\frac{\dot{k}(t)}{k(t)} - (1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\right\}$$

Substitute from the transition equation for $k(t)$ and from equations (A-67) and (A-70) and rearrange:

$$\begin{aligned}
&\alpha_E e_Y(t)^{-1}\frac{1}{1-\omega}\frac{\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{\omega}}{z(t)^{\omega}\kappa_E^{\frac{\sigma}{\sigma-1}}A_E}\left\{\alpha_E\frac{\dot{e}_Y(t)}{e_Y(t)}+(\rho-g_L)-\alpha_K k(t)^{-1}\nu(t)^{-1}\right\}\\
&=-\left[\dot{\mu}(t)-\dot{\gamma}(t)\right]\\
&\quad-\left[\mu(t)-\gamma(t)\right]\left\{\omega\frac{\dot{z}(t)}{z(t)}-\omega\frac{Q_D\dot{r}(t)-\dot{e}_Y(t)}{Q_D r(t)-e_Y(t)}\right.\\
&\qquad\left.-(\rho-g_L)-(1-\alpha_K)(\delta+g_L)+\alpha_K k(t)^{-1}\nu(t)^{-1}+(1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\right\}.
\end{aligned}$$

Substitute from the transition equations for $r(t)$ and $z(t)$ and solve for $\dot{\gamma}(t)$: for $\dot{\gamma}(t)$:

$$\begin{aligned}
\dot{\gamma}(t)=&\dot{\mu}(t)\\
&+\alpha_E e_Y(t)^{-1}\frac{1}{1-\omega}\frac{e_R(t)}{z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}}\left\{\alpha_E\frac{\dot{e}_Y(t)}{e_Y(t)}+(\rho-g_L)-\alpha_K k(t)^{-1}\nu(t)^{-1}\right\}\\
&+\left[\mu(t)-\gamma(t)\right]\left\{\omega(\Omega-g_L)-\omega z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}\left(\frac{1}{z(t)}+\frac{Q_D}{e_R(t)}\right)\right.\\
&\qquad+\omega(\lambda+g_L)\frac{e(t)}{e_R(t)}+\omega\frac{\dot{e}_Y(t)}{e_R(t)}-(\rho-g_L)-(1-\alpha_K)(\delta+g_L)+\alpha_K k(t)^{-1}\nu(t)^{-1}\\
&\qquad\left.+(1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\right\}.
\end{aligned}$$

Substitute for $\gamma(t)$ from (A-70):

$$\begin{aligned}
\dot{\gamma}(t)=&\dot{\mu}(t)\\
&+\alpha_E e_Y(t)^{-1}\frac{1}{1-\omega}\frac{e_R(t)}{z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}}\\
&\quad\left\{\alpha_E\frac{\dot{e}_Y(t)}{e_Y(t)}+(\rho-g_L)-\alpha_K k(t)^{-1}\nu(t)^{-1}\right.\\
&\qquad+\left[1+\nu(t)sAk(t)^{\alpha_K}e_Y(t)^{\alpha_E}\right]\\
&\qquad\left[\omega(\Omega-g_L)-\omega z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}}A_E[Q_D r(t)-e_Y(t)]\right)^{1-\omega}\left(\frac{1}{z(t)}+\frac{Q_D}{e_R(t)}\right)+\omega(\lambda+g_L)\frac{e(t)}{e_R(t)}\right.\\
&\qquad\left.\left.+\omega\frac{\dot{e}_Y(t)}{e_R(t)}-(\rho-g_L)-(1-\alpha_K)(\delta+g_L)+\alpha_K k(t)^{-1}\nu(t)^{-1}+(1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\right]\right\}.
\end{aligned} \tag{A-71}$$

Substitute for $\gamma(t)$ from (A-70) into (A-69):

$$\dot{\mu}(t) = \mu(t)(\lambda+\rho) - \alpha_E e_Y(t)^{-1} Q_D \left[1 + \nu(t) s A k(t)^{\alpha_K} e_Y(t)^{\alpha_E}\right]. \tag{A-72}$$

