Demand-Based Expected Returns

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Introduction

- (Subjective) Expected Returns are not observable
- Canonical estimates based on asset prices don't match survey data properties
    - High volatility, time-varying cyclicality
- This Paper: use asset prices + holdings

Recovery Problems

$$\text{subjective probability } \mathbb{P}^i = \frac{\text{risk-neutral probability } \mathbb{Q}}{\text{stochastic discount factor } M_i}$$

- $\mathbb{Q}$ is determined by sufficiently large arbitrage-free cross-section of options (Breeden and Litzenberger, 1978)
- look for a way to pin down M_i
- extract $\mathbb{P}^i$ and compute $\mathbb{E}^i[\cdot]$
- potentially recover any quantities

Key Idea

$$\text{subjective probability } \mathbb{P}^i = \frac{\text{risk-neutral probability } \mathbb{Q}}{\text{stochastic discount factor } M_i}$$

- $\mathbb{Q}$ is known from asset prices
- canonical estimates formulate assumptions on M_i based on asset prices
- prices are informative about beliefs on aggregate, quantities are available individually

$\Rightarrow$ use **demand-based** information to characterize $(M_i, \mathbb{P}^i)$ without relying on ad hoc asset pricing models

This Paper

- Theoretical framework for recovering subjective moments under data-driven beliefs
- Theoretical contributions:
    1. Demand-compatible SDF without ad hoc assumptions
    2. Demand information to identify beliefs of heterogeneous investors
    3. Extension to the case of measurement errors and convex frictions
- Empirical contributions:
    1. Subjective expectations may vary widely across investor classes
    2. Rationale to explain why statistical properties of survey data differ from price-based measures
    3. Reconcile non-monotonic SDF shapes

Theory

Demand-Based Recovery

- Risky asset, continuum set of options, risk-free asset
- Heterogeneous unconstrained investors with $M_i = 1/\boldsymbol{\theta}_i' \boldsymbol{R}$ maximizing long-run wealth under their subjective beliefs $\mathbb{P}^i$
- No arbitrage: $\mathbb{E}_t^i[M_i \boldsymbol{R}] = \boldsymbol{0}$
- $\mathbb{P}^i$ equivalent to $\mathbb{Q}$
- Subjective expected return:

$$\mathbb{E}_t^i[R_m] = \mathbb{E}_t^{\mathbb{Q}}[\boldsymbol{\theta}_i{}' \boldsymbol{R} R_m]$$

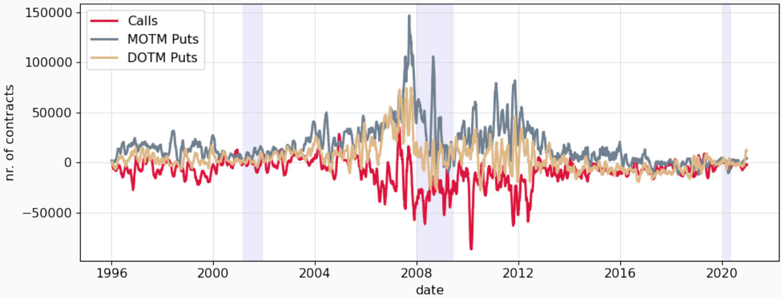
- $\boldsymbol{\theta}_i$ are investors portfolio holdings
- data-driven, real-time recovery

$$\mathbb{E}_t^i[R_m] = \theta_m \mathbb{V}\text{ar}_t^{\mathbb{Q}}(R_m) + \boldsymbol{\theta}'_{opt} \mathbb{C}\text{ov}_t^{\mathbb{Q}}(\boldsymbol{R}_{opt}, R_m) + 1$$

- Portfolio adjustment (with options) $\rightarrow$ Belief distortion
- Variance term is positive, counter-cyclical
- Covariance term is not predetermined in size, sign and cyclicality
- Generate data-compatible models for belief distortions
- For $M_i = 1/R_m$ we recover the SVIX (Martin, 2017)

