# Resolving New Keynesian Puzzles

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## Abstract

New Keynesian models generate puzzles when confronted with the zero lower bound (ZLB) on nominal interest rates (e.g. the forward guidance puzzle or the paradox of flexibility). We show that these puzzles are absent in simple and medium-scale models when monetary policy is history dependent. Standard approaches to modeling the ZLB do not capture the policy objectives assumed outside of the ZLB period, or approximate the policy a rational inflation targeting central bank would choose at the ZLB. It is this disconnect that is responsible for the puzzles. The puzzles, therefore, are best thought of as the plausible predictions of implausible monetary policy rather than implausible predictions to plausible monetary policy. We show how to write monetary policy rules that capture the same policy objective with and without the ZLB.


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## 1 Introduction

While the New Keynesian (NK) model provides an elegant theory of monetary policy during the Great Moderation, when confronted by the zero lower bound (ZLB) on nominal interest rates, it has yielded a litany of puzzles and paradoxes. The most well-known is the so-called forward guidance puzzle, which manifests in two ways. First, there is a qualitative aspect. The same monetary policy intervention is more powerful now if promised to be done one-year from now than one-day from now, when interest rates are fixed in the interim, and its power can grow without bound the further in the future it is promised, all else equal. We call this a *limit puzzle*. Second, there is also a quantitative aspect. NK models generate implausible predictions for credible promises of maintaining low interest rates, often resulting in explosive forecasts when applied to real-world central bank announcements. We call this a *quantitative puzzle*.

To illustrate a quantitative puzzle, consider the Federal Open Market Committee (FOMC) statement from August 9, 2011:

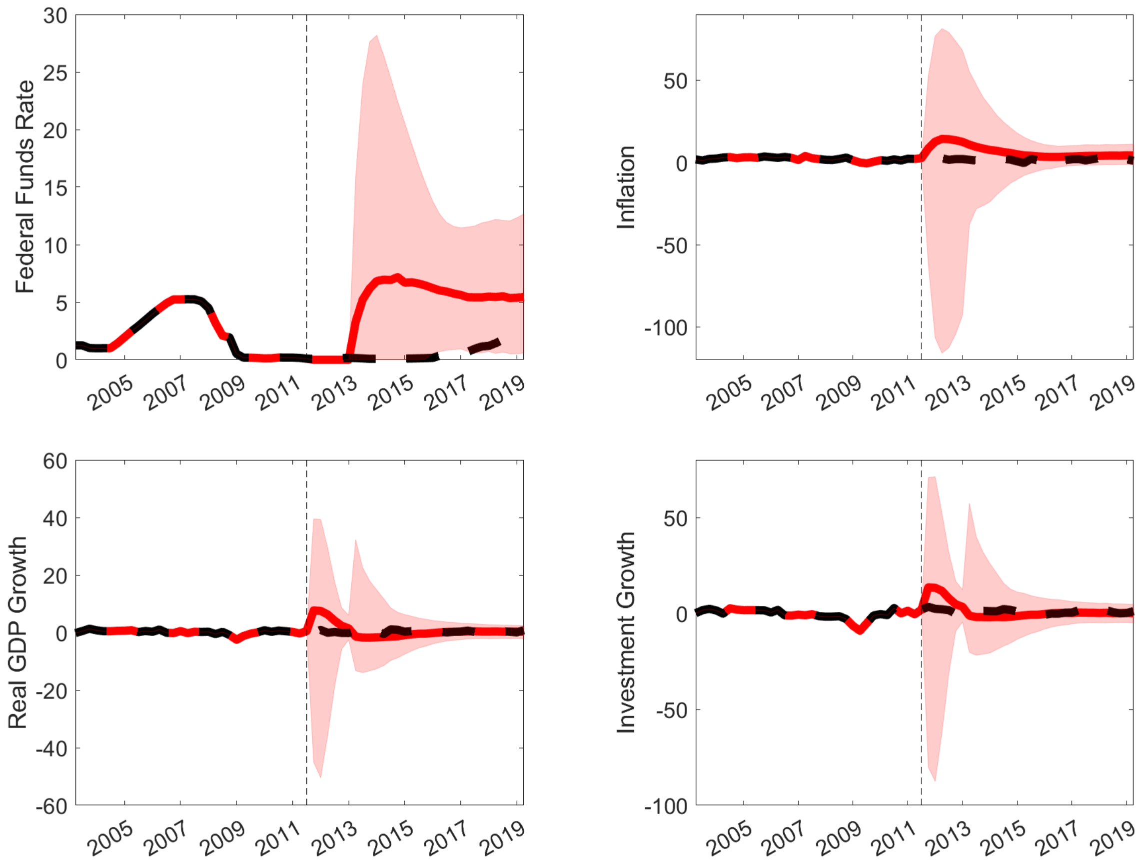
> *The Committee currently anticipates that economic conditions ... are likely to warrant exceptionally low levels for the federal funds rate at least through mid-2013.*
>
> \- FOMC Statement August, $9^{th}$ 2011

Figure 1 shows what forecasts of seven quarters of zero interest policy are predicted to do for inflation, real GDP growth, investment, and the Federal Funds rate using the model of Smets and Wouters (2007) estimated on data ending in 2004.[1] With forecasts like these, it is not surprising that some of the first papers documenting the forward guidance puzzle came from within the Federal Reserve System. For example, the work by Del Negro, Giannoni and Patterson (2012) or Carlstrom, Fuerst and Paustian (2015), which to our knowledge are among the first papers to study the issues in Figure 1.

In addition, there are three other ZLB NK puzzles with both quantitative and limit puzzle aspects widely studied in the literature: the fiscal multiplier puzzle – excessive responses to

[1] The forecasts are constructed using Smets and Wouter's original replication files and data (which ends in 2004) to estimate the posterior distribution of the structural parameters. We then follow Cagliarini and Kulish (2013), Jones (2017), Kulish, Morley and Robinson (2017), and Kulish and Pagan (2017) to generate forecasts by sampling from the estimated posterior distribution of the parameters while enforcing a fully credible and known policy of zero interest rates for seven quarters. After seven quarters, monetary policy is governed by an occasionally binding constraints algorithm, where the policy rate is the maximum of zero or the interest rate implied by the model's interest rate rule.

Figure 1: Calendar-based forward guidance and the forward guidance puzzle




*Notes:* Out-of-sample Q/Q forecasts of the August, $9^{th}$ 2011 Federal Reserve calendar-based forward guidance promise using the model of Smets and Wouters (2007). The dashed line is the actual data. The solid line is the median forecast. The shaded region represents the 84th and 16th percentiles of forecasts. Inflation and the Federal Funds Rates are annualized.

announced fiscal policy changes (e.g., Farhi and Werning, 2016), the paradox of toil – contractionary effects for *positive* productivity or labor supply shocks (e.g., Eggertsson, 2010), and the paradox of the flexibility – larger effects of monetary policy when prices are *more* flexible (e.g., Eggertsson and Krugman, 2012 or Kiley, 2016). Like with the forward guidance puzzle, economic intuition and empirical evidence is often at odds with many of these predictions.

We show that the NK puzzles have a single origin: a lack of history dependence to policy at the ZLB. The convention in the NK literature is to capture monetary policy using a reduced form Taylor (1993) type rule. These rules are justified on the grounds that they can approximate optimal policy under discretion or commitment – depending on the specification – and much research has been done to establish this connection without considering the ZLB (see, for example, Clarida, Gali and Gertler, 1999, Woodford, 2001 or Woodford, 2003b). We show, however, that when these rules are adapted to the ZLB they neither approximate optimal commitment policy of an optimizing central bank nor the objectives of the policy embedded in the standard rules that operate outside of the ZLB.

We show that the usual adaptation of Taylor-type rules to the ZLB environment inadvertently introduces a suboptimal policy regime switch, which materially changes the objectives of the policymaker relative to what is assumed before the ZLB is imposed. In effect, in most scenarios studied in the literature, policymakers are modeled to not respond to anything that occurs during the ZLB. Regardless of how high inflation may be in the quarter before interest rates lift off from zero, the central bank by construction ignores these outcomes. Requiring simple policy rules to maintain the same history dependence over output and inflation found empirically outside the ZLB, while at the ZLB, is sufficient to eliminate the quantitative aspects of the puzzles. Going further, if these empirical rules are replaced with rules that approximate the actual choices of an optimizing central bank at the ZLB, then all of the quantitative and limit puzzles are eliminated.

There are three important takeaways from our results. First, we show how any inertial Taylor rule can be written to eliminate the quantitative puzzles while preserving the same equilibrium outcomes outside of the ZLB period. Hence, we show how to solve the qualitative puzzles in nearly all policy relevant models in straightforward way using current policy rule parameter estimates. Therefore, the standard NK model can capture interest rate pegs, liquidity traps, and forward guidance policy much better than is currently recognized conditional on monetary policy having explicit history dependence over inflation (and/or output) outcomes during periods where interest rates are constrained. We show this explicitly in the next section using an estimated model on US data. Second, the recent adoption of average inflation target (AIT) policies make explicit the type of history dependence that we show eliminates quantitative aspects of the puzzle. For example, the formulations of AIT into the policy rule of the Federal Reserve Bank of New York's DSGE model captures the relevant type history dependence.[2] Finally, the NK puzzles are not a reason to explore bounded rationality or information frictions as alternative to the rational expectations in NK models such as in Angeletos and Lian (2018), Farhi and Werning (2019), and Gabaix (2020). Instead, these studies help reconcile the model with data at a more fundamental level. Eusepi, Gibbs and Preston (2024) show that even in the absence of the NK puzzles that standard models predict implausibly large general equilibrium effects to zero interest rate policies. Resolving the puzzles does not overturn the predictions

[2] Although, the model's rule also includes interest rate smoothing, which encodes the wrong history dependence at the ZLB.

of incredible control of the economy at the ZLB shown by Eggertsson and Woodford (2003). Those predictions, which are often ascribed to the puzzles, are actually obtained in a puzzle free environment. It is the incredible general equilibrium effects of policy writ large in NK models that justify these approaches.

**Related Literature.** The New Keynesian puzzles literature is large and has two main branches. The most significant branch from a quantitative and policy perspective is on the forward guidance puzzle.[3] The seminal papers are Del Negro et al. (2012) and Carlstrom et al. (2015), which both illustrate the problem – explosive responses of output and inflation to promised pegs of the interest rate – and posit solutions. The latter show that sticky information can reduce or eliminate puzzles while the former argues that the credibility of the policy at longer horizons is implausible. In fact, Del Negro et al. (2012)'s point on credibility of the policy makes a similar argument to ours: the forward guidance thought experiment implies future movements of interest rates that are implausible. However, we interpret these implausible interest rate movements as stemming from a logical disconnect between the assumption of credible commitment to forward guidance, and the subsequent adaptation of a rule that belies a core feature of commitment policy: history-dependence. Taming the puzzles is about making monetary policy during a liquidity trap logically consistent with monetary policy *after* a liquidity trap. Abandoning the assumption of full credibility is not essential.

Many papers have sought to eliminate the forward guidance puzzle by changing primitive assumptions of the NK model. For example, Del Negro, Giannoni and Patterson (2023) makes use of a perpetual youth model to micro found a reduction in the horizon of agents' expectations delivering the same ameliorative effect on forward guidance as imperfect credibility. Further refinements on using credibility to resolve the puzzle are studied by Haberis, Harrison and Waldron (2019) and Gibbs and McClung (2023). They both show that partial credibility of holding rates at zero resolves the forward guidance puzzle. Andrade, Gaballo, Mengus and Mojon (2019) study the case where some agents interpret forward guidance as worse economic conditions rather than a credible promise of stimulus, which also ameliorates the effect of such policies.

There are many studies on information frictions or bounded rationality as a way of resolving

[3]See Diba and Loisel (2021) and section 3 of this paper for formal definitions of the forward guidance puzzle.

the puzzle such as Kiley (2016) using sticky information; Angeletos and Lian (2018), Farhi and Werning (2019), and Evans, Gibbs and McGough (2025) using level-k reasoning; Gabaix (2020) myopia; and Eusepi et al. (2024) who use adaptive learning. Each these bounded rationality papers makes use of the fact that the significant impacts of forward guidance come through the general equilibrium effects of expectations. Bounded rationality lowers these effects and may eliminate the puzzles.

Another sub-strand of this literature is incomplete markets. McKay, Nakamura and Steinsson (2016) and Hagedorn, Luo, Manovskii and Mitman (2019) show that incomplete markets in a heterogeneous agent NK model can resolve the puzzle. Eggertsson, Mehrotra and Robbins (2019) show that the puzzle is absent in overlapping generation models with debt constraints. However, Farhi and Werning (2019) and Bilbiie (2020) show that incomplete market is not a robust way to eliminate the forward guidance puzzle.

In addition to these behavioral approaches, Cochrane (2017, 2023), McClung (2021), Gibbs and McClung (2023), and Diba and Loisel (2021) show that alternative monetary and fiscal policy frameworks also can eliminate the puzzles. The last paper listed shows that when monetary policy is conducted via money supply rules the economy is puzzle free under an interest rate peg. The remaining papers show that closing a model with active fiscal policy as under the Fiscal Theory of the Price Level provides a puzzle free equilibrium in the NK model. Our paper fits into this strand of the literature. The policies studied by these authors resolve the forward guidance puzzle because the relevant history dependence of policy is maintained during the period of constrained interest rates.

