Equilibrium Multiplicity: A Systematic Approach using Homotopies, with an Application to Chicago

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Multiple Equilibria in Spatial Models

What pins down the spatial distribution of populations and floor surface prices? Uniqueness/multiplicity of the spatial equilibrium affects the interpretation of treatment effects and the welfare analysis of place-based policies (Bhattacharya, Dupas & Kanaya 2024).

Since the early days of urban economics, equilibrium multiplicity has been at the **core of models**: Krugman (1991),Brock & Durlauf (2002) with important implications for policy design.

Applied work provides evidence for **multiple equilibria:** Brock & Durlauf (2001), Card, Mas & Rothstein (2008), revisited by DAVIS, Easton & Thies (2024), Bosker, Brakman, Garretsen & Schramm (2007), and current work at the frontier presents **conditions of equilibrium uniqueness:** Davis & Weinstein (2008), Allen & Donaldson (2020), Xu, Zenou & Zhou (2022), Zenou & Zhou (2024).

This paper

- A new approach in urban economics that enumerates the equilibria of the city.

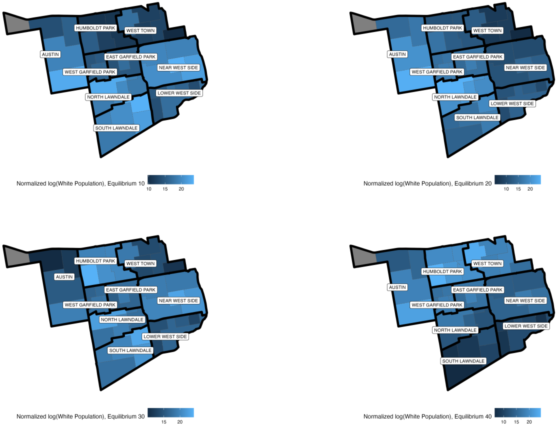
Relies on two observations:

1. recent QSMs can be written as polynomial systems when floor surface supply elasticity is infinite
2. the equilibria of a city with infinite supply elasticity can be smoothly transformed (using a multivariate ODE) into the equilibria of a city with finite supply elasticity.

Key concept of **homotopy** in which the equilibria of a simpler spatial model can be enumerated **and then** smoothly transformed into the equilibria of a more complex model.

$\Rightarrow$ Gives counterfactual **maps** of cities with different spatial distributions given the same set of structural parameters.

A Preview: Equilibria of the West Side Region



And a soon-public GitHub archive.

`https://github.com/aouazad/MultipleEquilibria.jl`

The simple idea

Write the equilibrium conditions $\mathbf{G}(\mathbf{z})$ for a quantitative spatial model. This includes land market clearing conditions for each location, and social equilibrium conditions as in Brock & Durlauf (2002).

$$\mathbf{G}(\mathbf{z}) = 0 \tag{1}$$

Typically a NP-hard problem without closed form solutions (more on this later). But there are simpler cities for which the equilibrium conditions $\mathbf{G}_{\text{simple}}(\mathbf{z}) = 0$ can be fully solved. Then write:

$$\mathbf{H}(\mathbf{z}, \theta) = \theta \mathbf{G}(\mathbf{z}) + (1 - \theta)\mathbf{G}_{\text{simple}}(\mathbf{z}) \tag{2}$$

Starting from all the equilibria at $\theta = 0$, update each initial equilibrium vector (multiple starting points) as:

$$\frac{d\mathbf{z}}{d\theta} = -\left(\frac{\partial \mathbf{H}}{\partial \mathbf{z}}\right)^{-1}\left(\frac{\partial \mathbf{H}}{\partial \theta}\right) \tag{3}$$

A path $\mathbf{z}(\theta)$ of equilibrium vectors, called a homotopy.

Key Conditions

Are we enumerating all equilibria?

- Yes, when G is a polynomial system, we can assert by Bezout's theorem that the number of (complex and real) roots is the product of the degrees.
  $\rightarrow$ We show that the equilibria of a QSM can be written as solutions to a polynomial system.
- We can show that the path will not diverge (homogeneous polynomials, factor by the norm of $\mathbf{z}^{p+q}$ and use the triangular inequality).

