# Robust Comparative Statics with Misspecified Bayesian Learning

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# Motivation

- We care about model misspecification in economic environments
  - Agents often work with misspecified models

    true model unknown, approx. with **incorrect** models

  - A departure from the traditional rational expectations framework

    cognitive biases, complexity, simplified perspectives...

- Selective examples
  - Maximum likelihood estimation of misspecified linear models (White (1982, ECMA))
  - Monopolist learning with misspecified demand model (Nyarko (1991, JET))
  - Portfolio choice with misspecified asset returns (Uppal-Wang (2003, JF))
  - Interest rate/GDP forecasting with misspecified models (Farmer et al. (2024, JPE))

# Motivation

- The Berk-Nash solution (Esponda and Pouzo (2016 (ECMA), 2021 (TE)))
  - Agent has **misspecified** models, takes action based on them, observes outcome, Bayesian updates on the models, and repeats...
  - The equilibrium/steady state characterization: optimal action/distribution and **best** incorrect model, both dependent on each other
  - Important because misspecification made explicit, allows for choice between different misspecified models to adjust for observed behavior
- Enriching economic environments with misspecified models
  - Captures limit outcomes of Bayesian learning when agents have misspecified models
  - One such environment of interest: Markov Decision Processes (MDPs)
- This paper: **Monotone comparative statics** with misspecified MDPs

# Motivation

- For e.g.- the infinite-horizon expected discounted utility problem

$$\max_{\{x_t\}_{t=0}^{\infty}} \mathbb{E}_Q \left[ \sum_{t=0}^{\infty} \delta^t u(s_t, x_t) \right], \; t = 0, 1, 2, \ldots$$

- $V$ is the solution to the Bellman equation in (1)

$$V(s) = \max_{x \in \mathbb{X}} \left\{ u(s, x) + \delta \int_{\mathbb{S}} V(s') \, Q(ds' \mid s, x) \right\} \quad (1)$$

# Motivation

$$V(s) = \max_{x \in \mathbb{X}} \left\{ u(s,x) + \delta \int_{\mathbb{S}} V(s') Q(ds' \mid s, x) \right\}$$

- Corresponding to (1), one can ask important comparative statics questions
  - $\uparrow$ in $Q \overset{?}{\to} \uparrow$ in **optimal policy**; $\uparrow$ in $Q \overset{?}{\to} \uparrow$ in **stationary distribution**
  - Results in the literature provide conditions, e.g. Hopenhayn-Prescott (1992)

**"Stochastic Monotonicity and Stationary Distributions for Dynamic Economies (ECMA)"**

- Instead misspecified models, $\{Q_\theta\}_{\theta \in \Theta}, Q \notin \{Q_\theta\}_{\theta \in \Theta}$, (Esponda-Pouzo (2021))

$$V(s,\mu) = \max_{x \in \mathbb{X}} \left\{ u(s,x) + \delta \int_{\mathbb{S}} V(s', \mu') \bar{Q}_\mu (ds' \mid s, x) \right\} \quad (2)$$

where $\bar{Q}_\mu = \int_\Theta Q_\theta \mu(d\theta)$ and $\mu'$ updated using Bayes' rule on models

# Motivation

- $$V(s, \mu) = \max_{x \in \mathbb{X}} \left\{ u(s, x) + \delta \int_{\mathbb{S}} V(s', \mu') \bar{Q}_\mu (ds' \mid s, x) \right\}$$

  where $\bar{Q}_\mu = \int_\Theta Q_\theta \mu(d\theta)$ and $\mu'$ updated using Bayes' rule on models
- Steady-state prediction: Berk-Nash solution (Esponda-Pouzo (2021), Berk (1966))
  - (a) stationary distribution over states and actions **dependent** on best-fit model
  - (b) best-fit model (KL divergence) **dependent** on stationary distribution
- **Monotone comparative statics of Berk-Nash solution w.r.t. primitives** $(u, \delta, Q, Q_\Theta, \Theta)$

