Quantile Local Projections: Identification, Smooth Estimation, and Inference

Josef Ruzicka

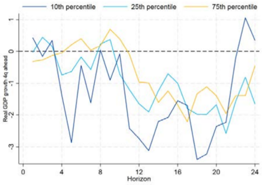
Department of Economics
School of Sciences and Humanities
Nazarbayev University

January 3, 2025

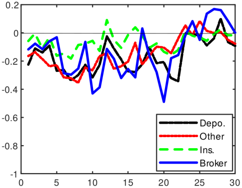
Introduction

- Dynamic causal effects of shocks – impulse response functions
- Local Projections (LP)
  - Simple and flexible alternative to VAR
  - Only capture the mean – lose information
  - Usually estimated by least squares – not robust
- Quantile Regression (QR)
  - Describes the entire conditional distribution
  - Special case – median regression – robust
- LP estimated by QR – quantile local projections (QLP)
  - Increasingly popular in empirical macroeconomics and finance
  - Researchers face three important challenges in practice.

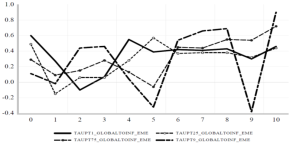
Open questions and contribution



Source: Lang et al. (2019)



Source: Han et al. (2020)



Source: Siklos (2020)

- QLP – open questions
  - How to identify quantile treatment effects?
  - How to construct closed-form confidence intervals?
  - How to smooth the impulse response functions?
- Contribution: coherent econometric framework for QLP
  - Solves the open questions – identification, inference, smoothing
  - Financial conditions affect the entire GDP growth distribution.
  - Heterogeneous effects of monetary policy

Local projections: very simple examples (OLS vs QR)

- Univariate time series $\{y_t\}$, forecast horizon $h \in \{1, 2, 3, \dots\}$
- Standard local projections

$$\mathbb{E}[y_{t+h}|y_t, y_{t-1}, \dots] = \alpha_h + \beta_h y_t$$
$$IR(h) = \frac{\partial \mathbb{E}[y_{t+h}|y_t, y_{t-1}, \dots]}{\partial y_t} = \beta_h$$

- Quantile local projections, quantile $\tau \in (0, 1)$

$$Q_\tau[y_{t+h}|y_t, y_{t-1}, \dots] = \alpha_h(\tau) + \beta_h(\tau) y_t$$
$$IR(h, \tau) = \frac{\partial Q_\tau[y_{t+h}|y_t, y_{t-1}, \dots]}{\partial y_t} = \beta_h(\tau)$$

Theoretical results – overview of contribution

- Causality in dynamic quantile models
  - Lemma: Structural quantile form of a stochastic process
    - Connects multivariate time series and IV-QR (Chernozhukov & Hansen, 2005, 2006, 2008)
    - Every multivariate higher-order Markov process can be represented in a nonlinear simultaneous equations model with iid disturbances.
  - Definition: Impulse response function
    - Quantile treatment effect of an exogenous impulse
    - Can be identified by recursive short-run restrictions or external instruments
- Recursive short-run restrictions – identification, estimation, inference
  - Proposition: Identification
  - Definition: SQLP estimator (QR + roughness penalties)
  - Propositions: Consistency, asymptotic normality
  - Information criteria for optimal smoothing
- External instruments – identification, estimation, inference
  - Proposition: Identification
  - Definition: SQLPI estimator (IV-QR + roughness penalties)
  - Proposition: Inference with weak instruments
  - Quasi-information criteria for optimal smoothing

SQLP application: financial conditions and GDP

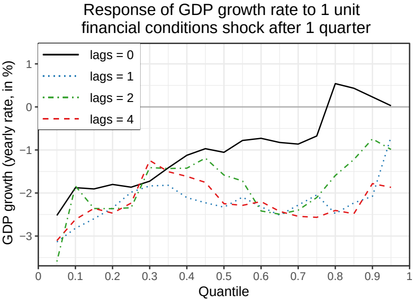
- Adrian et al. (2019, AER) study the effect of financial conditions on GDP in the US.
- Their main specification involves two time series:
  - Real GDP growth rate, denoted $gGDP_t$
  - National Financial Conditions Index, denoted $NFCI_t$
- Sample: 1973Q1 to 2015Q4 (172 observations)
- Their main specification is:

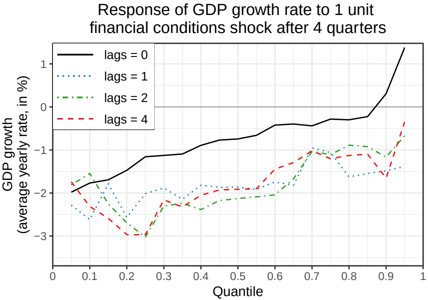
$$Q_\tau(gGDP_{t+h}|gGDP_t, NFCI_t) = \beta_{0,0,h}(\tau) + \beta_{1,0,h}(\tau)gGDP_t + \beta_{2,0,h}(\tau)NFCI_t$$

- Contrasting it with our Proposition 2, we notice that:
  - Their specification entails the following identification restriction: NFCI has no contemporaneous effect on GDP growth distribution.
  - Also, their specification does not control for any lags.

SQLP application: financial conditions and GDP

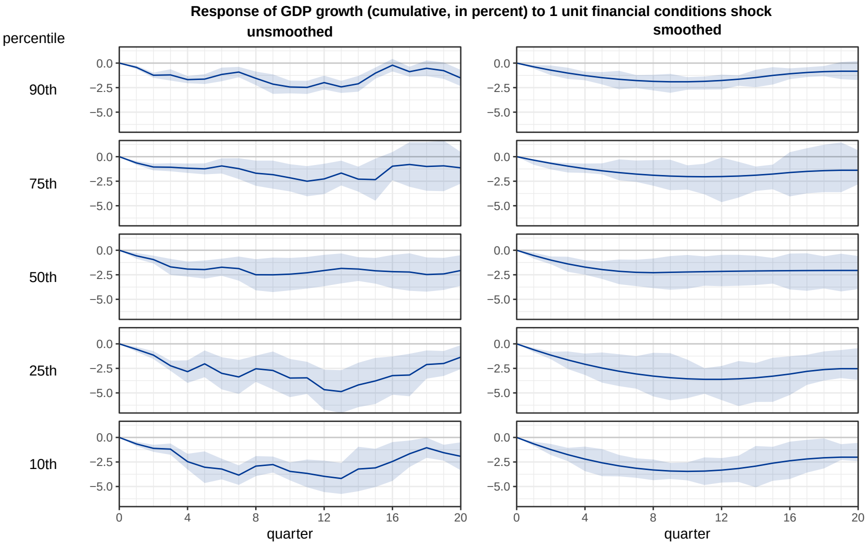
- Are the results of Adrian et al. (2019) sensitive to the number of lags? The black solid lines (no lags) are their main estimates:





- The figures reveal that if we add at least 1 lag, then:
  - the effect in the upper part of the distribution flips the sign from positive to negative, and
  - the heterogeneity across quantiles mostly disappears.

SQLP application: financial conditions and GDP



Conclusion

- Quantile regression local projections – increasingly popular
  - More robust and more informative than standard LP
  - Researchers face 3 important challenges in practice.
- Contribution: framework that overcomes the challenges
  - Identification: popular identification schemes
  - Smoothing: roughness penalties computationally convenient
  - Inference: closed-form and bootstrap confidence intervals
- Financial conditions affect the entire GDP growth distribution – not just the lower part as argued by Adrian et al. (2019).
- Statistically significant effect of monetary policy – in line with Gertler & Karadi (2015), contrary to Stock & Watson (2018)
- Heterogeneous effects of conventional monetary policy:
  - More effective at curbing inflation than at preventing deflation
  - More effective at taming excessive economic growth than at supporting recovery

Annex – Related literature

- LP – Jordà (2005)
- LP with instrumental variables – Stock & Watson (2018)
- Smooth LP – Barnichon & Brownlees (2019)
- Quantile Autoregression – Koenker & Xiao (2006)
- IV-QR – Chernozhukov & Hansen (2005, 2006, 2008)
- Quantile VAR – White et al. (2015), Montes-Rojas (2018, 2019), Ruzicka (2021b), Chavleishvili & Manganelli (2024)
- Examples of research that uses QLP
    - Distante et al. (2013) – technology shocks, business cycle reversals
    - Winkler & Linnemann (2015) – fiscal multipliers
    - Adrian et al. (2019, AER) – financial conditions and GDP
    - Boyarchenko et al. (2020) – bank capital and GDP
    - Han et al. (2020) – systemic risk of financial institutions
    - Lang & Forletta (2020) – risks to bank profitability
    - Mano & Sgherri (2020) – capital flow shocks
    - Siklos (2020) – inflation dynamics