Can the path exhibit singularities?

- We verify the singularity of the Jacobian $\frac{\partial \mathbf{H}}{\partial \mathbf{z}}$ numerically using its rank and its condition number.

A Simple City with Many Equilibria

Consider a population with two groups $G = 1, 2$ choosing across a set of J discrete locations, characterized by their amenity A_j, the price of housing q_j, and the population L_j^1 of group $g = 1$ around location j, weighted by the distance to the location.

$$\Psi_j = \sum_{k=1}^{J} e^{-\xi d_{jk}} L_j^1,$$

with d_{jk} the distance between location j and location k.

Using standard notations (Redding & Rossi-Hansberg 2017), the utility of household i of group 1 for location j is:

$$U_{ij} = A_j q_j^{-\alpha} \Psi_j^{\gamma} e_{ij}$$

where e_{ij} is Fréchet distributed with dispersion parameter set to 1. For the sake of simplicity we set $\gamma = 0$ for group 2.

Equilibrium Conditions

An equilibrium vector is a set of Ψ_js, and q_js such that:

(i) Social Interaction Condition:

$$\Psi_j = N^1 \sum_{k=1}^{J} e^{-\xi d_{jk}} \frac{A_k q_k^{-\alpha} \Psi_k^{\gamma}}{\sum_{\ell=1}^{J} A_\ell q_\ell^{-\alpha} \Psi_\ell^{\gamma}} \qquad \text{(EQ1)}$$

(ii) Market clearing Condition:

$$c_j q_j^{\eta} = N^1 \frac{A_j q_j^{-\alpha} \Psi_j^{\gamma}}{\sum_{\ell=1}^{J} A_\ell q_\ell^{-\alpha} \Psi_\ell^{\gamma}} \qquad \text{(EQ2)}$$

If:

- we set the inverse elasticity of supply $\frac{1}{\eta}$ to 0, and
- we express the social interaction preference parameter as a fraction $\gamma = p/q$,

The system becomes a polynomial system with integer powers p, q.

The Polynomial System For the Perfectly Elastic City

$$\eta = \infty$$

$$\left[\sum_{\ell=1}^{J} A_\ell \Psi_\ell^\gamma\right] \Psi_j = N^g \sum_{k=1}^{J} e^{-\xi d_{kj}} A_k \Psi_k^\gamma \qquad \text{(EQ1)}$$

And set $z_j = \psi_j^{1/q}$. This becomes:

$$\left[\sum_{\ell=1}^{J} A_\ell z_\ell^p\right] z_j^q = N^g \sum_{k=1}^{J} e^{-\xi d_{kj}} A_k z_k^p \qquad \text{(EQ1)}$$

This shows us:

1. That there aren't general closed form solutions by the **Abel Ruffini or the Galois theorem**.
2. That the number of solutions (complex and real) is the product of the degrees of the polynomials by **Bezout's Theorem**.
3. We can use **total degree homotopy** to obtain all solutions.

The Two Steps: Two Homotopies

#1: Total Degree Homotopy

We start with solutions to the system:

$$z_j^{p+q} = 1, \qquad j = 1, 2, \ldots, J \tag{4}$$

And transform them using the homotopy:

$$H_j(z, t) = (1 - t)(z_j^{p+q} - 1) + tP_j(\mathbf{z}) \tag{5}$$

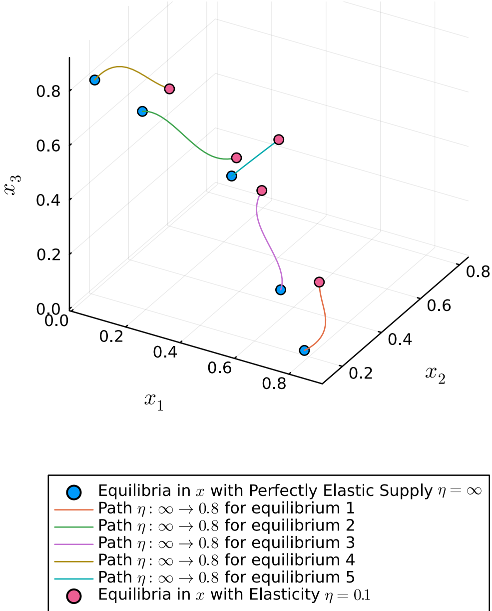
where $P()$ is the polynomial system of the previous slide.