The second smaller branch of this literature focuses on the other puzzles: fiscal multipliers, toil, and flexibility. The fiscal multiplier puzzle is essentially the same as the forward guidance puzzle in that anticipated fiscal policy has implausibly large effects. Hills and Nakata (2018) offer a resolution for this puzzle that involves tracking a shadow rate at the ZLB, which encodes the correct history dependence.[4] Eggertsson (2010) identified the paradox of toil as the result that negative supply shocks are expansionary at the ZLB. The paradox of flexibility is defined in Eggertsson and Krugman (2012) and is the result that reductions in price stickiness increases the relative strength of all the other puzzles at the ZLB, i.e., in the face of current and anticipated

[4] Although, other issues with using shadow rates remain in fully capturing the ZLB as explained in Krippner (2020).

shocks, the ZLB constraint is even more disruptive in the model as price flexibility increases. Bonciani and Oh (2023) note that a shadow rate here to can eliminate the flexibility puzzle. These puzzles are often, but not always, subordinate to the forward guidance puzzle. If you eliminate the forward guidance puzzle, then typically that removes the other puzzles as well but not vice versa. In addition, Wieland (2019) provides an empirical test for the paradox of toil. No empirical support is found for this puzzle, which is consistent with our argument that monetary policy in reality is not described well by standard adaptations of monetary policy to the ZLB environment.

**Outline.** Section 2 revisit standard results from the NK literature to establish the disconnect between conventional monetary policy rules and optimal policy at the ZLB. Section 3 then derives closed-form solutions to demonstrate how the severity of the puzzles relates to the degree of history dependence in policy. Finally, Section 4 presents quantitative results showing how our proposed reformulation of policy rules affects model outcomes in both theoretical and empirically relevant scenarios.

## 2 Interest Rate Rules and Optimal Policy Approximation

Standard Taylor-type rules lead to explosive dynamics at the ZLB. We show how to mitigate the explosiveness by choosing an alternative representation of the same form of monetary policy rule. We then revisit optimal policy away from the ZLB and at the ZLB. A monetary policy rule that includes explicit dependence on past inflation and output–such as the alternative representation of the Taylor rule we propose– can approximate features of optimal policy also in the presence of the ZLB.

### 2.1 Simple inertial rules at the ZLB

Typically, monetary policy is described in NK models by a rule of the form:

$$i_t = (1-\rho_i)\bar{r} + \rho_i i_{t-1} + (1-\rho_i)(\phi_\pi \pi_t + \phi_y y_t), \tag{1}$$

where $i_t$ is the policy rate, $\pi_t$ is inflation (assuming a zero percent inflation target), $y_t$ is a measure of real activity (usually the output gap), and $\bar{r}$ is the steady state real interest.

The parameters $\phi_\pi$ and $\phi_y$ capture the responsiveness of the central bank to current inflation and real activity, respectively, while $\rho_i$ captures a preference for gradual policy adjustment.[5] History dependence is communicated here through $0 < \rho_i < 1$, which encodes past movements in inflation and output shaping expectations of future policy. Hence, standard Taylor-type Rule (1) might appear to capture the property of history dependence. However, this is not the case when the rule is adapted to the ZLB.

To incorporate the ZLB, most researchers – including us in Figure 1 – retain Rule (1) and simply consider the following modification:

$$i_t = \max\left\{0, (1-\rho_i)\bar{r} + \rho_i i_{t-1} + (1-\rho_i)(\phi_\pi \pi_t + \phi_y y_t)\right\}. \tag{2}$$

This rule removes any history dependence to policy. Lagged interest rates no longer capture past economic conditions once the ZLB is binding. Regardless of what occurs at the ZLB under Rule (2), the monetary policymaker credibly commits to not respond to it. Bygones are bygones in the most strict sense when a model is closed with an interest rate rule like (2).

The lack of history dependence found in Rule (2) is counterfactual to how actual central banks behave at the ZLB. As evidenced by the earlier quote from the FOMC, policy is described as dependent on the evolution of economic data. It is also rejected by NK models when estimated on data outside of ZLB episodes. For example, both Smets and Wouters (2007) and Del Negro, Eusepi, Giannoni, Sbordone, Tambalotti, Cocci, Hasegawa and Linder (2013) find $\rho_i$ to be around 0.8, which indicates significant history dependence to policy from the perspective of the agents that inhabit the model. Therefore from the agents' perspective, the ZLB under Rule (2) is not just a constraint on the instrument of policy, it is a *policy* regime change.

To understand why Rule (2) represents policy regime change, consider a (seemingly) equivalent representation of Rule (2):

$$i_t = \max\left\{0, \bar{r} + \phi_\pi \omega_t^\pi + \phi_y \omega_t^y\right\}, \tag{3}$$

[5] The discussion here is based on the study of structural monetary models that are solved by standard first-order approximations using piece-wise solutions to capture the non-linearity at the ZLB. This assumption captures nearly the universe of models that are used in central banks and the simple environments in which the forward guidance policy is usually studied. Eggertsson and Singh (2019) shows considering the non-linear environments does not eliminate the puzzles. In other words, the NK puzzles are not an artifact of approximation method.

where

$$\omega_t^\pi = \omega_{t-1}^\pi + (1-\rho_i)(\pi_t - \omega_{t-1}^\pi)$$
$$\omega_t^y = \omega_{t-1}^y + (1-\rho_i)(y_t - \omega_{t-1}^y)$$

are weighted averages of past inflation and output (henceforth, we refer to Rule (3) as a *weighted average rule*). Models with either Rule (2) or Rule (3) implement the same equilibrium outcomes in standard linearized economies when the ZLB never binds.[6] However, Rule (3) makes clear the degree of history dependence that is implicitly assumed in Rule (1). The central bank responds to geometric averages of past inflation and output. High weight is placed on observations that occurred far in the past when $\rho_i$ is large. Quite naturally, the history that policymakers are responding to is that of past inflation and output, consistent with real world statutory mandates.

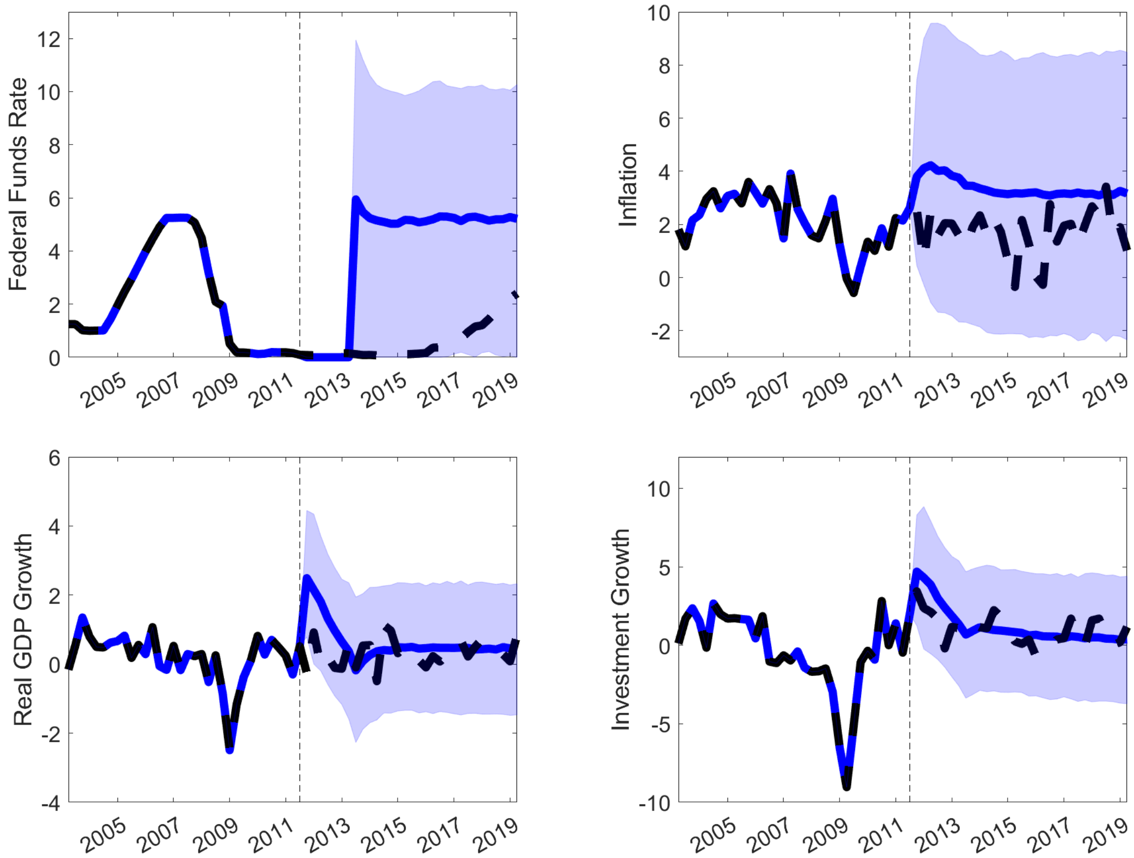
Rule (2) and Rule (3) make very different predictions for policy at the ZLB. Under Rule (2), setting $i_t = 0$ deletes the central bank's objectives from the model. In addition, upon lift off of the interest rate, Rule (2) does not return to approximating an optimizing central bank. Instead, the central bank down-weights its response to current and lagged output and inflation. Under Rule (3), however, exogenously setting $i_t = 0$ does not change what the central bank fundamentally cares about in the model. It simply implements exactly what the ZLB represents: a constraint on the choice of the instrument of policy. The central bank and the private sector do not lose their ability to track the evolution of output and inflation during the period interest rates are constrained; nor does the private sector lose its ability to formulate expectations about how policymakers will respond to current misses of their targets in the future.

Figure 2 shows how the out-of-sample forecasts from Figure 1 change when we transform the interest rate rule of the Smets and Wouters model to the form of Rule (3).[7] The forecasts no longer explode. The responses are large, but quite sensible. Forward guidance here provides a significant boost to the economy, larger than what occurred on average, but not unreasonable

[6] The equivalence holds for the rational solution of a first-order approximation of the economy in log-deviation from steady state form when initial conditions are the same, $\bar{r}$ is constant, and there is no monetary policy shock. Monetary policy shocks obviously propagate differently in equilibrium between these two rules. Likewise, if $\bar{r}$ tracks the natural rate shock, then the rules implement different policies, which is discussed in detail in Section 2.

[7] It is straightforward to verify that it implements the same rational solution when the ZLB is not considered. The exact formulation of the rule we use is given in Section 4.

Figure 2: Calendar-based forward guidance without the forward guidance puzzle




*Notes:* Out-of-sample Q/Q forecasts of the August, $9^{th}$ 2011 Federal Reserve calendar-based forward guidance promise using the model of Smets and Wouters (2007) with the interest rate rule written in the form of (3). The dashed line is the actual data. The solid line is the median forecast. The shaded region represents the 84th and 16th percentiles of forecasts. Inflation and the Federal Funds Rates are annualized.

given that the policy is assumed perfectly credible in the forecasts. For empirically relevant exercises like this one, the quantitative dimension of the forward guidance puzzle is absent at the posterior estimates of the model parameters applied to Rule (3). However, for extremely long expected ZLB episodes the limit puzzles remain. To resolve the limit puzzles, a central bank must follow a rule that more than makes-up for past misses in inflation from the target at the ZLB, which is what Eggertsson and Woodford (2003) show characterizes optimal policy.

## 2.2 Classic optimal policy in the NK model and inertial rules

Consider the standard optimal monetary policy problem with commitment from the timeless perspective described by Clarida et al. (1999) or Woodford (2003a). The central bank seeks to maximize household welfare by committing to a policy to offset shocks to the real interest rate and to markups. The central bank seeks to minimize

$$\min_{\pi_t, y_t, i_t} \left\{ \frac{1}{2} E_t \sum_{T=t}^{\infty} \beta^{T-t} (\pi_t^2 + \alpha y_t^2) \right\}, \tag{4}$$

which is a quadratic approximation to household utility where $\alpha$ is the weight given to variation in output ($y_t$) relative to inflation ($\pi_t$). Alternatively, we may view the loss function as that of an inflation targeting central bank with a dual mandate. The central bank seeks to minimize deviation of inflation from a target ($\bar{\pi} = 0$ in this case) while considering the output costs when implementing policy. We assert that most modelers roughly have in mind an objective of the form of (4) whenever they write down a reduced form monetary policy rule. The tacit assumption in the literature is that Taylor-type monetary policy rules approximately implement policy that minimizes (4).