# MCS Results in Markov Environments

| Environment | Change in Primitives | Sufficient Conditions |
|---|---|---|
| • Markov Processes<br>• **Positive Shocks** $\times$ | • $Q \uparrow \implies \Theta_Q(m) \uparrow$<br>• $\Theta \uparrow \implies \Theta_Q(m) \uparrow$ | • $Q \uparrow$: $\log(\frac{Q_{\theta_2}}{Q_{\theta_1}}) \uparrow$<br>• Milgrom-Shannon (94) |
| • Markov Decision Processes<br>• **Positive Shocks** $\checkmark$ | • Patience $(\delta \uparrow)$<br>• Utility primitives $(u \uparrow)$<br>• Beliefs $(\mu \uparrow)$ | • Assumptions 1 and 2<br>• Inc. diff. in $x$ and $p$<br>• Assumptions 1 and 2 |

**Table 1:** MCS and Berk-Nash solution

# Outline of the Paper

Results

- Theorem 1: Existence of such a Berk-Nash solution (a new proof based on monotonicity)
- Theorems 2-4: Robust MCS of Berk-Nash solution with primitives (identify a **positive** shock)
- Theorem 5: Bound on the cost of misspecification in terms of primitives (entropic bounds)

Technical Contribution

- Non-lattice fixed point techniques for endogenous MDPs with misspecification
  - Precursors: Smithson (1971), Acemoglu-Jensen (2015) (exogenous shocks with no misspecification in large economies)

Contribution to the literature

- Provide MCS for dynamic programming (MDPs) with misspecified learning and give robust predictions, without specific knowledge of primitives of the environment

# Plan of the Talk

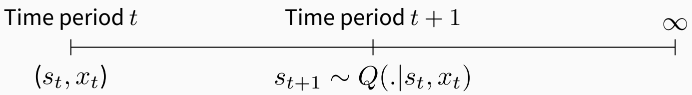
- Framework
- Examples
- Theorems
- Extensions
- Conclusion

# Model - Objective MDP

- Markov decision process (MDP) is a list $\langle \mathbb{S}, \mathbb{X}, u, Q, \delta \rangle$, where
    - (a) $\mathbb{S} \subseteq \mathbb{R}$ is a compact set of states, (b) $\mathbb{X} \subseteq \mathbb{R}$ is a compact set of actions, (c) $u : \mathbb{S} \times \mathbb{X} \to \mathbb{R}$ is a per-period payoff function, (d) $Q : \mathbb{S} \times \mathbb{X} \to \mathcal{M}_1(\mathbb{S})$ is a transition probability function, (e) $\delta \in [0, 1)$ is the discount factor

- Choose feasible policy rule $\{x_t\}_{t=1}^{\infty}$ to maximize expected discounted utility

$$\mathbb{E}_Q \left[ \sum_{t=0}^{\infty} \delta^t u\left(s_t, x_t\right) \right]$$



# Model - Objective MDP

- The Bellman for this problem

$$V(s) = \max_{x \in \mathbb{X}} \left\{ u(s, x) + \delta \int_{\mathbb{S}} V(s') \, Q(ds' \mid s, x) \right\} \tag{3}$$

- Corresponding to (3), action $\hat{x}$ is optimal given $s$ in the MDP$(Q)$ if

$$\hat{x} \in G(s) \equiv \arg\max_{x \in \mathbb{X}} \left\{ u(s, x) + \delta \int_{\mathbb{S}} V(s') \, Q(ds' \mid s, x) \right\} \tag{4}$$

# Model - Subjective MDP

- Uni-dimensional compact parameterized **misspecified** models $\Theta$

$$(\mathsf{MDP}(Q), \mathcal{Q}_\Theta)$$

where $\mathcal{Q}_\Theta = \{Q_\theta : \theta \in \Theta \subseteq \mathbb{R}\}, Q \notin \mathcal{Q}_\Theta$

