Flexible Moral Hazard Problems

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ASSA, January 2024
Overview

Classic moral hazard model:

- Effort is either binary, or belongs to an interval.
- Main result: contracts are motivated by informativeness.
- Need strong assumptions for wage to increase in output.
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Classic moral hazard model:

- Effort is either binary, or belongs to an interval.
- Main result: contracts are motivated by informativeness.
- Need strong assumptions for wage to increase in output.

Current paper:

- Allow agent to choose any output distribution.
- Contracts determined by agent’s marginal costs.
- Wages are increasing whenever costs increase in FOSD.
Two Examples
A principal (she) contracts with an agent (he).

- Compact set $X \subset \mathbb{R}$ of possible outputs.
- Principal offers agent a (bounded) contract: $w : X \rightarrow \mathbb{R}$.
- Agent can opt out and get $u_0$.
- If opts in, agent covertly chooses $\alpha \in A \subseteq \Delta(X)$.
- Effort costs: $C : A \rightarrow \mathbb{R}_+$, increasing in FOSD.
- Payoffs:

  Principal: $x - w$  
  Agent: $u(w) - C(\alpha)$.

$u$: strictly increasing, differentiable, unbounded, concave.
Standard Binary Effort Model

\[ X = [L, H], \quad \mathcal{A} = \{\alpha_l, \alpha_h\}. \]
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Suppose principal wants to implement \( \alpha_h \).

Then she offers a contract \( w \) that solves:

\[
\min_{w(\cdot)} \int w(x) \alpha_h(dx) \quad \text{s.t.} \quad \text{(IC) and (IR)}. 
\]
Standard Binary Effort Model

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The FOC from this cost minimization problem is:

\[
\frac{1}{u'(w(x))} = \lambda + \mu \left[ 1 - \frac{f_l(x)}{f_h(x)} \right]
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So: \( w \) is monotone \( \iff \) MLRP holds.
Flexible Binary Output Model

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To implement \( \alpha \in (0, 1) \), FOC is necessary & sufficient:

\[ u \circ w(H) - u \circ w(L) = C'(\alpha) \]
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To implement \( \alpha \in (0, 1) \), FOC is necessary & sufficient:

\[ u \circ w(H) - u \circ w(L) = C'(\alpha) \iff w(H) = u^{-1}(u \circ w(L) + C'(\alpha)). \]
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Implications:

- Cost minimization is trivial: \( \min w(L) \) s.t. IR.
- Shape of contract determined by \( C' \) and \( u \).
- IC contracts are monotone:

\[
w(H) = u^{-1}(u \circ w(L) + C'(\alpha)) \geq u^{-1}(u \circ w(L)) = w(L).
\]
Our Main Model

A principal (she) contracts with an agent (he).

- Compact set $X \subset \mathbb{R}$ of possible outputs.
- Principal offers agent a (bounded) contract: $w : X \to \mathbb{R}_+$. 
- Limited liability: $w(\cdot) \geq 0$. 
- Agent covertly chooses $\alpha \in \mathcal{A} = \Delta(X)$.
- Effort costs: $C : \mathcal{A} \to \mathbb{R}_+$, continuous, increasing in FOSD.
- Payoffs:
  
  Principal: $x - w$  
  Agent: $u(w) - C(\alpha)$, 

  $u$: increasing, continuous, unbounded & $u(0) = 0$. 

ASSUMPTIONS ON THE COST

Without loss:

- $C$ is convex. (if not, replace $\alpha$ with cheapest mixing that averages to $\alpha$)

Assumption. (smoothness)

- $C$ is Gateaux differentiable: every $\alpha$ admits a bounded $k_\alpha : X \to \mathbb{R}$ s.t.

\[
\lim_{\epsilon \to 0} \left[ C(\alpha + \epsilon(\beta - \alpha)) - C(\alpha) \right] = \int k_\alpha(x)(\beta - \alpha)(dx)
\]

for all $\beta \in A$.

- $k_\alpha(x)$: MC of increasing probability of output $x$.

- If $X$ is finite: smooth $\iff$ differentiable, which holds a.e.

- $C$ increases in FOSD $\iff$ $k_\alpha$ increasing $\forall \alpha$. 

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for all $\beta \in \mathcal{A}$. 
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- If $X$ is finite: smooth $\iff$ differentiable, which holds a.e.
- $C$ increases in FOSD $\iff$ $k_\alpha$ increasing $\forall \alpha$. 
**First-Order Approach**

**Lemma.** For a bounded $v : X \to \mathbb{R}$, and $\alpha \in \mathcal{A}$,

$$\alpha \in \arg \max_{\beta \in \mathcal{A}} \left[ \int v(x) \beta \,(dx) - C(\beta) \right]$$

if and only if

$$\alpha \in \arg \max_{\beta \in \mathcal{A}} \left[ \int v(x) \beta \,(dx) - \int k_\alpha(x) \beta \,(dx) \right]$$

(the "only if" direction also works if $C$ is not convex)
Consider the problem:

\[
\max_{a \in [0,1]} [av - c(a)]
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where \(v \in \mathbb{R}\) and \(c\) is convex and differentiable.
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Standard way of writing FOC for optimal \( a^* \in (0,1) \) is

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v - c'(a^*) = 0.
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Standard way of writing FOC for optimal $a^* \in (0, 1)$ is

$$v - c'(a^*) = 0.$$ 

An equivalent way of writing the above condition is:

$$a^* \in \arg\max_{a \in [0,1]} [av - ac'(a^*)].$$

The lemma generalizes the second formulation.
**First-Order Approach**

**Lemma.** For a bounded $\nu : X \to \mathbb{R}$, and $\alpha \in \mathcal{A}$,

$$\alpha \in \text{arg max}_{\beta \in \mathcal{A}} \left[ \int \nu(x) \beta (dx) - C(\beta) \right]$$

if and only if

$$\alpha \in \text{arg max}_{\beta \in \mathcal{A}} \left[ \int \nu(x) \beta (dx) - \int k_\alpha(x) \beta (dx) \right]$$