#2: From a Perfectly Elastic to a Finite-Elasticity City η

We start with a solution for $t = 1$, and perform the homotopy along $\eta = \infty$ to a finite η.

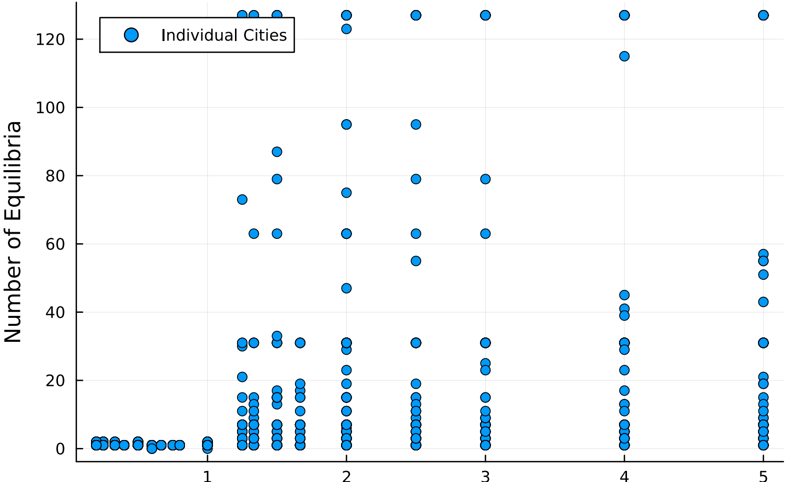
The equilibrium equations (EQ1) and (EQ2) are continuous as $\eta \to \infty$, as $q_j \to \mathrm{mc}_j$.

Example with 3 Locations, x_j population of group 1 in j.



Number of equilibria and Structural Parameters

For run simulations for different numbers of locations, different strengths of social interactions γ, different std(A), and different scopes ξ



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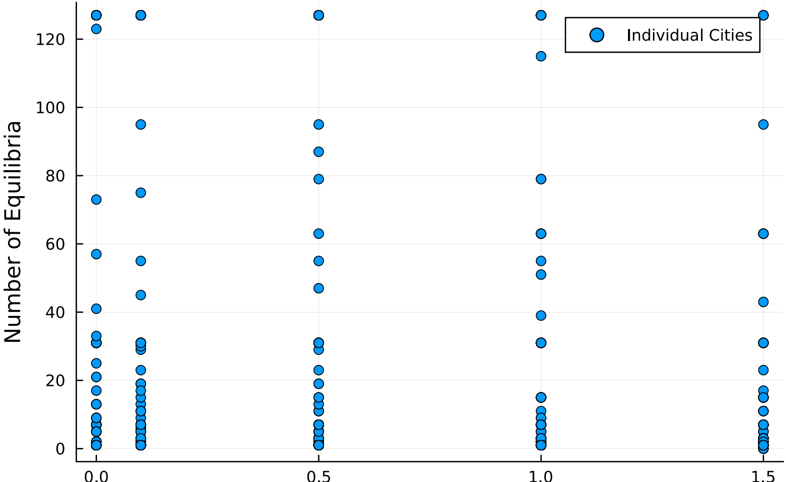

