Testing spatial correlation for spatial models with heterogeneous coefficients when both n and T are large

Shi Ryoung Chang Robert de Jong

Department of Economics
The Ohio State University

January 3, 2025

Introduction

- A natural first step in the spatial economic analysis is a **test for spatial correlation** (Greene, *Econometric Analysis*).
    - Moran (1950), Cliff and Ord (1973) for cross-sectional data

- Along with the advances in various spatial models, numerous contributions have been made to **tests for spatial correlation**.
    - The standard approach is to formulate a hypothesis on a spatial coefficient.

Traditional Test

- The pure spatial autoregressive (SAR) panel data model:

$$
\begin{aligned}
y_{1t} &= \lambda(w_{11}y_{1t} + w_{12}y_{2t} + \cdots + w_{1n}y_{nt}) + c_1 + \varepsilon_{1t}, \quad t = 1, \ldots, T \\
y_{2t} &= \lambda(w_{21}y_{1t} + w_{22}y_{2t} + \cdots + w_{2n}y_{nt}) + c_2 + \varepsilon_{2t}, \quad t = 1, \ldots, T \\
&\vdots \\
y_{nt} &= \lambda(w_{n1}y_{1t} + w_{n2}y_{2t} + \cdots + w_{nn}y_{nt}) + c_n + \varepsilon_{nt}, \quad t = 1, \ldots, T
\end{aligned}
$$

$$\Updownarrow$$

$$Y_{nt} = \lambda W_n Y_{nt} + \mathbf{c}_n + V_{nt}, \quad t = 1, \ldots, T$$

where $Y_{nt} = (y_{1t}, y_{2t}, \ldots, y_{nt})'$, W_n is an $n \times n$ spatial weights matrix, $\mathbf{c}_n$ is an $n \times 1$ vector of fixed effects, and $V_{nt} = (\varepsilon_{1t}, \varepsilon_{2t}, \ldots, \varepsilon_{nt})'$.

- To test $H_0 : \lambda = 0$, one may use a $N(0, 1)$ test $M = \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T} \tilde{Y}_{nt}' W_n \tilde{Y}_{nt}}{\sqrt{tr(W_n' W_n + W_n^2)}}$ when $(n, T) \to \infty$.

Data

- Let me look at the data!
  - y_{it} is the outcome variable for unit $i = 1, \ldots, 25$ at time $t = 1$

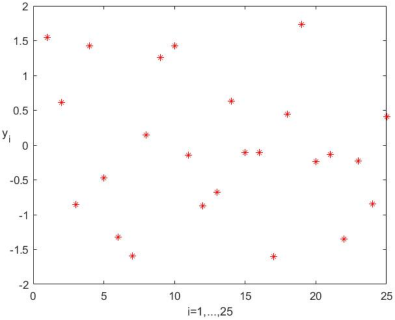

Figure: (A)

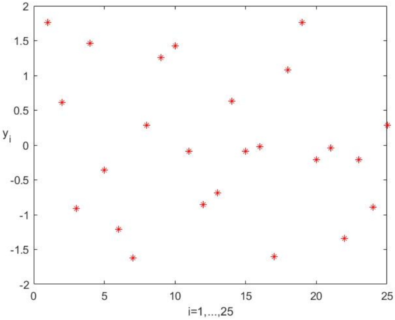

Figure: (B)

Data

- Let me look at the data!
  - y_{it} is the outcome variable for unit $i = 1, ..., 25$ at time $t = 1$

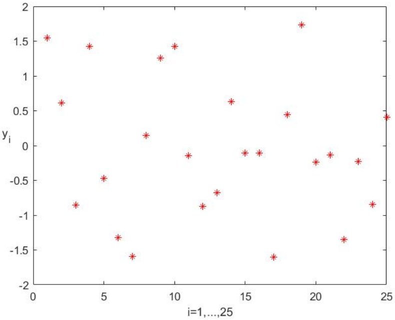

Figure: $y_{it} = \lambda \sum_{j=1}^{n} w_{ij} y_{jt} + c_i + \varepsilon_{it}$

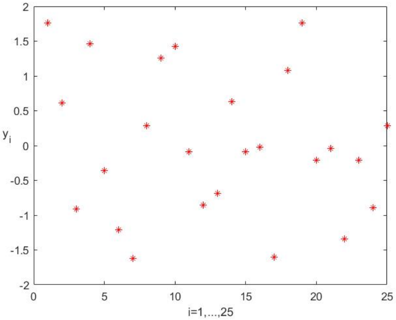

Figure: $y_{it} = \delta_i \sum_{j=1}^{n} w_{ij} y_{jt} + c_i + \varepsilon_{it}$

