Optimal Consumption and Investment Decisions with Disastrous Income Risk: Revisiting Rietz’s Rare Disaster Risk Hypothesis

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Motivation

- How should disastrous income risk affect the optimal consumption and investment decisions of individuals?
- Precautionary savings consistent with the permanent income hypothesis
In particular, by specifying Mehra and Prescott’s model to include a low-probability, depression-like third state, I can explain both high equity risk premia and low risk-free returns without abandoning the Arrow-Debreu paradigm (Rietz, 1988)

- We consider a version of the Merton (1969, 1971) model with the special feature that income can abruptly jump from a positive value to a smaller positive value or even to zero.
The currently available social securities and private insurance market are insufficient to perfectly hedge against disastrous income risk (Cocco et al., 2005; Bensoussan et al., 2016; Jang et al., 2019; Jang et al., 2020).

If there is an insurance market for (partially) hedging against disastrous income risk, the individual’s income is partly wiped out when a disastrous income shock occurs.
We shed new light on dynamic models of optimal consumption and investment decisions for individuals who exhibit constant absolute risk aversion (CARA) utility preferences by exploring insights into how possibility of a disastrous income shock combined with a non-negative constraint on borrowing affects both the consumption/savings and wealth allocation decisions between bonds and equity.
Main Findings

- A large precautionary savings motive
- A significant discontinuity and the dramatic change in the concavity of consumption
Main Findings (Cont’d)

- The precautionary savings terms’ role in the risky investment
- Risky assets as a partial hedging tool against disastrous income risk in view of agents’ liquidity
Main Findings (Cont’d)

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(A) consumption

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(B) risky investment

- The role of income recovery after the occurrence of income disaster in optimal decisions
Risk Management

- Significance of the low-probability, high-impact aspect of disastrous income risk
- Large and negative earnings losses are observed at job displacement (Low et al., 2010)
- Such substantial losses have a large impact on household investment and consumption decisions (Guvenen et al. 2015)
- Focus on the extremes of the probability distribution of income, deviating from log-normality substantially
Rare Disaster Risk Hypothesis


- The complete-markets general equilibrium economy v.s. the incomplete-markets partial equilibrium environment.

- Empirical regularities (e.g., the equity premium puzzle, the risk-free rate puzzle) with general equilibrium models v.s. optimal consumption/savings and investment behaviors with disastrous income risk with a partial equilibrium model.
Related Literature

- Cocco et al. (2005): the role of market incompleteness caused by uninsurable labor income risk in individuals’ optimal policies
- Bensoussan et al. (2016): the effects of the risk of forced unemployment on interdependent consumption/savings, portfolio selection retirement decisions
- Wang et al. (2016): the impact of stochastic income on optimal consumption and savings decisions with recursive utility
- Our paper: the relations among state-dependent and stochastically time-varying income disasters, consumption/savings, and portfolio choice
Model Settings

- The CARA utility preference
  \[ U = E \left[ \int_0^\infty e^{-\beta t \left( -\frac{1}{\gamma} e^{-\gamma c_t} \right)} dt \right] \]

- A riskless bond and a risky stock
  \[
  dB_t = rB_t dt, \\
  dS_t = \mu S_t dt + \sigma S_t dW_t
  \]

- The deterministic labor income stream
  \[ d\epsilon_t = \mu^\epsilon \epsilon_t dt, \quad \epsilon_0 = \epsilon > 0 \]
The value function

\[ V(x, \epsilon) \equiv \max_{(c, \pi)} E \left[ \int_0^{\infty} e^{-\beta t} \left( -\frac{1}{\gamma} e^{-\gamma c_t} \right) dt \right], \]

subject to

\[ dX_t = (rX_t - c_t + \epsilon_t) dt + \pi_t \sigma (dW_t + \theta dt), \quad \theta = \frac{\mu - r}{\sigma}, \quad X_0 = x > -\frac{\epsilon}{r^\epsilon}, \]

where

\[ r^\epsilon = r - \mu^\epsilon \]
Hamilton-Jacobi-Bellman (HJB) Equation

- The Hamilton-Jacobi-Bellman (HJB) equation

\[
\max_{(c, \pi)} \left[ -\beta V(x, \epsilon) + (rx - c + \epsilon) V_x(x, \epsilon) + \frac{1}{2} \pi^2 \sigma^2 V_{xx}(x, \epsilon) \right. \\
+ \pi \sigma \theta V_x(x, \epsilon) + \mu \epsilon \epsilon V_{\epsilon}(x, \epsilon) - \frac{1}{\gamma} e^{-\gamma c} \left. \right] = 0
\]

- Solution

\[
V(x, \epsilon) = -\frac{A}{\gamma r} e^{-\gamma r(x + a \epsilon)},
\]

where

\[
A = e^{-\frac{1}{r} \left( \frac{\theta^2}{2} + \beta - r \right)}, \quad a = \frac{1}{r^\epsilon}
\]
Optimal Strategies