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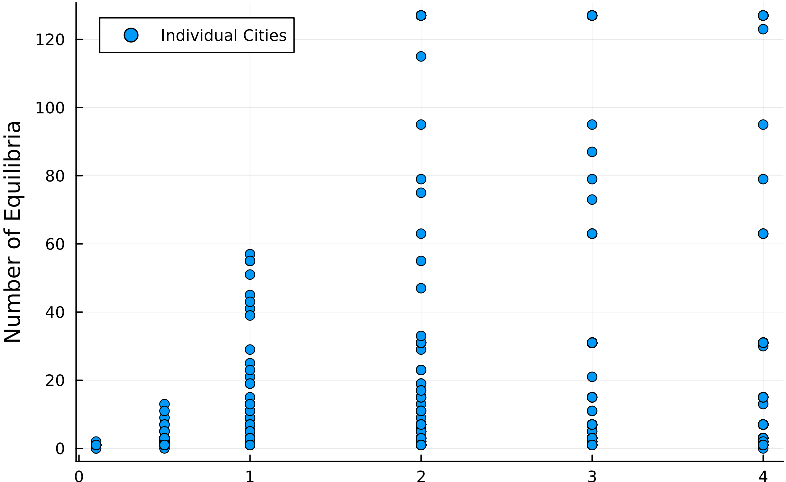
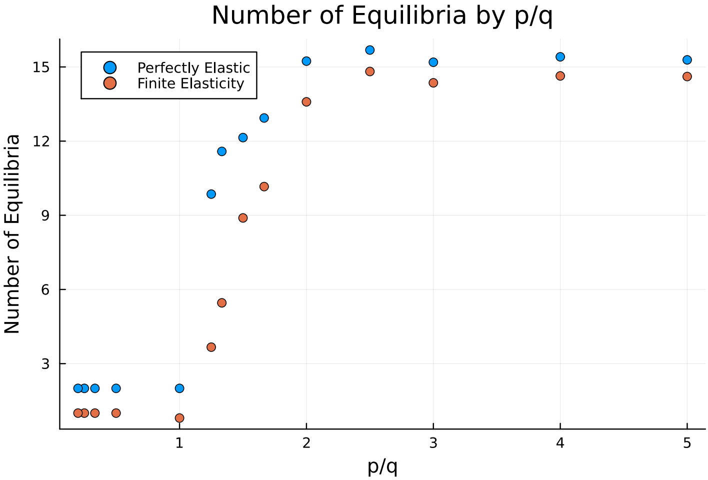
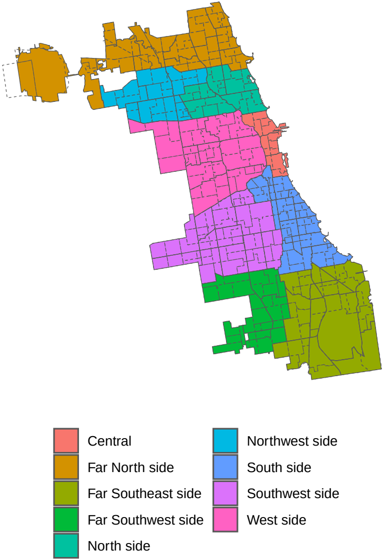


An inelastic supply reduces the number of equilibria



Intuitive! Social preferences γ^1 are a source of strategic complementarities, while prices are a source of strategic substitutabilities.

A Larger City: Chicago



A Larger City

We address the case of large J by setting up a model with local and distant scopes.

The utility of choosing neighborhood j in community i is:

$$U_{ij}^{g} = A_{ij} q_{ij}^{-\alpha} \Psi_{ij}^{\gamma^g} \Psi_{i}^{\gamma^g} \tag{6}$$

where Ψ_{ij} capture the '*near* interactions' across neighborhoods within communities (e.g. within Humboldt Park) and Ψ_i captures 'far interactions' between communities (e.g. between Humboldt Park and West Town).

$$\underbrace{\Psi_{ij} = \sum_{k=1}^{J_i} e^{-\xi d_{ijk}} \frac{L_{ik}^{g}}{L_{i}^{g}}}_{\text{Near interactions}}, \qquad \underbrace{\Psi_{i} = \sum_{\iota=1}^{n} e^{-\xi d_{i\iota}} \frac{L_{\iota}^{g}}{N}}_{\text{Far interactions}} \tag{7}$$