Motivation

$$y_{it} = \lambda(w_{i1}y_{1t} + w_{i2}y_{2t} + \cdots + w_{in}y_{nt}) + c_i + \varepsilon_{it}, \quad i = 1, ..., n, \; t = 1, ..., T$$

$$\Downarrow$$

$$y_{it} = \delta_i(w_{i1}y_{1t} + w_{i2}y_{2t} + \cdots + w_{in}y_{nt}) + c_i + \varepsilon_{it}, \quad i = 1, ..., n, \; t = 1, ..., T$$

- In practice, we have data (y_{it}, w_{ij}), but we may not know the true process—particularly whether it is *homogeneous* or *heterogeneous*.
- In the spatial literature, however, almost all contributions assume that spatial effects are *homogeneous* across units (λ for $\forall i = 1, ..., n$).

Motivation

$$y_{it} = \lambda(w_{i1}y_{1t} + w_{i2}y_{2t} + \cdots + w_{in}y_{nt}) + c_i + \varepsilon_{it}, \quad i = 1, ..., n, \; t = 1, ..., T$$

$$\Downarrow$$

$$y_{it} = \delta_i(w_{i1}y_{1t} + w_{i2}y_{2t} + \cdots + w_{in}y_{nt}) + c_i + \varepsilon_{it}, \quad i = 1, ..., n, \; t = 1, ..., T$$

- Tests for spatial correlation:

- To test $H_0 : \lambda = 0$ (for $\forall i$), the test statistic (**widely used** approach) is

$$M = \frac{\frac{1}{\tilde{\sigma}^2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\tilde{Y}_{nt}' W_n \tilde{Y}_{nt}}{\sqrt{tr(W_n' W_n + W_n^2)}}.$$

- To test $H_0 : \delta_i = 0$ for $\forall i$, the test statistic (**proposed** approach) is

$$S = ?$$

Q What if spatial processes are *heterogeneous* in nature?

Q Does the traditional M test behave *consistently* across all potential alternatives?

Motivation

$$y_{it} = \delta_i(w_{i1}y_{1t} + w_{i2}y_{2t} + \cdots + w_{in}y_{nt}) + c_i + \varepsilon_{it}, \quad i = 1, ..., n, \ t = 1, ..., T$$

- In fixed n ($n < T$), the hypothesis ($H_0 : \delta_i = 0$ for $\forall i = 1, ..., n$) can be tested.
  - QMLE for fixed n and $T \to \infty$ (Aquaro, Bailey and Pesaran, 2021)
  - Three classical tests: LR (Elhorst et al., 2021), Wald, and LM tests
- However, those exiting procedures are only valid for fixed n.
  - e.g., 10 European countries ($n = 10$) over 75 time periods ($T = 75$)

Motivation

- Small n? In many spatial or micro applications, n is often large!
- (**Q**) Can we let $\boldsymbol{n} \to \infty$ for the test?
- (**A**) Let me look at our setting.
  - $H_0 : \delta_i = 0$ for $\forall i = 1, ..., n$ is a set of n restrictions.
  - Do we need to estimate δ_i for all $i = 1, ..., n$?
  - For the LM test, the score vector is an $n \times 1$ vector.
  - Its variance structure is an $n \times n$ matrix.

Project goal

Project goal

- Propose the **test of spatial correlation** for spatial panel data models with fully **heterogeneous** coefficients **when both** n **and** T **are large**

$$y_{it} = \delta_i(w_{i1}y_{1t} + w_{i2}y_{2t} + \cdots + w_{in}y_{nt}) + c_i + \varepsilon_{it},\ i = 1, \ldots, n,\ t = 1, \ldots, T$$

  - $H_0 : \delta_i = 0$ for $\forall i = 1, \ldots, n$ against $H_1 : \delta_i \neq 0$ for a non-zero faction of units.

- $S \xrightarrow{d} N(0, 1)$ *as* $(n, T) \to \infty$ such that $\frac{n}{T} \to k$ where $0 \leq k < \infty$ under H_0.