- The optimal consumption and investment strategies

\[ c = r \left[ x + \frac{\epsilon}{r^\epsilon} + \frac{\theta^2}{2\gamma r^2} \left( 1 + \frac{2}{\theta^2} (\beta - r) \right) \right] \]

\[ \pi = \frac{\theta}{\gamma \sigma} \frac{1}{r} \]

- The affine structure of the optimal consumption in total wealth

- The *wealth effect* issue in the optimal investment
Three General Models

- **Mode 1**: The basic model with borrowing constraints
- **Model 2**: Model 1 with a one-time-only disastrous income shock
- **Model 3**: Model 2 with state-dependent and time-varying disastrous income risk
Model 1

- Borrowing constraints due to market frictions (e.g., informational asymmetry, agency conflicts, limited enforcement)

- In the presence of borrowing constraints,

  \[ X_t \geq 0 \quad \text{for all} \quad t \geq 0 \]

- In the presence of borrowing constraints, the HJB equation is no longer separable in wealth \( x \) and income \( \epsilon \) due to the wealth effect
Model 1: Convex-Dual Approach

- A modified convex-duality approach of Bensoussan et al. (2016)
- The dual variable and the convex-dual function
  \[ \lambda(x, \epsilon) \equiv V_x(x, \epsilon) \quad G(\lambda(x, \epsilon)) \equiv x + \frac{\epsilon}{r\epsilon} \]

- The dual HJB equation: for \(0 < \lambda < \overline{\lambda}\),
  \[ rG(\lambda) = \frac{1}{2} \theta^2 \lambda^2 G''(\lambda) + (\beta + \theta^2 - r)\lambda G'(\lambda) - \frac{1}{\gamma} \ln \lambda, \]
  subject to
  \[ G(\overline{\lambda}) = \frac{\epsilon}{r\epsilon}, \quad G'(\overline{\lambda}) = 0 \]
Model 1: Solution

Solution: for $0 < \lambda < \bar{\lambda}$,

$$G(\lambda) = -\frac{1}{\gamma r} \ln \lambda - \frac{\theta^2}{2\gamma r^2} \left(1 + \frac{2}{\theta^2} (\beta - r)\right) + B \lambda^{-\alpha^*},$$

where $-1 < \alpha^* < 0$ is the negative root of the following characteristic equation:

$$F(\alpha) \equiv -\frac{1}{2} \theta^2 \alpha (\alpha - 1) + \alpha (\beta - r) + r = 0,$$

$$B = -\frac{\bar{\lambda}^{\alpha^*}}{\gamma r \alpha^*} > 0,$$

$$\bar{\lambda} = \exp \left\{ -\frac{\theta^2}{2r} \left(1 + \frac{2}{\theta^2} (\beta - r)\right) - \frac{1}{\alpha^*} - \gamma \epsilon \right\} > 0$$
Model 1: Optimal Strategies

- The optimal consumption and investment strategies

\[ c = r \left[ x + \frac{\epsilon}{r} + \frac{\theta^2}{2\gamma r^2} \left( 1 + \frac{2}{\theta^2} (\beta - r) \right) - B \lambda^{-\alpha^*} \right], \]

\[ \pi = \frac{\theta}{\gamma\sigma} \left( \frac{1}{r} + \alpha^* B \lambda^{-\alpha^*} \right) \]

- The optimal consumption is no longer affine in total wealth
- Levels of wealth affect stock investments
In the presence of a one-time Poisson shock, the labor income dynamics are: \( \epsilon_0 = \epsilon > 0, \)

\[
d\epsilon_t = \mu \epsilon_t \, dt - (1 - k)\epsilon_t \, dN_t,
\]

where \( k \in [0, 1) \) is the income recovery parameter and \( N_t \) is the one-time Poisson shock with intensity \( \delta > 0 \).

The agent's income plummets from \( \epsilon_t \) to \( k\epsilon_t \) at the time when the disastrous Poisson shock occurs.

The positive income growth rate \( \mu \epsilon \).
Model 2: Role of Insurance

- Without any consideration of a potential role of insurance in the income recovery in the aftermath of the income disaster, the agent’s income would be completely wiped out reducing to nothing, i.e., $k = 0$

- Consider in a very reduced form the role of insurance for hedging the disastrous income shock

- With access to an insurance market to hedge against the income shock, the agent’s income can be partly recovered at the rate of $0 < k < 1$, so that she receives $k \epsilon$ post disaster
Model 2: HJB Equation

- The HJB equation

\[
\max_{(c, \pi)} \left[ - (\beta + \delta) V(x, \epsilon) + (rx - c + \epsilon) V_x(x, \epsilon) + \frac{1}{2} \pi^2 \sigma^2 V_{xx}(x, \epsilon) \right. \\
+ \pi \sigma \theta V_x(x, \epsilon) + \mu \epsilon V_{\epsilon}(x, \epsilon) - \frac{1}{\gamma} e^{-\gamma c} - \delta \frac{A}{\gamma r} e^{-\gamma r(x + k \epsilon / r^\epsilon)} \left. \right] = 0
\]