- The Kullback-Liebler (KL) divergence of a model $Q_\theta$ w.r.t. $Q$

$$\mathsf{KL}(Q_\theta || Q) \equiv \mathbb{E}_Q[\ln(Q/Q_\theta)] \text{ (finite)} \tag{5}$$

where the best-fit set is

$$\Theta_Q \equiv \arg\min_{\theta \in \Theta} \mathsf{KL}(Q_\theta || Q)$$

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- For infinite spaces, one uses the Radon-Nikodym derivative $D_\theta$ of $Q$ with respect to $Q_\theta$

## Model - Berk-Nash Solution

**Definition 1 (Esponda-Pouzo (2021)[1])**

A probability distribution $m^* \in \Delta(\mathbb{S} \times \mathbb{X})$ is a Berk-Nash solution of the regular-SMDP if there exists a belief $\mu^* \in \Delta(\Theta)$ such that the following conditions hold.

(i) Action $x^*$ optimal given $s$ in the MDP$(\bar{Q}_{\mu^*})$, $\bar{Q}_{\mu^*} = \int_\Theta Q_\theta \mu^*(d\theta), \forall (s,x)$ in support of $m^*$

(ii) Belief $\mu^* \in \Delta\left(\Theta_Q(m^*)\right)$ where $\Theta_Q(m^*) \equiv \arg\min_{\theta\in\Theta} \int_{\mathbb{S}\times\mathbb{X}}$ KL$(Q_\theta || Q) m^*(ds, dx)$

(iii) Invariant measure on state, for all $A \subseteq \mathbb{S}$, $m^*_{\mathbb{S}}(A) = \int_{\mathbb{S}\times\mathbb{X}} Q\left(A \,|\, s, x\right) m^*(ds, dx)$

Regular SMDP: continuity (absolute), compact parameter space, U.I. Radon-Nikodym derivatives

[1] Anderson-Duanmu-Ghosh-Khan (2024, JET), hereafter ADGK

# Model - Behavior of Fixed Points

- Fixed points of the Berk-Nash solution mapping

$$T : Z \times P \to 2^Z$$

  where $Z = \Delta(\mathbb{S} \times \mathbb{X}) \times \Delta(\Theta)$ and $P =< u, \delta, Q, Q_\Theta, \Theta >$ are our primitives

  The set of fixed points

$$\Lambda(p) \equiv \{z \in Z : z \in T(z, p)\},\ p \in P$$

- Question:

change in primitives $\overset{?}{\to}$ change in Berk-Nash solution

# Example I : Inference with No Role for Actions

Forecasting problem

- True process, $s_{t+1} \sim Q(\cdot|s_t)$, where

$$s_{t+1} = \rho s_t + \xi_{t+1}, \qquad \xi_{t+1} \sim 0.5F_{(\mu_1,\sigma^2)} + 0.5F_{(\mu_2,\sigma^2)} \tag{6}$$

  where $F$ denotes the cumulative density function for a normal distribution. The components have different means ($\mu_1 \neq \mu_2$) but identical variances ($\sigma_1^2 = \sigma_2^2$).

- Agent has a set of models $\{Q_\theta\}$, indexed by $|\theta| \in [0,1), Q \notin \{Q_\theta\}$

$$s_{t+1} = \theta s_t + \epsilon_{t+1}, \qquad \epsilon_{t+1} \sim N(0,\sigma^2)$$