Contributions

- Theoretical perspective: How can we address the theoretical challenges that arise as $n \to \infty$?
- Empirical perspective: It can solve a wide range of micro-empirical questions.
- Ambitious perspective: What if we use M (a widely used approach) when spatial processes are *heterogeneous* in nature? Our result
  - The power of the (widely used) tests for spatial correlation can be very low or vanish under certain circumstances (Krämer, 2005; Martellosio, 2010, 2012; Preinerstorfer and Pötscher, 2017; Preinerstorfer 2023).

Related literature

- There is little discussion of testing in the context of spatial models with fully heterogeneous spatial coefficients **when** $(\boldsymbol{n}, \boldsymbol{T}) \to \infty$.

$$y_{it} = \delta_i (w_{i1} y_{1t} + w_{i2} y_{2t} + \cdots + w_{in} y_{nt}) + c_i + \varepsilon_{it}, \quad i = 1, ..., n, \ t = 1, ..., T$$

  - $H_0 : \delta_i = 0$ for $\forall i = 1, ..., n$

- Pesaran and Yamagata (2008) propose the test of slope homogeneity for panel data models with exogenous regressors **when** $(\boldsymbol{n}, \boldsymbol{T}) \to \infty$.

$$y_{it} = \beta_i x_{it} + \alpha_i + \varepsilon_{it}, \qquad i = 1, ..., n, \ t = 1, ..., T$$

  - $H_0 : \beta_i = \beta$ for $\forall i = 1, ..., n$

Related literature

- There is little discussion of testing in the context of spatial models with fully heterogeneous spatial coefficients **when** $(\boldsymbol{n}, \boldsymbol{T}) \to \infty$.

$$y_{it} = \delta_i (w_{i1} y_{1t} + w_{i2} y_{2t} + \cdots + w_{in} y_{nt}) + c_i + \varepsilon_{it}, \quad i = 1, ..., n, \ t = 1, ..., T$$

  - $H_0 : \delta_i = 0$ for $\forall i = 1, ..., n$

- Pesaran and Yamagata (2008) propose the test of slope homogeneity for panel data models with exogenous regressors **when** $(\boldsymbol{n}, \boldsymbol{T}) \to \infty$.

$$y_{it} = \beta_i x_{it} + \alpha_i + \varepsilon_{it}, \qquad i = 1, ..., n, \ t = 1, ..., T$$

  - $H_0 : \beta_i = \beta$ for $\forall i = 1, ..., n$

$\Rightarrow$ This paper is key because it formulates a hypothesis on fully heterogeneous coefficients and derive limiting results when n is large.

P.&Y. (2008)

Related literature

- There is little discussion of testing in the context of spatial models with fully heterogeneous spatial coefficients **when** $(\boldsymbol{n}, \boldsymbol{T}) \to \infty$.

$$y_{it} = \delta_i(w_{i1}y_{1t} + w_{i2}y_{2t} + \cdots + w_{in}y_{nt}) + c_i + \varepsilon_{it}, \quad i = 1, \ldots, n, \ t = 1, \ldots, T$$

  - $H_0 : \delta_i = 0$ for $\forall i = 1, \ldots, n$

- Pesaran and Yamagata (2008) propose the test of slope homogeneity for panel data models with exogenous regressors **when** $(\boldsymbol{n}, \boldsymbol{T}) \to \infty$.

$$y_{it} = \beta_i x_{it} + \alpha_i + \varepsilon_{it}, \qquad i = 1, \ldots, n, \ t = 1, \ldots, T$$

  - $H_0 : \beta_i = \beta$ for $\forall i = 1, \ldots, n$

$\Rightarrow$ This paper is key because it formulates a hypothesis on fully heterogeneous coefficients and derive limiting results when n is large. P.&Y. (2008)

$\Rightarrow$ However, our setting requires **further consideration** due to the **endogeneity** by spatial dependence. C.&de J. (2024)

Conclusion

- This paper aims to establish econometric theory for *heterogeneous* spatial models in *large n* settings.
  - Without estimating heterogeneous coefficients ($\hat{\delta}_i$ for $i = 1, ..., n$), we can test the hypothesis on these coefficients ($H_0 : \delta_i = 0$ for $\forall i$).
  - Monte Carlo simulations and an empirical application are offered to show our key findings.