Causality in dynamic quantile models

- The observable macroeconomic variables follow a K-variate stochastic process $\{Y_t\}$. Denote $X_t = (Y'_{t-1}, \dots, Y'_{t-P})'$

Assumption 1 (Stationary Higher Order Markov Process)

Let $\{Y_t\}_{t\in\mathbb{Z}} = \{(y_{1t}, y_{2t}, \dots, y_{Kt})'\}_{t\in\mathbb{Z}}$ be a K-variate strictly stationary stochastic process with natural filtration $\{\mathcal{F}_t\}_{t\in\mathbb{Z}}$. For $P \geq 1$, let $\underline{Y}_{t-1} = (Y'_{t-1}, \dots, Y'_{t-P})'$. Assume that $\forall y \in \mathbb{R}^K$: $P(Y_t \leq y|\mathcal{F}_{t-1}) = P(Y_t \leq y|\underline{Y}_{t-1})$.

- The process $\{Y_t\}$ covers as a special case strictly stationary vector autoregressions with independent errors.
- To allow structural identification, we represent $\{Y_t\}$ in a suitable way (structural quantile form).

Causality in dynamic quantile models

Lemma 1 (Structural Quantile Form of a Stochastic Process)

Under Assumption 1 there exist functions $S_1, S_2, \ldots, S_K : \mathbb{R}^{K(P+1)} \mapsto \mathbb{R}$, non-decreasing in their first argument, and independent standard uniform random variables $\{u_{1t}, u_{2t}, \ldots, u_{Kt}\}_{t \in \mathbb{Z}}$ such that:

For all $\tau \in (0,1)$,

$$
\begin{aligned}
Q_\tau\left(y_{1t} - S_1(\tau | y_{2,t}, y_{3,t}, \ldots, y_{K,t}, \underline{Y}_{t-1}) \middle| \{u_{1t}, \ldots, u_{Kt}\} \setminus u_{1t}, \mathcal{F}_{t-1}\right) &= 0, \\
Q_\tau\left(y_{2t} - S_2(\tau | y_{1,t}, y_{3,t}, \ldots, y_{K,t}, \underline{Y}_{t-1}) \middle| \{u_{1t}, \ldots, u_{Kt}\} \setminus u_{2t}, \mathcal{F}_{t-1}\right) &= 0, \\
&\vdots \\
Q_\tau\left(y_{Kt} - S_K(\tau | y_{1,t}, y_{2,t}, \ldots, y_{K-1,t}, \underline{Y}_{t-1}) \middle| \{u_{1t}, \ldots, u_{Kt}\} \setminus u_{Kt}, \mathcal{F}_{t-1}\right) &= 0.
\end{aligned}
$$

Y_t is the a.s. unique solution of (1).

- $S_1, S_2, \ldots, S_K$ are structural quantile functions (Chernozhukov & Hansen, 2008).
- The disturbances $u_{1t}, u_{2t}, \ldots, u_{Kt}$ determine which quantiles occur.

Causality in dynamic quantile models

Assumption 2 (Counterfactual Stochastic Process)

Let $\{Y_t\}_{t\in\mathbb{Z}}, \{u_{1t}, u_{2t}, \dots, u_{Kt}\}_{t\in\mathbb{Z}}, S_1, S_2, \dots, S_K$ *be as in Lemma 1. Take* $i \in \{1, 2, \dots, K\}$, $s \in \mathbb{R}$ *such that* $\{Y_t\}_{t\in\mathbb{Z}}$ *has the identical support as* $\{\tilde{Y}_t\}_{t\in\mathbb{Z}} = \{(\tilde{y}_{1t}, \tilde{y}_{2t}, \dots, \tilde{y}_{Kt})'\}_{t\in\mathbb{Z}}$ *that is given by*