That assumption is mostly true when the ZLB is not present. Woodford (2001) shows this for non-inertial interest rate rules. Woodford (2003b) shows it is true of inertial interest rate rules that respond to lagged interest rates. It is not true, however, when the ZLB is considered. To understand why, consider the optimal policy problem of a central bank that seeks to minimize (4) taking as given the first-order conditions for household's and firm's decisions:

$$y_t = E_t y_{t+1} - \frac{1}{\sigma} (i_t - E_t \pi_{t+1} - r_t^n) \tag{5}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t + \mu_t \tag{6}$$

where Equations (5) and (6) are the standard NK IS and Phillips curves log-linearized around a zero inflation steady state, $i_t$ is the policy rate, $\sigma^{-1}$ is the intertemporal elasticity of substitution, $r_t^n$ is a potentially autoregressive real interest rate shock with mean $\bar{r}$, $\beta$ measures the rate of time preference, $\kappa$ is composite parameter capturing the degree of price rigidity, and $\mu_t$ is a potentially autoregressive cost push shock. When the ZLB is considered as a possible constraint, the solution must also satisfy $i_t \geq 0$.

#### 2.2.1 Approximating policy without the ZLB

The optimal target criterion that minimizes the central bank's loss function (4) from the timeless perspective, ignoring the ZLB constraint, is

$$y_t - y_{t-1} = -\frac{\kappa}{\alpha}\pi_t. \tag{7}$$

However, it is illustrative to study a refinement to the target criteria: the *unconditional* optimal target criteria proposed by Jensen and McCallum (2002) and Blake (2001). The former shows numerically and the latter analytically that welfare on average is improved when time discounting by the central bank is ignored resulting in the targeting criteria:

$$y_t - \beta y_{t-1} = -\frac{\kappa}{\alpha}\pi_t. \tag{8}$$

This criterion is convenient because we can write this as

$$(1-\beta L)y_t = -\frac{\kappa}{\alpha}\pi_t$$

where $L$ is the lag operator. The inverse of $(1-\beta L)$ exists provided $|\beta| < 1$. Therefore, we can write

$$y_t = -\frac{\kappa}{\alpha}\frac{\pi_t}{1-\beta L}.$$

**Proposition 1:** *The optimal target criterion (8) may be implemented by either of the following interest rate rules*

$$\textit{Optimal Rule 1} \quad \begin{cases} i_t &= \frac{\sigma\kappa}{\alpha(1-\beta)}\omega_t^\pi + \sigma E_t y_{t+1} + E_t\pi_{t+1} + r_t^n \\ \omega_t^\pi &= \omega_{t-1}^\pi + (1-\beta)(\pi_t - \omega_{t-1}^\pi) \end{cases} \tag{9}$$

$$\textit{Optimal Rule 2} \quad i_t = \beta i_{t-1} + \frac{\sigma\kappa}{\alpha}\pi_t + (1-\beta L)\left(\sigma E_t y_{t+1} + E_t\pi_{t+1} + r_t^n\right) \tag{10}$$

From Proposition 1, there are two things to note. First, optimal policy may be implemented with either a rule that includes lagged interest rates like Rule (1), or one that includes a weighted average of past inflation like Rule (3). Either formulation implements the same policy when the ZLB is not present. Second, history dependence is more complicated in the second formulation (10) than in the first (9). To properly implement optimal policy, the lagged interest rate is not a sufficient statistic for history. Past forecasts and shocks are also required. In contrast, past inflation is all that is required to implement the rule (9).

Consider what this means for approximating optimal policy with a rule like (1):

$$i_t = (1-\rho_i)\bar{r} + \rho_i i_{t-1} + (1-\rho_i)(\phi_\pi \pi_t + \phi_y y_t),$$

versus a rule like (3):

$$\begin{aligned} i_t &= \bar{r} + \phi_\pi \omega_t^\pi + \phi_y \omega_t^y \\ \omega_t^z &= \omega_{t-1}^z + (1-\rho_i)(\pi_t - \omega_{t-1}^z) \text{ for } z = \pi \text{ or } y \end{aligned}$$

when the central bank is faced only with demand shocks. Both rules fail to deliver Blanchard and Galí, 2007's Divine Coincidence, i.e., perfect stabilization of the output and inflation gaps in response to $r_t^n$. Each rule only responds to $r_t^n$ indirectly. However, only (3) offers a simple modification that engineers this result. Replace $\bar{r}$ with $r_t^n$ in the weighted average inflation rule and it more closely approximates (9). However, do this substitution in the second rule (1) and it may actually worsen the approximation to optimal policy if $r_t^n$ is persistent.

Why does the approximation to optimal policy worsen for Rule (1) when $\bar{r}$ is replaced with $r_t^n$? Because the lagged interest rate encodes the wrong history dependence when policy responds to $r_t^n$. The additional lag of the shock and expectations in (10) are there to undo the wrong history dependence that is encoded in lagged interest rates when responding to demand shocks.

#### 2.2.2 Approximating policy with the ZLB

The natural question here is if lagged interest rates are an issue, then why do the puzzles occur even when the policy rule includes no history dependence? Such as when a rule like

$$i_t = \bar{r} + \phi_\pi \pi_t + \phi_y y_t \tag{11}$$

is considered. The answer is that history dependence is an integral part of any sensible policy at the ZLB. The wrong history dependence or the omission of history dependence to policy is the problem.

Eggertsson and Woodford (2003) show that the optimal target criterion when $i_t \geq 0$ is a time-varying price level target:

$$\begin{aligned}
\tilde{p}_t &= p_t^* \\
\tilde{p}_t &= p_t + \frac{\alpha}{\kappa} x_t \\
p_{t+1}^* &= p_t^* + \beta^{-1}\left((1+\kappa\sigma^{-1}) - L\right)(p_t^* - \tilde{p}_t),
\end{aligned}$$

where $\tilde{p}_t$ adjusts according to past misses in that target $(p_t^* - \tilde{p}_t)$ and $L$ is again the lag operator. This criterion requires the central bank to be extremely history dependent. The central bank should not just be a price-level targeter – perfectly make-up for past misses. It should promise to permanently overshoot on the price level in response to shocks that cause the ZLB to bind – more than make-up for past misses.

Implementing or approximating the optimal target criterion using an interest rate rule requires forward guidance. A central bank that follows Rule (2) or Rule (3) may announce how long they intend to hold interest rates at zero in response to a shock. But that promise is explicitly paired with a promise to return to an interest rule that includes how policy will adjust to what occurred during the ZLB episode. Rule (2) or the simple rule (11) of course promises to not respond at all to what has occurred during the ZLB episode.

To illustrate the impact of either the wrong or an omission of history dependence to policy, we proceed in two steps. First, we replicate the ZLB thought experiment explored by Eggertsson and Woodford (2003) of optimally responding to a real interest rate shock of uncertain duration that causes the ZLB to bind. The optimal policy to this shock implies a state-contingent forward guidance promise, where the number of quarters of zero interest policy is indexed by the expected duration of the shock. Second, we use the optimal state-contingent forward guidance promise to simulate outcomes to the same shock under a promise that policy returns to either Rule (2) or Rule (3), rather than optimal policy. We then compare the equilibrium outcomes.

The real interest rate shock follows a two-state reducible Markov process. In period one, the shock occurs and $r_t^n = r_S < 0$. We call this state $S$. The shock remains in effect in each period with probability $(1-\delta)$. With complementary probability $\delta$, the real interest rate returns to steady state, $r_t^n = r_N = \bar{r} > 0$. We call this state $N$. The central bank and the private sector understand the shock process. The central bank responds in the period the shock occurs with a state-contingent forward guidance promise, where for every possible duration of the shock, $\tau = 1, 2, 3, ...$, the central bank provides a promised duration of additional periods of zero interest rate policy in state $N$, e.g., $k_\tau = \{1, 2, 2, 3, 3, 4, ...\}$, which generate the appropriate initial conditions for optimal policy forever after.

Figure 3 shows the equilibrium outcomes under optimal policy compared to the approximations using either Rule (2) or Rule (3) for realizations of the shocks lasting one through ten quarters. For comparison purposes, we have highlighted the paths associated with a shock lasting four quarters. We use the same calibration as Eggertsson and Woodford (2003) with $\beta = 0.99$, $\sigma = 2$, $\kappa = 0.02$, $r_N = 0.1$, $r_S = -0.005$ and $\delta = 0.1$.[8] For the policy rules, we assume $\rho_i = 0.8$, $\phi_\pi = 1.5$, and $\phi_y = 0.5$. Lastly, we set $\alpha = \kappa/(\sigma\phi_\pi)$ so that the weight placed on stabilizing inflation relative to output is comparable to the standard interest rate rule coefficients chosen.[9] By construction, the forward guidance policy is the same under all three specifications. The only difference is the policy pursued after the interest rate lifts off from zero.

The equivalence of Rule (2) and Rule (3) is broken. The two rules generate extremely different equilibrium dynamics and welfare outcomes in response to the same shock and forward guidance policy. The weighted average rule continues to approximate optimal policy, while the inertial rule exhibits the forward guidance puzzle.

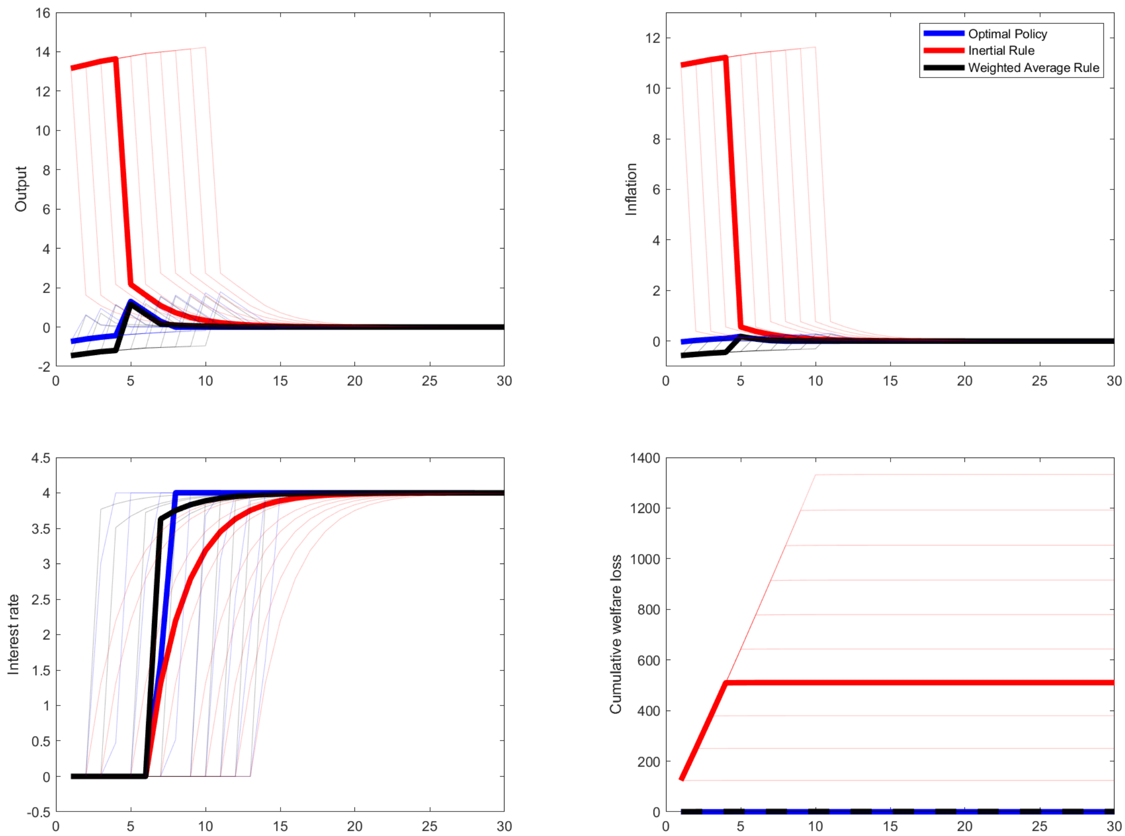
To understand the economics of the different outcomes, compare the highlighted paths of the interest rate for a shock that lasts four quarters in the bottom left panel of Figure 3. Optimal

[8] We follow the solution method described by Eusepi, Gibbs and Preston (2022), which is found in their online appendix. The method is based on the original solution algorithm proposed by Eggertsson and Woodford (2003) and described more recently in Eggertsson, Egiev, Lin, Platzer and Riva (2021).

[9] The same conclusions holds between the standard rules and optimal policy if we set $\phi_\pi = \kappa/(\sigma\alpha)$ and $\alpha$ at its welfare theoretic value. The welfare theoretic value of $\alpha$ is $\alpha = \theta/\kappa$ with $\theta = 7.87$.

policy calls for a return to the neutral rate two quarters after liftoff. The weighted average rule returns policy to neutral in about eight quarters after liftoff. The inertial rule, however, does not return policy to neutral for more than 24 quarters (six years) after liftoff. Moreover, this policy is pursued despite the fact it is known with certainty that no further shocks will occur. Policymakers are promising to systematically make errors for years because policy is dependent on past interest rate realizations rather than past inflation and output realizations. Policymakers here have explicitly abandoned their dual mandate.