- In future studies, our approach can be extended to testing **spatial lag homogeneity** in panel data models **when** $(\boldsymbol{n}, \boldsymbol{T}) \to \infty$.

$$y_{it} = \delta_i (w_{i1} y_{1t} + w_{i2} y_{2t} + \cdots + w_{in} y_{nt}) + c_i + \varepsilon_{it},\ i = 1, ..., n,\ t = 1, ..., T$$

  - $H_0 : \delta_i = \lambda$ for all i against $H_1 : \delta_i \neq \lambda$ for a non-zero faction of units
  - Testing spatial correlation ($\lambda = 0$) is a special case of testing spatial lag homogeneity.

Thank you

1. Assume one wants to test $H_0 : \theta_i = 0$ for $\forall i = 1, ..., n$ (n is fixed). Then, one can base a test on $\hat{\theta}' \Sigma^{D-1} \hat{\theta}$ (Swamy, 1970) as

$$\begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix}' \begin{pmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ 0 & 0 & \sigma_3^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{pmatrix}^{-1} \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix} = \Sigma_{i=1}^{n} (\frac{\hat{\theta}_i}{\sigma_i})^2 \xrightarrow{d} \chi_n^2 \;\; as \; T \to \infty$$

where $\Sigma^D = Asy.Var(\hat{\theta})$. [$D$: a diagonal matrix]

2. Pesaran and Yamagata (2008) propose a standardized version of $\hat{\theta}' \Sigma^{D-1} \hat{\theta}$ and show $\frac{1}{\sqrt{2n}} \left(\hat{\theta}' \Sigma^{D-1} \hat{\theta} - n \right) = \frac{1}{\sqrt{2n}} \Sigma_{i=1}^{n} \left(z_i - 1 \right) \xrightarrow{d} N(0, 1)$ as $(n, T) \to \infty$.

   - $z_i = (\frac{\hat{\theta}_i}{\sigma_i})^2$ is independent over i; A well-known CLT is applicable.

Challenges

Chang and de Jong (2024)

1. Assume we wants to test $H_0 : \theta_i = 0$ for $\forall i = 1, ..., n$ (n is fixed). Then, we derive $\hat{\theta}'\Sigma^{-1}\hat{\theta}$ as

$$\begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix}' \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \dots & \sigma_{2n} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \dots & \sigma_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \dots & \sigma_n^2 \end{pmatrix}^{-1} \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix} \to \chi_n^2 \ \ as\ T \to \infty$$

where $\Sigma = Asy.Var(\hat{\theta})$.

2. We propose $S = \frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{-1}\hat{\theta} - n\right)$ and show its limiting result as $(n, T) \to \infty$.

$\Rightarrow$ However, our setting requires **further consideration** due to the **endogeneity** by spatial dependence ($\sigma_{ij} \neq 0$).

- other limit theorems, the inverse of an $n \times n$ matrix (Σ^{-1})

Sketch of this paper

1. We construct $\hat{\theta}'\Sigma^{-1}\hat{\theta} \to \chi^2_n$ *as* $T \to \infty$.

- **LM** test [restricted estimate: $(0, ..., 0, \tilde{\sigma}^2)'$]

Sketch of this paper

1. We construct $\hat{\theta}'\Sigma^{-1}\hat{\theta} \to \chi^2_n$ *as* $T \to \infty$.

- **LM** test [restricted estimate: $(0, ..., 0, \tilde{\sigma}^2)'$]

2. We propose $S = \frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{-1}\hat{\theta} - n\right)$ and show its limiting result as $(n, T) \to \infty$ such that $\frac{n}{T} \to k$ where $0 \le k < \infty$.

$$S = \frac{1}{\sqrt{2n}}\left(\begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix}' \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \dots & \sigma_{2n} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \dots & \sigma_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \dots & \sigma_n^2 \end{pmatrix}^{-1} \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix} - n \right)$$

Sketch of this paper

1. We construct $\hat{\theta}'\Sigma^{-1}\hat{\theta} \to \chi^2_n$ *as* $T \to \infty$.

- **LM** test [restricted estimate: $(0, ..., 0, \tilde{\sigma}^2)'$]