- The post-disaster value function represented by the very last term on the right-hand side directly affects the pre-disaster value function, thus influencing optimal decisions pre disaster
Model 2: Convex-Duality Approach

- The dual HJB equation: for $0 < \lambda < \bar{\lambda}$,

$$ rG(\lambda) = \frac{1}{2} \theta^2 \lambda^2 G''(\lambda) + \left\{ \beta + \delta \left( 1 - \frac{A}{\lambda} e^{-\gamma r \left( G(\lambda) - \epsilon/r^\epsilon + k \epsilon/r^\epsilon \right)} \right) + \theta^2 - r \right\} \lambda G'(\lambda) - \frac{1}{\gamma} \ln \lambda, $$

subject to

$$ G(\bar{\lambda}) = \frac{\epsilon}{r^\epsilon}, \quad G'(\bar{\lambda}) = 0 $$

- The expected return compensation for the presence of the disastrous income shock and the disastrous income risk premium

$$ \beta + \delta \left( 1 - \frac{A}{\lambda} e^{-\gamma r \left( G(\lambda) - \epsilon/r^\epsilon + k \epsilon/r^\epsilon \right)} \right) + \theta^2 - r $$
Model 2: Optimal Strategies

The optimal consumption and investment strategies

\[ c = r \left[ x + \frac{\epsilon}{r} + \frac{\theta^2}{2\gamma r^2} \left( 1 + \frac{2}{\theta^2} (\beta + \delta - r) \right) - B\lambda^{-\alpha\delta} + PS \right], \]

\[ \pi = \frac{\theta}{\gamma \sigma} \left( \frac{1}{r} + \alpha\delta^* B\lambda^{-\alpha^*\delta} + \alpha\delta PS1 + \alpha^*\delta PS2 - RD \right), \]

where PS represents the precautionary savings driven by the disastrous income shock and it is given by

\[ PS = PS1 + PS2, \]

\[ PS1 = \frac{2\delta(\alpha\delta - 1)}{\theta^2(\alpha\delta - \alpha^*_\delta)} \lambda^{-\alpha\delta} \int_0^\lambda \mu^{\alpha\delta-2} \frac{A}{\gamma r} e^{-\gamma r \left( G(\mu) - \frac{\epsilon}{r^e} + \frac{k\epsilon}{r^e} \right)} > 0, \]

\[ PS2 = \frac{2\delta(\alpha^*_\delta - 1)}{\theta^2(\alpha\delta - \alpha^*_\delta)} \lambda^{-\alpha^*_\delta} \int_\lambda^\lambda \mu^{\alpha^*_\delta-2} \frac{A}{\gamma r} e^{-\gamma r \left( G(\mu) - \frac{\epsilon}{r^e} + \frac{k\epsilon}{r^e} \right)} < 0, \]

and RD represents the risk diversification demand driven by the disastrous income shock and it is given by

\[ RD = \frac{2\delta}{\theta^2 \lambda} \frac{A}{\gamma r} e^{-\gamma r (x + k\epsilon/r^e)} > 0. \]
Thinking about large, negative income shocks as recurring events that repeat over time (e.g., the great depression, the 2008 global financial crisis, the recent COVID-19 pandemic), the income shocks are state dependent disasters that fluctuate in extreme events.

Consider a general Poisson jump process with state-dependent and stochastically time-varying disaster intensity $\delta_t$ (instead of constant intensity $\delta$).

The income dynamics $\epsilon_t$ are then evolved by: $\epsilon_0 = \epsilon > 0$,

$$d\epsilon_t = \mu^\epsilon \epsilon_t^\epsilon dt - (1 - k)\epsilon_t^- dN_t^G,$$

where $N_t^G$ is the Poisson jump process with state-dependent and time-varying intensity $\delta_t$. 
Model 3 (Cont’d)

- State-dependent disastrous income shocks are modeled by a two-state Markov chain: the good state $G$ and the bad state $B$

- For a small time period $(t, t + dt)$, the state switches from the good state $G$ ($B$) to the bad state $B$ ($G$) with probability $\phi^G dt$ ($\phi^B dt$) when the current state is $G$ ($B$), and stays unchanged with the remaining probability $1 - \phi^G dt$ $(1 - \phi^B dt)$

- The intensity dynamics $\delta^i_t$ in the state $i$ are: $\delta^i_0 = \delta^i > 0$,

$$d\delta^i_t = -\delta^i_t dt + b^i \delta^i_t dZ_t,$$

where $b^i$ is the volatility on the intensity growth rate and $Z_t$ is a standard one-dimensional Brownian motion that is correlated with the market factor $W_t$ considered in the stock price dynamics

- The negative intensity growth rate
Conclusion

- The low-probability, depression-like additional state in the agent’s income caused by disastrous income risk significantly affects the agent’s optimal choices.

- Standard precautionary savings argument: consume less and save more.

- The precautionary savings turn out to contribute to an increase in risky investments: the role of partial hedging against disastrous income risk by dynamically trading in the stock market.

- The role of insurance for income recovery post disaster allows the agent to consume more than with no access to insurance.