Equilibria when $\eta = \infty$, $q_j = \mathrm{mc}_j$

Denoting by $V_i(L_i)$ the welfare in community i conditional on L_i, this is equal to the typical:

$$V_i(L_i) = \Gamma \left[\sum_{j=1}^{J_i} \left(A_{ij} q_{ij}^{-\alpha} \Psi_{ij}^{\gamma^g} \Psi_i^{\gamma^g} \right)^{\theta} \right]^{\frac{1}{\theta}} \tag{8}$$

where, as usual, Γ is Euler's gamma function. This welfare can be factored into:

$$V_i(L_i, \Psi_i) = \Psi_i^{\gamma} U_i(L_i) = \Psi_i^{\gamma} U_i(1), \tag{9}$$

Far interactions can be factored out at the community level and the equilibria are scale independent.

The probability of choosing community i is then driven by the

$$P_i = \frac{\Psi_i^{\gamma\theta} U_i(L_i)^{\theta}}{\sum_{k=1}^{N} \Psi_k^{\gamma\theta} U_k(L_k)^{\theta}} \tag{10}$$

Introducing Price Responses: $\zeta : 0 \to 1/\eta$

$\mathbf{q}_n$ the vector of prices in each neighborhood, the homotopy is the solution to an ordinary differential equation:

$$(1 - \Phi) \frac{d\mathbf{q}_n}{d\zeta} = B \tag{11}$$

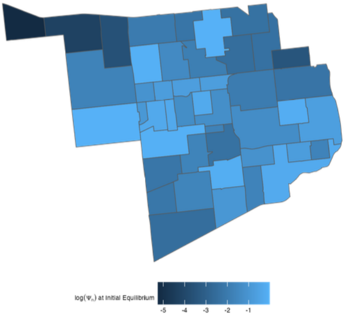
where Φ and B are two matrices changing along the path ζ. When $(\mathbf{1} - \Phi)$ is invertible, the path of equilbria is:

$$\mathbf{q}_n(\zeta) = \mathbf{mc} + \int_0^{\zeta} (\mathbf{1} - \Phi(s))^{-1} B(s) ds \tag{12}$$

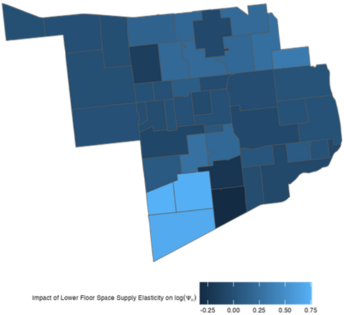
where the initial condition $\mathbf{q}_n(0) = \mathbf{mc}$ is the marginal cost $\mathbf{mc}$ of floor surface.

We show that the endogenous equilibrium vectors $\left[\mathbf{L}^w \ \mathbf{L}^b \ \mathbf{U}^w \ \mathbf{U}^b \ \boldsymbol{\Psi}_c \ \boldsymbol{\Psi}_n \ \mathbf{q}_n\right]$ are entirely determined by the path of equilibrium $\mathbf{q}_n(\zeta)$ prices (partitioned matrix).