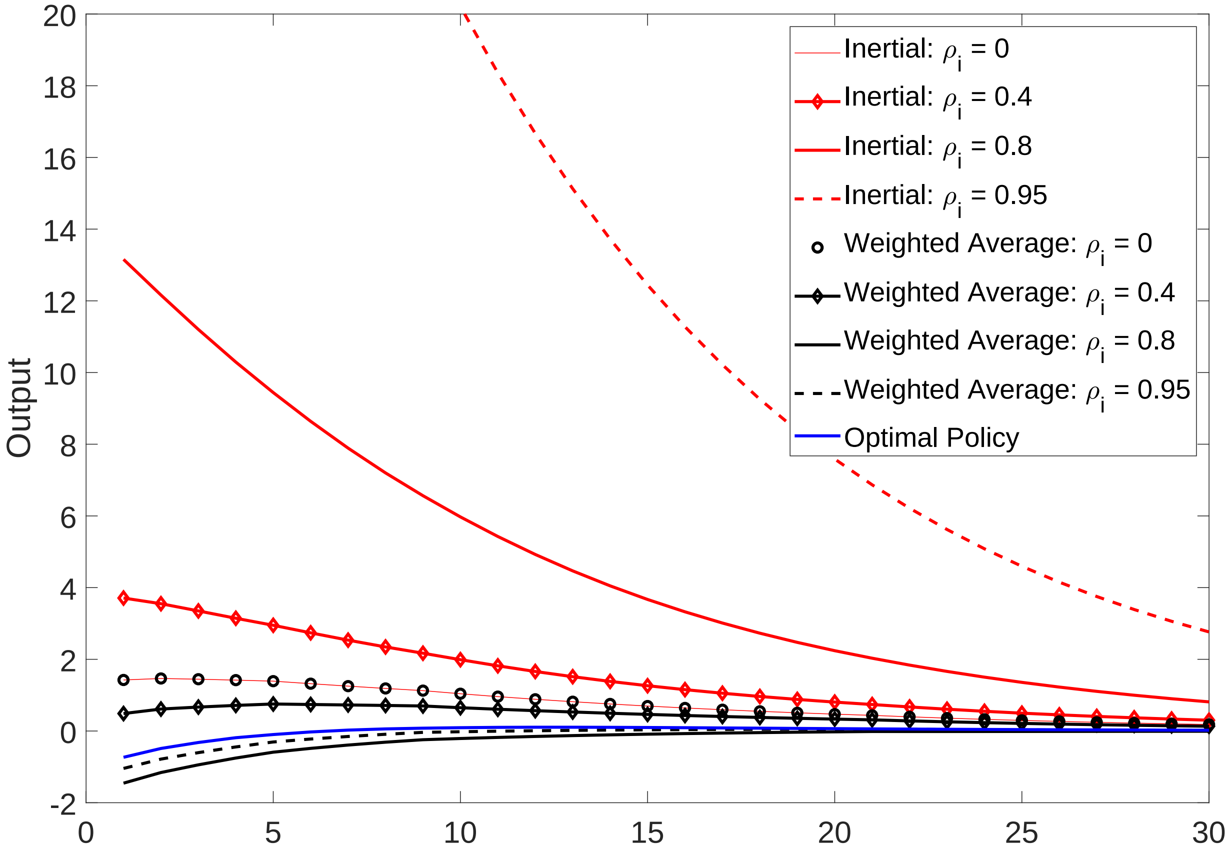
Figure 3: Standard rules versus optimal rules without the ZLB




*Notes:* Each line corresponds to a different realization of the Markov shock process. The outcomes for a shock lasting four quarters are highlighted to make comparisons across specifications easier. Inflation and interest rates are expressed in annual terms.

Figure 4 further illustrates the role of that history dependence plays. Here we plot the time zero expected paths for output under the inertial and weighted average rules for different values of $\rho_i$. That is we simulate each realization of uncertainty and then weight each realization by the probability it occurs to form the expected path of output at the time the shock and the policy are known. The solid blue line shows the outcomes under optimal policy. The dashed black line that is closest to it corresponds to the weighted average rule with $\rho_i = 0.95$, which is

Figure 4: Approximating optimal policy as a function of history dependence




*Notes:* Each line corresponds to the time zero expectation of the path of output. We simulate the model for many realizations of the shock and then weight the individual outcomes by the probability with which they occur.

close to a price level target. As we decrease $\rho_i$, the approximation of the policy rule to optimal policy is worse and the expected outcomes for output diverge.

The opposite relationship between $\rho_i$ and approximating optimal policy occurs for the inertial rule with a lagged interest rate. Less history dependence - $\rho_i$ closer to zero - generates outcomes closer to optimal policy. When $\rho_i = 0$, the weighted average rule and the inertial rule are the same and we have the simple rule (11). Inertial rules encode the wrong history dependence at the ZLB delivering better outcomes with *less* history dependence.

A further implication of Figure 4 is that the stabilization outcomes that are possible under optimal policy are not due to the forward guidance puzzle. Gibbs and McClung (2023) provide a sufficient condition to rule out the forward guidance puzzle in equilibrium. It is straightforward to numerically verify that optimal policy for these exercises satisfies it under standard calibrations. The power of forward guidance to stabilize the economy under optimal policy does not rely on any NK puzzle. Why this is true is made clear in the next section.

## 3 Resolving New Keynesian Puzzles

We now turn to quantifying how much of the *right* history dependence is required to eliminate the NK puzzles completely, i.e., to resolve the limit puzzles. We do so by adopting the definitions of the NK puzzles put forward by Diba and Loisel (2021) and analyzing the consequences of following a weighted average rule with different values for $\rho_i$.

To accommodate Diba and Loisel (2021) definitions, we modify the NK model defined by equations (5) and (6) in several ways. First, we remove the cost push shock from the model and replace it with a supply shock that represents variation in marginal cost from changes to labor supply ($a_t$). This shock allows us to explore the paradox of toil. Second, we add a government spending shock that may affect both supply and demand in the economy ($g_t$). This shock allows us to explore the fiscal multiplier puzzle. Finally, we assume that all shocks are i.i.d. without exogenous persistence. The NK model is now

$$y_t = E_t y_{t+1} - \sigma^{-1}\left(i_t - E_t\pi_{t+1} - r_t^n\right) + g_t - E_t g_{t+1} \quad (12)$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa\left(y_t - \delta_g g_t - a_t\right). \quad (13)$$

The NK puzzles are defined by way of a thought of experiment. Contemplate the effect of an anticipated shock that is known at time $t = T$ but occurs in time $t = T^* > T$ under the following passive interest rate setting regime for $t = T, \ldots, T^*$: $i_t = \bar{i} + \phi\pi_t$ where $0 \leq \phi < 1$. The proposed policy nests a transitory ZLB episode when $\bar{i} = -\bar{r}$ and $\phi = 0$. Diba and Loisel (2021) define the NK puzzles by studying the properties of dynamic multipliers for a shock as $\Delta_p := T^* - T \to \infty$,

$$\lim_{\Delta_p \to +\infty} \partial z_t / \partial X_{m,t+\Delta_p} = \infty \text{ where } z \in \{\pi, y\} \text{ and } X_m \in \{i^*, g, a\}.$$

The puzzles can thus be defined as follows.

**Definition 1 (forward guidance puzzle)** *When the policy rate is expected to be set passively during the next $\Delta_p > 0$ periods, the response of current inflation and output to an expected* *policy-rate shock* $i^* \neq \bar{i}$, $\Delta_p$ *periods ahead, goes to infinity with* $\Delta_p$ *i.e.*

$$\lim_{\Delta_p \to +\infty} |\partial z_t / \partial i_{t+\Delta_p}| = \infty \text{ where } z \in \{\pi, y\}.$$

**Definition 2 (fiscal multiplier puzzle)** *When the policy rate is expected to be set passively during the next* $\Delta_p > 0$ *periods, the response of current inflation and output to an expected expansionary government spending shock,* $g_t > 0$, $\Delta_p$ *periods ahead, goes to positive infinity with* $\Delta_p$, *i.e.,*

$$\lim_{\Delta_p \to +\infty} |\partial z_t / \partial g_{t+\Delta_p}| = \infty \text{ where } z \in \{\pi, y\}.$$

**Definition 3 (paradox of toil)** *When the policy rate is expected to be set passively during the next* $\Delta_p > 0$ *periods, the response of current output to a positive supply shock,* $a_t > 0$, $\Delta_p$ *periods ahead, is weakly contractionary with* $\Delta_p$, *i.e.,*

$$\partial y_t / \partial a_{t+\Delta_p} \leq 0.$$

**Definition 4 (paradox of flexibility)** *When the policy rate is expected to be set passively during the next* $\Delta_p > 0$ *periods, the response of current inflation and output to an expected shock* $\Delta_p$ *periods ahead goes to positive or negative infinity as* $\kappa$ *goes to infinity, i.e.,*

$$\lim_{\kappa \to +\infty} |\partial z_t / \partial v_{t+\Delta_p}| = \infty \text{ where } z \in \{\pi, y\} \text{ and } v = \{i^*, g, a\}.$$

In keeping with the literature, we defined the puzzles with respect to a transitory passive monetary policy regime (such as a zero interest peg expected to last $\Delta_p$ periods). We can only assess whether the puzzles emerge once we characterize how policy is conducted *after* the passive monetary policy regime ends.

### 3.1 The Puzzles under Standard Inertial Taylor Rule

Now we can determine whether the puzzles emerge under the standard Taylor-type rule. For brevity, we ignore inertia ($\rho = 0$). Combining the transitory passive interest rate regime with a policy of active interest rate setting for $t > T^*$, we have:

$$i_t = \begin{cases} \bar{i} + \phi\pi_t & \text{for } t = T, T+1, ..., T^* \\ \bar{i} + \phi^*\pi_t & \text{for } t > T^*, \end{cases} \tag{14}$$

where $0 \leq \phi < 1$ and $\phi^* > 1$.

We can derive a closed-form solution for the equilibrium effects of the shocks described immediately above by using the Phillips curve to eliminate $y_t$ in the IS curve. When $t < T^*$, the model may be expressed as

$$\left(\beta L^{-2} - (\beta + 1 + \frac{\kappa}{\sigma})L^{-1} + \left(1 + \frac{\kappa\phi}{\sigma}\right)\right)\pi_t = X_t$$

$$X_t \equiv -\frac{\kappa}{\sigma}(\bar{i} - r_t^n) - \kappa(a_t - E_t a_{t+1}) + \kappa(1 - \delta_g)\left(g_t - E_t g_{t+1}\right), \tag{15}$$

where $L$ is the lag operator. Factoring the lag polynomial we have

$$(L^{-1} - \lambda_1)(L^{-1} - \lambda_2)\pi_t = X_t$$

where the eigenvalues for the economically relevant parameters satisfy $0 \leq \lambda_1 < 1 < \lambda_2$. Using the method of partial fractions, we can write this as

$$\pi_t = \frac{1}{\lambda_2 - \lambda_1} E_t \left[\frac{\lambda_1^{-1}}{1 - (\lambda_1 L)^{-1}} - \frac{\lambda_2^{-1}}{1 - (\lambda_2 L)^{-1}}\right] X_t.$$

Finally, under the assumed monetary policy and shocks processes it follows that $E_t\pi_{T+T^*+j} = 0$ for all $j > 0$, which provides the necessary limit conditions to construct an unique rational expectation equilibrium by solving the model forward in time:

$$\begin{aligned}
\pi_t &= \frac{1}{\lambda_2 - \lambda_1} E_t \left[\sum_{T=t}^{T^*} \left(\left(\frac{1}{\lambda_1}\right)^{T-t+1} - \left(\frac{1}{\lambda_2}\right)^{T-t+1}\right) X_T\right] \\
y_t &= \frac{1}{\kappa(\lambda_2 - \lambda_1)} E_t \left[\sum_{T=t+1}^{T^*} \left((1 - \beta\lambda_1)\left(\frac{1}{\lambda_1}\right)^{T-t+1} - (1 - \beta\lambda_2)\left(\frac{1}{\lambda_2}\right)^{T-t+1}\right) X_T\right] \\
&+ \frac{\lambda_1^{-1} - \lambda_2^{-1}}{\kappa\left(\lambda_2 - \lambda_1\right)} X_t + \delta_g g_t + a_t
\end{aligned}$$

A well-known result is apparent from the above expression for $\pi_t, y_t$: the New Keynesian model combined with the standard Taylor-type rule *after* the transitory passive policy regime ends is susceptible to all four New Keynesian puzzles. Three of the four puzzles are direct consequences of the properties of $\lambda_1$. The forward guidance puzzle and the fiscal multiplier puzzles are caused by the fact that $\lambda_1 < 1$, which makes $\lambda_1^{-\Delta_p}$ grow without bound as $\Delta_p$ increases. The paradox of flexibility occurs because $\lambda_1 \to 0$ as $\kappa \to \infty$. The remaining puzzle occurs because the properties of $X_t$ with a negative sign always appearing in front of the anticipated supply shocks in equilibrium.

