A Method to Characterize Reduced-Form Auctions

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Figure 1: Plato’s Cave

Introduction

observed:

$$Q_{kj}(t_k) \quad = \int_{T_{-k}} q_{kj}(t_k, t_{-k}) d\mu_{-k}(t_{-k} | t_k) \quad \forall k \forall t_k \forall j$$

(bidder k of type t_k to have object j)

Introduction

observed: $?\exists q:$

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interim allocation: ex post allocation

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interim state ex post state

(k, j, t_k) $(t_1, \ldots, t_n) =: t$

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interim state ex post state

(k, j, t_k) $(t_1, \ldots, t_n) =: t$

feasibility of $(q_{kj}(t))_{k,j}$ $\forall t$

e.g., $\sum_k q_{kj}(t) \leq 1$ $\forall$ bidder k

(assignment problems)

Introduction

interim allocation:		ex post allocation
$Q_{kj}(t_k)$		$= \int_{T_{-k}} q_{kj}(t_k, t_{-k}) d\mu_{-k}(t_{-k} \vert t_k) \quad \forall k \forall t_k \forall j$
interim state (k, j, t_k)		ex post state $(t_1, \ldots, t_n) =: t$
$?$	$\Longleftrightarrow$	feasibility of $(q_{kj}(t))_{k,j}$ $\forall t$ $\sum_k q_{kj}(t) \leq 1$ $\forall$ bidder k (assignment problems)

Introduction

interim allocation: ex post allocation

$$Q_{kj}(t_k) = \int_{T_{-k}} q_{kj}(t_k, t_{-k}) d\mu_{-k}(t_{-k}|t_k) \quad \forall k \forall t_k \forall j$$

interim state ex post state

(k, j, t_k) $(t_1, \ldots, t_n) =: t$

$?$ $\Longleftrightarrow$ feasibility of $(q_{kj}(t))_{k,j}$ $\forall t$

What is the set of $(Q_{kj})_{k,j}$ that are the reduced forms of some $(q_{kj})_{k,j}$?

The literature

1. Border (1991)
   Border (2007), Manelli & Vincent (2010), Mierendorff (2011), Cai, Daskalakis & Weinberg (2011), Che, Kim & Mierendorff (2013), Goeree & Kushnir (2022), etc.
2. Majorization: Hart & Reny (2015), Kleiner, Moldovanu & Strack (2021), Kolesnikov, Sandomirskiy & Tsyvinski (2022), etc.
3. Contemporary: Lang & Yang (2022), Valenzuela-Stookey (2023)

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3. Contemporary: Lang & Yang (2022), Valenzuela-Stookey (2023)
4. Frontier
   Multiple objects with combinatorial constraints such as assignment problems with arbitrary numbers of types

This paper

Proposes a method to obtain an exact characterization with multiple objects & arbitrary distributions of types

1. Easy applications:
   (a) Extension of Che et al. to multiple objects
   (b) Counterpart to Lang and Lang
2. Application to assignment problems
   $N \geq 2$ objects, two bidders, arbitrary numbers of types
   (a) Full assignment
   (b) Partial assignment

Notations

1. I_1: set of bidders; I_2: set of objects; $I := I_1 \times I_2$
2. T_{i_1}: set of possible types of bidder i_1
3. $T := \prod_{i_1 \in I_1} T_{i_1}$; generic element $t := (t_{i_1})_{i_1 \in I_1}$; distribution μ
4. Ex post constraint X_t: nonempty compact $\subset \mathscr{R}^I$, $\mathscr{R}$ either $\mathbb{R}$ or $\mathbb{Z}$ generic element $x := (x_i)_{i \in I} := (x_{i_1,i_2})_{(i_1,i_2) \in I_1 \times I_2}$
5. Ex post allocation: $(q_i)_{i \in I}$: $(q_{i_1,i_2}(t))_{(i_1,i_2) \in I} \in \Delta X_t$ μ-a.e. $t \in T$
6. Interim allocation: $(Q_i)_{i \in I}$: $Q_{i_1,i_2} : T_{i_1} \to \mathbb{R}$ ($\forall i = (i_1, i_2) \in I$)
7. $(Q_i)_{i \in I}$ a *reduced form* iff: $\forall i \in I$ $\forall t_{i_1} \in T_{i_1}$,

$$Q_i(t_{i_1}) = \int_{T_{-i_1}} q_i(t_{i_1}, t_{-i_1}) d\mu_{-i_1}(t_{-i_1} | t_{i_1})$$