$$
\begin{aligned}
\tilde{y}_{1,t} &= \mathbb{1}(t<0)y_{1t} && + \mathbb{1}(t\geq 0)S_1(u_{1,t}|\tilde{y}_{2,t}, \tilde{y}_{3,t}, \dots, \tilde{y}_{K,t}, \underline{\tilde{Y}}_{t-1}),\\
&\vdots \\
\tilde{y}_{i-1,t} &= \mathbb{1}(t<0)y_{i-1,t} && + \mathbb{1}(t\geq 0)S_{i-1}(u_{i-1,t}|\tilde{y}_{1,t}, \dots, \tilde{y}_{i-2,t}, \tilde{y}_{i,t}, \dots, \tilde{y}_{K,t}, \underline{\tilde{Y}}_{t-1}),\\
\tilde{y}_{i,t} &= \mathbb{1}(t\leq 0)y_{it} + \mathbb{1}(t=0)s && + \mathbb{1}(t>0)S_i(u_{i,t}|\tilde{y}_{1,t}, \dots, \tilde{y}_{i-2,t}, \tilde{y}_{i,t}, \dots, \tilde{y}_{K,t}, \underline{\tilde{Y}}_{t-1}),\\
\tilde{y}_{i+1,t} &= \mathbb{1}(t<0)y_{i+1,t} && + \mathbb{1}(t\geq 0)S_{i+1}(u_{i+1,t}|\tilde{y}_{1,t}, \dots, \tilde{y}_{i,t}, \tilde{y}_{i+2,t}, \dots, \tilde{y}_{K,t}, \underline{\tilde{Y}}_{t-1}),\\
&\vdots \\
\tilde{y}_{K,t} &= \mathbb{1}(t<0)y_{Kt} && + \mathbb{1}(t\geq 0)S_K(u_{K,t}|\tilde{y}_{1,t}, \tilde{y}_{2,t}, \dots, \tilde{y}_{K-1,t}, \underline{\tilde{Y}}_{t-1}),
\end{aligned}
$$

where $\underline{\tilde{Y}}_{t-1} = (\tilde{Y}'_{t-1}, \dots, \tilde{Y}'_{t-P})'$. *We call* $\{\tilde{Y}_t\}_{t\in\mathbb{Z}}$ *the counterfactual process.*

Causality in dynamic quantile models

- QR models are random coefficient models, so responses to shocks are stochastic. We want to describe their distribution.
  - The QIR definition differs from other definitions in the literature – valid for any point-identification scheme and reflects the QTE of a shock.

Definition 1 (Impulse Response Function)

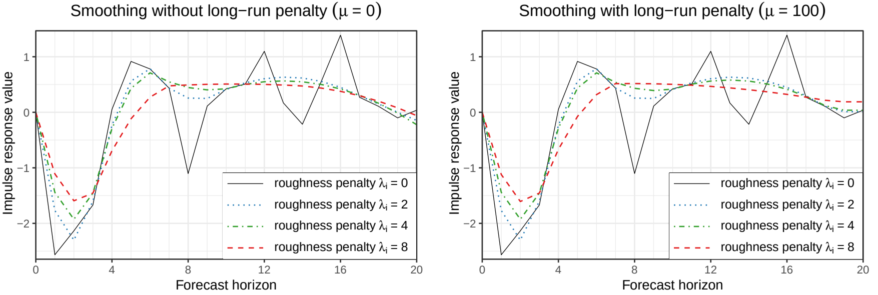
Let $\{Y_t\}_{t\in\mathbb{Z}}, \{u_{1t}, u_{2t}, \ldots, u_{Kt}\}_{t\in\mathbb{Z}}$, $\{\tilde{Y}_t\}_{t\in\mathbb{Z}}$, i and s be as in Assumption 2. Let $j \in \{1, 2, \ldots, K\}$ and $\tau \in (0, 1)$. Let $X_{it} = \underline{Y}_{t-1}$ or $X_{it} = (y_{1t}, \ldots, y_{it}, \underline{Y}'_{t-1})'$. Define the response of the jth variable to the ith impulse (or shock) s at quantile τ and horizon $h \in \mathbb{N}_0$ as

$$IR^{h,\tau,s}_{j,i}(Y_0, X_{i0}) = Q_\tau(Q_\tau(\tilde{y}_{jh}|u_{i0}, X_{i0})|Y_0, \underline{Y}_{-1}) - Q_\tau(Q_\tau(y_{jh}|u_{i0}, X_{i0})|Y_0, \underline{Y}_{-1}). \quad (1)$$

- *QIR* is the QTE of the ith shock on variable j after h periods.
- Under appropriate identification restrictions (formulated in the paper), it is possible to estimate *QIR* consistently by quantile local projections.