Empirics

Option Holdings



- CBOE data with daily-level transactions on SPX Options by Customers and Market Makers
- Invest θ_0 in S&P 500 and $1 - \theta_0$ in observed portfolio of OTM options; the rest in the risk-free

$$\theta_0 = \theta_0^{min} + \alpha \cdot (1 - \theta_0^{min})$$

Customers' Expected Returns

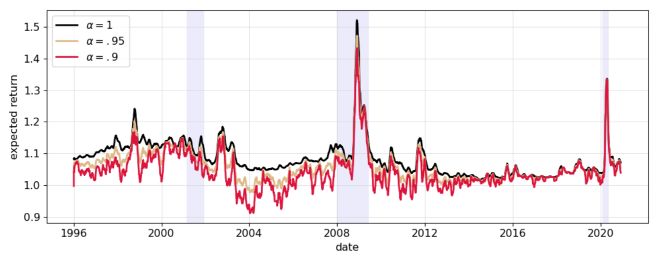


Table 2. Summary statistics of Customers' expected returns $\mathbb{E}^i[R]$, for various choices of α.

α	mean	std	min	median	max	corr (%)	AR(1)	index (avg.)
1	1.082	0.058	1.017	1.069	1.521	100	0.82	1
95%	1.061	0.056	0.977	1.049	1.472	96	0.77	0.99
90%	1.041	0.058	0.911	1.032	1.433	85	0.69	0.97
80%	1.004	0.071	0.792	1.006	1.366	59	0.58	0.94
50%	0.913	0.124	0.520	0.933	1.337	20	0.51	0.86
0	0.810	0.194	0.258	0.838	1.345	5	0.49	0.72

- More leveraged in options $\Longrightarrow$ exp. returns get smaller, more volatile, a-cyclical as in survey data

Market Makers' Expected Returns

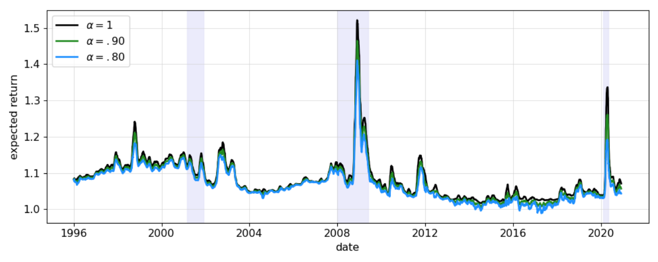


Table 3. Summary statistics of Market Makers' expected returns $\mathbb{E}^i[R]$, for various choices of α.

α	mean	std	min	median	max	corr (%)	AR(1)	index (avg.)
1	1.082	0.058	1.017	1.069	1.521	100	0.82	1
95%	1.078	0.056	1.011	1.067	1.493	100	0.83	0.93
90%	1.074	0.053	1.006	1.065	1.465	99	0.85	0.86
80%	1.045	0.049	0.989	1.059	1.410	97	0.87	0.72
50%	1.045	0.043	0.936	1.043	1.262	78	0.86	0.30
0	1.012	0.048	0.779	1.021	1.104	29	0.69	-0.40

- Δ-hedging neutralizes first-order belief corrections $\implies$ exp. returns aligned with price-based measures