### 3.2 The Puzzles under weighted average policy rule

We now show that replacing the standard Taylor rule with a weighted average rule *after* the transitory passive interest rate regime ends ($t > T^*$) resolves the puzzles. Consider the NK puzzle thought experiment under the following monetary policy

$$i_t = \begin{cases} \bar{i} + \phi\pi_t & \text{for } t = T, T+1, ..., T^* \\ \bar{i} + \phi^*\omega_t & \text{for } t > T^*, \end{cases} \tag{16}$$

$$\omega_t^\pi = \begin{cases} \rho\omega_{t-1} + \pi_t & \text{for } t = T, T+1, ..., T^* \\ \rho^*\omega_{t-1} + \pi_t & \text{for } t > T^*. \end{cases} \tag{17}$$

The idea is that agents understand that past inflation outcomes matter for policy in the future even if those past outcomes do not currently matter for the setting of the policy rate. As before, the proposed policy nests the ZLB experiment when $\bar{i} = -\bar{r}$ and $\phi = 0$. The parameter $\rho$ captures the degree to which policy responds to the outcomes in the first regime when in the second regime. It can approximate more-than-make-up policy required by optimal commitment at the ZLB by setting $\rho > 1$. For brevity, we assume the economy is in steady state at $t = T - 1$ ($\omega_{T-1} = 0$).

The question of interest is how large must $\rho$ be to remove the puzzles. It turns out the sufficient history dependence ameliorates the paradoxes.

**Proposition 2:** *The NK model (12)-(17) with $\phi^* > 1$, $\phi = 0$, and $0 < \rho^* < 1$ has the following properties*

1. *The equilibrium does not exhibit the forward guidance puzzle or fiscal multiplier puzzle if*

*$\rho > 1$.*

2. *The equilibrium exhibits the forward guidance puzzle for $0 \leq \rho < 1$, and the fiscal multiplier for $0 \leq \rho < 1$ if $\rho \neq \bar{\rho}$ where $\bar{\rho}$ is defined in the appendix.*

3. *The magnitude of the inflation/output response to anticipated monetary policy shocks is decreasing in $\rho$.*

4. *The equilibrium does not exhibit the paradox of toil if $\rho$ is sufficiently large.*

5. *The equilibrium exhibits the paradox of flexibility if and only if $\rho = 0$.*

Proposition 2 shows that the forward guidance and fiscal multiplier puzzles cannot be completely eliminated by a weighted average rule per se. It requires make-up policy as prescribed by optimal commitment policy (i.e., $\rho > 1$, see assertions 1 and 2). Additionally, it is shown in the proof of Proposition 2, that for large $\Delta_p$, the effect of an anticipated rate cut is increasing in magnitude in the horizon, $\Delta_p$, precisely when $\rho < 1$, but decreasing if $\rho > 1$. As such, $\rho > 1$ resolves the "anti-horizon" effect of forward guidance emphasized in Farhi and Werning (2019) and many others. Importantly, though, the size of the multiplier is decreasing in $\rho$ (assertion 3), which is why the forward guidance puzzle is so diminished in our quantitative example in Figure 2 when $\rho$ was large but less than one. Finally, a lack of history-dependence contributes to the paradoxes of toil and flexibility (assertions 4 and 5, respectively).

For brevity, Proposition 2 abstracts from the price-level targeting case: $\rho = 1$. Under $\rho = \rho^* = 1$, the central banks aims for price level stationarity, consistent with a Wicksellian price-level targeting rule. In the proof of Proposition 2, we show that the $\rho = 1$ case only ensures a "partial" resolution of the puzzle: $0 < \lim_{\Delta_p \to +\infty} |\partial z_t / \partial i_{t+\Delta_p}| < \infty$ where $z \in \{\pi, x\}$. That is, a monetary policy shock expected to occur in the infinite future has non-zero effects on inflation and output at the time of announcement (and therefore a sequence of anticipated shocks can have unbounded effects as the horizon is pushed to infinity). This shows that a standard price-level targeting policy tames ZLB puzzles, but lacks the proper history-dependence to fully resolve them.

### 3.3 Plausible Predictions vs. Plausible Policy

The forward guidance puzzle has called into question the plausibility of the RANK model. Our results may appear to absolve the New Keynesian model, by instead questioning the plausibility of various policy rules or even the relevance of the puzzle itself. A few points on this topic warrant some attention.

First, we emphasize that history dependence in monetary policy *after* a period of zero interest rates can ameliorate puzzles, even causing them to fully disappear. We show that the form of history dependence required is an implication of the same optimal policy framework that motivated zero interest rate forward guidance in the first place. As such, we view a history-dependent policy rule as the *plausible policy rule* to pair with an announcement to peg the interest rate. The Taylor rules most commonly assumed in ZLB analysis cannot be reconciled with a commitment to peg interest rates at zero. Viewed through this lens, the real puzzle is why a central bank should be fully expected to commit to zero interest rates *and* also be expected to commit to a suboptimal monetary policy rule immediately thereafter.

Additionally, we might perceive the forward guidance puzzle as a *plausible prediction* under the standard Taylor-type rule. In section 3.1, for example, the central bank promises to peg the interest rate in all periods, and deliver a rate cut $\Delta_p$ periods in the future (the fixed rate is baked into the promise for periods $t < T^*$, and determinacy implies steady state interest rate in expectation for all $t > T^*$). The structural equations of the New Keynesian model clearly support an anti-horizon effect of an announcement of this form. The puzzle represents a *plausible prediction* (in the model) of an *implausible policy rule*. The puzzle reflects strong general equilibrium effects, which are evident also in cases where history-dependence is present (e.g. see Figures 2-4).[10] Therefore, the large literature that has developed to eliminate the puzzles by dampening general equilibrium effects of expectations through bounded rationality or various sources of "discounting," as in Angeletos and Lian (2018), Farhi and Werning (2019), Gabaix (2020), Eusepi et al. (2022), Evans et al. (2025), among others, still has an important role to play in explaining economic dynamics and in the design of policy.

Finally, comparing Figures 2-4 to Proposition 2, it is apparent that while sufficient history

[10] For example, we showed that the optimal policy recommendation of Eggertsson and Woodford (2003), under which a central bank can almost perfectly stabilize the economy at the ZLB for a large shock, does not rely on, and is not an example of, the forward guidance puzzle.

dependence solves the quantitative puzzles, resolving the limit puzzles requires policy to commit to some price-level overshooting. To see that $\rho > 1$ in the rule (16)-(17) captures the price level overshooting property of optimal commitment, let's consider a ZLB episode that lasts from time-$t$ until time-$t + \Delta_p$. Figure 5 depicts how interest rates respond in the period of lift-off, $i_{t+\Delta_p+1}$, to lags of inflation during the ZLB episode under the rules we consider in this paper. Under the weighted average rule (16)-(17) with $\rho < 1$ there is history-dependence ($D_j i_{t+\Delta_p+1} := \partial i_{t+\Delta_p+1} / \partial \pi_{t+\Delta_p+1-j} > 0$), but only in the case $\rho > 1$ do we observe a stronger response of monetary policy to distant lags of inflation relative to current inflation, consistent with the interest rate rule that implements optimal commitment policy following a liquidity trap.

This strong response of post-ZLB policy to relatively long lags of inflation in the $\rho > 1$ case captures the commitment to raise the price level through especially accommodative policy responses to past ZLB misses of the target. Proposition 2 shows that $\rho > 1$ is necessary to resolve the qualitative limit puzzles from the literature. Thus, monetary policy must approximate this overshooting feature of the optimal commitment policy to completely resolve the New Keynesian paradoxes. Of course, optimal commitment may not be the best description of reality,[11] but our analysis shows that we only need a weaker form of history-dependence ($\rho < 1$) to rule out the quantitative puzzles. Under the ZLB-augmented Taylor rule (2), there is zero history-dependence: $D_j i_{t+\Delta_p+1} = 0$ for $j = 1, \ldots, \Delta_p + 1$. The level of the interest rate after liftoff is not affected by whether the economy witnessed -500% or 0% or +500% inflation during the liquidity trap. This utter and stark lack of history dependence stretches the limits of plausibility and gives rise to the puzzles.

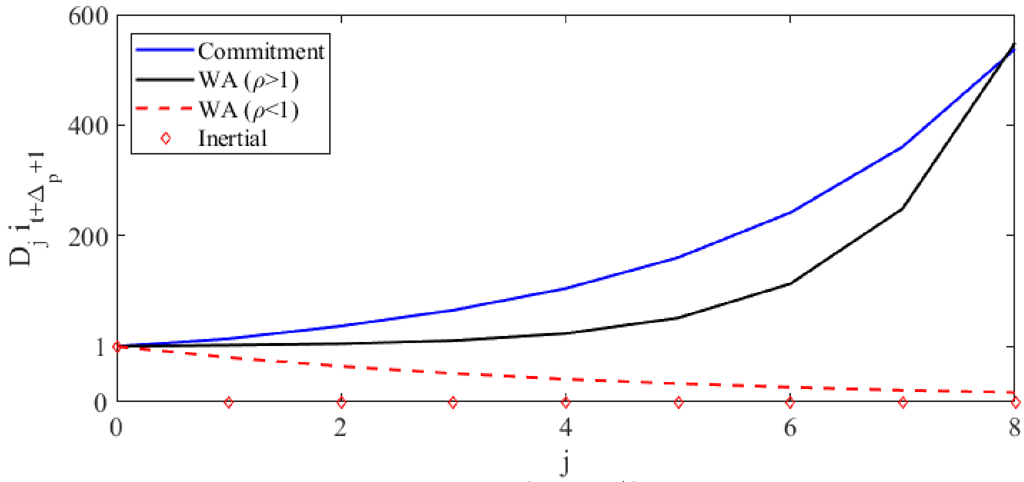
## 4 Quantitative Implications

We return to the model of Smets and Wouters (2007) to investigate the quantitative implications of our finding. We quantify how differently anticipated shocks transmit under an interest rate peg when the policy rule is written in weighted average form. Figures 1 and 2 in the Introduction previewed these results. Shocks transmit differently under the two formulations of policy.

We take the posterior distribution from the original Smets and Wouters (2007) paper as

[11] E.g., see Eggertsson and Egiev (2024).

Figure 5: History dependence after the ZLB




*Notes:* Each line corresponds depicts $D_j i_{t+\Delta_p+1} := \partial i_{t+\Delta_p+1} / \partial \pi_{t+\Delta_p+1-j} = 0$ for rules considered throughout the paper. "WA" denotes the weighted average rule (16)-(17).

given. We do not re-estimate the model. We want to hold everything fixed apart from the form of the policy rule. We convert the monetary policy rule in the model given by

$$i_t = \rho_i i_{t-1} + (1 - \rho_i)(\phi_\pi \pi_t + \phi_x x_t) + \phi_{dx}(x_t - x_{t-1}) + \epsilon_{r,t} \tag{18}$$

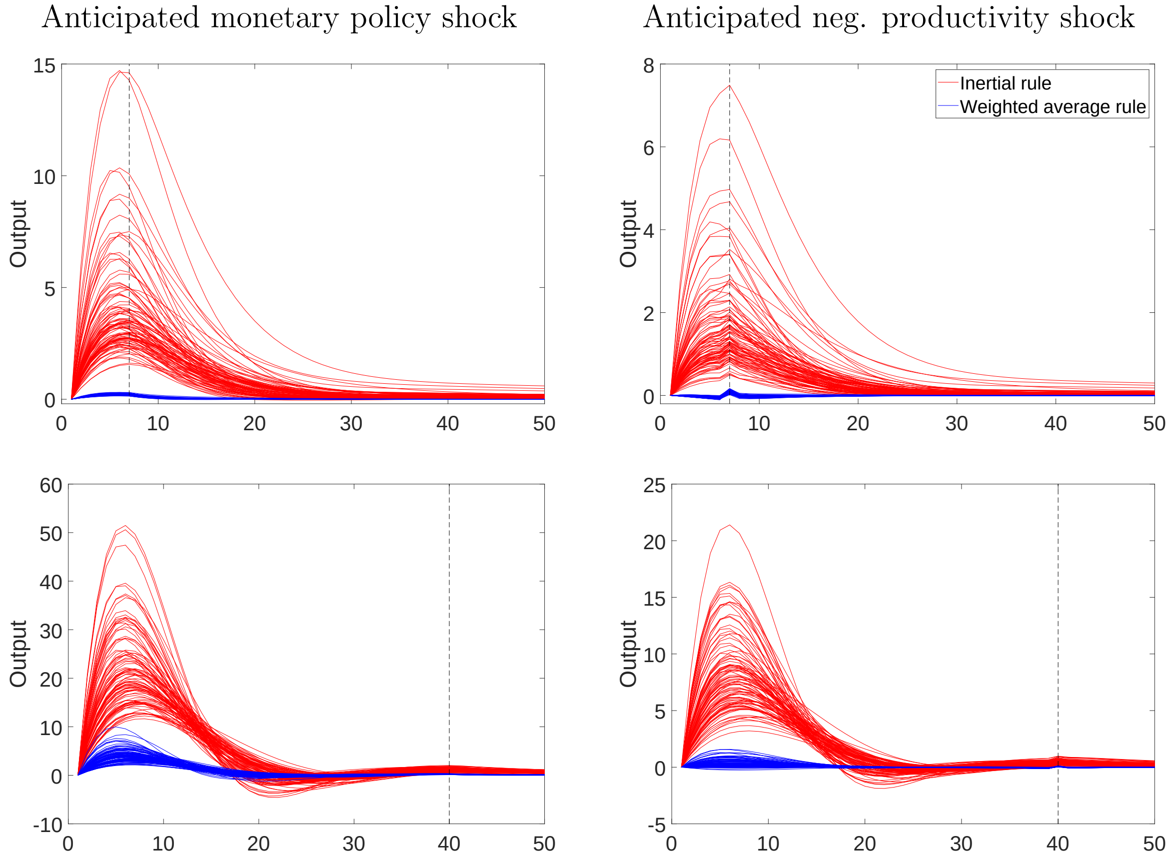
to a weighted average form given by

$$\begin{aligned} i_t &= \phi_\pi \omega_t^\pi + \phi_x \omega_t^x + \phi_{dx} \omega_t^{dx} + \omega_t^{\epsilon_r} \qquad (19) \\ \omega_t^\pi &= \omega_{t-1}^\pi + (1 - \rho_i)(\pi_t - \omega_{t-1}^\pi) \\ \omega_t^x &= \omega_{t-1}^x + (1 - \rho_i)(x_t - \omega_{t-1}^x) \\ \omega_t^{dx} &= \omega_{t-1}^{dx} + \rho_i(x_t - x_{t-1}) \\ \omega_t^{\epsilon_r} &= \omega_{t-1}^{\epsilon_r} + \rho_i \epsilon_{r,t} \end{aligned}$$

where $x_t$ is the output gap. These are the same rule used to create Figures 1 and 2.