(the “only if” direction also works if $C$ is not convex)
Say a contract-distribution pair \((w, \alpha)\) is IC if

\[\alpha \in \arg\max_{\beta \in \mathcal{A}} \left[ \int u \circ w(x) \beta(dx) - C(\beta) \right].\]
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\[
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**Proposition.** \((w, \alpha)\) is IC if and only if a \(m \in \mathbb{R}\) exists such that

\[
w(x) \leq u^{-1}(k_\alpha(x) + m)
\]

for all \(x\), and with equality \(\alpha\)-almost surely.
**Proposition.** \((w, \alpha)\) is IC if and only if a \(m \in \mathbb{R}\) exists such that

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**Proof.**
Proposition. \((w, \alpha)\) is IC if and only if a \(m \in \mathbb{R}\) exists such that

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Proof. By Lemma, \((w, \alpha)\) is IC if and only if \(\alpha\) solves

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\max_{\beta \in \Delta X} \int [u \circ w(x) - k_\alpha(x)] \beta(dx),
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**Proposition.** $(w, \alpha)$ is IC if and only if a $m \in \mathbb{R}$ exists such that

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**Proof.** By Lemma, $(w, \alpha)$ is IC if and only if $\alpha$ solves

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or equivalently, the following holds $\alpha$-almost surely:

$$u \circ w(x) - k_\alpha(x) = \sup(u \circ w - k_\alpha)(X)$$
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**Proposition.** \((w, \alpha)\) is IC if and only if a \(m \in \mathbb{R}\) exists such that

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Implications:

(i) wlog: set \(w = w_{m,\alpha}\), optimize \(m\).
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(iii) Cost minimizing \(m\) is:

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m_\alpha^* = -\inf k_\alpha(X).
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**Monotone Contracts**

**Claim.** $C$ is FOSD monotone $\iff w_{m,\alpha}$ is increasing $\forall \alpha, m$. 

**Proof of Claim.** Recall, $w_{m,\alpha}(x) = u - 1(k_\alpha(x) + m)$.

Therefore, $w_{m,\alpha}$ is increasing $\forall \alpha, m$ $\iff k_\alpha$ is increasing $\forall \alpha$ $\iff C$ is FOSD monotone.
**Monotone Contracts**

Claim. \( C \) is FOSD monotone \( \iff \) \( w_{m,\alpha} \) is increasing \( \forall \alpha, m \).

(Fact. \( C \) is FOSD monotone \( \iff \) \( k_\alpha \) is increasing \( \forall \alpha \)).
Claim. $C$ is FOSD monotone $\iff w_{m,\alpha}$ is increasing $\forall \alpha, m$.  

(Fact. $C$ is FOSD monotone $\iff k_\alpha$ is increasing $\forall \alpha$).

Proof of Claim. Recall,  
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**Monotone Contracts**

**Claim.** $C$ is FOSD monotone $\iff w_{m,\alpha}$ is increasing $\forall \alpha, m$.

**Fact.** $C$ is FOSD monotone $\iff k_\alpha$ is increasing $\forall \alpha$.

**Proof of Claim.** Recall,

$$w_{m,\alpha}(x) = u^{-1}(k_\alpha(x) + m).$$

Therefore,

$$w_{m,\alpha} \text{ is increasing } \forall \alpha, m \iff k_\alpha \text{ is increasing } \forall \alpha$$

**Explanation:** because $u^{-1}$ is increasing.
Claim. $C$ is FOSD monotone $\iff w_{m,\alpha}$ is increasing $\forall \alpha, m$.

(Fact. $C$ is FOSD monotone $\iff k_\alpha$ is increasing $\forall \alpha$).

Proof of Claim. Recall,

$$w_{m,\alpha}(x) = u^{-1}(k_\alpha(x) + m).$$

Therefore, $w_{m,\alpha}$ is increasing $\forall \alpha, m \iff k_\alpha$ is increasing $\forall \alpha \iff C$ is FOSD monotone.

Explanation: by the Fact.
Proposition. \( (w, \alpha) \) is IC if and only if a \( m \in \mathbb{R} \) exists such that
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w(x) \leq u^{-1}(k_\alpha(x) + m) =: w_{m,\alpha}(x)
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for all \( x \), and with equality \( \alpha \)-almost surely.

Implications:

(i) Without loss for principal to offer \( w_{m,\alpha} \) for some \( m \).

(ii) A cheapest contract implementing \( \alpha \) is \( w_{m^*,\alpha} \) for
\[
m^*_\alpha = - \inf k_\alpha(X).
\]

(iii) C FOSD increasing \( \implies \) wage is increasing without loss.
Related Literature


• **Flexible Monitoring:** Georgiadis and Szentes (2020), Mahzoon, Shourideh, and Zetlin-Joines (2022), Wong (2023).

Flexible Moral Hazard Problems

We show that in smooth & flexible moral hazard problems:

• Parameters driving contract: $k_\alpha$ and $u$.

• Cost minimization is trivial.

• Every distribution can be implemented.

• FOSD monotonicity $\implies$ wages increase in output.
Flexible Moral Hazard Problems

We show that in smooth & flexible moral hazard problems:

• Parameters driving contract: $k\alpha$ and $u$.

• Cost minimization is trivial.

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In paper, we also have results about principal optimality:

• FOC for the principal (1st order approach is valid).

• Optimality of single, binary, and discrete distributions.
Thanks!