Substitute $\dot{\mu}(t)$ into (A-71) and rearrange:

$$\begin{aligned}
\dot{\gamma}(t) =& \mu(t)(\lambda+\rho) - \alpha_E e_Y(t)^{-1} \frac{1}{1-\omega}\left(\left[1+\nu(t)sAk(t)^{\alpha_K} e_Y(t)^{\alpha_E}\right] Q_D + \omega \frac{e_R(t)}{z(t)}\right) \\
&+ \alpha_E e_Y(t)^{-1} \frac{1}{1-\omega} \frac{e_R(t)}{z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{1-\omega}} \\
&\quad \left\{\omega(\Omega - g_L) + \omega(\lambda + g_L)\frac{e(t)}{e_R(t)} + \omega \frac{\dot{e}_Y(t)}{e_R(t)} + \frac{\dot{e}_Y(t)}{e_Y(t)} \right. \\
&\quad\quad + \nu(t) s A k(t)^{\alpha_K} e_Y(t)^{\alpha_E} \\
&\quad\quad\quad \left[\omega(\Omega - g_L) - \omega z(t)^{\omega}\left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E [Q_D r(t) - e_Y(t)]\right)^{1-\omega} \frac{1}{z(t)}\right. \\
&\quad\quad\quad + \omega(\lambda+g_L)\frac{e(t)}{e_R(t)} + \omega\frac{\dot{e}_Y(t)}{e_R(t)} - (\rho - g_L) - (1-\alpha_K)(\delta + g_L) + \alpha_K k(t)^{-1}\nu(t)^{-1} \\
&\quad\quad\quad \left.\left. + (1-\alpha_E)\frac{\dot{e}_Y(t)}{e_Y(t)}\right]\right\}.
\end{aligned} \tag{A-73}$$

Solve (A-70) for $\gamma(t)$, substitute that and $\dot{\gamma}(t)$ from (A-73) into (A-68), and rearrange:

$$\begin{aligned}
&\frac{\dot{e}_Y(t)}{e_Y(t)}\left\{1 + \omega\frac{e_Y(t)}{e_R(t)} + \nu(t) s A k(t)^{\alpha_K} e_Y(t)^{\alpha_E}\left[(1-\alpha_E) + \omega \frac{e_Y(t)}{e_R(t)}\right]\right\} \\
=& -\frac{1}{\alpha_E} e_Y(t)\mu(t)\,(1-\omega)(\lambda+\Omega)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left(\frac{z(t)}{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E e_R(t)}\right)^{\omega} \\
&+ \left[1 + \nu(t) s A k(t)^{\alpha_K} e_Y(t)^{\alpha_E}\right] Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left(\frac{z(t)}{\kappa_E^{\frac{\sigma}{\sigma-1}} A_E e_R(t)}\right)^{\omega} \\
&- (\rho - \Omega) - \omega(\Omega - g_L) - \omega(\lambda + g_L)\frac{e(t)}{e_R(t)} \\
&- \nu(t) s A k(t)^{\alpha_K} e_Y(t)^{\alpha_E} \\
&\quad \left[\rho - \Omega + \omega(\Omega - g_L) + \omega(\lambda+g_L)\frac{e(t)}{e_R(t)} - (\rho - g_L) - (1-\alpha_K)(\delta+g_L) + \alpha_K k(t)^{-1}\nu(t)^{-1}\right]
\end{aligned} \tag{A-74}$$

Now consider a balanced growth path. From its transition equation, $k(t)$ grows at a constant rate if and only if $y(t)/k(t)$ is constant. So $g_k = g_y$. Substituting into the final good production function,

$$g_y = \frac{\alpha_E}{\alpha_E + \alpha_L} g_e. \tag{A-75}$$

From its transition equation, $z(t)$ grows at constant rate g_e if and only if:

$$\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \frac{e_R(t)}{z(t)} = (\Omega - g_L - g_e)^{\frac{1}{1-\omega}}. \tag{A-76}$$

Using this and the transition equation for $r(t)$, $r(t)$ grows at constant rate g_e if and only if:

$$\begin{aligned} g_e + \lambda + g_L =& \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E\right)^{1-\omega} \left(\frac{e_R(t)}{z(t)}\right)^{-\omega} \frac{e_R(t)}{r(t)} \\ =& \kappa_E^{\frac{\sigma}{\sigma-1}} A_E \left(\Omega - g_L - g_e\right)^{\frac{-\omega}{1-\omega}} \frac{e_R(t)}{r(t)} \\ \Leftrightarrow \frac{e_R(t)}{r(t)} =& [g_e + \lambda + g_L] \left(\kappa_E^{\frac{\sigma}{\sigma-1}} A_E\right)^{-1} (\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}}. \end{aligned} \tag{A-77}$$