The interim perspective

1. $\mathscr{Z} := \bigcup_{(i_1,i_2)\in I} \left(\{(i_1,i_2)\} \times T_{i_1}\right)$: the set of interim states
   $S \in \mathscr{Z}$: associated with an interim constraint
2. $\forall S \subseteq \mathscr{Z}\ \forall t := (t_{i_1})_{i_1 \in I_1} \in T$:
   (a) $I(S,t) := \{(i_1,i_2) \in I \mid (i_1,i_2,t_{i_1}) \in S\}$
   the set of bidder-object pairs due to which S is subject to some ex post constraints at ex post state t
   (b) $f(S,t) := \max_{x \in X_t} \sum_{i \in I(S,t)} x_i$
   upper bound of the total quantity that S can get for its members at ex post state t
   (c) $g(S,t) := \min_{x \in X_t} \sum_{i \in I(S,t)} x_i$
   lower bound thereof
3. Upper & lower bounds in expectation $\forall S \subseteq \mathscr{Z}$:
   $\int_T f(S,t)d\mu(t)$, $\int_T g(S,t)d\mu(t)$

The interim constraints

For all $S \subseteq \mathscr{Z}$:

$$\int_T g(S,t)d\mu(t) \leq \sum_{i \in I} \int_T Q_i(t_{i_1})\chi_S(i, t_{i_1}))d\mu(t) \leq \int_T f(S,t)d\mu(t)$$

Examples

1. Single-unit symmetric auction: for all $S \subseteq \mathcal{T}$:

$$0 \leq \int_{\mathcal{T}} Q(\tau)\chi_S(\tau)d\nu(\tau) \leq \frac{1}{|I_1|}\left(1-(1-\nu(S))^{|I_1|}\right)$$

2. Partial assignment: N objects and 2 bidders, then RHS is equal to

$$2 - \prod_{(k,j) \in I_1 \times I_2}(1-\mu_k(S_{kj})) - \prod_{j \in I_2}\left(1-\mu_1(S_{1j})+\mu_1(S_{1j})\prod_{j' \neq j}(1-\mu_2(S_{2j'}))\right)$$

and LHS is equal to zero

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Examples

2. Partial assignment: N objects and 2 bidders, then RHS is equal to

$$2 - \prod_{(k,j)\in I_1\times I_2} (1-\mu_k(S_{kj})) - \prod_{j\in I_2} \left(1 - \mu_1(S_{1j}) + \mu_1(S_{1j}) \prod_{j'\neq j} (1-\mu_2(S_{2j'})) \right)$$

   and LHS is equal to zero

3. Full assignment: Same N by two. RHS is same as above; LHS equals

$$\prod_{j\in I_2} \mu_1(S_{1j}) + \prod_{j\in I_2} \mu_2(S_{2j})$$

The characterization

$$\forall S \subseteq \mathscr{Z} : \int_T g(S,t)d\mu(t) \leq \sum_{i \in I} \int_T Q_i(t_{i_1})\chi_S(i,t_{i_1}))d\mu(t) \leq \int_T f(S,t)d\mu(t) \tag{1}$$

1. Trivial: Ineq. (1) is necessary for Q to be a feasible reduced form
2. Nontrivial: When is Ineq. (1) sufficient?
   I.e., when is $\mathscr{Q}_\text{B} \subseteq \mathscr{Q}$?
   - $\mathscr{Q}_\text{B} :=$ the set of interim allocations Q that satisfy (1)
   - $\mathscr{Q} :=$ the set of feasible reduced forms