2. We propose $S = \frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{-1}\hat{\theta} - n\right)$ and show its limiting result as $(n, T) \to \infty$ such that $\frac{n}{T} \to k$ where $0 \leq k < \infty$.

$$S = \frac{1}{\sqrt{2n}}\left(\begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix}' \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \dots & \sigma_{2n} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \dots & \sigma_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \dots & \sigma_n^2 \end{pmatrix}^{-1} \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix} - n\right)$$

(1) A special case ($\sigma_{ij} = 0$ for $\forall i, j$):
$S = \frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{D-1}\hat{\theta} - n\right) = \frac{1}{\sqrt{2n}}\Sigma_{i=1}^{n}\left(z_i - 1\right) \xrightarrow{d} N(0, 1)$ by the CLT under *near-epoch dependence* by Jenish and Prucha (2012).

Sketch of this paper

1. We construct $\hat{\theta}'\Sigma^{-1}\hat{\theta} \to \chi^2_n$ *as* $T \to \infty$.
   - **LM** test [restricted estimate: $(0, ..., 0, \tilde{\sigma}^2)'$]

2. We propose $S = \frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{-1}\hat{\theta} - n\right)$ and show its limiting result as $(n, T) \to \infty$ such that $\frac{n}{T} \to k$ where $0 \leq k < \infty$.

$$S = \frac{1}{\sqrt{2n}}\left(\begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix}' \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \dots & \sigma_{2n} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \dots & \sigma_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \dots & \sigma_n^2 \end{pmatrix}^{-1} \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix} - n\right)$$

   (1) A special case ($\sigma_{ij} = 0$ for $\forall i, j$):
   $S = \frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{D-1}\hat{\theta} - n\right) = \frac{1}{\sqrt{2n}}\Sigma_{i=1}^{n}\left(z_i - 1\right) \xrightarrow{d} N(0, 1)$ by the CLT under *near-epoch dependence* by Jenish and Prucha (2012).

   (2) General cases ($\sigma_{ij} \neq 0$ but is small enough for $\forall i, j$):
   $S = \frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{-1}\hat{\theta} - n\right) = \frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{D-1}\hat{\theta} - n\right) + A \xrightarrow{d} N(0, 1)$.

Sketch of this paper

- (A simple case) Under the special interaction ($\sigma_{ij} = 0$ for $\forall i, j$),

$$S = \frac{1}{\sqrt{2n}}\left(\begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix}' \begin{pmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ 0 & 0 & \sigma_3^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{pmatrix}^{-1} \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix} - n \right)$$

$$= \frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{D-1}\hat{\theta} - n\right) = \frac{1}{\sqrt{2n}}\Sigma_{i=1}^{n}\left(z_i - 1\right).$$

  - But, $z_i = (\frac{\hat{\theta}_i}{\sigma_i})^2$ is <u>not</u> independent over i; we use the CLT under *near-epoch dependence* by Jenish and Prucha (2012).

$\Rightarrow$ $\frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{D-1}\hat{\theta} - n\right) \xrightarrow{d} N(0,1)$ *as* $(n, T) \to \infty$ such that $\frac{n}{T} \to k$ where $0 \leq k < \infty$. [**Preliminary result**]

Sketch of this paper

- (General cases) Under general interactions ($\sigma_{ij} \neq 0$),

$$S = \frac{1}{\sqrt{2n}}\Big(\begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix}' \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \dots & \sigma_{2n} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \dots & \sigma_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \sigma_{n3} & \dots & \sigma_n^2 \end{pmatrix}^{-1} \begin{pmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \\ \hat{\theta}_3 \\ \vdots \\ \hat{\theta}_n \end{pmatrix} - n \Big)$$

$$= \frac{1}{\sqrt{2n}}\big(\hat{\theta}'\Sigma^{-1}\hat{\theta} - n\big).$$

  - If $\sigma_{ij} = 0$ for $\forall i, j$, we have $S \xrightarrow{d} N(0, 1)$. **[Preliminary result]**
  - If σ_{ij} is *small enough* for $\forall i, j$, we may expect $S \xrightarrow{d} N(0, 1)$.
  - But, we need the "analytical form" of Σ^{-1}; we obtain Σ^{-1} using a Neumann series.

