The Linear Algebra of Economic Geography Models

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Motivation

- A recent advance in spatial economics has been the development of quantitative spatial models
  - Rich enough to capture first-order features of the data
  - Tractable to permit an analytical characterization of general equilibrium
  - Parsimonious with a small number of structural parameters to estimate
  - Platform for undertake a wide range of counterfactuals, including productivity shocks and transport infrastructure improvements

- We show that comparative statics for productivity shocks in a constant elasticity economic geography model can be represented using a friend-enemy matrix
  - Stack conditions for first-order general equilibrium effects of productivity shocks in matrix form
  - Invert this matrix system to recover the full bilateral network of each location's exposure to productivity shocks in all locations

Motivation

- Friend-enemy representation has several attractive properties
  - Provides closed-form sufficient statistics for the impact of productivity shocks in terms of observed trade shares and parameters
  - Computationally efficient, allowing comparative statics to be computed almost instantaneously, even for high-dimensional states spaces
  - Simple and intuitive interpretation in terms of economic mechanisms
  - Friend-enemy measures provide theory-consistent measures of locations' exposure to productivity shocks that can be used as inputs in further economic and statistical analysis

Related Literature

- Quantitative models of international trade between countries
  - Eaton and Kortum (2002), Arkolakis, Costinot and Rodriguez-Clare (2012), Allen, Arkolakis and Takahashi (2020), Baqaee and Farhi (2019), Kleinman, Liu and Redding (2020)

- Research on economic geography
  - Krugman (1991), Helpman (1998), Fujita, Krugman and Venables (1999), Redding and Sturm (2008), Allen and Arkolakis (2014), Redding (2016), Monte, Redding and Rossi-Hansberg (2017), Redding and Rossi-Hansberg (2018), Caliendo, Parro, Rossi-Hansberg and Sarte (2018), Adão, Arkolakis and Esposito (2019)

- Abstract from dynamics from migration or capital accumulation
  - Caliendo, Dvorkin and Parro (2019), Kleinman, Liu and Redding (2020), Cai, Caliendo, Parro and Xiang (2023), Crews (2023), Dvorkin (2023)

- Abstract from input-output linkages
  - Caliendo and Parro (2015), Liu (2019), Liu and Tsyvinski (2023)

Model Outline

- The world economy consists of a set of locations indexed by $i, n \in \{1, \ldots, N\}$
- The economy as a whole has an exogenous supply of workers that we normalize to one ($\bar{\ell} = 1$)
- Denote the population share of each location by ℓ_n
- Each worker is endowed with one unit of labor that is supplied inelastically
- Workers are perfectly mobile across locations, but have idiosyncratic preferences for each location
- Goods are differentiated by location of origin (Armington)
- Goods are produced using labor under conditions of perfect competition and constant returns to scale
- Goods can be traded between locations subject to iceberg variable trade costs ($\tau_{ni} \geq 1$)

Preferences and Technology

- Preferences of worker ν who chooses to live in location n depend on common amenities (b_n), idiosyncratic amenities ($\epsilon_n(\nu)$) and wage (w_n)

$$u_n(\nu) = \frac{b_n \epsilon_n(\nu) w_n}{p_n}$$

- Consumption price index

$$p_n = \left[\sum_{i=1}^{N} p_{ni}^{1-\sigma} \right]^{\frac{1}{1-\sigma}}, \qquad \sigma > 1$$

- Idiosyncratic amenities

$$F(\epsilon) = \exp\left(-\epsilon^{-\kappa}\right), \qquad \kappa > 1$$

- Production technology (abstract from agglomeration forces)

$$p_{ni} = \frac{\tau_{ni} w_i}{z_i}$$

- Iceberg variable trade costs $\tau_{ni} \geq 1$

General Equilibrium

- Income in each location equals expenditure on the goods produced by that location

$$w_i \ell_i = \sum_{n=1}^{N} s_{ni} w_n \ell_n$$

$$s_{ni} = \frac{\left(\tau_{ni} w_i / z_i\right)^{-\theta}}{\sum_{m=1}^{N} \left(\tau_{nm} w_m / z_m\right)^{-\theta}}$$

- Probability that a worker chooses to live in location n

$$\ell_n = \frac{\left(b_n w_n / p_n\right)^{\kappa}}{\sum_{h=1}^{N} \left(b_h w_h / p_h\right)^{\kappa}}$$

- Expected utility

$$\bar{u} = \Gamma\left(\frac{\kappa - 1}{\kappa}\right) \left[\sum_{h=1}^{N} \left(b_h w_h / p_h\right)^{\kappa}\right]^{\frac{1}{\kappa}}$$

- Choose total income as the numeraire: $\sum_{i=1}^{N} q_i = \sum_{i=1}^{N} w_i \ell_i = 1$