The basic idea

1. To characterize feasible reduced forms is to replace ex post feasibility constraints by their interim counterparts
2. To validate the characterization, suffices to show that any allocation on the boundary of $\mathscr{Q}$ is just about to violate some interim constraint
3. Need a method to locate such binding interim constraints
4. An interim constraint corresponds to a set $S \subseteq \mathscr{X}$
5. Theorem 1: Need only to search among the $S \subseteq \mathscr{X}$ that satisfies a universal binding condition

Theorem 1 (universal binding condition)

$\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q} \iff \forall \alpha \in \mathbb{R}^{\mathscr{Z}}\ \exists (p_+, p_-) : 2^{\mathscr{Z}} \to \mathbb{R}^2_+$ and $(q_i^*)_{i \in I}$ such that

$$\forall z \in \mathscr{Z} : \alpha(z) = \sum_{S \subseteq \mathscr{Z}} \left(p_+(S) - p_-(S)\right) \chi_S(z),$$

$$\forall t := (t_{i_1})_{i_1 \in I_1} \in T : (q_i^*(t))_{i \in I} \in \arg\max_{(x_i)_{i \in I} \in X_t} \sum_{i \in I} x_i \alpha(i, t_{i_1})$$

and, for all $S \subseteq \mathscr{Z}$,

$$p_+(S) > 0 \ \Rightarrow\ \forall t \in T \left[f(S,t) = \textstyle\sum_{i \in I} q_i^*(t) \chi_S(i, t_{i_1})\right]$$
$$p_-(S) > 0 \ \Rightarrow\ \forall t \in T \left[g(S,t) = \textstyle\sum_{i \in I} q_i^*(t) \chi_S(i, t_{i_1})\right].$$

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$$\begin{aligned} p_+(S) > 0 \ &\Rightarrow\ \forall t \in T \left[f(S,t) = \textstyle\sum_{i \in I} q^*_i(t) \chi_S(i, t_{i_1}) \right] \\ p_-(S) > 0 \ &\Rightarrow\ \forall t \in T \left[g(S,t) = \textstyle\sum_{i \in I} q^*_i(t) \chi_S(i, t_{i_1}) \right]. \end{aligned}$$

- q^*: boundary point between the feasible set and supporting hyperplane normal to α; "the social planner's solution"

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- $p_+(S)$: shadow price for the expected upper bound $\sum_{t\in T} f(S,t)\mu\{t\}$
- $p_-(S)$: shadow price for the expected lower bound $\sum_{t\in T} g(S,t)\mu\{t\}$

Theorem 1 (universal binding condition)

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$$p_-(S) > 0 \Rightarrow \forall t \in T \left[g(S,t) = \textstyle\sum_{i \in I} q_i^*(t) \chi_S(i, t_{i_1}) \right]$$

- $p_+(S) > 0$ only if the total quantity that S gets for its members is maxed out to the ceiling at any ex post state t
- $p_-(S) > 0$ only if the total quantity that S gets for its members is reduced to the floor at any ex post state t

How to find universally binding sets?

1. A boundary point q^*: a choice function among interim states
2. Lemma 2: If $\succeq_Z$ is a preference relation that rationalizes q^* within $Z \subseteq \mathscr{X}$, then any upper (lower) contour set within Z with respect to $\succeq_Z$ is upward (downward) universally binding
3. Construct multiple $\succeq_Z$ so that the family of upper/lower contour sets covers every interim state

How to find universally binding sets?