Introducing Price Responses: $\zeta : 0 \to 1/\eta$



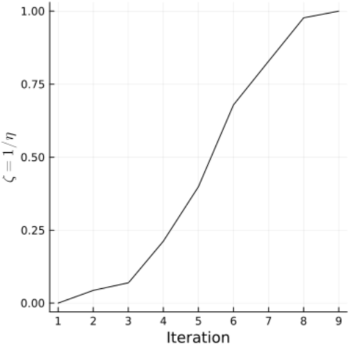
(c) Initial Ψ_n



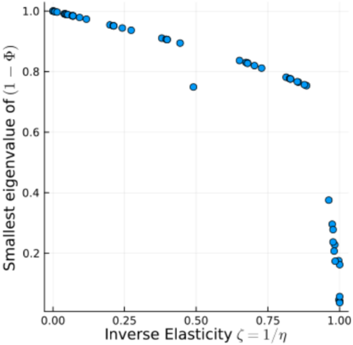
(d) Impact of Lower Elasticity η on Ψ_n

Introducing Price Responses: $\zeta : 0 \to 1/\eta$

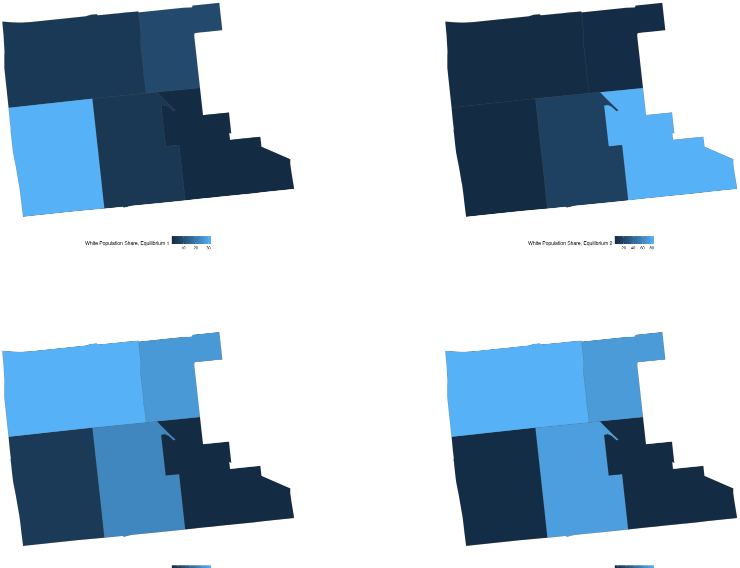
(a) Inverse Elasticity along the Path



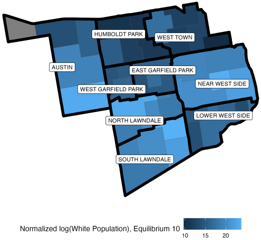
(b) Smallest Eigenvalue of the Jacobian

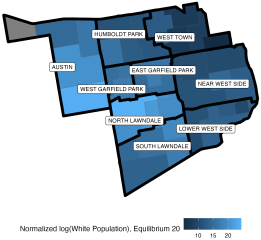


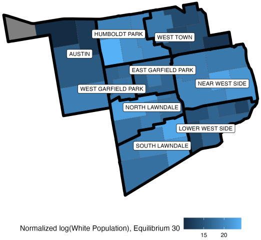
Equilibria of Humboldt Park

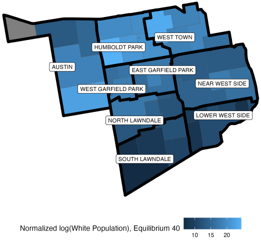


Equilibria of the West Side Region









Equilibria by Community

Community	# of Equilibria	Community	# of Equilibria
Albany Park	3	Lincoln Park	1
Archer Heights	1	Lincoln Square	3
Armour Square	1	Logan Square	3
Ashburn	5	Loop	1
Auburn Gresham	1	Lower West Side	1
Austin	9	Mckinley Park	1
Avalon Park	1	Montclare	1
Avondale	1	Morgan Park	1
Belmont Cragin	1	Mount Greenwood	3
Beverly	5	Near North Side	3
Bridgeport	1	Near South Side	1
Brighton Park	3	Near West Side	3
Burnside	1	New City	7
Calumet Heights	1	North Center	1
Chatham	1	North Lawndale	5
Chicago Lawn	1	North Park	1
Clearing	5	Norwood Park	5
Douglas	1	Oakland	1
Dunning	3	Ohare	7
East Garfield Park	3	Portage Park	7
East Side	1	Pullman	1