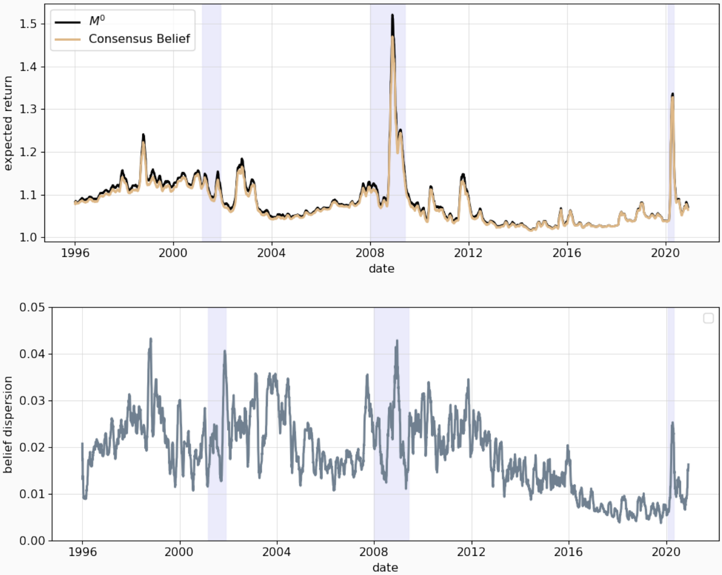
Definition

Given the subset $\mathcal{J}$ of unconstrained optimally-invested agents, and $\omega_i > 0 \ \forall i$, we define:

$$\text{(consensus belief)} \qquad CB_t := 1 + \mathbb{C}\mathrm{ov}_t^{\mathbb{Q}}\left(\sum_{i \in \mathcal{J}} \omega_i R_i, R_m\right)$$

$$\text{(belief dispersion)} \qquad D_t := \sum_{i \in \mathcal{J}} \omega_i \left|\mathbb{E}_t^i[R_m] - CB_t\right|$$

- $CB_t = SVIX$ if all the market participants are in $\mathcal{J}$
- ...but many agents (like market makers) are constrained!
- Interesting to measure deviations from CB_t (without frictions) or the market aggregate view CB_t (with frictions)


M^0
Consensus Belief
expected return
1.5
1.4
1.3
1.2
1.1
1.0
1996
2000
2004
2008
2012
2016
2020
date
belief dispersion
0.05
0.04
0.03
0.02
0.01
0.00
1996
2000
2004
2008
2012
2016
2020
date

Conclusions

- Holdings affect the SDF and hence measures of recovered moments significantly
- Time-series properties of beliefs vary across investors because they substantially depend on the holdings (in line with survey literature)
- Rich belief heterogeneity can be captured

Thank You for listening!

Backup Slides

Negative Covariance Condition

Suppose for some $\mathbb{P}^\star$ the NCC holds at order p

$$\text{Cov}^\star(M^\star \boldsymbol{\theta}' R, R_m^p) \leq 0$$

$$\implies \mathbb{E}^\star[R_m^p] \geq \left(\frac{\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\theta}' R R_m^p]}{\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\theta}' R]} \right)^p = \mathbb{E}^i[R_m^p]^p$$

For $p \to 1$, we naturally recognize

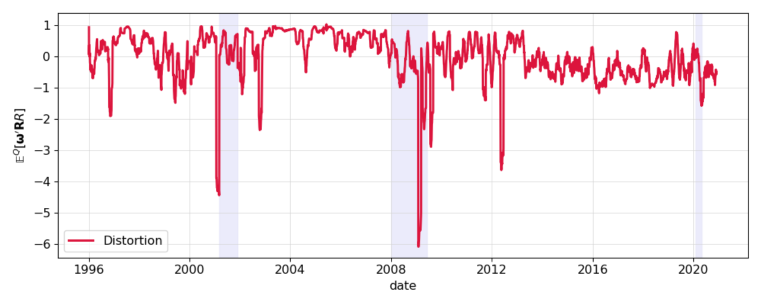
$$\frac{\boldsymbol{\theta}' R}{\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\theta}' R]} = \frac{1}{M_i}$$

as the change of measure from $\mathbb{Q}$ to $\mathbb{P}_i$. M_i has the form of a log-utility investor's SDF. $\mathbb{P}_i$ is the probability belonging to set spanned by $\boldsymbol{\theta}$ and p that supports the lower bound.

Proposition

The $\mathbb{E}^i[R]$ we extract under the log-utility assumption provides a lower bound for the subjective moment of the agent holding the same portfolio but with non-log utility.

Customers' Belief Distortion



- Max. distortion for $\theta_0 = 0$: always negative (avg. -0.7), very volatile (std. 3), a-cyclical (-12%)
- Implausible results do not support subjective beliefs because $M_i(\boldsymbol{\theta}_i)$ is **not** valid SDF

What if we don't observe θ?