1. A boundary point q^*: a choice function among interim states
   (a) E.g., $q^*_{1j}(r,s) = 1$ and $q_{1j}(r,s') = q_{1j'}(r,s') = 0$ for objects $j' \neq j$
   (b) $(1,j,r) \succ (1,j',r)$ when $t_2 = s$, and neither $\succ$ nor $\prec$ when $t_2 = s'$
2. Lemma 2: If $\succeq_Z$ is a preference relation that rationalizes q^* within $Z \subseteq \mathscr{Z}$, then any upper (lower) contour set within Z with respect to $\succeq_Z$ is upward (downward) universally binding
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   (a) $\succeq_Z$ rationalizing $q^*|_Z$: If $\varnothing = U^0 \subsetneq U^1 \subsetneq \cdots \subsetneq U^{n_*-1} \subsetneq U^{n_*} \subseteq Z$ are upper contour sets w.r.t. $\succeq_Z$, $t := (t_{i_1})_{i_1 \in I_1} \in T$ and $n = 1, \ldots, n_*$, then

$$\sum_{(i,t_{i_1}) \in U^n \setminus U^{n-1}} q_i^*(t) = f(U^n, t) - f(U^{n-1}, t);$$

   and symmetrically for lower contour sets
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How to find universally binding sets?

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3. Construct multiple $\succeq_Z$ so that the family of upper/lower contour sets covers every interim state
   (a) $\succeq_Z$ need not be total on $\mathscr{Z}$, nor directly from α
   (b) In assignment problems, two kinds of rivals among interim states:
      i. $\{(k, j, t_k) \in \mathscr{Z} \mid k \in I_1, t_k \in T_k\}$
      ii. $\{(k, j, t_k) \in \mathscr{Z} \mid j \in I_2\}$
   (c) For each set of rivals, derive from q^* a preference relation restricted therein

To verify the existence of the shadow prices

1. I.e., given a collection $\mathscr{S}_+$ ($\mathscr{S}_-$) of upper (lower) contour sets, prove existence of $(p_+, p_-) : \mathscr{S}_+ \times \mathscr{S}_- \to \mathbb{R}^2_+$ such that
$$\forall z \in \mathscr{Z} : \alpha(z) = \sum_{S \in \mathscr{S}_+} p_+(S)\chi_S(z) - \sum_{S \in \mathscr{S}_-} p_-(S)\chi_S(z) \qquad (2)$$
2. I.e., $\exists$ nonnegative solution of $\boldsymbol{p} := [(p_+(S))_{S \in \mathscr{S}_+}, (p_-(S))_{S \in \mathscr{S}_-}]^\intercal$ for
$$[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]\, \boldsymbol{p} = \mathbf{0}$$
   (a) $\mathbf{M}_+$: $|\mathscr{Z}|$-by-$|\mathscr{S}_+|$ matrix, $\mathbf{M}_+(z, S) := \chi_S(z)$
   (b) $\mathbf{M}_-$: $|\mathscr{Z}|$-by-$|\mathscr{S}_-|$ matrix, $\mathbf{M}_-(z, S) := -\chi_S(z)$
   (c) $\boldsymbol{\alpha} := [(\alpha(z))_{z \in \mathscr{Z}}]^\intercal$
3. Lemma 3: There exists $(p_+, p_-) : \mathscr{S}_+ \times \mathscr{S}_- \to \mathbb{R}^2_+$ satisfying (3) if no Gaussian elimination on the matrix $[\mathbf{M}_+, \mathbf{M}_-, -\boldsymbol{\alpha}]$ can produce any nonnegative row whose entry at the $-\boldsymbol{\alpha}$ position is (strictly) positive.

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$$\forall z \in \mathscr{Z} : \alpha(z) = \sum_{S \in \mathscr{S}_+} p_+(S)\chi_S(z) - \sum_{S \in \mathscr{S}_-} p_-(S)\chi_S(z) \tag{3}$$

2. I.e., $\exists$ nonnegative solution of $\boldsymbol{p} := [(p_+(S))_{S\in\mathscr{S}_+}, (p_-(S))_{S\in\mathscr{S}_-}]^\intercal$ for

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   (a) The condition makes it impossible for a Gaussian elimination to produce an equation contradictory to (3)
   (b) Suffices due to Chu et al.'s (2023) hyper-rectangle cover theory