Figure 6 plots the impulse response functions for two shocks using draws from the model's parameter posterior distribution. The first shock is an anticipated one-off 100-basis point monetary policy shock that occurs either seven (top) or forty (bottom) quarters in the future under an interest rate peg. The second shock is an anticipated one-off one standard deviation productivity shock that occurs either seven (top) or forty (bottom) quarters in the future under an interest rate peg. After the shocks occur, the policy rule becomes active again and it

Figure 6: An inertial rule vs a weighted average rule in the Smets and Wouters model




*Notes:* The black dashed line indicates when the shock occurs. Each solid line corresponds to the impulse response implied by one draw from the posterior distribution of parameters for a one-time 100-basis point monetary policy shock anticipated to occur seven or forty quarters in the future under an interest rate peg (left) or a one-time one standard deviation negative productivity shock anticipated to occur seven or forty quarters in the future under an interest rate peg (right). In period 8 (41) policy returns to either the inertial rule or the weighted average rule. Neither shock has persistence.

is parameterized according to the posterior draw. We show only the responses of output here for convenience. The other endogenous variables behave similarly to the shocks.

The inertial rule responses shown in red display the forward guidance puzzle, the paradox of toil, and the paradox of flexibility. Evidence of the latter puzzle is the significant variance in the magnitude of shock responses to different draws from the posterior. Much of this variation is explained by changes in wage and price stickiness with different posterior draws.

The weighted average rule responses shown in blue behave very differently compared to the inertial rule responses. The history dependence in the weighted average rule is insufficient to eliminate the puzzles since the posterior estimate of $\rho_i = 0.81$ with and 95% HPD interval of 0.77 to 0.85, consistent with Propositions 1 and 2 in the simple model. This is visible by comparing the top and bottom rows of plots in Figure 6. The response of output to the anticipated shocks increases in both cases when the shock is expected to occur farther into the future. However, the history dependence significantly dampens the effects of the puzzles such that they are much less relevant for policy at business cycle frequencies.

## 5 Conclusion

New Keynesian models of monetary policy are known to generate puzzling predictions for output and inflation in response to anticipated shocks when nominal interest rates are pegged or constrained by the zero lower bound. We show that all New Keynesian puzzles and paradoxes are a result of assuming that monetary policy has either the wrong or no history dependence. Adding an empirically plausible amount of history dependence to monetary policy over inflation and output during ZLB episodes significantly mitigates the New Keynesian puzzles or can eliminate them entirely.

# Appendix For Online Publication

## A1 Weighted average representations

We use lag operators to convert rules with interest rate smoothing to rule that are weighted averages of past output and inflation, where $LX_t = X_{t-1}$ and $L^n X_t = X_{t-n}$ for $n = ..., -2, -1, 0, 1, 2, ...$. Following Sargent (2009), we consider polynomials in the lag operator

$$A(L) = a_0 + a_1 L + a_2 L^2 + ... = \sum_{j=0}^{\infty} a_j L^j$$

where the $a_j$'s are constants and

$$A(L)X_t = (a_0 + a_1 L + a_2 L^2 + ...)X_t = \sum_{j=0}^{\infty} a_j X_{t-j}.$$

We exploit the following relationship

$$A(L) = \frac{1}{1 - \lambda L} = (1 + \lambda L + \lambda^2 L^2 + ...) = \sum_{j=0}^{\infty} \lambda^j,$$

where if $|\lambda| < 1$

$$\sum_{j=0}^{\infty} \lambda^j = \frac{1}{1 - \lambda}.$$

Applying these operations to a Taylor-type rule with interest rate smoothing we can derive equation (1):

$$\begin{aligned}
i_t - \rho_i i_{t-1} &= (1-\rho_i)\bar{r} + (1-\rho_i)(\phi_\pi \pi_t + \phi_y y_t) \\
(1-\rho_i L) i_t &= (1-\rho_i)\bar{r} + (1-\rho_i)(\phi_\pi \pi_t + \phi_y y_t) \\
i_t &= \frac{1}{1-\rho_i L}\left((1-\rho_i)\bar{r} + (1-\rho_i)(\phi_\pi \pi_t + \phi_y y_t)\right) \\
&= (1-\rho_i)\sum_{j=0}^{\infty}\rho_i^j \bar{r} + (1-\rho_i)\left(\sum_{j=0}^{\infty}\rho_i^j\left(\phi_\pi \pi_{t-j} + \phi_y y_{t-j}\right)\right) \\
&= (1-\rho_i)\frac{1}{1-\rho_i}\bar{r} + (1-\rho_i)\left(\sum_{j=0}^{\infty}\rho_i^j\left(\phi_\pi \pi_{t-j} + \phi_y y_{t-j}\right)\right) \\
&= \bar{r} + (1-\rho_i)\sum_{j=0}^{\infty}\rho_i^j\left(\phi_\pi \pi_{t-j} + \phi_y y_{t-j}\right).
\end{aligned}$$

We arrive at the representation of rule with auxiliary variables $\omega_t^\pi$ and $\omega_t^y$ by way of the following calculations:

$$\begin{aligned}
\omega_t^z &= \omega_{t-1}^z + (1-\rho_i)(z_t - \omega_{t-1}^z) \\
\omega_t^z &= \rho_i \omega_{t-1}^z + (1-\rho_i) z_t \\
\omega_t^z - \rho_i \omega_{t-1}^z &= (1-\rho_i) z_t \\
(1-\rho_i L)\omega_t^z &= (1-\rho_i) z_t \\
\omega_t^z &= \frac{(1-\rho_i)}{1-\rho_i L} z_t = (1-\rho_i)\sum_{j=0}^{\infty}\rho_i^j z_{t-j} \qquad \text{(A1)}
\end{aligned}$$

Applying this representation for $z = y, \pi$ yields equation (3).

## A2 Proofs of propositions

**Proposition 1:** The unconditional optimal targeting criteria is

$$y_t = -\frac{\kappa}{\alpha}\frac{\pi_t}{1-\beta L}.$$

Using the IS equation, we can write this as

$$E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1} - r_t^n) = -\frac{\kappa}{\alpha} \frac{\pi_t}{1-\beta L}.$$

Solving for $i_t$ we have

$$i_t = \frac{\sigma\kappa}{\alpha} \frac{\pi_t}{1-\beta L} + \sigma E_t y_{t+1} + E_t \pi_{t+1} + r_t^n.$$

The first representation of the optimal rule comes from using (A1) where

$$\begin{aligned} (1-\beta)\frac{\pi_t}{1-\beta L} &= \omega_t^\pi \\ \omega_t^\pi &= \omega_{t-1}^\pi + (1-\beta)(\pi_t - \omega_{t-1}^\pi). \end{aligned}$$

The second representation of the optimal rule comes from multiplying both sides of the equation by $(1-\beta L)$:

$$\begin{aligned} (1-\beta L) i_t &= (1-\beta L)\left(\frac{\sigma\kappa}{\alpha}\frac{\pi_t}{1-\beta L} + \sigma E_t y_{t+1} + E_t \pi_{t+1} + r_t^n\right) \\ i_t - \beta i_{t-1} &= \frac{\sigma\kappa}{\alpha}\pi_t + (1-\beta L)\left(\sigma E_t y_{t+1} + E_t \pi_{t+1} + r_t^n\right). \end{aligned}$$

**Proposition 2:** Consider first the calculation of the solution in the terminal regime. It is assumed, without loss of generality, that $a_t = g_t = 0$ for $t > T^* + 1$. The unique bounded equilibrium law of motion for $t > T^*$ assumes the form:

$$z_t = b_z^* \omega_{t-1}$$

where $z = \pi, y, \omega$, and

$$b_y^* = (b_y^* + \sigma^{-1} b_\pi^*) b_\omega^* - \phi^* b_\omega^* \quad \text{(A2)}$$

$$b_\pi^* = \kappa b_y^* + \beta b_\pi^* b_\omega^* \quad \text{(A3)}$$

$$b_\omega^* = \rho^* + b_\pi^* \quad \text{(A4)}$$

which can be verified through the method of undetermined coefficients. $b_\omega^*$ is the only real root in $(-1,1)$ of the third-order polymonial:[12]

$$P(b) = b^3 - \frac{\beta\rho^* + \beta + 1 + \kappa\sigma^{-1}}{\beta}b^2 + \frac{\beta\rho^* + \rho^* + 1 + \kappa\rho^*\sigma^{-1} + \kappa\sigma^{-1}\phi}{\beta}b - \frac{\rho^*}{\beta}$$

where $P(b)$ is obtained by substituting (A3) and (A4) into (A2). Because $P(0) < 0 < P(\rho^*) = \frac{\kappa\rho^*\phi^*}{\beta\sigma}$, $b_\omega^* \in (0,\rho^*)$ which implies that $-\rho^* < b_\pi^* < 0$. It follows that $E_t\pi_{T^*+2} = b_\pi^*\omega_{T^*+1} = b_\omega^*\pi_{T^*+1}$. For $t = T,\ldots,T^*$ inflation must satisfying the following second-order difference equation in expected inflation:

$$\begin{aligned} E_t\pi_{t+2} - \gamma_1 E_t\pi_{t+1} + \gamma_0\pi_t = X_t \\ X_t := \frac{\kappa}{\beta}\left((1-\delta_g)(g_t - E_t g_{t+1}) + E_t a_{t+1} - a_t - \sigma^{-1}i_t\right) \end{aligned}$$

which is obtained by combining the Euler equation and Phillips curve, and in which $\gamma_0 = \beta^{-1}$, $\gamma_1 = \beta^{-1} + 1 + \beta^{-1}\kappa\sigma^{-1} > 1$. For $t = T^*$:

$$E_t\pi_{T^*+2} - \gamma_1 E_t\pi_{T^*+1} + \gamma_0 E_t\pi_{T^*} = (b_\omega^* - \gamma_1)E_t\pi_{T^*+1} + \gamma_0\pi_{T^*} = X_{T^*}$$

which implies:

$$\begin{aligned} E_t\pi_{T^*+1} &= \lambda_{T^*+1}\pi_{T^*} - (\gamma_1 - b_\omega^*)^{-1}X_{T^*} = \lambda_{T^*+1}(\pi_{T^*} - \beta X_{T^*}) \\ \lambda_{T^*+1} &:= \frac{\gamma_0}{\gamma_1 - b_\omega^*}. \end{aligned}$$

For $t = T^* - 1$:

$$\begin{aligned} E_t\pi_{T^*+1} - \gamma_1 E_t\pi_{T^*} + \gamma_0\pi_{T^*-1} &= X_{T^*-1} \\ (\lambda_{T^*+1} - \gamma_1)E_t\pi_{T^*} + \gamma_0\pi_{T^*-1} - \beta\lambda_{T^*+1}X_{T^*} &= X_{T^*-1} \end{aligned}$$

[12] A straightforward application of Blanchard-Kahn to the terminal regime model proves that there can only be one real root of $P$ inside the unit circle.

which implies

$$E_t\pi_{T^*} = \lambda_{T^*}\pi_{T^*-1} - \beta\lambda_{T^*}\left(\beta\lambda_{T^*+1}X_{T^*} + X_{T^*-1}\right)$$
$$\lambda_{T^*} = \frac{\gamma_0}{\gamma_1 - \lambda_{T^*+1}}.$$

Proceeding recursively backward in time for $t = T^*-1, \ldots, T$, imposing $X_t = 0$ for $t < T^*-1$, we have:

$$E_t\pi_{t+1} = \lambda_{t+1}E_t\pi_t - \left(\beta\lambda_{T^*+1}X_{T^*} + X_{T^*-1}\right)\beta^{T^*-t}\Pi_{k=0}^{T^*-t-1}\lambda_{T^*-k}$$

where $\lambda_{t+1} = \frac{\gamma_0}{\gamma_1-\lambda_{t+2}}$. Now define $h(\lambda) := \frac{\gamma_0}{\gamma_1-\lambda}$ and $\tilde{\lambda}_j := \lambda_{T^*-t}$. There are two roots of $h(\lambda)$: $0 \le \lambda^Z < 1 < \lambda^U$. Further, $h'(\lambda) = \frac{\gamma_0}{(\gamma_1-\lambda)^2} > 0, h''(\lambda) = \frac{2\gamma_0}{(\gamma_1-\lambda)^3}$ if $\lambda \neq \gamma_1$, where $h''(\lambda) > 0$ if $\lambda < 1$ since $\gamma_1 > 1$. In addition, $h(0) = \gamma_0/\gamma_1 > 0$ and $0 < h(1) = 1/(1+\kappa\sigma^{-1}) < 1$, and $h$ is continuous and twice differentiable on $[0,1]$. Hence, $\tilde{\lambda}_j \in (0,1)\ \forall j$ and $\tilde{\lambda}_j \to \lambda^z$ as $j \to \infty$ given $\tilde{\lambda}_{-1} = b^*_\omega \in [0,1)$. Therefore, $\lim_{j\to\infty}\tilde{\lambda}_j = \lim_{\Delta_p\to\infty}\lambda_{T+1} = \lambda^Z$.