Substituting for $y(t)/k(t)$ from the capital transition equation into (A-67), we find:

$$\frac{\dot{\nu}(t)}{\nu(t)} = (\rho + \delta) - \alpha_K k(t)^{-1} \nu(t)^{-1} - \alpha_K [g_k + \delta + g_L].$$

$\nu(t)k(t)$ is constant if and only if $\dot{\nu}(t)/\nu(t) = -g_k$. Using that condition, $g_k = g_y$, and (A-75) in the foregoing equation yields

$$\nu(t)k(t) = \frac{\alpha_K}{\alpha_E g_e + (\delta + \rho) - \alpha_K(\delta + g_L)}. \tag{A-78}$$

Substitute $y(t)/k(t)$ from the capital transition equation, $g_k = g_y$, g_y from (A-75), $\nu(t)k(t)$ from (A-78), and $\dot{e}_Y(t)/e_Y(t) = g_e$ into (A-74):

$$\begin{aligned} & g_e \left[1 + \omega \frac{e_Y(t)}{e_R(t)}\right] \frac{1}{1 - \alpha_K} \frac{\alpha_E g_e + (1 - \alpha_K)(\delta + \rho)}{\alpha_E g_e + (\delta + \rho) - \alpha_K(\delta + g_L)} \\ =& -\frac{1}{\alpha_E} e_Y(t) \mu(t) (1 - \omega)(\lambda + \Omega) \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E\right]^{1-\omega} \left(\frac{z(t)}{e_R(t)}\right)^{\omega} \\ & - \frac{1}{1 - \alpha_K} \frac{\alpha_E g_e + (1 - \alpha_K)(\delta + \rho)}{\alpha_E g_e + (\delta + \rho) - \alpha_K(\delta + g_L)} \\ & \quad \left[(\rho - \Omega) + \omega(\Omega - g_L) + \omega(\lambda + g_L) \frac{e(t)}{e_R(t)} - Q_D \left[\kappa_E^{\frac{\sigma}{\sigma-1}} A_E\right]^{1-\omega} \left(\frac{z(t)}{e_R(t)}\right)^{\omega}\right]. \end{aligned} \tag{A-79}$$

Observe that

$$\frac{e_R(t)}{r(t)} = \frac{e_R(t)}{e_Y(t)}\frac{e_Y(t)}{r(t)} = \frac{e_R(t)}{e_Y(t)} Q_D \frac{e_Y(t)}{e(t)} = \frac{e_R(t)}{e_Y(t)} Q_D \left(1 - \frac{e_R(t)}{e(t)}\right) = \frac{e_R(t)}{e_Y(t)}\left(Q_D - \frac{e_R(t)}{r(t)}\right)$$
.

Therefore:

$$\frac{e_Y(t)}{e_R(t)} = \frac{Q_D - \frac{e_R(t)}{r(t)}}{\frac{e_R(t)}{r(t)}}.$$

Using that and (A-77),

$$\frac{e_Y(t)}{e_R(t)} + \frac{1}{\omega} = \frac{Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E + \frac{1-\omega}{\omega}\left[g_e + \lambda + g_L\right](\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}}}{\left[g_e + \lambda + g_L\right](\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}}}.$$

Substitute into (A-79), and also use $e_R(t)/r(t)$ from (A-77) and $e_R(t)/z(t)$ from (A-76):

$$\begin{aligned}
&\omega g_e \frac{Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E + \frac{1-\omega}{\omega}\left[g_e + \lambda + g_L\right](\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}}}{\left[g_e + \lambda + g_L\right](\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}}}\Big[\alpha_E g_e + (1-\alpha_K)(\delta+\rho)\Big] \\
&= -\frac{1-\alpha_K}{\alpha_E} e_Y(t)\mu(t)\,(1-\omega)(\lambda+\Omega)\kappa_E^{\frac{\sigma}{\sigma-1}} A_E \Big[\alpha_E g_e + (\delta+\rho) - \alpha_K(\delta + g_L)\Big](\Omega - g_L - g_e)^{\frac{-\omega}{1-\omega}} \\
&\quad - \Big[\alpha_E g_e + (1-\alpha_K)(\delta+\rho)\Big] \\
&\qquad \left[(\rho - \Omega) + \omega(\Omega - g_L) - Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E (\Omega - g_L - g_e)^{\frac{-\omega}{1-\omega}} \frac{g_e + (1-\omega)(\lambda + g_L)}{g_e + \lambda + g_L}\right]. \quad \text{(A-80)}
\end{aligned}$$