Sketch of this paper

- We obtain the inverse of $\Sigma = \Sigma^D(I - B)$ where I is the identity matrix and $||B||_\infty < 1$.

$$\Sigma^{-1} = (I - B)^{-1}\Sigma^{D-1} = (I + \sum_{k=1}^{\infty} B^k)\Sigma^{D-1} = \Sigma^{D-1} + \sum_{k=1}^{\infty} B^k \Sigma^{D-1}$$

- Therefore, we have

$$\begin{aligned} S &= \frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{-1}\hat{\theta} - n\right) \\ &= \frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{D-1}\hat{\theta} - n\right) + \sum_{k=1}^{\infty} \frac{1}{\sqrt{2n}}\hat{\theta}' B^k \Sigma^{D-1}\hat{\theta}. \end{aligned}$$

- $\frac{1}{\sqrt{2n}}\left(\hat{\theta}'\Sigma^{D-1}\hat{\theta} - n\right) \xrightarrow{d} N(0,1)$ [**Preliminary result**]
- $\sum_{k=1}^{\infty} \frac{1}{\sqrt{2n}}\hat{\theta}' B^k \Sigma^{D-1}\hat{\theta} \xrightarrow{p} 0$ when σ_{ij} is *small enough* for $\forall i, j$

$\Rightarrow$ $S \xrightarrow{d} N(0,1)$ *as* $(n, T) \to \infty$ such that $\frac{n}{T} \to k$ where $0 \leq k < \infty$ under $\sup_{i,j} \frac{w_{ij}}{\sqrt{\sum_{j=1}^{n} w_{ij}^2}} = O(\frac{1}{h_n^*})$. [**Main result**]

Power properties of S

Comparison with M

- Consider the following local alternatives:

  $H_{1,nT} : \delta_i = \frac{\Delta_i}{n^\alpha T^\beta}$ for $i = 1, ..., n$

  where Δ_i is a fixed constant ($\Delta_i \neq 0$).

<Power of the S test>

$$S \xrightarrow{d} N(\frac{\Phi}{\sqrt{2}}, 1)$$

$$\Phi \propto \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Delta_i^2 \sum_{j=1}^{n} w_{ij}^2$$

<Power of the M test>

$$M \xrightarrow{d} N(\phi, 1)$$

$$\phi \propto \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Delta_i \sum_{j=1}^{n} w_{ij}^2$$

  - If the sign of Δ_i is different across i, the power of the traditional M test may be low in general, or vanish under certain circumstances.

Power properties of S

Comparison with M

- Consider the following local alternatives:

  $H_{1,nT} : \delta_i = \frac{\Delta_i}{n^\alpha T^\beta}$ for $i = 1, ..., n$

  where Δ_i is a fixed constant ($\Delta_i \neq 0$).

<Power of the S test>

$$S \xrightarrow{d} N(\frac{\Phi}{\sqrt{2}}, 1)$$

$$\Phi \propto \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \Delta_i^2 \sum_{j=1}^n w_{ij}^2$$

<Power of the M test>

$$M \xrightarrow{d} N(\phi, 1)$$

$$\phi \propto \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \Delta_i \sum_{j=1}^n w_{ij}^2$$

  - If the sign of Δ_i is different across i, the power of the traditional M test may be low in general, or vanish under certain circumstances.

$\Rightarrow$ The low power of M may happen when spatial lag coefficients are heterogeneous and can be more severe when sample size is small.

Power properties of S

Comparison with M

- Consider the following local alternatives:

  $H_{1,nT} : \delta_i = \frac{\Delta_i}{n^{\alpha} T^{\beta}}$ for $i = 1, ..., n$

  where Δ_i is a fixed constant ($\Delta_i \neq 0$).

<Power of the S test>

$$S \xrightarrow{d} N(\frac{\Phi}{\sqrt{2}}, 1)$$

$$\Phi \propto \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Delta_i^2 \sum_{j=1}^{n} w_{ij}^2$$

<Power of the M test>

$$M \xrightarrow{d} N(\phi, 1)$$

$$\phi \propto \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \Delta_i \sum_{j=1}^{n} w_{ij}^2$$

  - If the sign of Δ_i is different across i, the power of the traditional M test may be low in general, or vanish under certain circumstances.

  ⇒ The low power of M may happen when spatial lag coefficients are heterogeneous and can be more severe when sample size is small.

  ⇒ M does not behave *consistently* across all potential alternatives.