The method

1. For any linear valuation $\alpha \in \mathbb{R}^{\mathscr{Z}}$, find a solution q^* to the social planner's problem
2. Derive from q^* some partial orders $\succeq_Z$ on $\mathscr{Z}$ that partially rationalizes q^*; the upper or lower contour sets with respect to $\succeq_Z$ satisfy the universal binding condition.
3. Prove that (3) has a nonnegative solution for (p_+, p_-) such that p_+ is supported by the upper contour sets, and p_- supported by the lower contour sets

Applications

1. If a constraint structure is paramodular and if $|T|$ is finite, $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ (Theorem 2)
2. If a constraint structure is decomposable, linear and it fully characterizes $\mathrm{cv} X_t$, and if $|T|$ is finite, then $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$ (Theorem 3)
3. In the assignment model with $N \geq 2$ objects and two bidders, and $|T|$ finite, $\mathscr{Q}_{\mathrm{B}} \subseteq \mathscr{Q}$
   (a) Full assignment: each bidder gets exactly one object (Theorem 4)
   (b) Partial assignment: each gets at most one object (Theorem 5)
4. Theorems 2 and 5 are extended to allow for infinite $|T|$

Paramodularity (Theorem 2)

1. The application is easy
2. Paramodularity includes
   (a) Single-unit auctions
   (b) Multiunit auctions (Che et al., 2013)
   (c) Two-player bargaining
   (d) Multiple-object auctions subject to paramodularity
3. Paramodularity guarantees that the social planner's problem is solved by the greedy-generous algorithm wrt the α-values, hence easy

Decomposability (Theorem 3)

1. The application is relatively easy
2. Decomposability and full characterization of $\mathrm{cv}X_t$ are implications of Lang and Yang's (2023) total unimodularity assumption
3. With the linearity assumption, Lang and Yang's characterization reduces to mine; without it, their conclusion is slightly weaker
4. Decomposability decomposes the social planner's revealed preferences into multiple ones, each partitioning the interim states into only three indifference sets, the good, the bad), and the neutral; then it is trivial to construct the upper or lower contour sets and prove existence of shadow prices

Full assignment

1. The social planner's solution q^*
   (a) Pair the first- or second-highest $\alpha(1, j, t_1)$ among $j \in I_2$ with the first- or second-highest $\alpha(2, j, t_2)$ among $j \in I_2$ so that the pair refer to different j
   (b) E.g., q^*: good 1 $\to$ bidder 2, and good 2 $\to$ bidder 1 at (t_1, t_2); good 2 $\to$ bidder 2, and good 3 $\to$ bidder 1 at (t_1, t'_2)

	$(1, t_1)$	$(2, t_2)$	$(1, t_1)$	$(2, t'_2)$
1	-1	3	-1	$1/2$
2	4	0	4	3
3	2	$1/2$	2	0

   (c) α-value differential: $\delta(k, j, t_k) := \alpha(k, j, t_k) - \max_{j' \in I_2 \setminus \{j\}} \alpha(k, j', t_k)$

Full assignment

1. The social planner's solution q^*
2. Revealed preferences
   (a) $\succeq_{k,t_k}$ among those referring to the same bidder-type (k, t_k): ranked by their α-values except the top two contenders, which are $\sim_{k,t_k}$
   (b) $\succ_j$ among those referring to the same object j: ranked by $\delta(k, j, t_k)$
   (c) Upper/lower contour sets