This last equation requires $\mu(t)e_Y(t)$ be constant, which in turn requires that $g_\mu = -g_e$. Substitute that into the left-hand side of (A-72) and, in the right-hand side of (A-72), substitute $\dot{k}(t)$ from its transition equation, $g_k = g_y$ and g_y from (A-75), and $\nu(t)k(t)$ from (A-78), and then solve for $\mu(t)e_Y(t)$:

$$\mu(t)e_Y(t) = \alpha_E \frac{Q_D}{g_e + \lambda + \rho}\frac{1}{1-\alpha_K}\frac{\alpha_E g_e + (1-\alpha_K)(\delta+\rho)}{\alpha_E g_e + (\delta+\rho) - \alpha_K(\delta + g_L)}. \quad \text{(A-81)}$$

Substitute into (A-80):

$$(g_e + \rho - \Omega) + \omega(\Omega - g_L - g_e) = (1-\omega) Q_D \kappa_E^{\frac{\sigma}{\sigma-1}} A_E (\Omega - g_L - g_e)^{\frac{-\omega}{1-\omega}} \frac{g_e + \rho - \Omega}{g_e + \rho + \lambda} \quad \text{(A-82)}$$

The left-hand side of (A-82) increases in g_e. The right-hand side of (A-82) is real-valued for $g_e < \Omega - g_L$ and, in those case, increases in g_e for $g_e > -(\rho + \lambda)$. As g_e approaches

$-(\rho-\Omega)$ from above, the left-hand side of (A-82) approaches $\omega(\rho-g_L)>0$ and the right-hand side of (A-82) approaches zero. As g_e approaches $\Omega-g_L$ from below, the left-hand side of (A-82) approaches $\rho-g_L>0$ and the right-hand side of (A-82) approaches positive infinity. Therefore there exists $g_e\in\big(-(\rho-\Omega),\Omega-g_L\big)$ such that (A-82) holds. An intersection occurs at strictly negative g_e if the left-hand side of (A-82) is strictly less than the right-hand side of (A-82) when each is evaluated at $g_e=0$. That sufficient condition is:

$$\begin{aligned}0<&(1-\omega)Q_D\kappa_E^{\frac{\sigma}{\sigma-1}}A_E\left(\Omega-g_L\right)^{\frac{-\omega}{1-\omega}}\frac{\rho-\Omega}{\rho+\lambda}-(\rho-\Omega)-\omega(\Omega-g_L)\\ \Leftrightarrow Q_DA_E>&\frac{1}{1-\omega}\left(\Omega-g_L\right)^{\frac{\omega}{1-\omega}}\frac{\rho+\lambda}{\kappa_E^{\frac{\sigma}{\sigma-1}}(\rho-\Omega)}\Big\{(\rho-\Omega)+\omega(\Omega-g_L)\Big\}.\end{aligned}\tag{A-83}$$

Defining χ as in Proposition 5, this last inequality is equivalent to

$$Q_DA_E>\left[\frac{\lambda+\rho}{\lambda+g_L}\frac{(1-\omega)(\rho-\Omega)+\omega(\rho-g_L)}{(1-\omega)(\rho-\Omega)}\right]^{\frac{-1}{\sigma-1}}\chi.\tag{A-84}$$

Define X as the right-hand side of (A-84). If $Q_DA_E>X$, then the g_e that solves (A-82) is strictly negative. Observing that the terms inside square brackets in (A-84) are strictly greater than 1 and recalling that $\sigma>1$ under the conditions of the proposition, we find: (i) $X\leq\chi$; (ii) χ/X is strictly decreasing in σ; and (iii) $\lim_{\sigma\to\infty}X=\lim_{\sigma\to\infty}\chi$.

Increasing Q_DA_E increases the right-hand side of (A-82) when it is strictly positive, without affecting the left-hand side of (A-82). Because the right-hand side of (A-82) cuts the left-hand side of (A-82) from below around the first intersection in g_e (which is the one whose existence is implied by (A-83)), an increase in Q_DA_E moves that intersection to smaller g_e.