Recursive short-run restrictions – smoothing illustration

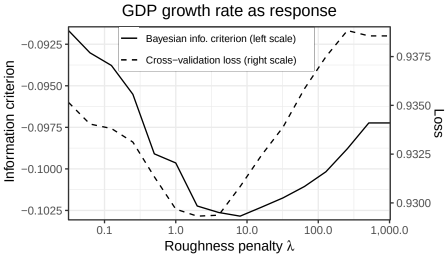
- The data and specification are taken from an empirical application.



- Higher roughness penalty means more smoothing.
- Long-run penalty straightens the function at the end.

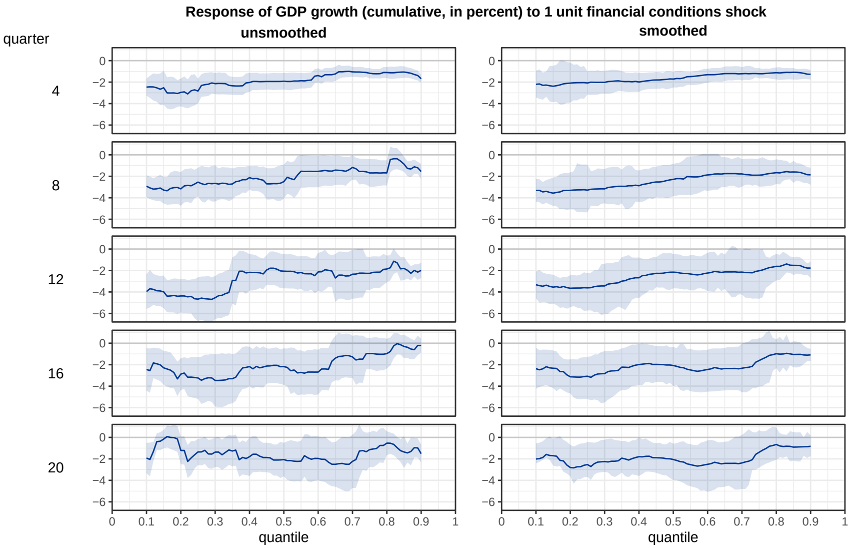
Recursive short-run restrictions – optimal smoothing

- The roughness penalty value can be chosen by information criteria.
- Let us illustrate the roughness penalty selection on a real example.

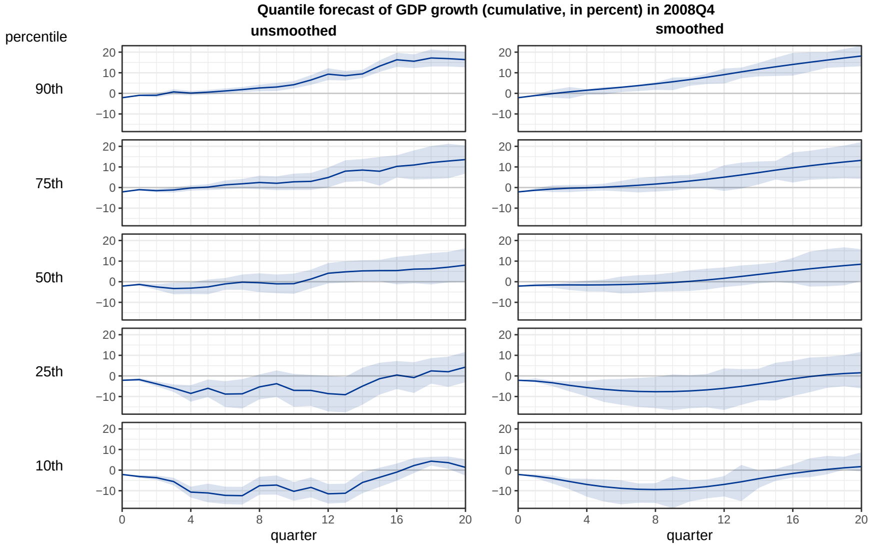


- Bias vs variance tradeoff: $\lambda \uparrow \Rightarrow \downarrow$ variance, $\uparrow$ bias
- Intuitively, the smoothing makes the estimates more accurate by reducing their variance, at the cost of a small bias.