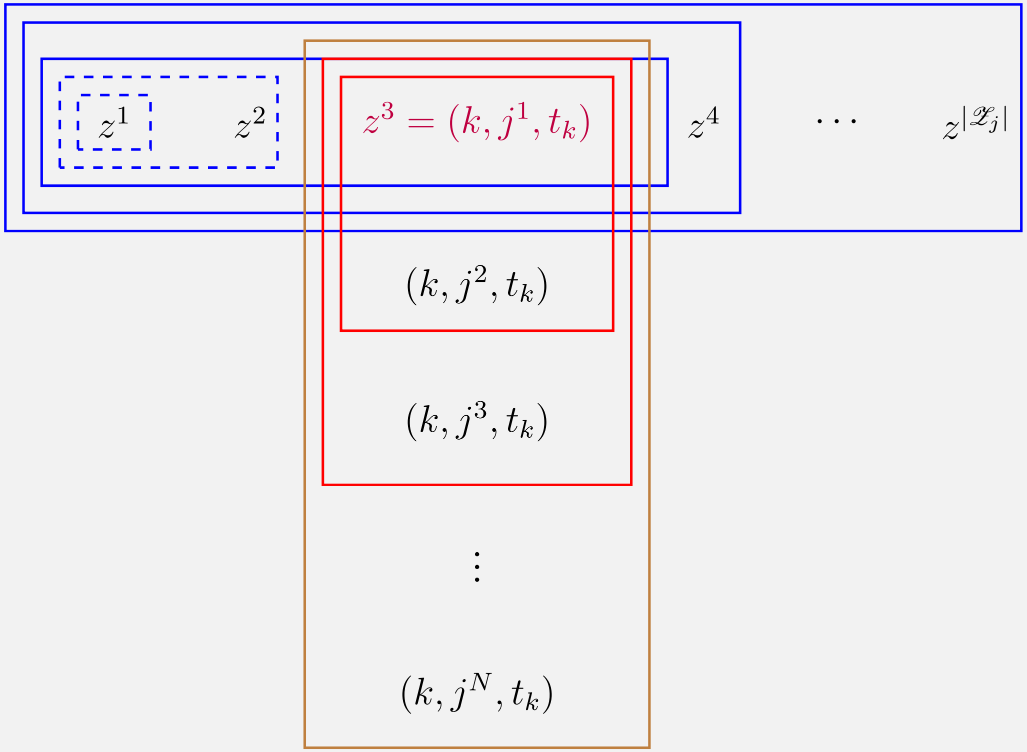

Figure 2: Regarding interim state z^3, top in the "column" and 3rd in the "row"