# Example I : Inference with No Role for Actions

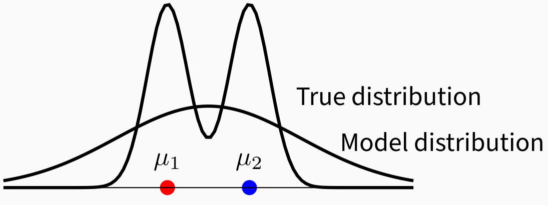



**Figure 1:** AR(1) models with misspecified Gaussian noise

- Best-fit inferred AR(1) parameter $\theta^*$ in the Berk-Nash equilibrium $m^*_{\mathbb{S}}$ has the following form,

$$\theta^* = \int_{\mathbb{S}} \hat{\theta}(s) m^*_{\mathbb{S}} = \rho + \int_{\mathbb{S}} \frac{(\mu_1 + \mu_2)}{s} m^*_{\mathbb{S}}$$

- A digression: notice when $\mu_1 + \mu_2 = 0$? The comparative statics of true persistence $\rho$ and inferred persistence at the steady state $\theta^*$ is one-to-one

# Example II: Inference and Actions Together - Misinference Channel

Savings with misperceived wealth process (based on Esponda-Pouzo (2021), ADGK)

- The agent learns about the return on their wealth process while optimally deciding consumption and savings
- Each period, the agent realizes wealth $y_t$, an i.i.d. preference shock $z_t$, and chooses savings $x_t \in [0, y_t] = \mathbb{X} \subseteq \mathbb{R}_+$
- State variables $s = (y, z)$ belong to $\mathbb{S} = \mathbb{R}_+ \times [0, 1]$
- Period $t$ payoffs are $u(y_t, z_t, x_t) = z_t \ln(y_t - x_t)$, with discount factor $\delta$
- **True** process:

$$\ln y_{t+1} = \alpha^* + \beta^* \ln x_t + \varepsilon_t,$$

where the unobserved productivity shock $\varepsilon_t = \boldsymbol{\gamma}^* z_t + \xi_t$, with $\xi_t \sim \mathcal{N}(0, 1)$, $z_t \sim U[0, 1]$, and $\boldsymbol{\gamma}^* > \mathbf{0}$ (correlated shocks)

# Example II: Inference and Actions Together - Misinference Channel

- **Misspecified** process:

$$\ln y_{t+1} = \alpha + \beta \ln x_t + \varepsilon_t,$$

  where $\varepsilon_t \sim \mathcal{N}(0,1)$, ignoring the correlation between productivity and preference shocks. Higher $\boldsymbol{\gamma}^*$, **starker** the misspecification

- A key comparative static for the Berk-Nash solution: an increase in $\boldsymbol{\gamma}^*$ leads to:
  - $m^*$: The long-run **perceived** distribution of the wealth process (↓)
  - $\hat{\beta}$: The best-fit parameter inferred for the return on the process (↓)

- Misinference channel: A higher $\boldsymbol{\gamma}^*$ leads to a larger negative bias in the inferred return, driven by lower preference shocks and higher savings

# Model - Orders on Primitives and Eqm. Objects

- Set $Y$ dominates $X$ in the strong-set order if for any $x$ in $X$ and $y$ in $Y$, we have $\sup\{x, y\}$ in $Y$ and $\inf\{x, y\}$ in $X$.
- Parameter space $\Theta \subseteq \mathbb{R}$ (strong-set order), utility and discount factor (natural order)
- $f : \mathbb{R}^n \to \mathbb{R}$, is increasing if for $x \geqslant y$ in the component-wise order, $f(x) \geqslant f(y)$
- For all bounded, increasing, and measurable $f's$,

$$m_2 \succsim_{st} m_1 \equiv \int_{\mathbb{S}\times\mathbb{X}} f(s,x)m_2(ds,dx) \geqslant \int_{\mathbb{S}\times\mathbb{X}} f(s,x)m_1(ds,dx)$$

- Poset $(X, \succsim)$ is a lattice if for any $x, x' \in X$, the meet $x \wedge x'$ and the join $x \vee x'$ are in $X$
  - For e.g. $(\mathbb{R}, \succsim_{st})$ is a lattice. However, the poset $(\mathbb{S} \times \mathbb{X}, \succsim_{st})$ is not a lattice (Kamae, Krengel, O'Brien (1977))

# Model - Assumptions

**Assumption 1 (Standard)**

$\mathbb{S}, \mathbb{X}, \Theta$ *are lattices,* $u(s, x)$ *is supermodular in* $(s, x)$*, increasing in* $s$.