Now consider the feasibility of a balanced growth path. From the capital transition equation, $g_k=g_y$, and (A-75), $y(t)/k(t)>0$ if and only if

$$0<\frac{1}{s}\left[\frac{\alpha_E}{\alpha_E+\alpha_L}g_e+\delta+g_L\right].$$

This is satisfied if and only if

$$g_e>-\frac{\alpha_E+\alpha_L}{\alpha_E}(\delta+g_L).\tag{A-85}$$

From (A-78), $\nu(t)k(t)>0$ if and only if

$$g_e>-\frac{1}{\alpha_E}\left[(1-\alpha_K)\delta+\rho-\alpha_Kg_L\right].$$

Using that $\rho > g_L$, this inequality holds whenever inequality (A-85) holds. From (A-77), $e_R(t)/e(t) > 0$ if and only if

$$0 < [g_e + \lambda + g_L] (\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}} , \tag{A-86}$$

which holds if and only if $g_e \in (-(\lambda + g_L), \Omega - g_L)$. From (A-81), $\mu(t) e_Y(t) > 0$ if and only if

$$0 < \frac{1}{g_e + \lambda + \rho} \frac{\alpha_E g_e + (1 - \alpha_K)(\delta + \rho)}{\alpha_E g_e + (\delta + \rho) - \alpha_K(\delta + g_L)}.$$

The first fraction is strictly positive for $g_e > -(\rho - \Omega)$, which is met by the analyzed solution to (A-82). Both the denominator and the numerator in the second fraction are strictly positive when inequality (A-85) holds. From (A-76), $e_R(t)/z(t) > 0$ if and only if $g_e < \Omega - g_L$, which is met by the analyzed solution to (A-82). Finally, from (A-77), $e_R(t)/e(t) < 1$ if and only if

$$Q_D A_E > \kappa_E^{-\frac{\sigma}{\sigma-1}} [g_e + \lambda + g_L] (\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}} . \tag{A-87}$$

At $g_e = 0$, the right-hand side of (A-87) is strictly less than the right-hand side of (A-83), so that inequality (A-87) is implied by inequality (A-83) when $g_e = 0$. The derivative of the right-hand side of (A-87) with respect to g_e is

$$\kappa_E^{-\frac{\sigma}{\sigma-1}} (\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}} \left[1 - \frac{\omega}{1-\omega} \frac{g_e + \lambda + g_L}{\Omega - g_L - g_e}\right].$$

The right-hand side of (A-87) is zero at $g_e = -(\lambda + g_L)$ (which is a lower bound on g_e from inequality (A-86)) and increases until it reaches a maximum and then decreases to 0 at $g_e = \Omega - g_L$. That maximum is at strictly positive g_e if and only if the above derivative is strictly positive when evaluated at $g_e = 0$, and thus if and only if

$$\Omega - g_L > \frac{\omega}{1-\omega}(\lambda + g_L). \tag{A-88}$$

When inequality (A-88) holds, we know that inequality (A-87) holds at all $g_e < 0$ because its right-hand side increases in g_e up to at least $g_e = 0$ and we showed that the inequality does hold at $g_e = 0$.

From inequalities (A-85) and (A-86), a balanced growth path with $g_e \in \big(-(\rho - \Omega), \Omega - g_L\big)$ is feasible if

$$g_e > \max\left\{-\frac{\alpha_E + \alpha_L}{\alpha_E}(\delta + g_L), -(\lambda + g_L)\right\}. \tag{A-89}$$

If $\rho - \Omega$ is not too large, then this inequality holds for all $g_e > -(\rho - \Omega)$.

We have found a path along which $\ell_R(t), k_R(t) = 0$ with $e_R(t) > 0$ and all variables grow at a constant rate. The conditions of the proposition ensure that the path is feasible, since inequalities (A-84), (A-87), and (A-89) hold. Part i of the proposition follows from (A-75). Part ii of the proposition follows from the analysis of (A-82). Part iii of the proposition follows from that same analysis and from (A-76).

Using (A-77), (7) becomes:

$$EROI(t) = 1 + Q_D A_E \kappa_E^{\frac{\sigma}{\sigma-1}} \frac{g_e + g_L}{[g_e + \lambda + g_L] (\Omega - g_L - g_e)^{\frac{\omega}{1-\omega}}}.$$

We have proved part iv of the proposition.

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