Equilibria by Community

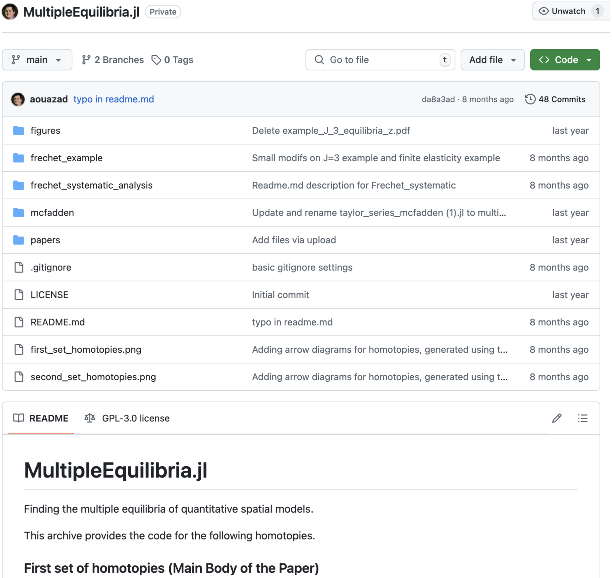
Community	# of Equilibria	Community	# of Equilibria
Edgewater	1	Riverdale	3
Edison Park	1	Rogers Park	1
Englewood	3	Roseland	5
Forest Glen	5	South Chicago	1
Fuller Park	3	South Deering	7
Gage Park	1	South Lawndale	3
Garfield Ridge	7	South Shore	1
Grand Boulevard	1	Uptown	1
Greater Grand Cg	3	Washington Heights	3
Hegewisch	3	Washington Park	1
Hermosa	1	West Elsdon	1
Humboldt Park†	5	West Englewood	1
Hyde Park	1	West Garfield Park	1
Irving Park	3	West Lawn	1
Jefferson Park	3	West Pullman	5
Kenwood	1	West Ridge	3
Lake View	1	West Town	3
		Woodlawn	1

The Technical Details

- When performing a homotopy, the smooth transformation of the equilibrium vector $d\mathbf{z}$ **may not be unique**. $\rightarrow$ We provide a method in the paper to solve for such bifurcations.
- A **numerical test** of this is to measure the lowest eigenvalue of the Jacobian. If it goes close to zero, a likely bifurcation.
- Solving for the equilibrium $\mathbf{z}$ for $\theta = 1$ given the equilibrium vector $\mathbf{z}$ at $\theta = 0$ is solving a first-order ordinary differential equation. Runge-Kutta.

Soon this GitHub archive will move from Private to Public.

Plan to Release GitHub Public Repo



MultipleEquilibria.jl Private

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aouazad typo in readme.md da8a3ad · 8 months ago 48 Commits

figures	Delete example_J_3_equilibria_z.pdf	last year
frechet_example	Small modifs on J=3 example and finite elasticity example	8 months ago
frechet_systematic_analysis	Readme.md description for Frechet_systematic	8 months ago
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papers	Add files via upload	last year
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first_set_homotopies.png	Adding arrow diagrams for homotopies, generated using t...	8 months ago
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MultipleEquilibria.jl

Finding the multiple equilibria of quantitative spatial models.

This archive provides the code for the following homotopies.

First set of homotopies (Main Body of the Paper)

Extensions

- The paper describes and implements an approach to solve for equilibria with **McFadden unobservable heterogeneity** using a Taylor expansion (example in the slides).
- The paper also shows that **dynamic models** are similarly polynomial systems, whose solutions are not closed form but can be obtained by Bezout. Introducing dynamics significantly increases the algorithmic complexity.
- A numerical approach for singularities (ongoing work).