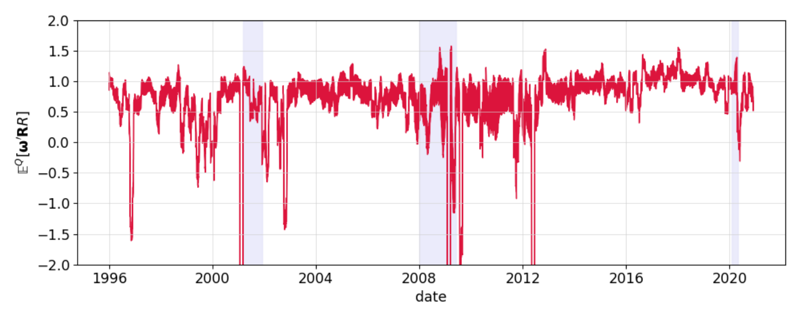
- The true market exposure is likely different from what we measure
- We can still characterize economically meaningful $\mathbb{P}^i$
- Approximate holdings with $\boldsymbol{\theta}$ solving:

$$\inf_{\boldsymbol{\theta}} \left\{ \mathbb{E}_t^{\mathbb{Q}}[\boldsymbol{\theta}' \boldsymbol{R} R_m] + \lambda \left(\frac{1}{2} \|\boldsymbol{\theta}^\star - \boldsymbol{\theta}\|_2^2 - \delta \right) \right\}$$

Proposition (Bounds on Subjective Expected Returns)

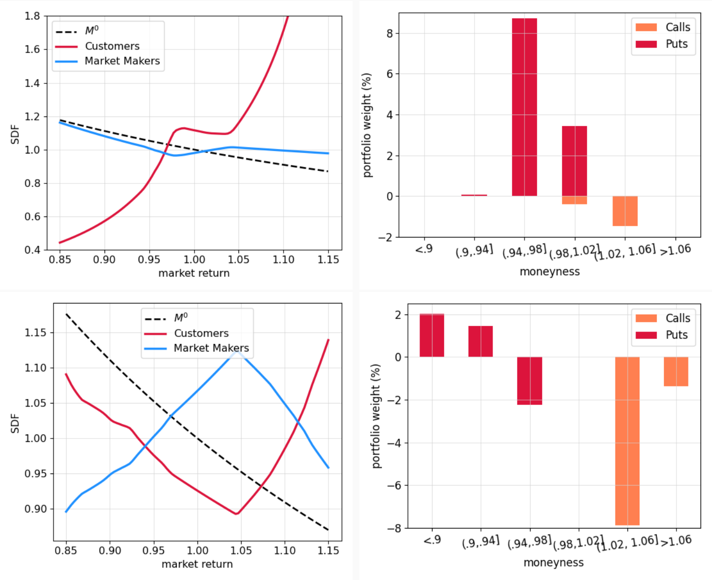
$$\mathbb{E}^i[R_m] \geq \mathbb{E}^{\mathbb{Q}}[R_m] + \boldsymbol{\theta}' \mathbb{E}^{\mathbb{Q}}[\boldsymbol{R}^e R_m] - \sqrt{2\delta} \|\mathbb{E}^{\mathbb{Q}}[\boldsymbol{R}^e R_m]\|_2$$

$$\mathbb{E}^i[R_m] \leq \mathbb{E}^{\mathbb{Q}}[R_m] + \boldsymbol{\theta}' \mathbb{E}^{\mathbb{Q}}[\boldsymbol{R}^e R_m] + \sqrt{2\delta} \|\mathbb{E}^{\mathbb{Q}}[\boldsymbol{R}^e R_m]\|_2$$



- Belief heterogeneity increases in bad times
- Pro-cyclical lower bound ("most pessimistic" investor / "worst-case" expectation) (corr. -34%)
- More leveraged in options

Holdings affect SDF



Subjective Measures of Risk