Supemodularity: $u(s_1, x_1) + u(s_2, x_2) \leqslant u((s_1, x_1) \vee (s_2, x_2)) + u((s_1, x_1) \wedge (s_2, x_2))$
For an increasing $f : \mathbb{S} \to \mathbb{R}$ :

**Assumption 2 (Models)**

*The following are true for all models* $\theta$ *in the family of models,* $\mathcal{Q}_\Theta = \{Q_\theta : \theta \in \Theta\}$ .

(i) $Q_\theta$ *is stochastically increasing in* $(s, x)$ *i.e.* $\int_{\mathbb{S}} f(s')Q_\theta(ds'|s, x)$ *is increasing in* $(s, x)$

(ii) $Q_\theta$ *is stochastically supermodular in* $(s, x)$ *i.e.* $\int_{\mathbb{S}} f(s')Q_\theta(ds'|s, x)$ *is supermodular in* $(s, x)$

Examples 1 and 2 satisfy such requirements, for e.g., AR (1) process. Further, $Q$ is assumed to be monotone in $(s, x)$.

# Model - Assumptions

**Assumption 3 (Point Identification)**

*For any given* $m \in \Delta(\mathbb{S} \times \mathbb{X})$, *a SMDP* $(Q, Q_\Theta)$ *is point-identified, i.e.*

$$\theta, \theta' \in \Theta(m; Q) \implies \theta = \theta^{'}$$

**Assumption 4 (Single Crossing)**

$K_Q(\theta; m)$ *satisfies the single crossing property in* $(\theta; m)$, $\theta_1 \leqslant \theta_2, m_2 \succeq_{st} m_1$

$$K_Q(\theta_2; m_1) - K_Q(\theta_1; m_1) \geqslant 0 \implies K_Q(\theta_2; m_2) - K_Q(\theta_1; m_2) \geqslant 0.$$

$$\Theta_Q(m) \equiv \arg\min_{\theta \in \Theta} K_Q(m, \theta)$$

Invoke Milgrom-Shannon (1994): quasi-supermodularity trivially satisfied

# Results - Existence

Under standard (lattice and increasing payoffs), increasing and supermodular models, point identification : Assumptions 1-3

**Theorem 1 (Existence and Compactness)**

Under assumptions 1-3, every regular SMDP $(Q, \mathcal{Q}_\Theta)$ with a bounded and continuous utility function has a Berk-Nash equilibrium and the set of such equilibria is compact.

A new existence proof:

- Esponda-Pouzo (2021): Only for finite spaces
- ADGK: Uses nonstandard analysis for infinite (compact and non-compact) spaces
- Theorem 1: A standard proof for compact spaces using ADGK and monotonicity assumptions

# Main Result - Comparative Statics

Fix any belief $\mu$ over models, define the optimal policy correspondence $G$ :

$$G(s,\mu,p) \equiv \arg\max_{x\in\mathbb{X}} \left\{ u\left(s,x\right) + \delta \int_{\mathbb{S}} V\left(s'\right) \bar{Q_{\mu}}\left(ds' \mid s,x\right) \right\} \tag{7}$$

**Positive Shock:**

A $\Delta$ in a primitive from $p_1$ to $p_2$ is a positive shock (SSO) if:

For all $y_1 \in G\left(s,\mu,p_1\right)$ and $y_2 \in G\left(s,\mu,p_2\right), y_1 \vee y_2 \in G\left(s,\mu,p_2\right)$ and $y_1 \wedge y_2 \in G\left(s,\mu,p_1\right)$

Fix $p \in P$. A $\Delta$ in the model distribution from $\mu_1$ to $\mu_2$ is a positive shock (SSO) if:

$G(s,\mu,p)$ is ascending in $\mu$ from $\mu_1$ to $\mu_2$

# Main Result - Comparative Statics (via Positive Shock)

Under standard (lattice and increasing payoffs), increasing and supermodular models, point identification (assumptions 1-3), and single-crossing differences (assumption 4)

**Theorem 2 (Main Result)**

Suppose assumptions 1-4 hold. Then a positive shock to the primitives of the regular SMDP will lead to an increase in the least and the greatest equilibrium best-fit models. Further, a positive shock to the primitives will lead to

(a) an increase in the least and greatest Berk-Nash equilibrium in the usual stochastic order dominance if changes in beliefs over models are positive shocks.

(b) a decrease in the least and greatest Berk-Nash equilibrium in the usual stochastic order dominance if changes in beliefs over models are negative shocks.

Think of the unique Berk-Nash solution!

# Identifying Positive Shocks in Misspecified Environments

| Environment | Change in Primitives | Sufficient Conditions |
|---|---|---|
| • Markov Processes<br>• Positive Shocks $\times$ | • $Q \uparrow \implies \Theta_Q(m) \uparrow$<br>• $\Theta \uparrow \implies \Theta_Q(m) \uparrow$ | • $Q \uparrow$: $\log(\frac{Q_{\theta_2}}{Q_{\theta_1}}) \uparrow$<br>• Milgrom-Shannon (94) |
| • Markov Decision Processes<br>• Positive Shocks ✓ | • Patience $(\delta \uparrow)$<br>• Utility primitives $(u \uparrow)$<br>• Beliefs $(\mu \uparrow)$ | • Assumptions 1 and 2<br>• Inc. diff. in $x$ and $p$<br>• Assumptions 1 and 2 |

**Table 2:** MCS and Berk-Nash solution

# Other Results - (more) Comparative Statics, Welfare Costs

**Theorem 3 (Increasing Models)**

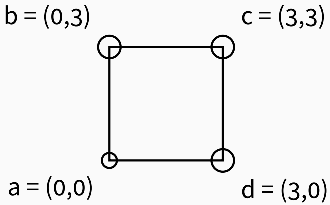
Suppose the hypothesis in Theorem 1 continue to hold. If a change in beliefs over models is a positive shock, then an increase in the parameter set under the strong set order leads to an increase in the least and the greatest equilibrium best-fit models.

**Theorem 4 (Increasing and Convex Order)**

Suppose assumptions 1-3 and single-crossing holds for increasing and convex order. Then a positive (negative) shock to the primitives will lead to an increase in the least and greatest Berk-Nash equilibrium in the increasing and convex order if changes in beliefs over models are positive (negative) shocks.

# Technical Contribution

- Berk-Nash equilibrium map $T : W \to 2^W$, $W = \Delta(\mathbb{S} \times \mathbb{X}) \times \Delta(\Theta)$
- Space of probability measures ordered by $\succsim_{st}$ not lattice - Tarski/Topkis/Knaster-Tarski/Hopenhayn-Prescott $\times$
- But it is chain-complete: a chain that has both infimum and supremum $p_1 = 0.5(\epsilon_a + \epsilon_b), p_2 = 0.5(\epsilon_a + \epsilon_c), p_3 = 0.5(\epsilon_c + \epsilon_b), p_4 = 0.5(\epsilon_a + \epsilon_d).$




- Apply non-lattice techniques that we tailor for endogenous MDPs with misspecification
    - Endogenous MDPs require stronger conditions of supermodularity on the Bellman