McFadden Case

The Equilibrium conditions can be written as:

$$\Psi_j = \sum_{k=1}^{J} e^{-d_{jk}} x_k = \sum_{k=1}^{J} e^{-\xi d_{jk}} \frac{N^1}{s_k} \frac{\exp(A_k - \alpha q_k + \gamma^1 \Psi_k)}{\sum_\ell \exp(A_\ell - \alpha q_\ell + \gamma^1 \Psi_\ell)} \quad (13)$$

And the homotopy can be performed by "stopping" the Taylor expansion at degree n.

$$\begin{aligned} H_j^n(\boldsymbol{\Psi}, t) \quad = \quad & t\Psi_j^{n+1} \\ & +(1-t)\left\{ \sum_{\ell=1}^{J} \sum_{p=0}^{n} \frac{(A_\ell + \gamma^1 \Psi_\ell)^p}{p!} \Psi_j \right. \\ & \left. \quad - \sum_{k=1}^{J} e^{-\xi d_{jk}} \frac{N^1}{s_k} \sum_{p=0}^{n} \frac{(A_k + \gamma^1 \Psi_k)^p}{p!} \right\} \quad (14) \end{aligned}$$

The Taylor expansion of the exponential converges uniformly to the exponential.

McFadden Case: An Example

A city with 4 locations, homogeneous amenities, and an 8th degree expansion of the exponential.

	Equilibrium Social Demographics			
Equilibrium #	x_1	x_2	x_3	x_4
1	0.008	0.008	0.009	0.974
2	0.008	0.009	0.973	0.009
3	0.009	0.973	0.009	0.008
4	0.068	0.072	0.429	0.431
5	0.072	0.428	0.428	0.072
6	0.078	0.417	0.084	0.420
7	0.126	0.292	0.282	0.300
8	0.262	0.238	0.238	0.262
9	0.286	0.275	0.146	0.293
10	0.293	0.146	0.275	0.286
11	0.300	0.282	0.293	0.126
12	0.420	0.084	0.418	0.078
13	0.420	0.080	0.080	0.420
14	0.431	0.429	0.072	0.068
15	0.974	0.009	0.008	0.008

Singularity Along a Path of Equilibria

At the singular point, the Jacobian is non invertible and its kernel is therefore of dimension ≥ 1. There can be either no solution or multiple solutions to:

$$\frac{\partial H}{\partial \mathbf{z}} \frac{d\mathbf{z}}{dt} + \frac{\partial H}{\partial t} = 0 \tag{15}$$

Two cases:

1. The derivative $\frac{\partial H}{\partial t}$ of H w.r.t. t is in the image of the Jacobian $\frac{\partial H}{\partial \mathbf{z}}$. There are multiple solutions in $\frac{d\mathbf{z}}{dt}$.
   $\rightarrow$ We have a potential bifurcation.
2. The derivative $\frac{\partial H}{\partial t}$ is **not** in the image of $\frac{\partial H}{\partial \mathbf{z}}$ and thus 15 cannot be solved.
   $\rightarrow$ We have a potential cusp.

Addressing the Cusp and the Bifurcation

Case #1: The Bifurcation

There are multiple potential solutions, all expressed as the (direct) sum of one solution of (15) and an element of the null space of $\frac{\partial H}{\partial \mathbf{z}}$.
$\rightarrow$ Add a condition to define this second term, typically a second-order Taylor expansion as in Keller (1977).

Case #2: The Cusp

Here we want to overcome the singularity. In this case we can use homotopy path as:

$$H(\mathbf{z}(\theta), t(\theta)) = 0 \tag{16}$$

and impose a condition on the size of the changes in $\mathbf{z}$ and t for each change $d\theta$. One option is:

$$\sqrt{\frac{d\mathbf{z}}{d\theta}^2 + \frac{dt}{d\theta}^2} = 1 \tag{17}$$

Conclusion

- Exciting questions around the determinacy of our quantitative spatial models.
- Techniques could also be useful for a second order expansion instead of the first-order expansion of Kleinman, Liu, Redding (2023) and Ouazad and Kahn (2024): tipping in impulse response functions?
- We are not the first to use these techniques in economics (Judd 2012, Balasko 2009), but we show here that QSMs can be straightforwardly expressed to take advantage of these approaches.
- Exciting intellectual questions ahead!
- Deep questions of Redding (2024): are multiple equilibria a consequence of a model missing out on some key details? Are cities always about history and not about expectations (Krugman 1991)?

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