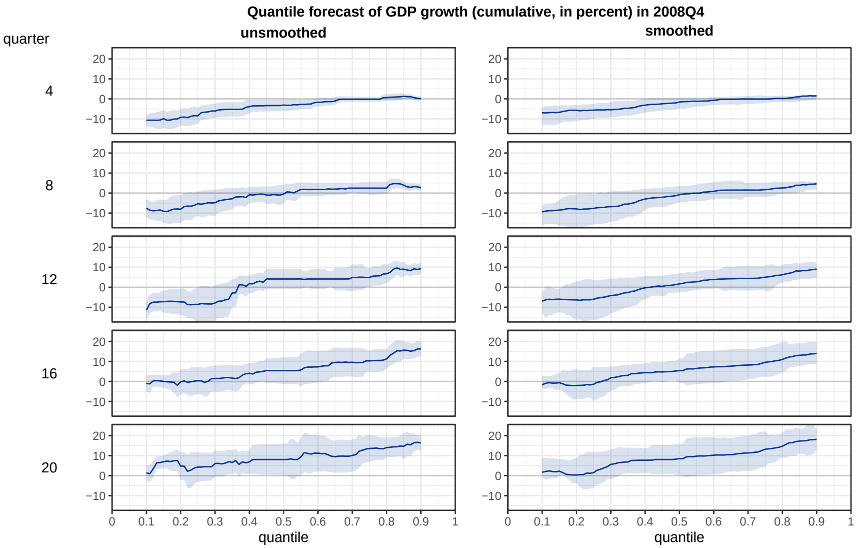
SQLP application: financial conditions and GDP



SQLP application: quantile forecasts



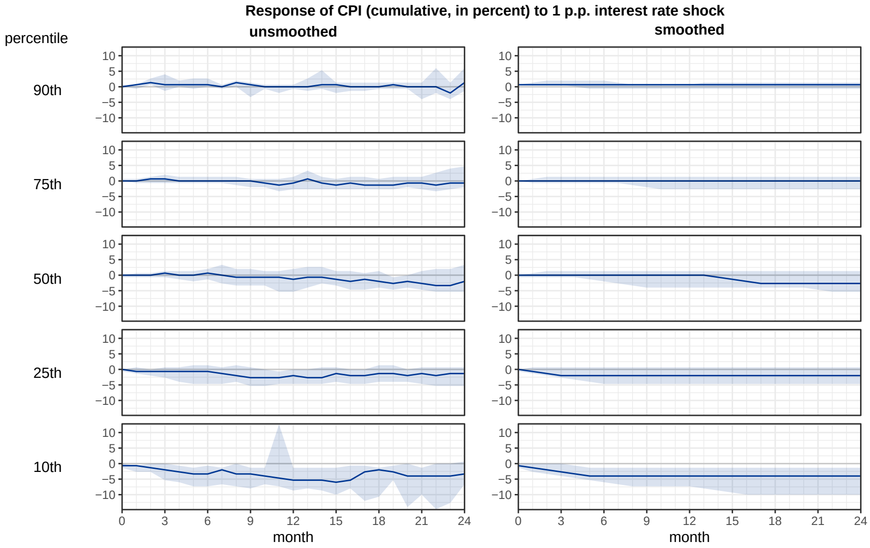
SQLP application: quantile forecasts



SQLPI application: monetary policy shocks

- Specification and data from Stock & Watson (2018) (except for the interest rate)
- Four time series:
  - Federal Funds rate (instead of one-year treasury rate)
  - Growth of industrial production index
  - Growth of consumer price index (CPI)
  - Excess bond premium
- Identification by an external instrument – changes in Federal Funds futures rates around FOMC announcement dates
- US monthly data, 1990m1 to 2012m6 (270 observations)

SQLPI application: monetary policy shocks



SQLPI application – effect of 1 p.p. interest rate shock