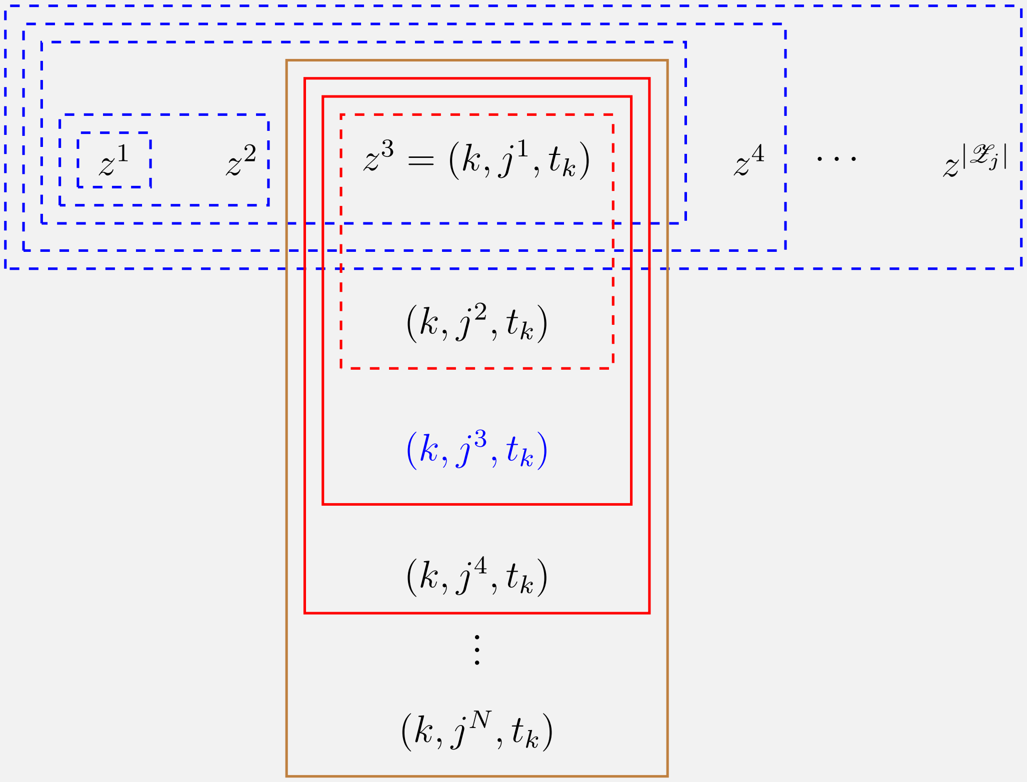

Figure 3: Regarding interim state (k, j^3, t_k), 3rd in the “column”

Full assignment

1. The social planner's solution q^*
2. Revealed preferences
3. Existence of the shadow prices
   (a) $\forall z \in \mathscr{Z}$, $[z] :=$ the row in $[\mathbf{M}_+, \mathbf{M}_-, \boldsymbol{\alpha}]$ corresponding to z
   (b) By Lemma 2, suffices to prove there exist no $Z \subseteq \mathscr{Z}$ and $(\beta_z)_{z \in Z} \in (\mathbb{R} \setminus \{0\})^Z$ for which

$$\sum_{z \in Z} \beta_z [z](S) \ \geq \ 0 \quad \forall S \in \mathscr{S}_+ \sqcup \mathscr{S}_- \quad \text{and}$$
$$\sum_{z \in Z} \beta_z \alpha(z) \ < \ 0.$$

Full assignment

1. The social planner's solution q^*
2. Revealed preferences
3. Existence of the shadow prices

   c. Intuition: "differences of differences" quadruples

      i. Suppose $[z'] - [z] \geqq \mathbf{0}$ and $([z'] - [z])(-\boldsymbol{\alpha}) > 0$

      ii. Then $\alpha(z) > \alpha(z')$, and z and z' refer to the same (k, t_k), and so z is the top contender, and z' the second, for the same (k, t_k)

      iii. But then $([z'] - [z])(U_j^n) = -1$ for some $U_j^n \ni z$; then nonnegavity requires $+[z'_*]$ to $[z'] - [z]$ for some $z'_* \succeq_j z$

      iv. Then $\delta(z'_*) \geq \delta(z)$, so even if the α-value added of z'_* is minimized by subtracting from it its highest rival z_*, the "difference of differences" $[z'_*] - [z_*] - ([z] - [z'])$ still has a nonnegative net value, to the opposite of (4).

Partial assignment

1. The social planner's solution q^*
   (a) Pair the first- or second-highest positive $\alpha(1, j, t_1)$ among $j \in I_2$ with the first- or second-highest positive $\alpha(2, j, t_2)$ among $j \in I_2$ so that the two refer to different j
   (b) E.g., q^*: good 1 to bidder 2, and none t bidder 1 at (t_1, t_2)

	$(1, t_1)$	$(2, t_2)$
1	2	3
2	-4	0
3	-1	$1/2$

   (c) α differential: $\delta(k, j, t_k) := \alpha(k, j, t_k) - \max_{j' \in I_2 \setminus \{j\}} \max\{0, \alpha(k, j', t_k)\}$

Partial assignment

1. The social planner's solution q^*
2. Partial revealed preferences
   (a) $\succeq_{k,t_k}$: similar to that in the full assignment model
   (b) $\succeq_j$: similar to that in the full assignment model except that only those with $\alpha(k, j, t_k) > 0$ need to be ranked
3. Upper/lower contour sets: similar to those in the full assignment model except that the lower contour sets are $\{z\}$ for any $\alpha(z) \leq 0$
4. Existence of shadow prices
   (a) Can exclude any $[z]$ for which $\alpha(z) \leq 0$
   (b) The rest mimics its counterpart in the full assignment model

Conclusion

1. A method to characterize feasible reduced forms
2. Generalization of the mainstream result (paramodularity) and a counterpart to a contemporary result (total unimodularity)
3. New results in assignment models, making the mechanism design method available to a nontrivial set of assignment problems