Now, given the terminal solution above and following Gibbs and McClung (2023), it can be shown that the solution for $t = T, \ldots, T^*$ is of the form:

$$z_t = b_{z,t}\omega_{t-1} + d_{z,t}$$

for $z = \pi, y, \omega$. We first construct the sequence: $\{b_{\pi,t}, d_{\pi,t}\}$. For $t = T^*$

$$E_t\pi_{T^*+1} = b^*_\pi\omega_{T^*} = b^*_\pi(\rho\omega_{T^*-1} + \pi_{T^*}) = \lambda_{T^*+1}(\pi_{T^*} - \beta X_{T^*})$$

which implies $b_{\pi,T^*} = \frac{\rho b^*_\pi}{\lambda_{T^*+1}-b^*_\pi}$, $d_{\pi,T^*} = \frac{\beta\lambda_{T^*+1}}{\lambda_{T^*+1}-b^*_\pi}X_{T^*}$. For $t = T^*-1$:

$$E_t\pi_{T^*} = b_{\pi,T^*}\omega_{T^*-1} + d_{\pi,T^*} = b_{\pi,T^*}(\rho\omega_{T^*-2} + \pi_{T^*-1}) + d_{\pi,T^*} = \lambda_{T^*}\pi_{T^*-1} - \beta\lambda_{T^*}(\beta\lambda_{T^*+1}X_{T^*} + X_{T^*-1})$$

which implies $b_{\pi,T^*-1} = \frac{\rho b_{\pi,T^*}}{\lambda_{T^*}-b_{\pi,T^*}}$, $d_{\pi,T^*-1} = \frac{d_{\pi,T^*}+\beta\lambda_{T^*}(\beta\lambda_{T^*+1}X_{T^*}+X_{T^*-1})}{\lambda_{T^*}-b_{\pi,T^*}}$. Proceeding recursively backward in time:

$$b_{\pi,t} = \frac{\rho b_{\pi,t+1}}{\lambda_{t+1} - b_{\pi,t+1}}, d_{\pi,t} = \frac{d_{\pi,t+1} + \left(\beta\lambda_{T^*+1}X_{T^*} + X_{T^*-1}\right)\beta^{T^*-t}\Pi_{k=0}^{T^*-t-1}\lambda_{T^*-k}}{\lambda_{t+1} - b_{\pi,t+1}}.$$

Define $\tilde{b}_j := (b_{\pi,T^*-j})^{-1}$. Given $b^*_\pi < 0$, we have $b_{\pi,t} < 0$ for all $T \le t \le T^*$ and hence $\tilde{b}_j < 0$

for all $j \geq 0$. Therefore

$$\tilde{b}_j = \frac{\tilde{\lambda}_{j-1}}{\rho}\tilde{b}_{j-1} - \rho^{-1}$$

is defined for all $j$, and where $\tilde{\lambda}_j$ is independent of $\tilde{b}_j, \rho$, as shown above. As $j \to \infty$, $\tilde{b}_j \to -\infty$ if $\rho \leq \lambda^Z$, and $\tilde{b}_j \to (\lambda^Z - \rho)^{-1}$ if $\rho > \lambda^Z$. Therefore, $\lim_{\Delta_p \to \infty} b_{\pi,T} = 0$ if $\rho \leq \lambda^Z$, and $\lim_{\Delta_p \to \infty} b_{\pi,T} = \lambda^Z - \rho$ if $\rho > \lambda^Z$.[13] Additionally, we have:

$$b_{\omega,t} = \rho + b_{\pi,t}, b_{y,t} = \kappa^{-1}(1 - \beta\lambda_{t+1})b_{\pi,t}$$

and hence $\lim_{\Delta_p \to \infty} b_{\omega,T} = \lambda^Z$ and $\lim_{\Delta_p \to \infty} b_{y,T} = \kappa^{-1}(1-\beta\lambda^Z)(\lambda^Z - \rho)$ if $\rho > \lambda^Z$ and otherwise, $\lim_{\Delta_p \to \infty} b_{\omega,T} = \rho$ and $\lim_{\Delta_p \to \infty} b_{y,T} = 0$. Finally,

$$\begin{aligned}
d_{\omega,t} &= d_{\pi,t} \\
d_{y,T^*} &= \kappa^{-1}(1 - \beta b_\pi^*)d_{\pi,T^*} + a_{T^*} + \delta_g g_{T^*} \\
d_{y,t} &= \kappa^{-1}(1 - \beta\lambda_{t+1})d_{\pi,t} + (\beta\lambda_{T^*+1}X_{T^*} + X_{T^*-1})\, \beta^{T^*-t+1}\Pi_{k=0}^{T^*-t-1}\lambda_{T^*-k} \text{ for } t = T, \ldots, T^* - 1
\end{aligned}$$

This completes the derivation of the equilibrium law-of-motion of $\pi_t, y_t, \omega_t$. We now consider the effects of individual shocks determining $X_{T^*}, X_{T^*-1}$. In what follows, we will make use of the result: $\partial b_{\pi,t}/\partial\rho < 0$ (i.e., $\partial|b_{\pi,t}|/\partial\rho > 0$). That is:

$$\frac{\partial b_{\pi,T^*}}{\partial\rho} = b_\pi^*(\lambda_{T^*+1} - b_\pi^*)^{-1} < 0$$

and therefore, by induction:

$$\frac{\partial b_{\pi,t}}{\partial\rho} = \rho^{-1}b_{\pi,t+1} + b_{\pi,t+1}^{-1}b_{\pi,t}\lambda_{t+1}(\lambda_{t+1} - b_{\pi,t+1})^{-1}\frac{\partial b_{\pi,t+1}}{\partial\rho} < 0$$

for $t = T, \ldots, T^* - 1$.

[13] There are two real MSV solutions (i.e., solutions of the form: $z_t = b_z\omega_{t-1}$ with $b_z \in \mathbb{R}$) in the case of an interest rate peg. Whether $b_{\pi,T}$ converges to one or the other depends on $\rho$ in relation to $\lambda^Z$.

**Interest Rate Shock:** The dynamic multiplier for a forward guidance shock is:

$$\partial\pi_t/\partial i_{t+\Delta_p} = \partial d_{\pi,t}/\partial i_{t+\Delta_p} = \frac{\partial d_{\pi,t+1}/\partial i_{t+\Delta_p} - \kappa\sigma^{-1}\beta^{\Delta_p}\Pi_{k=-1}^{\Delta_p-1}\lambda_{t+\Delta_p-k}}{\lambda_{t+1} - b_{\pi,t+1}}.$$

given $\partial\pi_{t+\Delta_p}/\partial i_{t+\Delta_p} = \partial d_{\pi,t+\Delta_P}/\partial i_{t+\Delta_p} = -\kappa\sigma^{-1}\lambda_{t+\Delta_p+1}(\lambda_{t+\Delta_p} - b^*_\pi)^{-1} < 0$. It immediately follows that $\partial\pi_{t+k}/\partial i_{t+\Delta_p} < 0$ for $k = 0, \ldots, \Delta_p$, and further:[14]

$$\lim_{\Delta_p\to\infty} \partial\pi_t/\partial i_{t+\Delta_p} = \rho^{-1} \lim_{\Delta_p\to\infty} \partial\pi_{t+1}/\partial i_{t+\Delta_p}$$

if $\rho > \lambda^Z$, and otherwise:

$$\lim_{\Delta_p\to\infty} \partial\pi_t/\partial i_{t+\Delta_p} = (\lambda^Z)^{-1} \lim_{\Delta_p\to\infty} \partial\pi_{t+1}/\partial i_{t+\Delta_p}.$$

Therefore,

$$\lim_{\Delta_p\to\infty} \partial\pi_t/\partial i_{t+\Delta_p} = -\infty$$

if $\rho < 1$,

$$-\infty < \lim_{\Delta_p\to\infty} \partial\pi_t/\partial i_{t+\Delta_p} < 0$$

if $\rho = 1$, and

$$\lim_{\Delta_p\to\infty} \partial\pi_t/\partial i_{t+\Delta_p} = 0$$

if $\rho > 1$. Also,

$$\frac{\partial}{\partial\rho}\left(\frac{\partial d_{\pi,t+\Delta_p}}{\partial i_{t+\Delta_p}}\right) = 0$$

$$\frac{\partial}{\partial\rho}\left(\frac{\partial d_{\pi,t+\Delta_p-1}}{\partial i_{t+\Delta_p}}\right) = \frac{\partial d_{\pi,t+\Delta_p-1}}{\partial i_{t+\Delta_p}}(\lambda_{t+\Delta_p} - b_{\pi,t+\Delta_p})^{-1}\frac{\partial b_{\pi,t+\Delta_p}}{\partial\rho} > 0$$

and hence, by induction:

$$\frac{\partial}{\partial\rho}\left(\frac{\partial d_{\pi,t+k}}{\partial i_{t+\Delta_p}}\right) = (\lambda_{t+k+1} - b_{\pi,t+k+1})^{-1}\left(\frac{\partial d_{\pi,t+k}}{\partial i_{t+\Delta_p}}\frac{\partial b_{\pi,t+k+1}}{\partial\rho} + \frac{\partial}{\partial\rho}\left(\frac{\partial d_{\pi,t+k+1}}{\partial i_{t+\Delta_p}}\right)\right) > 0$$

[14] Note that $\lambda_t$ and $b_{\pi,t}$ do not depend on $d_{\pi,t}$ or $i_{t+\Delta_p}$, which simplifies the expressions for the limits.

Therefore, $|\frac{\partial d_{\pi,t+\Delta_p}}{\partial i_{t+\Delta_p}}|$ is decreasing in $\rho$ (i.e., the magnitude of the inflation response to a forward guidance shock is decreasing in $\rho$). The output response is given by:

$$\begin{aligned} \partial y_{t+\Delta_p}/\partial i_{t+\Delta_p} &= \partial d_{y,t+\Delta_p}/\partial i_{t+\Delta_p} = \kappa^{-1}(1-\beta b^*_\omega)(\partial d_{\pi,t+\Delta_p}/\partial i_{t+\Delta_p}) \\ \partial y_t/\partial i_{t+\Delta_p} = \partial d_{y,t}/\partial i_{t+\Delta_p} &= \kappa^{-1}(1-\beta\lambda_{t+1})(\partial d_{\pi,t}/\partial i_{t+\Delta_p}) - \sigma^{-1}\beta^{\Delta_p+1}\Pi_{k=0}^{\Delta_p-1}\lambda_{t+\Delta_p-k} < 0. \end{aligned}$$

which implies that $\lim_{\Delta_p\to\infty}\partial y_t/\partial i_{t+\Delta_p}$ exists if and only if $\lim_{\Delta_p\to\infty}\partial\pi_t/\partial i_{t+\Delta_p}$ exists, and $\frac{\partial}{\partial\rho}\left(\frac{\partial d_{y,t+k}}{\partial i_{t+\Delta_p}}\right) > 0$ (i.e, $\frac{\partial}{\partial\rho}\left(\left|\frac{\partial d_{y,t+k}}{\partial i_{t+\Delta_p}}\right|\right) < 0$) for $k = 0,\ldots,\Delta_p-1$. If $\rho > 1$, then $\lim_{\Delta_p\to\infty}\partial y_t/\partial i_{t+\Delta_p} = 0$.