# Technical Contribution

- Non-lattice structure of the Berk-Nash solution:
  - Endogenous misspecified MDPs require stronger conditions for uniqueness, including supermodularity of the Bellman function (assumptions 1 and 2).
- The proof technique follows a three-step structure:
  - Step 1: For Theorem 2, show stationary distributions $m^*$ induced by $G$ are Type I (Type II) monotonic in $p$ for $\mu \in \Delta(\Theta)$
  - Step 2: Construct a mapping $\hat{\theta}$ that, for each $\mu$ and $p$, gives model distributions $\mu'$. Fixed points are equilibrium distributions $\mu^*$
  - Step 3: Show least and greatest selections of the map increase in $p$, also provides a new existence proof for Theorem 1 based on monotonicity and identification of $T$

# Back to Example II - Analysis

An increase (decrease) in $\gamma^*$ is a **negative** (positive) shock

- The state, action, and parameter spaces are lattices; utility is increasing in $y$ and $z$. The concave payoff function with $\frac{\mathrm{d}^2 u(y,z,x)}{\mathrm{d}x\ \mathrm{d}y} > 0$ is supermodular, satisfying Assumption 1
- Model distributions are Gaussian with mean $\alpha + \beta \ln x$ and unit variance, satisfying Assumption 2 via stochastic dominance of higher $x$
- Assumption 3 holds as Gaussian distributions are strictly log-concave, ensuring unique identification. Thus, Theorem 1 guarantees the Berk-Nash equilibrium
- Assumption 4 is verified via the sufficient condition

# Concluding Remarks

- We establish new results on monotone comparative statics for misspecified dynamic programs and provide novel predictions for misspecified behavior
- The results are of applied interest across a variety of domains, including forecasting, consumption-saving models, and effort-choice problems (In paper)
- The machinery to establish the results are powerful and relies on non–lattice characterizations
- Paper link: Here!

# References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# Other Results - Welfare Ranking

- Objective welfare

$$W(s,\bar{\theta}) = \mathbb{E}_{Q(\cdot|s,g(s,\bar{\theta}))}\left[\sum_{t=0}^{\infty}\beta^t u(s_t, g(s_t,\bar{\theta}))\right],\ t = 0,1,2,\ldots. \tag{8}$$

- $\bar{\theta} = \theta^*$ (correct), $\theta_*$ (misspecified)
- Approximation error in optimal policy

$$||g(s,\theta^*) - g(s,\theta_*)|| < \gamma$$

- $u : \mathbb{S} \times \mathbb{X} \to \mathbb{R}$ is continuously differentiable in actions

**Theorem 5**

- $W(s, \theta^*) \geqslant W(s, \theta_*)$
- For a given approximation error $\gamma$,

$$||W(s, \theta^*) - W(s, \theta_*)|| \leqslant \frac{2\beta m_0(1 - e^{-k^*}) + m_1\gamma}{1 - \beta}, \tag{9}$$

where $m_0$ and $m_1$ denote the absolute upper bound on the utility and the marginal utility function, respectively, and $k^*$ is the supremum on the KL entropy between $Q$ with the optimal policy and the $Q$ with the misspecified policy

Inspired by Santos (2000), Theorem 5 is potentially useful in the numerical approximation of the Berk-Nash equilibria.

# Related Literature

## Learning with Misspecified models

Berk (1966), Arrow-Green (1973), Nyarko (1991), Hansen-Sargent (1999, 2001), Esponda-Pouzo (2016, 2021), Heidheus-Koszegi-Strack (2018), Esponda-Pouzo-Yamamoto (2021), Molavi (2022), Frick-Iijima-Ishii (2022), Farmer-Nakamura-Steinsson (2024), Lanzani (2024), Anderson-Duanmu-Ghosh-Khan (2024)

## Monotone Comparative statics

Smithson (1971), Höft (1987), Amir (1991), Hopenhayn-Prescott (1992), Milgrom-Shannon (1994), Topkis (1998), Huggett (2003), Torres (2005), Acemoglu-Jensen (2013, 2015), Light (2021), Balbus-Dziewulski-Reffett-Wozny (2022), Dziewulski-Quah (2023)

## Miscellaneous

Santos (2000), Koulovatianos-Mirman-Santugini (2009)