Comparative Statics

- Consider small productivity shocks, holding constant amenities ($\mathrm{d}\ln b_i = 0$), trade costs ($\mathrm{d}\ln \tau_{ni} = 0$), and total population ($\mathrm{d}\ln \bar{\ell} = 0$)
- Totally differentiate the conditions for general equilibrium and represent them in matrix form
- Goods market clearing condition
  - Market-size effect
  - Cross-substitution effect
  - Population mobility affects market size (differs from trade models)

$$\mathrm{d}\ln \mathbf{w} + \mathrm{d}\ln \boldsymbol{\ell} = \mathbf{T}\left(\mathrm{d}\ln \mathbf{w} + \mathrm{d}\ln \boldsymbol{\ell}\right) + \theta\left(\mathbf{TS} - \mathbf{I}\right)\left(\mathrm{d}\ln \mathbf{w} - \mathrm{d}\ln \mathbf{z}\right)$$

- Population shares
  - Nominal wage
  - Consumption price index

$$\mathrm{d}\ln \boldsymbol{\ell} = \kappa(\mathbf{I} - \mathbf{1}\ell')\left[\mathrm{d}\ln \mathbf{w} - \mathbf{S}\left(\mathrm{d}\ln \mathbf{w} - \mathrm{d}\ln \mathbf{z}\right)\right]$$

- Population mobility equalizes expected utility
  - Population share-weighted average of real wage changes
  - Differs from trade models

$$\mathrm{d}\ln \bar{u} = \boldsymbol{\ell}'\left[\ \mathrm{d}\ln \mathbf{w} - \mathbf{S}\left(\mathrm{d}\ln \mathbf{w} - \mathrm{d}\ln \mathbf{z}\right)\ \right]$$

Comparative Statics

- Goods market clearing condition again

$$d\ln \mathbf{w} + d\ln \boldsymbol{\ell} = \mathbf{T}\left(d\ln \mathbf{w} + d\ln \boldsymbol{\ell}\right) + \theta\left(\mathbf{TS} - \mathbf{I}\right)\left(d\ln \mathbf{w} - d\ln \mathbf{z}\right)$$

- Our choice of numeraire implies

$$\sum_{i=1}^{N} q_i = \sum_{i=1}^{N} w_i \ell_i = 1$$

$$\mathbf{Q}\left(d\ln \mathbf{w} + d\ln \boldsymbol{\ell}\right) = 0$$

- Impose this numeraire in the goods market clearing condition

$$\left(\mathbf{I} + \mathbf{Q}\right)\left(d\ln \mathbf{w} + d\ln \boldsymbol{\ell}\right) = \mathbf{T}\left(d\ln \mathbf{w} + d\ln \boldsymbol{\ell}\right) + \theta\left(\mathbf{TS} - \mathbf{I}\right)\left(d\ln \mathbf{w} - d\ln \mathbf{z}\right)$$

Nominal Wage Exposure

- Elasticity of the nominal wage in each location with respect to productivity shocks in all locations

$$\mathrm{d} \ln \boldsymbol{w} = \mathbf{W} \mathrm{d} \ln \boldsymbol{z}$$

- where $\mathbf{W}$ is our friend-enemy matrix of nominal wage exposure

$$\mathbf{W} \equiv -\left((1+\kappa)\, \mathbf{I} - \kappa \mathbf{1} \boldsymbol{\ell}' - \mathbf{V} \right)^{-1} \mathbf{V},$$

$$\mathbf{V} \equiv \left[\kappa \left(\mathbf{I} - \mathbf{1} \boldsymbol{\ell}' \right) + \left(\mathbf{I} - \mathbf{T} + \mathbf{Q} \right)^{-1} \theta \left(\mathbf{TS} - \mathbf{I} \right) \right]$$

- Presence of $\mathbf{Q}$ ensures that the matrices $(\mathbf{I} - \mathbf{T} + \mathbf{Q})$ and $((1+\kappa)\, \mathbf{I} - \kappa \mathbf{1} \boldsymbol{\ell}' - \mathbf{V})$ are invertible
- Recover entire bilateral network of bilateral nominal wage exposure through matrix inversion
  - Computationally efficient even with high-dimensional state spaces
  - Provides exposure measures as inputs for further economic and statistical analysis

Real Wage Exposure

- Common change in expected utility across all locations

$$\mathrm{d}\ln \bar{u} = \boldsymbol{\ell}' \mathbf{U} \mathrm{d} \ln \boldsymbol{z}$$

- where **U** is our friend-enemy matrix of real wage exposure

$$\mathbf{U} \equiv \left[\left(\mathbf{I} - \mathbf{S} \right) \mathbf{W} + \mathbf{S} \right],$$

Conclusions

- We provide sufficient statistics for nominal and real wage exposure in a constant elasticity economic geography measures
  - Summarize first-order general equilibrium elasticity of nominal and real wages in each location to productivity shocks in all locations
- Readily computed using commonly-available trade data and trade and migration elasticities
- Intuitive interpretation in terms of economic mechanisms
- Compute them for all bilateral pairs of locations through matrix inversion
  - Computationally efficient even with high-dimensional state spaces
- Provide theory-consistent measures of exposure to productivity shocks that can be used for further economic and statistical analysis

Thank You