**Fiscal Multiplier:** The dynamic multiplier for a fiscal shock is:

$$\partial\pi_t/\partial g_{t+\Delta_p} = \partial d_{\pi,t}/\partial g_{t+\Delta_p} = \frac{\partial d_{\pi,t+1}/\partial g_{t+\Delta_p} - \kappa(1-\delta_g)(1-\beta\lambda_{t+\Delta_p+1})\beta^{\Delta_p-1}\Pi_{k=0}^{\Delta_p-1}\lambda_{t+\Delta_p-k}}{\lambda_{t+1}-b_{\pi,t+1}}.$$

given $\partial\pi_{t+\Delta_p}/\partial g_{t+\Delta_p} = \partial d_{\pi,t+\Delta_P}/\partial g_{t+\Delta_p} = \kappa(1-\delta_g)\lambda_{t+\Delta_p+1}(\lambda_{t+\Delta_p}-b^*_\pi)^{-1} > 0$. Thus,

$$\lim_{\Delta_p\to\infty}\partial\pi_t/\partial g_{t+\Delta_p} = \rho^{-1}\lim_{\Delta_p\to\infty}\partial\pi_{t+1}/\partial g_{t+\Delta_p}$$

if $\rho > \lambda^Z$, and otherwise:

$$\lim_{\Delta_p\to\infty}\partial\pi_t/\partial g_{t+\Delta_p} = (\lambda^Z)^{-1}\lim_{\Delta_p\to\infty}\partial\pi_{t+1}/\partial g_{t+\Delta_p}.$$

Therefore,

$$\lim_{\Delta_p\to\infty}\partial\pi_t/\partial g_{t+\Delta_p} = 0$$

if $\rho > 1$,

$$0 < \lim_{\Delta_p\to\infty}|\partial\pi_t/\partial g_{t+\Delta_p}| < \infty$$

if $\rho = 1$.

For the case: $\rho < 1$, we note that if $\partial d_{\pi,t+j}/\partial g_{t+\Delta_p} < 0$ for some $j\in[1,\ldots,\Delta_p]$ then $\partial d_{\pi,t+k}/\partial g_{t+\Delta_p} < 0$ for all $0\le k < j$. Hence, if $\partial d_{\pi,t+j}/\partial g_{t+\Delta_p}\ge 0$ for some $j\in[1,\ldots,\Delta_p]$, then $\partial d_{\pi,t+k}/\partial g_{t+\Delta_p} \geq 0$ for all $j \leq k \leq \Delta_p$. Thus,

$$\lim_{\Delta_p \to \infty} \partial \pi_t / \partial g_{t+\Delta_p} = -\infty$$

if $\rho < 1$ and $\partial d_{\pi,t+j}/\partial g_{t+\Delta_p} < 0$ for some $j \in [1, \ldots, \Delta_p]$. If, instead, $\partial d_{\pi,t+j}/\partial g_{t+\Delta_p} \geq 0$ for all $j \in [1, \ldots, \Delta_p]$ then

$$\lim_{\Delta_p \to \infty} \frac{\partial}{\partial \rho} \left( \frac{\partial d_{\pi,t}}{\partial g_{t+\Delta_p}} \right) = \sum_{k=1}^{\infty} \frac{\partial d_{\pi,t+k}}{\partial g_{t+\Delta_p}} \Pi_{i=1}^{k} (\lambda_{t+i} - b_{\pi_{t+i}})^{-1} + \frac{\partial}{\partial \rho} \left( \frac{\partial d_{\pi,T^*}}{\partial g_{T^*}} \right) \Pi_{i=1}^{\infty} (\lambda_{t+i} - b_{\pi_{t+i}})^{-1} < 0$$

which implies existence of $\bar{\rho}$ such that $\lim_{\Delta_p \to \infty} \partial \pi_t / \partial g_{t+\Delta_p} = 0$ if $\rho = \bar{\rho}$. Therefore,

$$\lim_{\Delta_p \to \infty} \partial \pi_t / \partial g_{t+\Delta_p} = \pm\infty$$

if $\rho < 1$ and $\rho \neq \bar{\rho}$.[15] The output response is given by:

$$\begin{aligned} \partial y_{t+\Delta_p} / \partial g_{t+\Delta_p} &= \partial d_{y,t+\Delta_p} / \partial g_{t+\Delta_p} = \kappa^{-1} (1 - \beta b_\omega^*) (\partial d_{\pi,t+\Delta_p} / \partial g_{t+\Delta_p}) + \delta_g \\ \partial y_t / \partial g_{t+\Delta_p} &= \partial d_{y,t} / \partial g_{t+\Delta_p} = \kappa^{-1} (1 - \beta \lambda_{t+1}) \frac{\partial d_{\pi,t}}{\partial g_{t+\Delta_p}} - \beta^{\Delta_p + 1} (1 - \delta_g)(1 - \beta \lambda_{t+\Delta_p+1}) \Pi_{k=0}^{\Delta_p - 1} \lambda_{t+\Delta_p - k} \end{aligned}$$

which implies that $\lim_{\Delta_p \to \infty} \partial y_t / \partial g_{t+\Delta_p}$ exists if and only if $\lim_{\Delta_p \to \infty} \partial \pi_t / \partial g_{t+\Delta_p}$ exists. If $\rho > 1$, then $\lim_{\Delta_p \to \infty} \partial y_t / \partial g_{t+\Delta_p} = 0$.

**Supply shock:** The dynamic multiplier for a supply shock is:

$$\partial \pi_t / \partial a_{t+\Delta_p} = \partial d_{\pi,t} / \partial a_{t+\Delta_p} = \frac{\partial d_{\pi,t+1} / \partial a_{t+\Delta_p} + \kappa (1 - \beta \lambda_{t+\Delta_p+1}) \beta^{\Delta_p - 1} \Pi_{k=0}^{\Delta_p - 1} \lambda_{t+\Delta_p - k}}{\lambda_{t+1} - b_{\pi,t+1}}.$$

given $\partial \pi_{t+\Delta_p} / \partial a_{t+\Delta_p} = \partial d_{\pi,t+\Delta_p} / \partial a_{t+\Delta_p} = -\kappa \lambda_{t+\Delta_p+1} (\lambda_{t+\Delta_p} - b_\pi^*)^{-1} < 0$. The output response for $t = T^*$ is:

$$\frac{\partial y_{t+\Delta_p}}{\partial a_{t+\Delta_p}} = \frac{\partial d_{y,t+\Delta_p}}{\partial a_{t+\Delta_p}} = \kappa^{-1} (1 - \beta b_\omega^*) (\partial d_{\pi,t+\Delta_p} / \partial a_{t+\Delta_p}) + 1 = \frac{b_\pi^* (\beta \lambda_{t+\Delta_p+1} - 1)}{\lambda_{t+\Delta_p+1} - b_\pi^*} > 0$$

[15] It is only necessary to assume $\rho \neq \bar{\rho}$ if $\bar{\rho} \in [0, 1)$.

and for $\Delta_p > 0$ :

$$\partial y_t/\partial a_{t+\Delta_p} = \partial d_{y,t}/\partial a_{t+\Delta_p} = \kappa^{-1}(1-\beta\lambda_{t+1})\frac{\partial d_{\pi,t}}{\partial a_{t+\Delta_p}} + \beta^{\Delta_p+1}(1-\beta\lambda_{t+\Delta_p+1})\Pi_{k=0}^{\Delta_p-1}\lambda_{t+\Delta_p-k}$$

It is apparent that (a) if $\partial d_{\pi,t+j}/\partial a_{t+\Delta_p} \geq 0$ for some $j \in [1,\ldots,\Delta_p]$ then $\partial d_{\pi,t+k}/\partial a_{t+\Delta_p} \geq 0$ for all $0 \leq k < j$; (b) $\partial d_{y,t+j}/\partial a_{t+\Delta_p} > 0$ if $\partial d_{\pi,t+j}/\partial a_{t+\Delta_p} \geq 0$ or $\partial d_{\pi,t+j}/\partial a_{t+\Delta_p} < 0$ but sufficiently small in magnitude; and (c):

$$\frac{\partial}{\partial\rho}\left(\frac{\partial d_{\pi,t+\Delta_p}}{\partial a_{t+\Delta_p}}\right) = 0$$

$$\frac{\partial}{\partial\rho}\left(\frac{\partial d_{\pi,t+\Delta_p-1}}{\partial a_{t+\Delta_p}}\right) = \frac{\partial d_{\pi,t+\Delta_p-1}}{\partial a_{t+\Delta_p}}(\lambda_{t+\Delta_p} - b_{\pi,t+\Delta_p})^{-1}\frac{\partial b_{\pi,t+\Delta_p}}{\partial\rho} > 0$$

if $\frac{\partial d_{\pi,t+\Delta_p-1}}{\partial a_{t+\Delta_p}} < 0$, and hence, by induction:

$$\frac{\partial}{\partial\rho}\left(\frac{\partial d_{\pi,t+k}}{\partial a_{t+\Delta_p}}\right) = (\lambda_{t+k+1} - b_{\pi,t+k+1})^{-1}\left(\frac{\partial d_{\pi,t+k}}{\partial a_{t+\Delta_p}}\frac{\partial b_{\pi,t+k+1}}{\partial\rho} + \frac{\partial}{\partial\rho}\left(\frac{\partial d_{\pi,t+k+1}}{\partial a_{t+\Delta_p}}\right)\right) > 0$$

if $\frac{\partial d_{\pi,t+k}}{\partial a_{t+\Delta_p}} < 0$ (which implies $\frac{\partial}{\partial\rho}\left(\frac{\partial d_{\pi,t+k+1}}{\partial a_{t+\Delta_p}}\right) > 0$). Assertions (a)-(c) imply $\partial y_t/\partial a_{t+\Delta_p} > 0$ if $\rho > 0$ is sufficiently large.

**Paradox of Flexibility:** To assess the paradox of flexibility, we rely on the following results, which follow directly from the derivations above:

$$\lim_{\kappa\to\infty} b_\pi^* = -\rho^*, \lim_{\kappa\to\infty} b_{\pi,t} = -\rho, \lim_{\kappa\to\infty} \lambda_t = \lim_{\kappa\to\infty} \lambda^* = 0$$

for $t = T,\ldots,T^*$. Define $R_t := \kappa^{-1}X_t$, $Q_t := \beta^{T^*-t}(\beta\lambda_{T^*+1}X_{T^*} - X_{T^*-1})\Pi_{k=0}^{T^*-t-1}\lambda_{T^*-k} = \beta\lambda_{t+1}Q_{t+1}$, and $Q_{T^*-1} := \beta\lambda_{T^*}(\beta\lambda_{T^*+1}X_{T^*} + X_{T^*-1})$. We have $\lim_{\kappa\to\infty} Q_{T^*-1} = \sigma\beta R_{T^*-1}$ which implies

$$\lim_{\kappa\to\infty} Q_t = \left(\lim_{\kappa\to\infty} \beta\lambda_{t+1}\right)\left(\lim_{\kappa\to\infty} Q_{t+1}\right) = 0$$

for $t = T, \ldots, T^* - 2$. Finally, we note that when considering the effects of an individual shock to $a$, $g$ or $i$, $R_{T^*} \neq -\rho^* R_{T^*-1}$ given $\rho^* < 1$. For $t = T^*$

$$\begin{aligned}\lim_{\kappa\to\infty} d_{\pi,T^*} &= \lim_{\kappa\to\infty} \frac{\beta\lambda_{T^*+1}\kappa}{\lambda_{T^*+1} - b_\pi^*} R_{T^*} = \lim_{\kappa\to\infty} \left( \frac{\beta^{-1} + b_\pi^*\lambda^* - b_\pi^*(1+\beta^{-1})}{\kappa} - b_\pi^*(\sigma\beta)^{-1} \right)^{-1} R_{T^*} \\ &= (\rho^*)^{-1}\sigma\beta R_{T^*}\end{aligned}$$

Then for $t = T^* - 1$:

$$\lim_{\kappa\to\infty} d_{\pi,T^*-1} = \left((\rho^*)^{-1}\sigma\beta R_{T^*} + \sigma\beta R_{T^*-1}\right) \left(\lim_{\kappa\to\infty} (\lambda_{T^*} - b_{\pi,T^*})^{-1}\right) = \frac{(\rho^*)^{-1}\sigma\beta R_{T^*} + \sigma\beta R_{T^*-1}}{\rho}$$

if $\rho > 0$ and $\lim_{\kappa\to\infty} d_{\pi,T^*-1} = \pm\infty$ if $\rho = 0$. Proceeding recursively for $t = T^* - 2, \ldots, T$:

$$\begin{aligned}\lim_{\kappa\to\infty} d_{\pi,t} &= \lim_{\kappa\to\infty} \left( \frac{d_{\pi,t+1} + Q_t}{\lambda_{t+1} - b_{\pi,t+1}} \right) \\ &= \rho^{t-T^*}\sigma\beta\left((\rho^*)^{-1}R_{T^*} + R_{T^*-1}\right)\end{aligned}$$

if $\rho > 0$ and $\lim_{\kappa\to\infty} d_{\pi,t} = \pm\infty$ if $\rho = 0$. Further, $\lim_{\kappa\to\infty} d_{y,t}$ exists if $\rho > 0$ and does not exist otherwise. We conclude that the paradox of flexibility is resolved if and only if $\rho > 0$.

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